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Improved Stability and Stabilization Results for Stochastic Synchronization of Continuous-Time Semi-Markovian Jump Neural Networks with Time-varying Delay

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***Abstract*—Continuous-time semi-Markovian jump neural networks (semi-MJNNs) are those MJNNs whose transition states (TRs) are not constant but depend on the random sojourn-time. Addressing stochastic synchronization of semi-MJNNs with time-varying delay, an improved stochastic stability criterion is derived in this paper to guarantee stochastic synchronization of the response systems with the drive systems. This is achieved through constructing a semi-Markovian Lyapunov-Krasovskii functional (LKF) together as well as making use of a novel integral inequality and the characteristics of cumulative distribution functions (CDFs). Then, with a linearization procedure, controller synthesis is carried out for stochastic synchronization of the drive-response systems. The desired state-feedback controller gains can be determined by solving a linear matrix inequality (LMI)-based optimization problem. Simulation studies are carried out to demonstrate the effectiveness and less conservatism of the presented approach.**

***Index Terms*—Semi-Markovian jump neural networks, Stochastic synchronization, Sojourn-time-dependent transition rates, Time-varying delay.**

I. Introduction

Neural networks (NNs) have been comprehensively investigated in recent decades in both mathematics and control communities. Various mathematical models have been presented for NNs, e.g., local field NNs and static NNs. They have been successful applied in a variety of areas, e.g., associative memory [1], pattern recognition [2], image and signal processing [3] and affine invariant matching [4]. NNs usually face the difficulty of keeping long-term dependencies in the input stream. For example, the information latching phenomenon usually occurs in NNs, and can be treated via extracting finite-state representations (e.g., clusters, patterns or modes) from trained networks [5], [6], [7], [8], [9]. Another example is found in pathological states of the brain, i.e., epileptic seizures [10].

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In NNs, transitions from one state to another can be modelled by using a stochastic process [11]. The stochastic process relies on the duration between two successive transitions. This duration is known as *sojourn-time*. It is a random variable following a probability distribution. In some cases, the distribution is an exponential distribution, and thus the transition states (TRs) are constant according to the memoryless characteristic of the exponential distribution. This implies that the switches are only related with the latest state. Such a stochastic process is a Markov process. With the advances in general Markovian jump systems, progress has also been made in Markovan jump NNs (MJNNs) [12], [13], [14], [15].

In practice, however, network mode transitions do not always comply with the memoryless restriction. In other words, the TRs are usually not constant but time-varying. A continuous-time stochastic process with sojourn-time obeying a non-exponential distribution is often addressed as a semi-Markov process [16], [17], [18]. MJNNs whose mode transitions obey a semi-Markov process are referred to as semi-MJNNs. Semi-MJNNs model and describe a broader class of practical stochastic systems. Traditional MJNNs are a special case of semi-MJNNs. Therefore, investigation into the analysis and synthesis of semi-MJNNs has a potential of wide applications. This motivates our research in the present paper.

For NNs, time delays are often present in their dynamics due to the finite transition speed as in amplifiers in electronic NNs and finite signal propagation in biological networks [19], [20], [21], [22], [23], [24], [25]. It is known that time delays may cause degraded system performance, unexpected oscillations and even system instability. Therefore, studies on analysis and synthesis of NNs with time delay have become significant. While theories have been established for maintenance of the stability of NNs with time delay, a further reduction in the conservatism of the stability criteria available from the literature is still a significant issue.

Among various stability analysis methods, the direct Lyapunov function approach is a powerful tool to deal with systems with time delay. It relies on construction of LKFs and employment of some tight techniques for manipulating the time-derivative or difference of the LKFs [26], [27], [28], [29], [30], [31]. A key step is to construct LKFs to involve more useful information on time delay such that the inherent conservatism can be reduced. Several attempts have been pursued with regard to the structure of the functional by extending state-based LKFs [32], [33], discretized Lyapunov functions or discontinuous Lyapunov functions [34], [35]. Then, by virtue of some more or less tight techniques to bound the crossing terms of the derivative or difference of the LKF, tractable stability analysis criteria can be established. As an example, by using the Jensen's inequality, the sampled-data stochastic synchronization problem is solved for MJNNs with time delays [36]; By employing the free-weighting matrix approach, the stability analysis problem for Hopfield NNs with Markovian jumping parameters and time delay has been investigated [6]. by adopting the reciprocally convex inequality, the stochastic stability analysis problem is studies for generalized NNs with Markovian jumping parameters and time-varying delays [15]. Via the Wirtinger-based integral inequality, the global asymptotic stability analysis problem is tackled for NNs with interval time-varying delays [37]. All these examples show that the choice of LKFs and over-bounding techniques inevitably induce some degree of conservatism. Therefore, there is a room to further reduce the conservatism of existing approaches for stochastic synchronization of delayed semi-MJNNs. This also motivates our work in the present paper.

This paper aims to improve the stability and stabilization results for stochastic synchronization of continuous-time semi-MJNNs with time-varying delay. More specifically, by constructing a semi-Markovian LKF, combined with carefully exploring the characteristics of CDFs and the employment of a new integral inequality, an improved stochastic stability analysis criterion will be established for the error systems of the semi-MJNNs. It guarantees that the response systems are stochastically synchronized with the drive systems. Then, by using a linearizition technique, the controller synthesis problem is investigated for stochastic synchronization of the drive-response systems. It will be shown that the desired state-feedback controller gains can be derived from a convex optimization scheme. Simulation results will be conducted to demonstrate the effectiveness and less conservatism of the proposed scheme. In comparison with existing literature, this paper shows two unique features: 1) The considered NNs with semi-Markovian jumping parameters, where the TRs are sojourn-time dependent, can be employed to describe a broader class of practical stochastic systems; 2) By introducing an improved integral inequality, together with Projection lemma and a linearization procedure, new delay-dependent stochastic synchronization conditions are derived with less conservatism for semi-MJNNs with time-varying delay; and 3) a new method is presented for controller gains synthesis from the derived stability conditions.

The paper is organized as follows. Section II presents system modelling. Section III develops improved stability analysis. This is followed by controller synthesis in Section IV. Simulation studies are conducted in Section V. Finally, Section VI concludes the paper.

Notations. $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the sets of non-negative real numbers and non-negative integers, respectively. $\mathbb{S}^\iota$ represents the set of $\iota \times \iota$ real symmetric and positive definite matrices. $\mathtt{Sym}\{A\} := A + A^\top$. $\mathcal{E}[\cdot]$ means mathematical expectation.

II. Model description

To formally define a semi-Markov process, we introduce the following three stochastic processes:

1) Stochastic process $\{r_n\}_{n\in\mathbb{Z}_+}$ takes values in $\mathcal{I} := \{1, 2, \cdots, N\}$, where r_n refers to the index of the system mode at the nth transition;

2) Stochastic process $\{t_n\}_{n\in\mathbb{Z}_+}$ takes values in $\mathbb{R}_+$, where t_n represents the time at the nth transition; $t_0 = 0$, and t_n increases monotonically with n; and

3) Stochastic process $\{h_n\}_{n\in\mathbb{Z}_+}$ takes values in $\mathbb{R}_+$, where $h_n = t_n - t_{n-1}$, $\forall n \in \mathbb{Z}_{\geq 1}$ represents the sojourn-time of mode r_{n-1} between the $(n-1)$th and nth transitions; $h_0 = 0$.

A possible evolution of the stochastic processes is shown in Fig. 1.

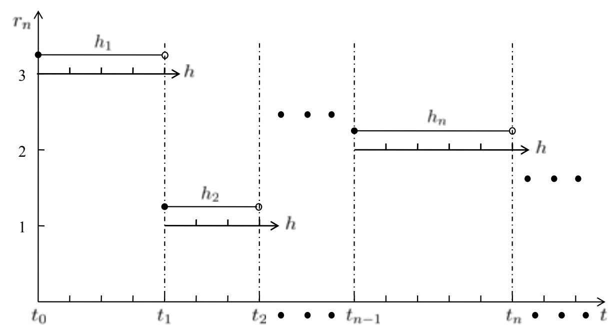

Fig. 1. A possible evolution of stochastic processes r_n, t_n and h_n for $N = 3$.

Consider a stochastic switched system:

$$\Sigma: \ \dot{x}(t) = A(r(t))x(t), \ t_n \leq t < t_{n+1}, \tag{1}$$

where $x(0) = x_0 \in \Re^\iota$ is a constant vector; $A(r(t)) \in \Re^{n\times n}$, $r(t) \in \mathcal{I}$, are real matrices. Also assume that the initial condition $t_0 = 0$ and $r(0)$ is a constant.

Definition 2.1 [16], [18]. We say that the stochastic process $r(t) := r_n$, $t \in [t_n, t_{n+1})$, is a homogeneous semi-Markov process, and Σ is a continuous-time homogeneous semi-MJLS if the following two conditions hold $\forall i, j \in \{1, \cdots, N\}$, $t_0, t_1, \cdots, t_n \geq 0$:

(i) It holds that

$$\begin{aligned} &\Pr(r_{n+1} = j, h_{n+1} \leq h | r_n, \cdots, r_0, t_n, \cdots, t_0) \\ &= \Pr(r_{n+1} = j, h_{n+1} \leq h | r_n). \end{aligned} \tag{2}$$

(ii) The probability

$$\Pr(r_{n+1} = j, h_{n+1} \leq h | r_n = i) \tag{3}$$

is independent of n.

Conditions (i) and (ii) show that the process $\{(r_n, t_n)\}_{n=0}^\infty$ is a time-homogeneous Markov renewal process, and therefore $\{r_n\}_{n=0}^\infty$ is a time-homogeneous Markov process. It is also known from Definition 2.1 that the transition probabilities of homogeneous semi-Markov process $r(t) := r_n$, $t \in [t_n, t_{n+1})$, $n \in \mathbb{Z}_{\geq 1}$, are merely dependent on the sojourn-time h_n instead of system-operation time t. Thus, the TRs of homogeneous semi-Markov process are characterized by sojourn-time h only.

In this paper, we focus on the following continuous-time semi-MJNNs with time-varying state delay:

$$\begin{cases} \dot{x}(t) = -A(r(t))x(t) + B(r(t))\psi(x(t)) \\ \qquad\quad + B_d(r(t))\psi(x(t - d(t))) + V(t) \\ x(t) = \phi_t, \ t \in [-d_2, 0], \end{cases} \tag{4}$$

where $x(t) = [x_1^\top(t), x_2^\top(t), \cdots, x_\iota^\top(t)]^\top \in \Re^\iota$ refers to the state vector associated with the ι neutrons; $\psi(x(t)) = [\psi_1^\top(x_1(t)), \psi_2^\top(x_2(t)), \cdots, \psi_\iota^\top(x_\iota(t))]^\top \in \Re^\iota$ is the neuron activation function, where each activation function $\psi_l(\cdot)$ is continuous and bounded, and satisfies

$$F_l^- \le \frac{\psi_l(\tau_2) - \psi_l(\tau_1)}{\tau_2 - \tau_1} \le F_l^+, \; l = 1, 2, \cdots, \iota \tag{5}$$

where $\tau_1, \tau_2 \in \Re$, and $\tau_1 \neq \tau_2$; $V(t)$ refers to an external input vector; and $d(t)$ is a time-varying delay with $0 \le d_1 \le d(t) \le d_2 < \infty$ and $\dot{d}(t) \le \mu < \infty$, $\{d_1, d_2\} \in \mathbb{R}_+$ represent the lower and upper delay bounds, respectively. In (4), ϕ_t is a real-valued initial condition defined on $[-d_2, 0]$; $\{r(t), h\}_{t\ge 0} := \{r_n, h_n\}_{n\in\mathbb{N}_{\ge 1}}$ is a continuous-time and discrete-state homogeneous semi-Markov process with right continuous trajectories and with values in a finite set $\mathcal{I} := \{1, \cdots, N\}$ with TR matrix $\Lambda(h) := [\lambda_{ij}(h)]_{N\times N}$ characterized with [16]:

$$\begin{cases} \Pr\{r_{n+1} = j, h_{n+1} \le h + \delta | r_n = i, h_{n+1} > h\} \\ \qquad = \lambda_{ij}(h)\delta + o(\delta), \qquad i \neq j \\ \Pr\{r_{n+1} = j, h_{n+1} > h + \delta | r_n = i, h_{n+1} > h\} \\ \qquad = 1 + \lambda_{ii}(h)\delta + o(\delta), \; i = j \end{cases} \tag{6}$$

where $\delta > 0$, $\lim_{\delta\to 0}(o(\delta)/\delta) = 0$, and $\lambda_{ij}(h) \ge 0$, for $j \neq i$, refers to the TR from mode i at time t to mode j at time $t+\delta$, and $\lambda_{ii}(h) = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}(h)$. Subsequently, for each possible $r(t) = i$, $i \in \mathcal{I}$, the system matrices of the i-th mode are signified by (A_i, B_i, B_{di}).

For stochastic stability, we give the following definition.

Definition 2.2 [16], [17], [18]. System (4) is stochastically stable (SS) if there exists a finite positive constant $T(x_0, r_0)$ to make the subsequent inequality hold for any initial condition (x_0, r_0):

$$\mathcal{E}\left[\int_0^\infty \|x(t)\|^2 dt \middle| (x_0, r_0) \right] \le T(x_0, r_0).$$

In this paper, we address the delay-dependent stochastic synchronization problem for continuous-time semi-MJNNs (4) with time-varying delay. In particular, we take system (4) as the drive system, and then from the drive-response concept, a response system for (4) is obtained with the following state equation:

$$\begin{cases} \dot{\bar{x}}(t) = -A(r(t))\bar{x}(t) + B(r(t))\psi(\bar{x}(t)) \\ \qquad + B_d(r(t))\psi(\bar{x}(t - d(t))) + V(t) + u(t) \\ \bar{x}(t) = \varphi_t, \; t \in [-d_2, 0], \end{cases} \tag{7}$$

where $\bar{x}(t) = [\bar{x}_1^\top(t), \bar{x}_2^\top(t), \cdots, \bar{x}_\iota^\top(t)]^\top \in \Re^\iota$ represents the response state vector; $A(r(t))$, $B(r(t))$, and $B_d(r(t))$ are matrices prescribed in (4), and $u(t) \in \Re^\iota$ is the appropriate control input which will be synthesized in the sequel. The drive system with state variable $x(t)$ pushes the response system having identical dynamical equations with state variable $\bar{x}(t)$. Although the system parameters are the same, the initial condition on the drive system is different from that of the response system. In fact, even an infinitesimal differential in the initial condition in (4) and (7) will lead to different chaotic phenomena in those systems. By defining the synchronization error vector $e(t) = \bar{x}(t) - x(t)$ with $e(t) = [e_1^\top(t), e_2^\top(t), \cdots, e_\iota^\top(t)]^\top$, the error dynamics between (4) and (7) can be expressed by

$$\begin{aligned} \dot{e}(t) = &-A(r(t))e(t) + B(r(t))g(e(t)) \\ &+ B_d(r(t))g(e(t - d(t))) + u(t) \end{aligned} \tag{8}$$

where $g(e(t)) := \psi(\bar{x}(t)) - \psi(x(t))$. From Assumption (5), it is known that the functions g_l are subject to the following condition,

$$F_l^- \le \frac{g_l(\tau)}{\tau} \le F_l^+, \; l = 1, 2, \cdots, \iota \tag{9}$$

where $\tau \in \Re$ and $\tau \neq 0$.

For dynamic error system (8), the control input $u(t)$ is suitably designed as:

$$u(t) = K(r(t))e(t), \tag{10}$$

where $K(r(t)) \in \Re^{\iota\times\iota}$, $r(t) \in \mathcal{I}$ are the controller gain matrices to be synthesized.

With controller (10), the closed-loop dynamic error system can be formulated as

$$\begin{aligned} \dot{e}(t) = &-\bar{A}(r(t))e(t) + B(r(t))g(e(t)) \\ &+ B_d(r(t))g(e(t - d(t))). \end{aligned} \tag{11}$$

where $\bar{A}(r(t)) := A(r(t)) - K(r(t))$.

This paper will determine the control input $u(t)$ associated with the state feedback for stochastic synchronization of the drive-response semi-MJNNs with the same system parameters but different initial conditions. It will search for a group of state-feedback controller gain matrices K_i, $i \in \mathcal{I}$ such that the dynamic error system (11) is SS.

For a reduction in the conservatism of delay-dependent stability analysis for semi-MJNNs (11), we introduce a new integral inequality as follows.

Proposition 2.1. Given a matrix $Z \in \mathbb{S}^{n_1}$, for all continuous function $\omega \in [a, b] \to \Re^{n_1}$ and any constant matrices $\{W_1, W_2\} \in \Re^{n_1\times n_2}$, the following inequality holds:

$$\begin{aligned} -\int_a^b \omega^\top(s) Z \omega(s) \mathrm{d}s \le \bar{\zeta}^\top (&\mathtt{Sym}\{W_1^\top \Pi_1 + W_2^\top \Pi_2\} \\ &+ (b - a)\bar{W}^\top \mathcal{Z}^{-1} \bar{W})\bar{\zeta} \end{aligned} \tag{12}$$

where $\{\Pi_1, \Pi_2\} \in \Re^{n_1\times n_2}$, the vector $\bar{\zeta} \in \Re^{n_2}$, and

$$\begin{cases} \int_a^b \omega(s)\mathrm{d}s = \Pi_1\bar{\zeta}, \\ -\int_a^b \omega(s)\mathrm{d}s + \frac{2}{b-a}\int_a^b \int_a^s \omega(\alpha)\mathrm{d}\alpha \mathrm{d}s = \Pi_2\bar{\zeta}, \\ \bar{W} := \begin{bmatrix} W_1^\top & W_2^\top \end{bmatrix}^\top, \\ \mathcal{Z} := \mathrm{diag}\{Z, 3Z\}. \end{cases} \tag{13}$$

Proof: Define

$$\begin{cases} p(s) := \frac{2s-b-a}{b-a}, \\ \bar{W} := \begin{bmatrix} W_1^\top & W_2^\top \end{bmatrix}^\top, \\ \eta(s) := \begin{bmatrix} \bar{\zeta}^\top & p(s)\bar{\zeta}^\top \end{bmatrix}^\top. \end{cases} \tag{14}$$

For any matrix $Z \in \mathbb{S}^{n_1}$, it is known from Schur complement that

$$\begin{bmatrix} Z & * \\ \bar{W} & \bar{W}^\top Z^{-1}\bar{W} \end{bmatrix} \ge 0, \tag{15}$$

which directly leads to

$$\begin{bmatrix} \omega(s) \\ \eta(s) \end{bmatrix}^\top \begin{bmatrix} Z & * \\ \bar{W} & \bar{W}^\top Z^{-1}\bar{W} \end{bmatrix} \begin{bmatrix} \omega(s) \\ \eta(s) \end{bmatrix} \geq 0, \tag{16}$$

or equivalently,

$$-2\eta^\top(s)\bar{W}\omega(s) \leq \eta^\top(s)\bar{W}^\top Z^{-1}\bar{W}\eta(s) + \omega^\top(s)Z\omega(s). \tag{17}$$

Integrating both sides of the latter inequality from a to b yields

$$\begin{aligned} &-2\bar{\zeta}^\top W_1 \int_a^b \omega(s)\mathrm{d}s - 2\bar{\zeta}^\top W_2 \int_a^b p(s)\omega(s)\mathrm{d}s \\ &\quad \leq \int_a^b \eta^\top(s)\bar{W}^\top Z^{-1}\bar{W}\eta(s)\mathrm{d}s + \int_a^b \omega^\top(s)Z\omega(s)\mathrm{d}s. \end{aligned} \tag{18}$$

Then, with some mathematical manipulations, e.g., integration by parts, we further obtain

$$\begin{aligned} &-2\bar{\zeta}^\top W_1 \int_a^b \omega(s)\mathrm{d}s \\ &\quad -2\bar{\zeta}^\top W_2\Big(\int_a^b \omega(s)\mathrm{d}s - \frac{2}{b-a}\int_a^b\int_a^s \omega(\alpha)\mathrm{d}\alpha\mathrm{d}s\Big) \\ &\leq (b-a)\bar{\zeta}^\top W_1^\top Z^{-1}W_1\bar{\zeta} + \frac{b-a}{3}\bar{\zeta}^\top W_2^\top Z^{-1}W_2\bar{\zeta} \\ &\qquad + \int_a^b \omega^\top(s)Z\omega(s)\mathrm{d}s, \end{aligned} \tag{19}$$

which renders the conclusion in (12). ■

Remark 2.1. Proposition 2.1 provides a new integral inequality for computing the upper bound of integral quadratic terms in the form of $-\int_a^b \omega^\top(s)Z\omega(s)\mathrm{d}s$. Actually, depending on the selection of parameters in Proposition 2.1, the proposed integral inequalities can be extended to various cases. Firstly, when

$$\begin{cases} \omega(s) = \dot{x}(s), \\ n_2 = 2n_1, \\ W_1 = \begin{bmatrix} Z & \mathbf{0}_{n_1\times n_1} \end{bmatrix}, \\ W_2 = \begin{bmatrix} \mathbf{0}_{n_1\times n_1} & Z \end{bmatrix}, \\ \bar{\zeta} := \begin{bmatrix} x^\top(b) & x^\top(a) \end{bmatrix}^\top, \\ \Pi_1 = \begin{bmatrix} \mathbf{0}_{n_1\times n_1} & -(b-a)\mathbf{I} \end{bmatrix}, \\ \Pi_2 = \begin{bmatrix} \mathbf{0}_{n_1\times n_1} & \frac{2}{3}(b-a)\mathbf{I} \end{bmatrix}, \end{cases} \tag{20}$$

the corresponding condition in Proposition 2.1 degenerates into the traditional Jensen's inequality. Secondly, when

$$\begin{cases} \omega(s) = \dot{x}(s), \\ n_2 = 3n_1, \\ W_1 = \frac{1}{b-a}\begin{bmatrix} -Z & Z & \mathbf{0}_{n_1\times n_1} \end{bmatrix}, \\ W_2 = \frac{3}{b-a}\begin{bmatrix} -Z & -Z & 2Z \end{bmatrix}, \\ \bar{\zeta} := \begin{bmatrix} x^\top(b) & x^\top(a) & \frac{1}{b-a}\int_a^b x^\top(s)\mathrm{d}s \end{bmatrix}^\top, \\ \Pi_1 = \begin{bmatrix} \mathbf{I} & -\mathbf{I} & \mathbf{0}_{n_1\times n_1} \end{bmatrix}, \\ \Pi_2 = \begin{bmatrix} \mathbf{I}_{n_1} & \mathbf{I} & -2\mathbf{I} \end{bmatrix}, \end{cases} \tag{21}$$

the underlying condition in Proposition 2.1 evolves to the Wirtinger-based integral inequality (Corollary 4 in [38]). Thirdly, when

$$\begin{cases} \omega(s) = x(s), \\ n_2 = 2n_1, \\ W_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0}_{n_1\times n_1} \end{bmatrix}, \\ W_2 = \begin{bmatrix} -\mathbf{I}_{n_1} & 2\mathbf{I}_{n_1} \end{bmatrix}, \\ \bar{\zeta} := \begin{bmatrix} \frac{1}{b-a}\int_a^b x^\top(s)\mathrm{d}s & \frac{1}{b-a}\int_a^b\int_a^s x^\top(\alpha)\mathrm{d}\alpha\mathrm{d}s \end{bmatrix}^\top, \\ \Pi_1 = -\frac{1}{b-a}ZW_1, \\ \Pi_2 = -\frac{3}{b-a}ZW_2, \end{cases} \tag{22}$$

the condition in Proposition 2.1 is reduced to the free-matrix-based integral inequality (Lemma 4 in [39]). Therefore, the integral inequality in (12) is more general to cover many existing integral inequalities. Moreover, due to more slack parameters in Proposition 2.1, it is also expected that applications of the integral inequality (12) will lead to less conservatism for delay-dependent stability analysis criteria of time-delay systems.

III. Delay-dependent stochastic stability analysis

This section presents an improved stochastic stability condition for closed-loop dynamic error system (11). This will be achieved through constructing a new semi-Markovian LKF, together with making use of the new integral inequality given in Proposition 2.1. We have the following theorem.

Theorem 3.1. The system in (11) is SS if there exist matrices $P_i \in \mathbb{S}^{3\iota}$, $\{Q_{1i}, Q_{2i}, Q_{3i}, R_1, R_2\} \in \mathbb{S}^{\iota}$, $\{Z_1, Z_2\} \in \mathbb{S}^{2\iota}$, $W_k \in \Re^{13\iota\times 4\iota}$, $k = 1,2,3$, $\mathcal{Y}_i \in \Re^{13\iota\times\iota}$, and diagonal matrices $\{V_{1i}, V_{2i}\} \in \mathbb{S}^{\iota}$, $i \in \mathcal{I}$, such that the following matrix inequalities hold,

$$\begin{aligned} &\mathtt{Sym}\{\Lambda_1 P_i \Lambda_2^\top(d(t)) + \mathcal{Y}_i\mathcal{A}_i + W_1\Lambda_5^\top + W_2\Lambda_6^\top(d(t)) \\ &+W_3\Lambda_7^\top(d(t))\} + \Lambda_2(d(t))\Big(\sum_{j=1}^N \bar{\lambda}_{ij}P_j\Big)\Lambda_2^\top(d(t)) \\ &+\Lambda_3\mathcal{Q}_i\Lambda_3^\top + E_2\mathcal{R}E_2^\top + \Lambda_4\mathcal{Z}_0\Lambda_4^\top + d_1W_1\mathcal{Z}_1^{-1}W_1^\top \\ &+(d(t)-d_1)W_2\mathcal{Z}_2^{-1}W_2^\top + (d_2-d(t))W_3\mathcal{Z}_2^{-1}W_3^\top \\ &\qquad\qquad +\Lambda_8\mathcal{V}_{1i}\Lambda_8^\top + \Lambda_9\mathcal{V}_{2i}\Lambda_9^\top < 0, \end{aligned} \tag{23}$$

$$\mathscr{Q}_\nu := \sum_{j=1}^N \bar{\lambda}_{ij}Q_{\nu j} - R_\nu \leq 0,\ \ \nu = 1, 2, \tag{24}$$

$$\mathscr{Q}_3 := \sum_{j=1}^N \bar{\lambda}_{ij}(Q_{2j} + Q_{3j}) - R_2 \leq 0, \tag{25}$$

where $i \in \mathcal{I}$, and

$$\begin{cases} \Lambda_1 := \begin{bmatrix} E_1 & E_2 - E_4 & E_4 - E_5 \end{bmatrix}, \\ \Lambda_2(d(t)) := \begin{bmatrix} E_2 & d_1E_8 & (d(t)-d_1)E_9 + (d_2-d(t))E_{10} \end{bmatrix}, \\ \Lambda_3 := \begin{bmatrix} E_2 & E_3 & E_4 & E_5 \end{bmatrix}, \\ \Lambda_4 := \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \\ \Lambda_5 := \begin{bmatrix} d_1E_8 & E_2 - E_4 & d_1E_8 - 2E_{11} & E_2 + E_4 - 2E_8 \end{bmatrix}, \end{cases}$$

$$\begin{cases} \Lambda_6(d(t)) := \big[\ (d(t)-d_1)E_9 \quad E_4-E_3 \\ \qquad\qquad (d(t)-d_1)E_9-2E_{12} \\ \qquad\qquad\qquad E_4+E_3-2E_9\ \big], \\ \Lambda_7(d(t)) := \big[\ (d_2-d(t))E_{10} \quad E_3-E_5 \\ \qquad\qquad (d_2-d(t))E_{10}-2E_{13} \\ \qquad\qquad\qquad E_3+E_5-2E_{10}\ \big], \\ \Lambda_8 := \big[\ E_2 \quad E_6\ \big], \\ \Lambda_9 := \big[\ E_2 \quad E_7\ \big], \\ \mathcal{Q}_i := \mathrm{diag}\{Q_{1i}+Q_{2i}+Q_{3i}, -(1-\mu)Q_{3i}, Q_{1i}, Q_{2i}\}, \\ \mathcal{R} := d_1R_1+d_2R_2,\ \mathcal{Z}_0 := d_1Z_1+dZ_2, \\ \mathcal{Z}_1 := \mathrm{diag}\{Z_1, 3Z_1\},\ \mathcal{Z}_2 := \mathrm{diag}\{Z_2, 3Z_2\}, \\ \mathcal{V}_{1i} := \begin{bmatrix} -F_1V_{1i} & F_2V_{1i} \\ * & -V_{1i} \end{bmatrix}, \\ \mathcal{V}_{2i} := \begin{bmatrix} -F_1V_{2i} & F_2V_{2i} \\ * & -V_{2i} \end{bmatrix}, \\ F_1 := \mathrm{diag}\{F_1^-F_1^+, F_2^-F_2^+, \cdots, F_\iota^-F_\iota^+\}, \\ F_2 := \mathrm{diag}\{\frac{F_1^-+F_1^+}{2}, \frac{F_2^-+F_2^+}{2}, \cdots, \frac{F_\iota^-+F_\iota^+}{2}\}, \\ \mathcal{A}_i := \big[\ -\mathbf{I} \quad \bar{A}_i \quad \mathbf{0}_{\iota\times 3\iota} \quad B_i \quad B_{di} \quad \mathbf{0}_{\iota\times 6\iota}\ \big], \\ E_\kappa := \Big[\ \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{\kappa-1} \quad \mathbf{I}_\iota \quad \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{13-\kappa}\ \Big]^\top \in \Re^{13\iota\times\iota}, \\ \qquad\qquad \kappa = 1, \cdots, 13. \end{cases} \tag{26}$$

Proof: Our proof is given in Appendix. ■

Remark 3.1. From the semi-Markovian LKF in (46), together with the novel integral inequality given in Proposition 2.1, an improved stability analysis criterion for the semi-MJNNs in (11) is presented in Theorem 3.1. Compared with existing results on the stability analysis of time-delay systems, the originality of this improvement is on the construction of Lyapunov functional $V_4(e_s(t), r(t), t) := \int_{-d_1}^{0}\int_{t+s}^{t}\tilde{e}^\top(\alpha)Z_1\tilde{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s + \int_{-d_2}^{-d_1}\int_{t+s}^{t}\tilde{e}^\top(\alpha)Z_2\tilde{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s$, where the term $\tilde{e}(\alpha) := \big[\ e^\top(\alpha) \quad \dot{e}^\top(\alpha)\ \big]^\top$ is constructed to contain more information about state delays (i.e., the delayed states and *their derivatives*, and the integral of the states/derivatives over the period of the delay). Meantime, Proposition 2.1 is applied to deal with the crossing terms generated by the derivative of $V_4(e_s(t), r(t), t)$. Especially, the augmented terms $\frac{1}{d_1}\int_{t-d_1}^{t}e(s)\mathrm{d}s$, $\frac{1}{d(t)-d_1}\int_{t-d(t)}^{t-d_1}e(s)\mathrm{d}s$, $\frac{1}{d_2-d(t)}\int_{t-d_2}^{t-d(t)}e(s)\mathrm{d}s$, $\frac{1}{d_1}\int_{-d_1}^{0}\int_{t-d_1}^{t+s}e(\alpha)\mathrm{d}\alpha\mathrm{d}s$, $\frac{1}{d(t)-d_1}\int_{-d(t)}^{-d_1}\int_{t-d(t)}^{t+s}e(\alpha)\mathrm{d}\alpha\mathrm{d}s$, and $\frac{1}{d_2-d(t)}\int_{-d_2}^{-d(t)}\int_{t-d_2}^{t+s}e(\alpha)\mathrm{d}\alpha\mathrm{d}s$, together with some slack variables $W_k \in \Re^{13\iota\times 4\iota}$, $k = 1, 2, 3$, are simultaneously introduced to offer a lower bound of quadratic integral terms as in (60). In view of those procedures, the resulting delay-dependent stability analysis criterion for time-delay systems is expected to be less conservative.

Remark 3.2. For stability analysis of semi-MJNNs within the semi-Markovian-Lyapunov-functional framework, semi-Markovian Lyapunov matrices P_i in (46) are introduced. As a result, the stability analysis conditions in (23) are affine with respect to the time-varying and nonlinear delay term $d^2(t)$, i.e., the quadratic term $\Lambda_2(d(t))\left(\sum_{j=1}^{N}\bar{\lambda}_{ij}P_j\right)\Lambda_2^\top(d(t))$ in (23). This will cause difficulties in the numerical tractability of the stability analysis problem. However, due to the *non-positive property* of TRs $\bar{\lambda}_{ii}$, $i \in \mathcal{I}$, Schur complement cannot be directly applied to perform this decoupling. Alternatively, the powerful Projection lemma [40] will be utilized. Thus, in the sequel, a decoupling between the nonlinear time-varying delay term will be initiated by introducing a free matrix variable. This decoupling technique enables us to acquire a more readily tractable condition for stability analysis.

Theorem 3.2. The system in (11) is SS if there exist matrices $P_i \in \mathbb{S}^{3\iota}$, $\{Q_{1i}, Q_{2i}, Q_{3i}, R_1, R_2\} \in \mathbb{S}^\iota$, $\{Z_1, Z_2\} \in \mathbb{S}^{2\iota}$, $\mathcal{J} \in \Re^{3\iota\times 13\iota}$, $W_k \in \Re^{13\iota\times 4\iota}$, $k = 1, 2, 3$, $\mathcal{Y}_i \in \Re^{13\iota\times\iota}$, and diagonal matrices $\{V_{1i}, V_{2i}\} \in \mathbb{S}^\iota$, $i \in \mathcal{I}$, such that the conditions (24), (25) and the following matrix inequalities hold,

$$\begin{bmatrix} \Xi_i^{(\ell)} & * & * & * \\ -\mathcal{J} & \bar{\lambda}_{ii}P_i & * & * \\ \mathscr{P}_i\Pi_i^\top\Lambda_2^{(\ell)\top} & \mathbf{0} & -\mathscr{P}_i & * \\ \mathcal{W}^{(\ell)} & \mathbf{0} & \mathbf{0} & -\mathscr{Z} \end{bmatrix} < 0,\ i \in \mathcal{I},\ \ell = 1, 2, \tag{27}$$

where

$$\begin{cases} \Xi_i^{(\ell)} := \mathtt{Sym}\{\Lambda_1P_i\Lambda_2^{(\ell)\top} + \Lambda_2^{(\ell)}\mathcal{J} + \mathcal{Y}_i\mathcal{A}_i + W_1\Lambda_5^\top \\ \qquad + W_2\Lambda_6^{(\ell)\top} + W_3\Lambda_7^{(\ell)\top}\} + \Lambda_3\mathcal{Q}_i\Lambda_3^\top + E_2\mathcal{R}E_2^\top \\ \qquad + \Lambda_4\mathcal{Z}_0\Lambda_4^\top + \Lambda_8\mathcal{V}_{1i}\Lambda_8^\top + \Lambda_9\mathcal{V}_{2i}\Lambda_9^\top, \\ \Lambda_2^{(1)} := \big[\ E_2 \quad d_1E_8 \quad dE_9\ \big], \\ \Lambda_2^{(2)} := \big[\ E_2 \quad d_1E_8 \quad dE_{10}\ \big], \\ \Lambda_6^{(1)} := \big[\ \mathbf{0} \quad E_4-E_3 \quad -2E_{12} \quad E_4+E_3-2E_9\ \big], \\ \Lambda_6^{(2)} := \big[\ dE_9 \quad E_4-E_3 \quad dE_9-2E_{12} \\ \qquad\qquad E_4+E_3-2E_9\ \big], \\ \Lambda_7^{(1)} := \big[\ dE_{10} \quad E_3-E_5 \quad dE_{10}-2E_{13} \\ \qquad\qquad E_3+E_5-2E_{10}\ \big], \\ \Lambda_7^{(2)} := \big[\ \mathbf{0} \quad E_3-E_5 \quad -2E_{13} \quad E_3+E_5-2E_{10}\ \big], \\ \mathcal{W}^{(1)} := \big[\ W_1^\top \quad W_2^\top\ \big]^\top, \\ \mathcal{W}^{(2)} := \big[\ W_1^\top \quad W_3^\top\ \big]^\top, \\ \mathscr{Z} := \mathrm{diag}\{\frac{1}{d_1}Z_1, \frac{3}{d_1}Z_1, \frac{1}{d}Z_2, \frac{3}{d}Z_2\}, \\ \mathscr{P}_i := \mathrm{diag}\{P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_N\}, \\ \Pi_i := \big[\ \sqrt{\bar{\lambda}_{i1}}\mathbf{I} \quad \cdots \quad \sqrt{\bar{\lambda}_{i,i-1}}\mathbf{I} \\ \qquad\qquad \sqrt{\bar{\lambda}_{i,i+1}}\mathbf{I} \quad \cdots \quad \sqrt{\bar{\lambda}_{iN}}\mathbf{I}\ \big], \\ d := d_2-d_1 \end{cases} \tag{28}$$

and all other notations are defined the same as in (26).

Proof: Our proof is given in Appendix. ■

With the results given in Theorem 3.2 on a new delay-dependent stability analysis criterion, we are ready to develop a stochastic synchronization procedure in the next section.

IV. Stochastic Synchronization

From Theorem 3.2, this section derives stochastic synchronization conditions for the drive-response dynamic systems. This is achieved through adopting a linerization technique for a stochastic stabilization procedure.

Theorem 4.1. Consider semi-MJNNs (4). If there exist matrices $P_i \in \mathbb{S}^{3\iota}$, $\{Q_{1i}, Q_{2i}, Q_{3i}, R_1, R_2\} \in \mathbb{S}^\iota$, $\{Z_1, Z_2\} \in \mathbb{S}^{2\iota}$, $\mathcal{J} \in \Re^{3\iota\times 13\iota}$, $W_k \in \Re^{13\iota\times 4\iota}$, $k = 1, 2, 3$, $\{Y_i, \bar{K}_i\} \in$

$\Re^{\iota\times\iota}$, and diagonal matrices $\{V_{1i}, V_{2i}\} \in \mathbb{S}^{\iota}$, $i \in \mathcal{I}$, such that the conditions in (24), (25) and the following LMIs hold,

$$\begin{bmatrix} \bar{\Xi}_i^{(\ell)} & * & * & * \\ -\mathcal{J} & \bar{\lambda}_{ii}P_i & * & * \\ \mathscr{P}_i\Pi_i^{\top}\Lambda_2^{(\ell)\top} & \mathbf{0} & -\mathscr{P}_i & * \\ \mathcal{W}^{(\ell)} & \mathbf{0} & \mathbf{0} & -\mathscr{Z} \end{bmatrix} < 0,\ i\in\mathcal{I},\ \ell=1,2, \tag{29}$$

where

$$\begin{cases} \bar{\Xi}_i^{(\ell)} := \mathtt{Sym}\{\Lambda_1 P_i \Lambda_2^{(\ell)\top} + \Lambda_2^{(\ell)}\mathcal{J} + \mathcal{Y}_i\mathscr{A}_i + \mathcal{H}\mathcal{K}_i + W_1\Lambda_5 \\ \qquad + W_2\Lambda_6^{(\ell)\top} + W_3\Lambda_7^{(\ell)\top}\} + \Lambda_3\mathcal{Q}_i\Lambda_3^{\top} + E_2\mathcal{R}E_2^{\top} \\ \qquad + \Lambda_4\mathcal{Z}_0\Lambda_4^{\top} + \Lambda_8\mathcal{V}_{1i}\Lambda_8^{\top} + \Lambda_9\mathcal{V}_{2i}\Lambda_9^{\top}, \\ \mathcal{Y}_i := \begin{bmatrix} Y_i^{\top} & \rho Y_i^{\top} & \mathbf{0}_{\iota\times 11\iota} \end{bmatrix}^{\top}, \\ \mathscr{A}_i := \begin{bmatrix} -\mathbf{I} & A_i & \mathbf{0}_{\iota\times 3\iota} & B_i & B_{di} & \mathbf{0}_{\iota\times 6\iota} \end{bmatrix}, \\ \mathcal{H} := \begin{bmatrix} \mathbf{I} & \delta\mathbf{I} & \mathbf{0}_{\iota\times 11\iota} \end{bmatrix}^{\top}, \\ \mathcal{K}_i := \begin{bmatrix} \mathbf{0}_{\iota\times\iota} & \bar{K}_i & \mathbf{0}_{\iota\times 11\iota} \end{bmatrix}, \end{cases} \tag{30}$$

with the other notations defined the same as in (26) and (28), then the closed-loop dynamic error system in (11) is SS. Specifically, the desired controller gains can be obtained as

$$K_i = Y_i^{-1}\bar{K}_i,\ i\in\mathcal{I}. \tag{31}$$

Proof: The result in this theorem follows the conditions in (24), (25) and (27) given in Theorems 3.1 and 3.2. For tractability of the controller synthesis procedure, and by inspection of the inner structure of system matrices in (11), we prescribe the slack matrices $\mathcal{Y}_i$ implicitly in (27) as

$$\mathcal{Y}_i := \begin{bmatrix} Y_i^{\top} & \delta Y_i^{\top} & \mathbf{0}_{\iota\times 11\iota} \end{bmatrix}^{\top},\ i\in\mathcal{I}, \tag{32}$$

where $Y_i \in \Re^{\iota\times\iota}$. It follows that if we introduce

$$\bar{K}_i := Y_i K_i,\ i\in\mathcal{I}, \tag{33}$$

the freedom variable Y_i can be absorbed by the controller gain variable K_i. Then, substituting matrices $\mathcal{Y}_i$ defined in (32) into (27) yields (29).

Moreover, the conditions in (29) result in $-Y_i - Y_i^{\top} < 0$, implying that Y_i is invertible. Thus, the controller gains can be calculated from (31). This completes the proof. ■

The conditions presented in Theorem 4.1 are derived based on the new integral inequality (12). For comparison, in the sequel, we propose another stochastic stability analysis method based on a similar LKF to (46), but with $V_4(e_s(t), r(t), t) := \int_{-d_1}^{0}\int_{t+s}^{t}\dot{e}^{\top}(\alpha)\hat{Z}_1\dot{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s + \int_{-d_2}^{-d_1}\int_{t+s}^{t}\dot{e}^{\top}(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s$, where $\{\hat{Z}_1, \hat{Z}_2\} \in \mathbb{S}^{\iota}$. By applying the integral inequality (12) with parameters given in (22), together with the utilization of Projection lemma (Lemma A1), the corresponding stochastic stability analysis result is elaborated in the following corollary.

Corollary 4.1. Consider the semi-MJNNs (4). If there exist matrices $P_i \in \mathbb{S}^{3\iota}$, $\{Q_{1i}, Q_{2i}, Q_{3i}, R_1, R_2, \hat{Z}_1, \hat{Z}_2\} \in \mathbb{S}^{\iota}$, $\hat{\mathcal{J}} \in \Re^{3\iota\times 10\iota}$, $\hat{W}_k \in \Re^{10\iota\times 2\iota}$, $k=1,2,3$, $\hat{\mathcal{Y}}_i \in \Re^{10\iota\times\iota}$, and diagonal matrices $\{V_{1i}, V_{2i}\} \in \mathbb{S}^{\iota}$, $i\in\mathcal{I}$, such that the conditions (24), (25) and the following matrix inequalities hold,

$$\begin{bmatrix} \hat{\Xi}_i^{(\ell)} & * & * & * \\ -\hat{\mathcal{J}} & \bar{\lambda}_{ii}P_i & * & * \\ \mathscr{P}_i\Pi_i^{\top}\hat{\Lambda}_2^{(\ell)\top} & \mathbf{0} & -\mathscr{P}_i & * \\ \hat{\mathcal{W}}^{(\ell)} & \mathbf{0} & \mathbf{0} & -\hat{\mathscr{Z}} \end{bmatrix} < 0,\ i\in\mathcal{I},\ \ell=1,2, \tag{34}$$

where

$$\begin{cases} \hat{\Xi}_i^{(\ell)} := \mathtt{Sym}\{\hat{\Lambda}_1 P_i \hat{\Lambda}_2^{(\ell)\top} + \hat{\Lambda}_2^{(\ell)}\hat{\mathcal{J}} + \hat{\mathcal{Y}}_i\hat{\mathcal{A}}_i + \hat{\Lambda}_5\hat{W}_1 \\ \qquad + \hat{\Lambda}_6^{(\ell)}\hat{W}_2 + \hat{\Lambda}_7^{(\ell)}\hat{W}_3\} + \hat{\Lambda}_3\mathcal{Q}_i\hat{\Lambda}_3^{\top} + \hat{E}_2\mathcal{R}\hat{E}_2^{\top} \\ \qquad + \hat{\Lambda}_4\hat{\mathcal{Z}}_0\hat{\Lambda}_4^{\top} + \hat{\Lambda}_8\mathcal{V}_{1i}\hat{\Lambda}_8^{\top} + \hat{\Lambda}_9\mathcal{V}_{2i}\hat{\Lambda}_9^{\top}, \\ \hat{\Lambda}_1 := \begin{bmatrix} \hat{E}_1 & \hat{E}_2 - \hat{E}_4 & \hat{E}_4 - \hat{E}_5 \end{bmatrix}, \\ \hat{\Lambda}_2^{(1)} := \begin{bmatrix} \hat{E}_2 & d_1\hat{E}_8 & d\hat{E}_9 \end{bmatrix}, \\ \hat{\Lambda}_2^{(2)} := \begin{bmatrix} \hat{E}_2 & d_1\hat{E}_8 & d\hat{E}_{10} \end{bmatrix}, \\ \hat{\Lambda}_3 := \begin{bmatrix} \hat{E}_2 & \hat{E}_3 & \hat{E}_4 & \hat{E}_5 \end{bmatrix}, \\ \hat{\Lambda}_4 := \begin{bmatrix} \hat{E}_1 & \hat{E}_2 \end{bmatrix}, \\ \hat{\Lambda}_5 := \begin{bmatrix} d_1\hat{E}_8 & \hat{E}_2 - \hat{E}_4 \end{bmatrix}, \\ \hat{\Lambda}_6^{(1)} := \begin{bmatrix} \mathbf{0} & \hat{E}_4 - \hat{E}_3 \end{bmatrix}, \\ \hat{\Lambda}_6^{(2)} := \begin{bmatrix} d\hat{E}_9 & \hat{E}_4 - \hat{E}_3 \end{bmatrix}, \\ \hat{\Lambda}_7^{(1)} := \begin{bmatrix} d\hat{E}_{10} & \hat{E}_3 - \hat{E}_5 \end{bmatrix}, \\ \hat{\Lambda}_7^{(2)} := \begin{bmatrix} \mathbf{0} & \hat{E}_3 - \hat{E}_5 \end{bmatrix}, \\ \hat{\Lambda}_8 := \begin{bmatrix} \hat{E}_2 & \hat{E}_6 \end{bmatrix}, \\ \hat{\Lambda}_9 := \begin{bmatrix} \hat{E}_2 & \hat{E}_7 \end{bmatrix}, \\ \mathcal{Q}_i := \mathrm{diag}\{Q_{1i} + Q_{2i} + Q_{3i}, -(1-\mu)Q_{3i}, Q_{1i}, Q_{2i}\}, \\ \mathcal{R} := d_1R_1 + d_2R_2,\ \hat{\mathcal{Z}}_0 := d_1\hat{Z}_1 + d\hat{Z}_2, \\ \hat{\mathcal{Z}}_1 := \mathrm{diag}\{\hat{Z}_1, 3\hat{Z}_1\},\ \hat{\mathcal{Z}}_2 := \mathrm{diag}\{\hat{Z}_2, 3\hat{Z}_2\}, \\ \hat{\mathcal{W}}^{(1)} := \begin{bmatrix} \hat{W}_1^{\top} & \hat{W}_2^{\top} \end{bmatrix}^{\top}, \\ \hat{\mathcal{W}}^{(2)} := \begin{bmatrix} \hat{W}_1^{\top} & \hat{W}_3^{\top} \end{bmatrix}^{\top}, \\ \hat{\mathscr{Z}} := \mathrm{diag}\{\frac{1}{d_1}\hat{Z}_1, \frac{3}{d_1}\hat{Z}_1, \frac{1}{d}\hat{Z}_2, \frac{3}{d}\hat{Z}_2\}, \\ \hat{E}_\kappa := \begin{bmatrix} \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{\kappa-1} & \mathbf{I}_\iota & \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{10-\kappa} \end{bmatrix}^{\top} \in \Re^{10\iota\times\iota}, \\ \qquad\qquad \kappa = 1,2,\cdots,10. \end{cases} \tag{35}$$

Proof: To use the integral inequality in (12) with notations defined in (22), which is indeed the free-matrix-based integral inequality as proposed in [39], we reconstruct the semi-Markovian LKF similar to (46) with some slight modifications. Specifically, $V_m(e_s(t), r(t), t)$, $m=1,2,3$, are defined the same as in (46), however,

$$V_4(e_s(t), r(t), t) := \int_{-d_1}^{0}\int_{t+s}^{t}\dot{e}^{\top}(\alpha)\hat{Z}_1\dot{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s + \int_{-d_2}^{-d_1}\int_{t+s}^{t}\dot{e}^{\top}(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha\mathrm{d}s, \tag{36}$$

where $\{\hat{Z}_1, \hat{Z}_2\} \in \mathbb{S}^{\iota}$.

Applying the weak infinitesimal generator in (49) gives

$$\mathscr{V}_1 = \hat{\zeta}^{\top}(t)\Bigg[\mathtt{Sym}\{\hat{\Lambda}_1 P_i \hat{\Lambda}_2(d(t))\} + \hat{\Lambda}_2(d(t))\left(\sum_{j=1}^{N}\bar{\lambda}_{ij}P_j\right)\hat{\Lambda}_2^{\top}(d(t))\Bigg]\hat{\zeta}(t), \tag{37}$$

(58) and (59), where

$$\begin{cases} \hat{\zeta}(t) := \big[\ \dot{e}^\top(t) \quad e^\top(t) \quad e^\top(t-d(t)) \quad e^\top(t-d_1) \\ \qquad e^\top(t-d_2) \quad f^\top(e(t)) \quad f^\top(e(t-d(t))) \\ \qquad \frac{1}{d_1}\int_{t-d_1}^{t} e^\top(s)\mathrm{d}s \quad \frac{1}{d(t)-d_1}\int_{t-d(t)}^{t-d_1} e^\top(s)\mathrm{d}s \\ \qquad \frac{1}{d_2-d(t)}\int_{t-d_2}^{t-d(t)} e^\top(s)\mathrm{d}s \ \big]^\top, \\ \hat{\Lambda}_1 := \big[\ \hat{E}_1 \quad \hat{E}_2-\hat{E}_4 \quad \hat{E}_4-\hat{E}_5 \ \big], \\ \hat{\Lambda}_2(d(t)) := \big[\ \hat{E}_2 \quad d_1\hat{E}_8 \\ \qquad (d(t)-d_1)\hat{E}_9+(d_2-d(t))\hat{E}_{10} \ \big], \\ \hat{E}_\kappa := \Big[\ \underbrace{\mathbf{0}\cdots\mathbf{0}}_{\kappa-1} \quad \mathbf{I}_\iota \quad \underbrace{\mathbf{0}\cdots\mathbf{0}}_{10-\kappa} \ \Big]^\top \in \Re^{10\iota\times\iota}, \\ \qquad \kappa=1,2,\cdots,10. \end{cases} \tag{38}$$

For the derivative of $V_4(e_s(t), r(t), t)$, we have

$$\begin{aligned} \mathscr{V}_4 \ = \ & d_1\dot{e}^\top(t)\hat{Z}_1\dot{e}(t) - \int_{t-d_1}^{t} \dot{e}^\top(\alpha)\hat{Z}_1\dot{e}(\alpha)\mathrm{d}\alpha \\ & + d\dot{e}^\top(t)\hat{Z}_2\dot{e}(t) - \int_{t-d(t)}^{t-d_1} \dot{e}^\top(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha \\ & - \int_{t-d_2}^{t-d(t)} \dot{e}^\top(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha. \end{aligned} \tag{39}$$

Using Proposition 2.1 to the second, fourth, and fifth term, respectively, in the RHS of equation (39), we get

$$-\int_{t-d_1}^{t} \dot{e}^\top(\alpha)\hat{Z}_1\dot{e}(\alpha)\mathrm{d}\alpha \le \hat{\zeta}^\top(t)(\mathtt{Sym}\{\hat{W}_1\hat{\Lambda}_5^\top\} + d_1\hat{W}_1\hat{\mathcal{Z}}_1\hat{W}_1^\top)\hat{\zeta}(t) \tag{40}$$

$$-\int_{t-d(t)}^{t-d_1} \dot{e}^\top(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha \le \hat{\zeta}^\top(t)(\mathtt{Sym}\{\hat{W}_2\hat{\Lambda}_6^\top(d(t))\} + (d(t)-d_1)\hat{W}_2\hat{\mathcal{Z}}_2\hat{W}_2^\top)\hat{\zeta}(t) \tag{41}$$

$$-\int_{t-d_2}^{t-d(t)} \dot{e}^\top(\alpha)\hat{Z}_2\dot{e}(\alpha)\mathrm{d}\alpha \le \hat{\zeta}^\top(t)(\mathtt{Sym}\{\hat{W}_3\hat{\Lambda}_7^\top(d(t))\} + (d_2-d(t))\hat{W}_3\hat{\mathcal{Z}}_2\hat{W}_3^\top)\hat{\zeta}(t) \tag{42}$$

for suitable matrices $\hat{W}_k$, $k=1,2,3$.

Then, we use the proof as in that for Theorems 3.1 and 3.2 to derive the final result. Taking the similar procedures as in (64)-(69), and (71)-(77) yields the resultant stochastic stability analysis condition in (34). This completes the proof. ■

Remark 4.1. On the basis of the integral inequality presented in [39], Corollary 4.1 offers another approach to the stochastic synchronization problem for semi-MJNNs with time-varying delay. It is noted that the integral inequality in [39] can only be employed to deal with the quadratic term in the form of $\int_{t-a}^{t-b}\dot{e}^\top(s)Z\dot{e}(s)\mathrm{d}s$. Hence, the Lyapunov functional $V_4(e_s(t), r(t), t)$ should be constructed as in (36). Then, performing the time-derivative to $V_4(e_s(t), r(t), t)$, together with the utilization of the integral inequality in (12) with parameters given in (22), leads to conditions (40)-(42), where only the augmented terms $\frac{1}{d_1}\int_{t-d_1}^{t}e(s)\mathrm{d}s$, $\frac{1}{d(t)-d_1}\int_{t-d(t)}^{t-d_1}e(s)\mathrm{d}s$, and $\frac{1}{d_2-d(t)}\int_{t-d_2}^{t-d(t)}e(s)\mathrm{d}s$, together with some free matrices $\hat{W}_k \in \Re^{10\iota\times 2\iota}$, $k=1,2,3$, are utilized to evaluate the lower bound of the integral quadratic terms in (39). Because more time-delay information is involved in the Lyapunov functional in (46) than that in (36), it is expected that application of the new integral inequality in (12) will lead to a reduction in the resultant analysis conservatism. This will be demonstrated later in simulation studies.

Remark 4.2. It is also worth mentioning that the conditions developed in Theorems 3.1, 3.2, 4.1 and Corollary 4.1 rely on the derivative of the time-varying delay with $\dot{d}(t) \le \mu < \infty$. Nevertheless, for a delay-derivative-independent scenario, these results can be easily extended by imposing $Q_{3i}=0$, $i\in\mathcal{I}$ in Theorems 3.1, 3.2, 4.1 and Corollary 4.1 for the underlying systems.

V. Simulation results

This section conducts simulations to demonstrate the effectiveness and less conservatism of the presented conditions.

Example 4.1. Consider a three-mode continuous-time semi-MJNN with time-varying delay (4)

$$\left[\begin{array}{c|c|c} A_1 & B_1 & B_{d1} \\ \hline A_2 & B_2 & B_{d2} \\ \hline A_3 & B_3 & B_{d3} \end{array}\right] = \left[\begin{array}{cc|cc|cc} 2.2 & 0 & 0.8 & 0.4 & 1.2 & 1 \\ 0 & 1.8 & -0.2 & 0.1 & -0.2 & 0.3 \\ \hline 0.2 & 0 & 0.7 & 1.1 & -2.4 & -4.8 \\ 0 & 3.4 & 0.2 & -0.05 & -0.32 & 2 \\ \hline 1 & 0 & 1 & 1 & 0.88 & 1 \\ 0 & 0.8 & -1 & -1 & 1 & 1 \end{array}\right].$$

The activation functions are $\psi_i(\alpha)$, $i=1,2,3$, which satisfy the condition in (5) with parameters $F_l^-=0$ and $F_l^+=0.4, l=1,2$. Hence,

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Transitions among the three modes are determined by a semi-Markov process with its TR matrix described as

$$\begin{aligned} \Lambda(h) &= \begin{bmatrix} \lambda_{11}(h) & \lambda_{12}(h) & \lambda_{13}(h) \\ \lambda_{21}(h) & \lambda_{22}(h) & \lambda_{23}(h) \\ \lambda_{31}(h) & \lambda_{32}(h) & \lambda_{33}(h) \end{bmatrix} \\ &= \begin{bmatrix} -2h & h & h \\ 0.5h & -h & 0.5h \\ \frac{2}{9}h & \frac{2}{9}h & -\frac{4}{9}h \end{bmatrix}. \end{aligned} \tag{43}$$

Considering the properties of Weibull distribution, we conclude that the TR function in (43) can be termed as an approximation in the context that the sojourn-time is subject to Weibull distribution with its PDF $f(h) = \frac{\beta}{\alpha^\beta}h^{\beta-1}\exp\left[-\left(\frac{h}{\alpha}\right)^\beta\right]$, $h\ge 0$. In particular, when $i=1$, the TR function h can be characterized by Weibull distribution with the scale parameter $\alpha=1$ and the shape parameter $\beta=2$, implying that $f_1(h)=2h\mathrm{e}^{-h^2}$. For $i=2$, the TR function $0.5h$ can be modelled by Weibull distribution with $\alpha=2$ and $\beta=2$, giving $f_2(h)=0.5h\mathrm{e}^{-0.25h^2}$. When $i=3$, the TR function $\frac{2}{9}h$ can be described by Weibull distribution with $\alpha=3$ and $\beta=2$, thus $f_3(h)=\frac{2}{9}h\mathrm{e}^{-\frac{1}{9}h^2}$. Therefore, the mathematical expectation of TR $\lambda_{12}(h)$ can be expressed as $\mathcal{E}\{\lambda_{12}(h)\} = \int_0^\infty hf_1(h)\mathrm{d}h = \int_0^\infty 2h^2\mathrm{e}^{-h^2}\mathrm{d}h = 0.8862\mathrm{s}$.

TABLE I
ALLOWABLE UPPER BOUND d_2 OF DELAY UNDER VARIOUS VALUES OF LOWER DELAY BOUND d_1 AND DELAY DERIVATIVE μ.

Method	$\mu =$	0.2		0.5		0.8		unknown μ value	
	$d_1 =$	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5
Theorem 3.2	$d_2 =$	1.410	1.146	0.984	0.853	0.918	0.858	0.902	0.865
Corollary 4.1	$d_2 =$	1.322	1.080	0.937	0.814	0.879	0.802	0.865	0.797

Taking the same vein for other elements in the TR matrix (43), we have the mathematical expectation of the TR matrix as

$$\mathcal{E}\{\Lambda(h)\} = \begin{bmatrix} -1.7724 & 0.8862 & 0.8862 \\ 1.7725 & -3.5450 & 1.7725 \\ 2.6587 & 2.6587 & -5.3174 \end{bmatrix}. \quad (44)$$

To compare the delay-dependent stability analysis results proposed in Theorem 3.2 and Corollary 4.1, some numerical tests are carried out. The resulting maximum allowable upper bounds d_2 with various lower bounds d_1 of time delay and delay derivatives are shown in Table I. The scenario of unknown μ value in Table I refers to delay-derivative-independent stability analysis as discussed in Remark 4.2.

It is seen from Table I that under various scenarios, the maximum allowable upper delay bound d_2 values from Theorem 3.2 are generally larger than those from Corollary 4.1. This indicates the superiority of the proposed delay-dependent stochastic stability conditions for semi-MJNNs with time-varying delay.

The following example aims to demonstrate the effectiveness of the delay-dependent stochastic synchronization algorithm proposed in Theorem 4.1.

Example 4.2. Consider the drive system (4) and response system (7) with the following parameters:

$$\left[\begin{array}{c|c|c} A_1 & B_1 & B_{d1} \\ \hline A_2 & B_2 & B_{d2} \\ \hline A_3 & B_3 & B_{d3} \end{array}\right] = \left[\begin{array}{cc|cc|cc} 1 & 0 & 2.7 & -0.6 & 1.2 & 1 \\ 0 & 1 & 2.1 & 3 & -0.2 & 0.3 \\ \hline 1.1 & 0 & 2.8 & -0.4 & -2.7 & -1.1 \\ 0 & 0.9 & 1.9 & 2.8 & -0.7 & -2.3 \\ \hline 1.2 & 0 & 2.5 & -0.5 & -2.8 & -1.2 \\ 0 & 1 & 2.2 & 2.6 & -0.5 & -2.1 \end{array}\right].$$

The activation functions are

$$\psi_i(\alpha) = \tanh(\alpha),\ i = 1, 2, 3,$$

which satisfy the condition in (5) with parameters $F_l^- = 0$ and $F_l^+ = 1$, $l = 1, 2$. Hence,

$$F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

The TR matrix is the same as in (43), and thus the corresponding mathematical expectation is expressed in (44). The time-varying delay is assumed to be $d(t) = 1 + 0.3\sin(2t)$. This gives $0.7 \le d(t) \le 1.3$ and $\dot{d}(t) \le 0.6$.

Our goal is to synthesize a state-feedback controller (10) to synchronize the drive-response system with guaranteed stochastic stability of the resultant closed-loop error system. Adopting Theorem 3.3 with $\rho = 5$, we obtain the following admissible state-feedback controller gains

$$\begin{aligned} K_1 &= \begin{bmatrix} -13.0337 & -4.0111 \\ -5.2076 & -12.3806 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -13.9123 & -4.7402 \\ -4.7882 & -13.5715 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -15.0876 & -5.0173 \\ -4.9914 & -14.6842 \end{bmatrix}. \end{aligned} \quad (45)$$

To verify the design procedure and the effectiveness of the proposed results, simulations are conducted with the initial condition $x(t) = \begin{bmatrix} 5e^{2t}\sin(3.14t) & 0 \end{bmatrix}^\top$ and $\bar{x}(t) = \begin{bmatrix} 0.4e^{10t}\cos(0.01t) & 2\sin(3.3t) \end{bmatrix}^\top$, $t \in [-1.3, 0]$, and $V(t) = 0$. For control input $u(t) = 0$, Fig. 2 shows the chaotic behaviours of the drive system (upper plot) and response system (lower plot). With the controller in (45), Fig. 3 depicts the state responses of the closed-loop dynamic error system. It is seen from Figure 3 that the state-feedback controller derived from this paper synchronizes well the response system with the drive system.

VI. CONCLUSION

The stability analysis and stabilization problems for delay-dependent stochastic synchronization of continuous-time semi-MJNNs with time-varying delay have been investigated in this paper. By constructing a semi-Markovian LKF, combined with a new integral inequality, an improved stochastic stability analysis condition has been established for the semi-MJNN error systems. With less conservatism than existing methods, it guarantees that the response system is stochastically synchronized with the response systems. From the stability analysis, the stochastic synchronization controller has been synthesized with a linearization technique. Simulation studies have been carried out to demonstrate the effectiveness of the results derived in this paper.

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APPENDIX

A. Proof of Theorem 3.1

Let $\mathbb{C}[-d_2, 0]$ denote the space of continuous functions evolving on $[-d_2, 0]$. Define $e_s(t) := e(t+s), e_s(t) \in$

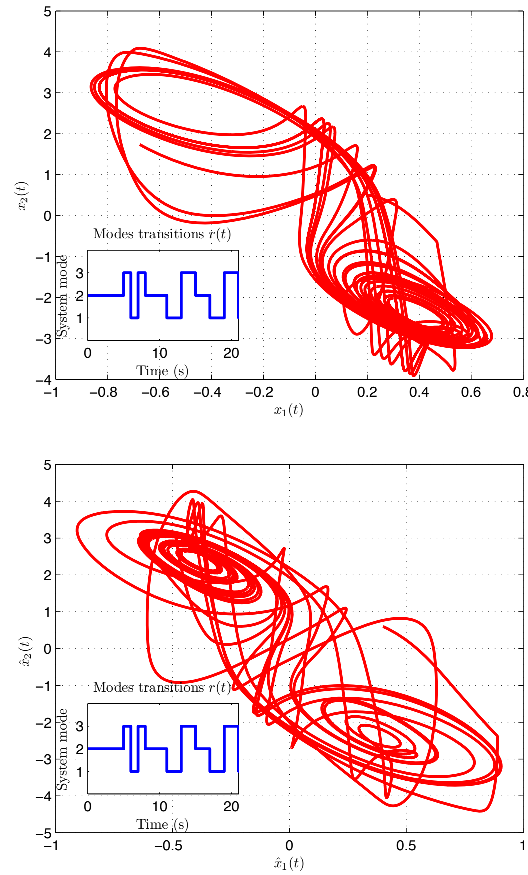

Fig. 2. The chaotic dynamics of the semi-MJNNs. Upper plot: drive system (4); lower plot: response system (7).

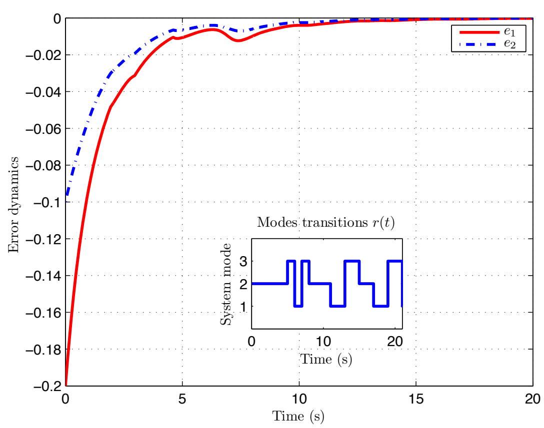

Fig. 3. Synchronization error dynamics (8) in Example 4.2.

$\mathbb{C}[-d_2, 0], s \in [-d_2, 0]$. Then, $\{(e_s(t), r(t)), t \geq 0\}$ is a semi-Markov process with initial state $(\phi_t - \varphi_t, r(0))$. Construct the following semi-Markovian LKF for the dynamic error system in (11),

$$V(e_s(t), r(t)) := \sum_{m=1}^{4} V_m(e_s(t), r(t)), \tag{46}$$

where

$$\begin{cases} V_1(e_s(t), r(t)) := \bar{e}^\top(t) P(r(t)) \bar{e}(t), \\ V_2(e_s(t), r(t)) := \int_{t-d(t)}^{t} e^\top(\alpha) Q_3(r(t)) e(\alpha) \mathrm{d}\alpha \\ \qquad\qquad + \sum_{\nu=1}^{2} \int_{t-d_\nu}^{t} e^\top(\alpha) Q_\nu(r(t)) e(\alpha) \mathrm{d}\alpha, \\ V_3(e_s(t), r(t)) := \sum_{\nu=1}^{2} \int_{-d_\nu}^{0} \int_{t+s}^{t} e^\top(\alpha) R_\nu e(\alpha) \mathrm{d}\alpha \mathrm{d}s, \\ V_4(e_s(t), r(t)) := \int_{-d_1}^{0} \int_{t+s}^{t} \tilde{e}^\top(\alpha) Z_1 \tilde{e}(\alpha) \mathrm{d}\alpha \mathrm{d}s \\ \qquad\qquad + \int_{-d_2}^{-d_1} \int_{t+s}^{t} \tilde{e}^\top(\alpha) Z_2 \tilde{e}(\alpha) \mathrm{d}\alpha \mathrm{d}s. \end{cases} \tag{47}$$

with $\bar{e}(t) := \begin{bmatrix} e^\top(t) & \int_{t-d_1}^{t} e^\top(s)\mathrm{d}s & \int_{t-d_2}^{t-d_1} e^\top(s)\mathrm{d}s \end{bmatrix}^\top$, $\tilde{e}(t) := \begin{bmatrix} e^\top(t) & \dot{e}^\top(t) \end{bmatrix}^\top$, $P(r(t)) \in \mathbb{S}^{3\iota}$, $\{Q_1(r(t)), Q_2(r(t)), Q_3(r(t)), R_1, R_2\} \in \mathbb{S}^{\iota}$, and $\{Z_1, Z_2\} \in \mathbb{S}^{2\iota}$.

With the semi-Markovian LKF defined in (46), the following condition

$$\mathscr{D}[V(e_s(t), r(t))] < 0, \tag{48}$$

assures that the dynamic error system in (11) is SS [17], where $\mathscr{D}$ refers to the weak infinitesimal generator. Considering the definition of $\mathscr{D}$ [18], we have

$$\begin{aligned} \mathscr{D}[V(e_s(t), r(t))] := \lim_{\delta \to 0^+} \frac{1}{\delta} [\mathcal{E}\{V(e_s(t+\delta), r(t+\delta)) | \\ e_s(t), r(t) = i\} - V(e_s(t), r(t))], \\ \mathscr{V}_m := \mathscr{D}[V_m(e_s(t), r(t))], \ m = 1, 2, 3, 4. \end{aligned} \tag{49}$$

For each $r(t) = i \in \mathcal{I}$, adopting the law of total probability and conditional expectation, we have

$$\begin{aligned} \mathscr{V}_1 := \lim_{\delta \to 0^+} \frac{1}{\delta} \Bigg[\mathcal{E}\Bigg\{ \sum_{j=1, j\neq i}^{N} \Pr\{r_{n+1} = j, h_{n+1} \leq h + \delta | \\ r_n = i, h_{n+1} > h\} \times \bar{e}^\top(t+\delta) P_j \bar{e}(t+\delta) \\ + \Pr\{r_{n+1} = i, h_{n+1} > h + \delta | r_n = i, \\ h_{n+1} > h\} \bar{e}^\top(t+\delta) P_i \bar{e}(t+\delta) \Bigg\} - \bar{e}^\top(t) P_i \bar{e}(t) \Bigg] \\ = \lim_{\delta \to 0^+} \frac{1}{\delta} \Bigg[\mathcal{E}\Bigg\{ \sum_{j=1, j\neq i}^{N} \frac{\Pr\{r_{n+1} = j, r_n = i\}}{\Pr\{r_n = i\}} \\ \times \frac{\Pr\{h < h_{n+1} \leq h + \delta | r_{n+1} = j, r_n = i\}}{\Pr\{h_{n+1} > h | r_n = i\}} \\ \times \bar{e}^\top(t+\delta) P_j \bar{e}(t+\delta) + \frac{\Pr\{h_{n+1} > h + \delta | r_n = i\}}{\Pr\{h_{n+1} > h | r_n = i\}} \\ \times \bar{e}^\top(t+\delta) P_i \bar{e}(t+\delta) \Bigg\} - \bar{e}^\top(t) P_i \bar{e}(t) \Bigg] \\ = \lim_{\delta \to 0^+} \frac{1}{\delta} \Bigg[\mathcal{E}\Bigg\{ \sum_{j=1, j\neq i}^{N} \frac{q_{ij}(G_i(h+\delta) - G_i(h))}{1 - G_i(h)} \\ \times \bar{e}^\top(t+\delta) P_j \bar{e}(t+\delta) + \frac{1 - G_i(h+\delta)}{1 - G_i(h)} \\ \times \bar{e}^\top(t+\delta) P_i \bar{e}(t+\delta) \Bigg\} - \bar{e}^\top(t) P_i \bar{e}(t) \Bigg], \end{aligned} \tag{50}$$

where $G_i(h)$ represents the CDF of the sojourn-time when the system stays in mode i, and $q_{ij} := \frac{\Pr\{r_{n+1}=j, r_n=i\}}{\Pr\{r_n=i\}} =$

$\Pr\{r_{n+1}=j|r_n=i\}$ refers to the probability intensity of the system switching from mode i to mode j. With a small δ, the first-order approximation of $\bar{e}(t+\delta)$ is

$$\begin{aligned}\bar{e}(t+\delta) &= \bar{e}(t)+\delta\dot{\bar{e}}(t)+o(\delta)\\ &= (\delta\Lambda_1+\Lambda_2(d(t)))\zeta(t)+o(\delta),\end{aligned} \tag{51}$$

where

$$\begin{cases}\zeta(t):=\big[\ \dot{e}^\top(t)\quad e^\top(t)\quad e^\top(t-d(t))\quad e^\top(t-d_1)\\ \quad e^\top(t-d_2)\quad f^\top(e(t))\quad f^\top(e(t-d(t)))\\ \quad \frac{1}{d_1}\int_{t-d_1}^{t}e^\top(s)\mathrm{d}s\quad \frac{1}{d(t)-d_1}\int_{t-d(t)}^{t-d_1}e^\top(s)\mathrm{d}s\\ \quad \frac{1}{d_2-d(t)}\int_{t-d_2}^{t-d(t)}e^\top(s)\mathrm{d}s\\ \quad \frac{1}{d_1}\int_{-d_1}^{0}\int_{t-d_1}^{t+s}e^\top(\alpha)\mathrm{d}\alpha\mathrm{d}s\\ \quad \frac{1}{d(t)-d_1}\int_{-d(t)}^{-d_1}\int_{t-d(t)}^{t+s}e^\top(\alpha)\mathrm{d}\alpha\mathrm{d}s\\ \quad \frac{1}{d_2-d(t)}\int_{-d_2}^{-d(t)}\int_{t-d_2}^{t+s}e^\top(\alpha)\mathrm{d}\alpha\mathrm{d}s\ \big]^\top,\\ \Lambda_1:=\big[\ E_1\quad E_2-E_4\quad E_4-E_5\ \big],\\ \Lambda_2(d(t)):=\big[\ E_2\quad d_1E_8\\ \quad (d(t)-d_1)E_9+(d_2-d(t))E_{10}\ \big],\\ E_\kappa:=\Big[\ \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{\kappa-1}\quad \mathbf{I}_\iota\quad \underbrace{\mathbf{0}\ \cdots\ \mathbf{0}}_{13-\kappa}\ \Big]^\top\in\Re^{13\iota\times\iota},\\ \qquad\kappa=1,2,\cdots,13\end{cases} \tag{52}$$

From (50) - (52), we have

$$\begin{aligned}\mathscr{V}_1 \;=\; &\lim_{\delta\to 0^+}\frac{1}{\delta}\bigg[\mathcal{E}\bigg\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\delta)-G_i(h))}{1-G_i(h)}\zeta^\top(t)\\ &\times(\delta\Lambda_1+\Lambda_2(d(t)))P_j(\delta\Lambda_1+\Lambda_2(d(t)))^\top\zeta(t)\\ &+\frac{1-G_i(h+\delta)}{1-G_i(h)}\zeta^\top(t)(\delta\Lambda_1+\Lambda_2(d(t)))\\ &\times P_i(\delta\Lambda_1+\Lambda_2(d(t)))^\top\zeta(t)\bigg\}-\bar{e}^\top(t)P_i\bar{e}(t)\bigg].\end{aligned}$$

From the condition that $\lim_{\delta\to 0^+}\frac{G_i(h+\delta)-G_i(h)}{1-G_i(h)}=0$, it follows that

$$\begin{aligned}\mathscr{V}_1=\zeta^\top(t)\bigg[\mathcal{E}\bigg\{&\lim_{\delta\to 0^+}\bigg(\frac{1-G_i(h+\delta)}{1-G_i(h)}\mathtt{Sym}\{\Lambda_1P_i\Lambda_2^\top(d(t))\}\\ &+\Lambda_2(d(t))\mathscr{P}_i\Lambda_2^\top(d(t))\bigg)\bigg\}\bigg]\zeta(t),\end{aligned}$$

where

$$\begin{aligned}\mathscr{P}_i=\sum_{j=1,j\neq i}^{N}&\frac{q_{ij}(G_i(h+\delta)-G_i(h))}{\delta(1-G_i(h))}P_j\\ &+\frac{G_i(h)-G_i(h+\delta)}{\delta(1-G_i(h))}P_i.\end{aligned}$$

In view of the characteristics of the CDF, we obtain

$$\begin{aligned}&\lim_{\delta\to 0^+}\frac{1-G_i(h+\delta)}{1-G_i(h)}=1,\\ &\lim_{\delta\to 0^+}\frac{G_i(h+\delta)-G_i(h)}{\delta(1-G_i(h))}=\lambda_i(h),\end{aligned} \tag{53}$$

where $\lambda_i(h)$ stands for the TR of the system switching from mode i.

Define

$$\begin{aligned}\lambda_{ij}(h)\quad &:=\quad q_{ij}\lambda_i(h),\ i\neq j\\ \lambda_{ii}(h)\quad &:=\quad -\sum_{j=1,j\neq i}^{N}\lambda_{ij}(h).\end{aligned} \tag{54}$$

Then, it follows that

$$\begin{aligned}\mathscr{V}_1=\zeta^\top(t)\bigg[&\mathtt{Sym}\{\Lambda_1P_i\Lambda_2(d(t))\}\\ &+\Lambda_2(d(t))\bigg(\sum_{j=1}^{N}\bar{\lambda}_{ij}P_j\bigg)\Lambda_2^\top(d(t))\bigg]\zeta(t),\end{aligned} \tag{55}$$

with $\bar{\lambda}_{ij}:=\mathcal{E}\{\lambda_{ij}(h)\}=\int_0^\infty\lambda_{ij}(h)f_i(h)\mathrm{d}h$, where $f_i(h)$ is the probability density function (PDF) of sojourn-time h staying at mode i.

For the first term of $V_2(e_s(t),r(t))$, we have

$$\begin{aligned}&\mathcal{E}\bigg\{\int_{t+\delta-d(t+\delta)}^{t+\delta}e^\top(\alpha)Q_3(r(t+\delta))e(\alpha)\mathrm{d}\alpha\bigg\}\\ &=\mathcal{E}\bigg\{\int_{t}^{t+\delta}e^\top(\alpha)Q_3(r(t+\delta))e(\alpha)\mathrm{d}\alpha\\ &\qquad+\int_{t-d(t)}^{t}e^\top(\alpha)Q_3(r(t+\delta))e(\alpha)\mathrm{d}\alpha\\ &\quad+\int_{t+\delta-d(t+\delta)}^{t-d(t)}e^\top(\alpha)Q_3(r(t+\delta))e(\alpha)\mathrm{d}\alpha|e(t),r(t)=i\bigg\}\\ &=\mathcal{E}\bigg\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(h+\delta)-G_i(h))}{1-G_i(h)}\bigg(\delta e^\top(t)Q_{3j}e(t)\\ &\quad+\int_{t-d(t)}^{t}e^\top(\alpha)Q_{3j}e(\alpha)\mathrm{d}\alpha\\ &\quad-\delta(1-\dot{d}(t))e^\top(t-d(t))Q_{3j}e(t-d(t))+o(\delta)\bigg)\\ &\quad+\frac{1-G_i(h+\delta)}{1-G_i(h)}\bigg(\delta e^\top(t)Q_{3i}e(t)\\ &\quad+\int_{t-d(t)}^{t}e^\top(\alpha)Q_{3i}e(\alpha)\mathrm{d}\alpha\\ &\quad-\delta(1-\dot{d}(t))e^\top(t-d(t))Q_{3i}e(t-d(t))+o(\delta)\bigg)\bigg\}.\end{aligned} \tag{56}$$

Following a similar procedure to (56) for the second term of $V_2(e_s(t),r(t))$, we obtain

$$\begin{aligned}&\mathcal{E}\bigg\{\sum_{\nu=1}^{2}\int_{t+\delta-d_\nu}^{t+\delta}e^\top(\alpha)Q_\nu(r(t+\delta))e(\alpha)\mathrm{d}\alpha\bigg\}\\ &=\sum_{\nu=1}^{2}\bigg[\sum_{j=1,j\neq i}^{N}\mathcal{E}\bigg\{\frac{q_{ij}(G_i(h+\delta)-G_i(h))}{1-G_i(h)}\bigg\}\\ &\quad\times\bigg(\delta e^\top(t)Q_{\nu j}e(t)+\int_{t-d_\nu}^{t}e^\top(\alpha)Q_{\nu j}e(\alpha)\mathrm{d}\alpha\end{aligned}$$

$$-\delta e^\top(t-d_\nu)Q_{\nu j}e(t-d_\nu)+o(\delta)\Big)$$
$$+\mathcal{E}\Big\{\frac{1-G_i(h+\delta)}{1-G_i(h)}\Big\}\Big(\delta e^\top(t)Q_{\nu i}e(t)$$
$$+\int_{t-d_\nu}^{t}e^\top(\alpha)Q_{\nu i}e(\alpha)\mathrm{d}\alpha$$
$$-\delta e^\top(t-d_\nu)Q_{\nu i}e(t-d_\nu)+o(\delta)\Big)\Big]. \quad (57)$$

From (49), (56) and (57), it follows that

$$\begin{aligned}\mathscr{V}_2 \ \le\ & e^\top(t)Q_{3i}e(t)-(1-\mu)e^\top(t-d(t))Q_{3i}e(t-d(t))\\ &+\int_{t-d(t)}^{t}e^\top(\alpha)\Big(\sum_{j=1}^{N}\bar{\lambda}_{ij}Q_{3j}\Big)e(\alpha)\mathrm{d}\alpha\\ &+\sum_{\nu=1}^{2}\big(e^\top(t)Q_{\nu i}e(t)-e^\top(t-d_\nu)Q_{\nu i}e(t-d_\nu)\big)\\ &+\int_{t-d_1}^{t}e^\top(\alpha)\Big(\sum_{j=1}^{N}\bar{\lambda}_{ij}Q_{1j}\Big)e(\alpha)\mathrm{d}\alpha\\ &+\int_{t-d_2}^{t-d(t)}e^\top(\alpha)\Big(\sum_{j=1}^{N}\bar{\lambda}_{ij}Q_{2j}\Big)e(\alpha)\mathrm{d}\alpha\\ &+\int_{t-d(t)}^{t}e^\top(\alpha)\Big(\sum_{j=1}^{N}\bar{\lambda}_{ij}Q_{2j}\Big)e(\alpha)\mathrm{d}\alpha. \end{aligned}\quad (58)$$

In addition, we also have

$$\begin{aligned}\mathscr{V}_3 \ =\ & \sum_{\nu=1}^{2}d_\nu e^\top(t)R_\nu e(t)-\int_{t-d_1}^{t}e^\top(\alpha)R_1e(\alpha)\mathrm{d}\alpha\\ &-\int_{t-d(t)}^{t}e^\top(\alpha)R_2e(\alpha)\mathrm{d}\alpha\\ &-\int_{t-d_2}^{t-d(t)}e^\top(\alpha)R_2e(\alpha)\mathrm{d}\alpha, \end{aligned}\quad (59)$$

$$\begin{aligned}\mathscr{V}_4 \ =\ & d_1\tilde{e}^\top(t)Z_1\tilde{e}(t)-\int_{t-d_1}^{t}\tilde{e}^\top(\alpha)Z_1\tilde{e}(\alpha)\mathrm{d}\alpha\\ &+d\tilde{e}^\top(t)Z_2\tilde{e}(t)-\int_{t-d(t)}^{t-d_1}\tilde{e}^\top(\alpha)Z_2\tilde{e}(\alpha)\mathrm{d}\alpha\\ &-\int_{t-d_2}^{t-d(t)}\tilde{e}^\top(\alpha)Z_2\tilde{e}(\alpha)\mathrm{d}\alpha. \end{aligned}\quad (60)$$

Applying Proposition 2.1 to the second, fourth, and fifth term, respectively, on the right-hand side (RHS) of Equation (60), we obtain

$$-\int_{t-d_1}^{t}\tilde{e}^\top(\alpha)Z_1\tilde{e}(\alpha)\mathrm{d}\alpha\le\zeta^\top(t)(\mathtt{Sym}\{W_1\Lambda_5^\top\}+d_1W_1\mathcal{Z}_1^{-1}W_1^\top)\zeta(t) \quad (61)$$

$$-\int_{t-d(t)}^{t-d_1}\tilde{e}^\top(\alpha)Z_2\tilde{e}(\alpha)\mathrm{d}\alpha\le\zeta^\top(t)(\mathtt{Sym}\{W_2\Lambda_6^\top(d(t))\}+(d(t)-d_1)W_2\mathcal{Z}_2^{-1}W_2^\top)\zeta(t) \quad (62)$$

$$-\int_{t-d_2}^{t-d(t)}\tilde{e}^\top(\alpha)Z_2\tilde{e}(\alpha)\mathrm{d}\alpha\le\zeta^\top(t)(\mathtt{Sym}\{W_3\Lambda_7^\top(d(t))\}+(d_2-d(t))W_3\mathcal{Z}_2^{-1}W_3^\top)\zeta(t) \quad (63)$$

for suitable matrices W_k, $k=1,2,3$.

Furthermore, by considering equation (11), for any appropriately dimensioned matrix $\mathcal{Y}_i$, we have

$$2\zeta^\top(t)\mathcal{Y}_i[-\dot{e}(t)-\bar{A}(r(t))e(t)+B(r(t))g(e(t))+B_d(r(t))g(e(t-d(t)))]=0. \quad (64)$$

In addition, according to the condition in (9), we have

$$(g_l(e_l(t))-F_l^-e_l(t))(g_l(e_l(t))-F_l^+e_l(t))\le 0, \quad (65)$$

with $l=1,2,\cdots,\iota$, which is equivalent to

$$\begin{bmatrix} e(t)\\ f(e(t))\end{bmatrix}^\top\begin{bmatrix} F_l^-F_l^+\chi_l\chi_l^\top & -\frac{F_l^-+F_l^+}{2}\chi_l\chi_l^\top\\ * & \chi_l\chi_l^\top\end{bmatrix}(*)\le 0 \quad (66)$$

where χ_l refers to the unit column vector with one element on its lth row and zeros elsewhere. Thus, for any diagonal matrix $V_{1i}>0$ of appropriate dimension, the following inequality holds

$$\begin{bmatrix} e(t)\\ f(e(t))\end{bmatrix}^\top\begin{bmatrix} -F_1V_{1i} & F_2V_{1i}\\ * & -V_{1i}\end{bmatrix}(*)\ge 0, \quad (67)$$

where $F_1:=\mathrm{diag}\{F_1^-F_1^+,F_2^-F_2^+,\cdots,F_\iota^-F_\iota^+\}$ and $F_2:=\mathrm{diag}\{\frac{F_1^-+F_1^+}{2},\frac{F_2^-+F_2^+}{2},\cdots,\frac{F_\iota^-+F_\iota^+}{2}\}$. Similarly, for any diagonal matrix $V_{2i}>0$ of appropriate dimension, we obtain

$$\begin{bmatrix} e(t)\\ f(e(t-d(t)))\end{bmatrix}^\top\begin{bmatrix} -F_1V_{2i} & F_2V_{2i}\\ * & -V_{2i}\end{bmatrix}(*)\ge 0. \quad (68)$$

Now, adding the terms on the left-hand side (LHS) of equation (64) and inequality (67) and (68) to the LHS of (48), together with (55) and (58)-(63), we have

$$\begin{aligned}\mathrm{LHS}(48)\le\ & \zeta^\top(t)(\mathtt{Sym}\{\Lambda_1P_i\Lambda_2^\top(d(t))+\mathcal{Y}_i\mathcal{A}_i+W_1\Lambda_5^\top\\ &+W_2\Lambda_6^\top(d(t))+W_3\Lambda_7^\top(d(t))\}\\ &+\Lambda_2(d(t))\Big(\sum_{j=1}^{N}\bar{\lambda}_{ij}P_j\Big)\Lambda_2^\top(d(t))\\ &+\Lambda_3\mathcal{Q}_i\Lambda_3^\top+E_2\mathcal{R}E_2^\top+\Lambda_4\mathcal{Z}_0\Lambda_4^\top\\ &+d_1W_1\mathcal{Z}_1^{-1}W_1^\top+(d(t)-d_1)W_2\mathcal{Z}_2^{-1}W_2^\top\\ &+(d_2-d(t))W_3\mathcal{Z}_2^{-1}W_3^\top\\ &+\Lambda_8\mathcal{V}_{1i}\Lambda_8^\top+\Lambda_9\mathcal{V}_{2i}\Lambda_9^\top)\zeta(t)+\bar{\mathscr{Q}}(t), \end{aligned}\quad (69)$$

where

$$\begin{cases}\bar{\mathscr{Q}}(t):=\int_{t-d_1}^{t}e^\top(\alpha)\mathscr{Q}_1e(\alpha)\mathrm{d}\alpha\\ \qquad +\int_{t-d_2}^{t-d(t)}e^\top(\alpha)\mathscr{Q}_2e(\alpha)\mathrm{d}\alpha\\ \qquad +\int_{t-d(t)}^{t}e^\top(\alpha)\mathscr{Q}_3e(\alpha)\mathrm{d}\alpha,\\ \mathscr{Q}_\nu:=\sum_{j=1}^{N}\bar{\lambda}_{ij}Q_{\nu j}-R_\nu,\ \nu=1,2,\\ \mathscr{Q}_3:=\sum_{j=1}^{N}\bar{\lambda}_{ij}(Q_{2j}+Q_{3j})-R_2.\end{cases} \quad (70)$$

From (23)-(25), we have that $\mathrm{LHS}(48)<0$, which means that the error system (11) is SS on the basis of Lyapunov stability theory. This completes the proof. ■

B. Proof of Theorem 3.2

From Theorem 3.1, the system in (11) is SS if the conditions in (23)-(25) hold.

Rewrite (23) as

$$\Upsilon(d(t))\Theta_i\Upsilon^\top(d(t)) < 0, \tag{71}$$

where

$$\begin{cases} \Upsilon(d(t)) := \begin{bmatrix} \mathbf{I}_{13\iota} & \Lambda_2(d(t)) \end{bmatrix}, \\ \Theta_i := \text{diag}\{\vec{\Theta}_i, \bar{\lambda}_{ii}P_i\}, \end{cases}$$

$$\begin{cases} \vec{\Theta}_i := \mathtt{Sym}\{\Lambda_1 P_i \Lambda_2^\top(d(t)) + W_1\Lambda_5^\top + W_2\Lambda_6^\top(d(t)) \\ \qquad + W_3\Lambda_7^\top(d(t)) + \mathcal{Y}_i\mathcal{A}_i\} \\ \quad + \Lambda_2(d(t))\Big(\sum\limits_{j=1, j\neq i}^{N} \bar{\lambda}_{ij}P_j\Big)\Lambda_2^\top(d(t)) \\ \quad + \Lambda_3\mathcal{Q}_i\Lambda_3^\top + E_2\mathcal{R}E_2^\top + \Lambda_4\mathcal{Z}_0\Lambda_4^\top \\ \quad + d_1W_1\mathcal{Z}_1^{-1}W_1^\top + (d(t)-d_1)W_2\mathcal{Z}_2^{-1}W_2^\top \\ \quad + (d_2 - d(t))W_3\mathcal{Z}_2^{-1}W_3^\top \\ \quad + \Lambda_8\mathcal{V}_{1i}\Lambda_8^\top + \Lambda_9\mathcal{V}_{2i}\Lambda_9^\top. \end{cases} \tag{72}$$

By applying Projection lemma [40] to (71), the following inequality implies (71):

$$\Theta_i + \mathtt{Sym}\{\Upsilon_\perp(d(t))\bar{\mathcal{J}}\} < 0. \tag{73}$$

Now, it follows from (73) that by introducing the free matrix $\bar{\mathcal{J}}$, the nonlinear coupling of time-delay $d(t)$ in the quadratic term $\Lambda_2(d(t))\bar{\lambda}_{ii}P_i\Lambda_2^\top(d(t))$ has been eliminated. It is worth mentioning that, however, the matrix $\bar{\mathcal{J}}$ is intrinsically with a high dimension, which will incur heavy computational burden. To balance the computational complexity and conservatism, we specify $\bar{\mathcal{J}} := \begin{bmatrix} \mathcal{J} & \mathbf{0}_{3\iota\times 3\iota} \end{bmatrix}$, where $\mathcal{J} \in \Re^{3\iota\times 13\iota}$. Furthermore, with respect to Schur complement, we can rewrite the condition in (73) as,

$$\begin{bmatrix} \breve{\Xi}_i(d(t)) & * & * \\ -\mathcal{J} & \bar{\lambda}_{ii}P_i & * \\ \mathscr{P}_i\Pi_i^\top\Lambda_2^\top(d(t)) & \mathbf{0} & -\mathscr{P}_i \end{bmatrix} < 0, \ i \in \mathcal{I}, \tag{74}$$

where

$$\begin{aligned} \breve{\Xi}_i(d(t)) := \mathtt{Sym}\{&\Lambda_1 P_i\Lambda_2^\top(d(t)) + \Lambda_2(d(t))\mathcal{J} + \mathcal{Y}_i\mathcal{A}_i \\ &+ W_1\Lambda_5^\top + W_2\Lambda_6^\top(d(t)) + W_3\Lambda_7^\top(d(t))\} \\ &+ \Lambda_3\mathcal{Q}_i\Lambda_3^\top + E_2\mathcal{R}E_2^\top + \Lambda_4\mathcal{Z}_0\Lambda_4^\top \\ &+ d_1W_1\mathcal{Z}_1^{-1}W_1^\top + (d(t)-d_1)W_2\mathcal{Z}_2^{-1}W_2^\top \\ &+ (d_2-d(t))W_3\mathcal{Z}_2^{-1}W_3^\top \\ &+ \Lambda_8\mathcal{V}_{1i}\Lambda_8^\top + \Lambda_9\mathcal{V}_{2i}\Lambda_9^\top. \end{aligned} \tag{75}$$

Notice that the condition in (74) is affine with respect to the time-varying delay $d(t)$, which satisfies

$$d_1 \leq d(t) \leq d_2. \tag{76}$$

This implies that $d(t)$ may take any value in $[d_1, d_2]$. Then, $d(t)$ can be further expressed as a convex combination in the following form,

$$d(t) = \eta d_1 + (1-\eta)d_2, \tag{77}$$

where $0 \leq \eta \leq 1$. Since $d(t)$ in (77) varies with respect to η linearly, it is only required that (74) holds with $\eta = 0$ and $\eta = 1$, respectively. This leads to (27) after twice Schur complements.

Hence, we conclude that the closed-loop error system in (11) is SS if (24), (25) and (27) hold. This completes the proof. ■

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