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Modalities in Vector Logic

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Abstract Vector logic is a mathematical model of logic in which the truth values are mapped on elements of a vector space. The binary logical functions are performed by rectangular matrices operating on the Kronecker product of their vectorial arguments. The binary operators acting on vectors representing ambiguous (fuzzy) truth values, generate many-valued logics. In this article we show that, within the formalism of vector logic, it becomes possible to obtain truth-functional definitions of the modalities "possibility" and "necessity". These definitions are based on the matrix operators that represent disjunction and conjunction respectively, and each modality emerges by means of an iterative process. The construction of these modal operators was inspired in Tarski's truth-functional definition of possibility for the 3-valued logic of Łukasiewicz. The classical Aristotelian link between possibility and necessity becomes, in the basic vector logic, a corollary of the De Morgan's connection between disjunction and conjunction. Finally, we describe extensional versions of the existential and universal quantifiers for vector logics.

1 Introduction The mathematical representations of logic have opened illuminating perspectives for the understanding of the logical constructions created by the humans. These representations have also provided us with powerful technical instruments that present a wide spectrum of applications. A profound analysis of the relations between mathematics and logic can be found in Curry [4].

Recently, an algebraic representation of the propositional calculus in which the truth values are mapped on the elements of a vector space has been described. In this representation, the logical computations are performed by matrix operators. In particular, binary operations are executed by rectangular matrices that act over the Kronecker product of their vectorial arguments. This algebraic model of logic has been denominated "vector logic", and has been discovered investigating a neural model (see Mizraji [11] and [12]).

An interesting property of this formalism is the following: once a binary vector logic isomorphic in the binary domain with the classical propositional calculus is defined, the binary operators can compute ambiguous truth values,

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generating a many-valued logic. These ambiguous truth values are defined as a linear combination of the basic truth values. Moreover, this vectorial formalism allows to generate non-standard many-valued logics (an example is the Shefferian vector logic, described in Mizraji [12], that presents a non-involutive negation).

Copilowish has published a pioneer paper applying linear algebra to formal logic (see Copilowish [3]). And a recent contribution, Stern [14], provides with a representation of the logical calculus in which square matrices operate on two adjoint vector spaces. This system produces scalar outputs (a difference with vector logics that generate vectorial outputs).

The original contributions by Łukasiewicz to the field of many-valued logics had been guided by his research on modal logics (see Łukasiewicz [9]). He found that a truth-functional definition of the modal operations was not possible within a binary system of logic. The very natural emergence of many-valued logics in the framework of the basic vector logic suggests that it can constitute a suitable formalism to construct modal operators. In this article we show how these constructions can be performed. The "possibility" and the "necessity" operators are built up by means of an iterative process that uses the matrix operators disjunction and conjunction, respectively. It is then shown that the classical Aristotelian relations between "possibility" and "necessity" are corollaries of the De Morgan's relations between disjunction and conjunction. We describe, also, an extensional representation of the existential and universal quantifiers in the frame of vector logic.

2 Basic vector logic The investigations about the vector structures of the logical systems are just in their beginnings. The first goal of the research program on vector logics is the statement of some classical logical operations in terms of matrix formalisms. Usually, the vectorial representation of the classical results implies the "collapse" of an arbitrary Q-dimensional vector space in a 1-dimensional subspace (even in the case of some probabilistic many-valued logics, as we will see in Section 3). The development of multidimensional logical operators will be the subject of future communications. In this section, we describe the basic matrix operators of classical propositional calculus.

Given a pair of truth values T, "true", and F, "false", and a set $\tau = \{T,F\}$, the monadic logical operators are functions of the class $M : \tau \rightarrow \tau$, and the binary operations correspond to functions of the class $B : \tau \times \tau \rightarrow \tau$ ($\tau \times \tau$: Cartesian product of τ). The basic vector logic assigns to T the column vector $\mathbf{s}$ and to F the column vector $\mathbf{n}$, with $\mathbf{s},\mathbf{n} \in V_Q$, a Q-dimensional vector space defined over the field of real numbers. Given a matrix $\mathbf{W}$, we represent the transpose as $\mathbf{W}^T$. Let us assume that $\mathbf{s}$ and $\mathbf{n}$ are orthonormal vectors. Hence, we have the following scalar products $\langle \mathbf{s},\mathbf{n} \rangle = \langle \mathbf{n},\mathbf{s} \rangle = 0$ and $\langle \mathbf{s},\mathbf{s} \rangle = \langle \mathbf{n},\mathbf{n} \rangle = 1$ (the scalar product between two Q-dimensional column vectors $\mathbf{a},\mathbf{b}$ is $\mathbf{a}^T\mathbf{b} = \langle \mathbf{a},\mathbf{b} \rangle$).

We will use in this work an operation named the Kronecker product (also named direct or tensor product) defined as follows (see Bellman [2] and Graham [5]):

Definition 2.1 Given the matrices $\mathbf{A} = [a_{ij}]$ of order $m \times n$ and $\mathbf{B} = [b_{ij}]$ of order $p \times q$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of order $(mp) \times (nq)$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = [a_{ij}\mathbf{B}].$$

This product is associative and distributive:

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{A}' = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{A}').$$

$$(\mathbf{A} + \mathbf{B}) \otimes (\mathbf{A}' + \mathbf{B}') = \mathbf{A} \otimes \mathbf{A}' + \mathbf{A} \otimes \mathbf{B}' + \mathbf{B} \otimes \mathbf{A}' + \mathbf{B} \otimes \mathbf{B}'.$$

An important property of this product is the following: if the dimensions are appropriate, the matrices $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{B}$, and $\mathbf{B}'$ satisfy

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{A}' \otimes \mathbf{B}') = (\mathbf{AA}') \otimes (\mathbf{BB}').$$

For Q-dimensional column vectors $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ this property implies

$$(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}}(\mathbf{a}' \otimes \mathbf{b}') = \langle \mathbf{a},\mathbf{a}' \rangle \langle \mathbf{b},\mathbf{b}' \rangle.$$

Definition 2.2 The Kronecker power (see Bellman [2]) is defined by the following properties:

$$\mathbf{A}^{[2]} \equiv \mathbf{A} \otimes \mathbf{A}$$

$$\mathbf{A}^{[k]} = \mathbf{A}^{[k-1]} \otimes \mathbf{A}$$

and satisfies

$$\mathbf{A}^{[k]} \otimes \mathbf{A}^{[l]} = \mathbf{A}^{[k+l]}$$

$$(\mathbf{AB})^{[k]} = \mathbf{A}^{[k]}\mathbf{B}^{[k]}.$$

We define now the four monadic matrices:

$$\mathbf{I} = \mathbf{ss}^{\mathrm{T}} + \mathbf{nn}^{\mathrm{T}},$$

$$\mathbf{N} = \mathbf{ns}^{\mathrm{T}} + \mathbf{sn}^{\mathrm{T}},$$

$$\mathbf{K} = \mathbf{ss}^{\mathrm{T}} + \mathbf{sn}^{\mathrm{T}},$$

$$\mathbf{M} = \mathbf{ns}^{\mathrm{T}} + \mathbf{nn}^{\mathrm{T}}.$$

The matrix $\mathbf{I}$ acts as an identity operator for the set $\{\mathbf{s},\mathbf{n}\}$, $\mathbf{Is} = \mathbf{s}$, $\mathbf{In} = \mathbf{n}$. On the other hand, the matrix $\mathbf{N}$ corresponds to the classical negations ($\neg \mathrm{T} = \mathrm{F}$, $\neg \mathrm{F} = \mathrm{T}$), since $\mathbf{Ns} = \mathbf{n}$ and $\mathbf{Nn} = \mathbf{s}$.

These square matrices are linked by the following identities:

$$\mathbf{NN} = \mathbf{I}$$

$$\mathbf{KI} = \mathbf{KN} = \mathbf{KM} = \mathbf{KK} = \mathbf{K}$$

$$\mathbf{MI} = \mathbf{MN} = \mathbf{MM} = \mathbf{MK} = \mathbf{M}.$$

We will now show the representation of the conjunction and disjunction operations in the basic vector logic. The classical conjunction $p \wedge q$ and disjunction $p \vee q$ between two arbitrary propositions p and q are described in the following table:

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Given the pair of orthonormal vectors $(\mathbf{s},\mathbf{n})$, a matrix version of these logical connectives can be immediately constructed using the Kronecker product (Mizraji [12]).

The conjunction is executed by the following matrix $\mathbf{C}$,

$$\mathbf{C} = \mathbf{s}(\mathbf{s} \otimes \mathbf{s})^{\mathrm{T}} + \mathbf{n}(\mathbf{s} \otimes \mathbf{n})^{\mathrm{T}} + \mathbf{n}(\mathbf{n} \otimes \mathbf{s})^{\mathrm{T}} + \mathbf{n}(\mathbf{n} \otimes \mathbf{n})^{\mathrm{T}},$$

since

$$\mathbf{C}(\mathbf{s} \otimes \mathbf{s}) = \mathbf{s}$$

$$\mathbf{C}(\mathbf{s} \otimes \mathbf{n}) = \mathbf{C}(\mathbf{n} \otimes \mathbf{s}) = \mathbf{C}(\mathbf{n} \otimes \mathbf{n}) = \mathbf{n}.$$

The disjunction is computed by the matrix $\mathbf{D}$,

$$\mathbf{D} = \mathbf{s}(\mathbf{s} \otimes \mathbf{s})^{\mathrm{T}} + \mathbf{s}(\mathbf{s} \otimes \mathbf{n})^{\mathrm{T}} + \mathbf{s}(\mathbf{n} \otimes \mathbf{s})^{\mathrm{T}} + \mathbf{n}(\mathbf{n} \otimes \mathbf{n})^{\mathrm{T}},$$

with

$$\mathbf{D}(\mathbf{s} \otimes \mathbf{s}) = \mathbf{D}(\mathbf{s} \otimes \mathbf{n}) = \mathbf{D}(\mathbf{n} \otimes \mathbf{s}) = \mathbf{s}$$

$$\mathbf{D}(\mathbf{n} \otimes \mathbf{n}) = \mathbf{n}.$$

Using the monadic operators, these rectangular matrices can be represented as follows:

$$\mathbf{C} = \mathbf{I} \otimes \mathbf{s}^{\mathrm{T}} + \mathbf{M} \otimes \mathbf{n}^{\mathrm{T}},$$

$$\mathbf{D} = \mathbf{K} \otimes \mathbf{s}^{\mathrm{T}} + \mathbf{I} \otimes \mathbf{n}^{\mathrm{T}}.$$

It can be easily demonstrated that $\mathbf{C}$ and $\mathbf{D}$ are connected by the matrix version of De Morgan's relations:

$$\mathbf{D} = \mathbf{NC}(\mathbf{N} \otimes \mathbf{N}),$$

$$\mathbf{C} = \mathbf{ND}(\mathbf{N} \otimes \mathbf{N}).$$

It can be shown (see Mizraji [12]) that:

(a) the matrix $\mathbf{L}$ corresponding to the implication $p \rightarrow q$ is $\mathbf{L} = \mathbf{D}(\mathbf{N} \otimes \mathbf{I})$,
(b) the matrix $\mathbf{S}$ corresponding to the Sheffer's connective $p \mid q$ is $\mathbf{S} = \mathbf{NC}$,
(c) the matrix $\mathbf{P}$ corresponding to the Peirce's connective $p \downarrow q$ is $\mathbf{P} = \mathbf{ND}$.

3 Logical subspaces Given a vector space V_Q, spanned by an orthonormal basis $B_Q = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_Q\}$, we can define, for any ordered pair $(\mathbf{x}_i, \mathbf{x}_j)$, the following truth values: $\mathbf{s} \equiv \mathbf{x}_i$, $\mathbf{n} \equiv \mathbf{x}_j$. Consequently, a Q-dimensional vector space allows construction of $Q(Q-1)$ different sets of matrix propositional operators. Once the pair $\mathbf{s}$, $\mathbf{n}$, is selected, a class of many valued logic is generated by the operators of the basic vector logic, provided that the inputs have the form $\mathbf{w} = \gamma\mathbf{s} + (1-\gamma)\mathbf{n}$, $\gamma \in [0,1]$.

Let us define the set of these inputs as follows:

Definition 3.1 $\mathcal{F} = \{\gamma\mathbf{s} + (1-\gamma)\mathbf{n} : \gamma \in [0,1]\}$.

"Fuzzy" inputs happen when $\gamma \in (0,1)$.

This set $\mathcal{F}$ is one among the $Q(Q-1)$ 1-dimensional logical subspaces of V_Q supported by the pairs $\{\mathbf{s},\mathbf{n}\}$. These logical subspaces allow us to contact with the probabilistic logics defined in the scalar domain, as we show in what follows.

If $\mathbf{h}$ is the output vector of a matrix operator $\boldsymbol{\Omega}$ the segmental projections are: $\mu_\Omega = \mathbf{s}^T \mathbf{h}$. These segmental projections generate a scalar version of the vectorial many-valued logic produced when the inputs belong to the set $\mathcal{T}$. Given $\mathbf{u} = \alpha\mathbf{s} + (1-\alpha)\mathbf{n}$, $\mathbf{v} = \beta\mathbf{s} + (1-\beta)\mathbf{n}$, $\alpha, \beta \in [0,1]$, we obtain:

$$\mu_N = 1 - \alpha$$

$$\mu_C = \alpha\beta$$

$$\mu_D = \alpha + \beta - \alpha\beta.$$

These last equations define a truth-functional probabilistic logic (again see Mizraji [12]).

We must emphasize that the fundamental property of these 1-dimensional logical subspaces is that they are closed with respect to the propositional operations; that is, if the inputs of a matrix logical operator belong to their particular $\mathcal{T}$, then the output belongs also to this $\mathcal{T}$. Remark that these 1-dimensional vector subspaces support a vectorial many-valued logic with the cardinality of continuum. This fact illustrates an important issue: vector logics require one to distinguish between "dimension" and "many-valuation". In our example, a bidimensional basis generates an infinite-valued vector logic. The potentialities of logics with further dimensionalities remain to be explored.

4 Recursive generation of modalities An important area of research in modal logic is concerned with the definitions of the "possibility" and "necessity" operators. An exhaustive description of this area can be found in the text of Hughes and Cresswell [6].

Given a proposition p, the modalities "p is possible", $\Diamond p$, and "p is necessary", $\Box p$, are linked by the following Aristotelian equivalence: $\Box p \; eq \; \neg \Diamond(\neg p)$. One of the objectives of the following constructions is to define, within the framework of vector logic, matrix modal operators that satisfy the Aristotelian equivalences. A complementary objective is to built up these modal operators using the $\mathbf{D}$ and $\mathbf{C}$ operators of propositional vector logic, defined in Section 2.

Let us mention an important historical antecedent. In a classical work published in 1930, Łukasiewicz showed that the existence of truth-functional modal operators needs the construction of many-valued logics (Łukasiewicz [9],[10]). In the case of the Łukasiewicz 3-valued logic, $Ł_3$, the truth values for a proposition p could be 0, $\frac{1}{2}$, and 1. The negation was symbolically defined by: $\neg 1 = 0$, $\neg 0 = 1$, $\neg\frac{1}{2} = \frac{1}{2}$. Representing the truth value of a proposition X as $|X|$, the implication $p \to q$ satisfies in $Ł_3$ the usual relations for the values 0,1 (corresponding to F and T), and for the third truth value we have $|0 \to \frac{1}{2}| = |\frac{1}{2} \to \frac{1}{2}| = |\frac{1}{2} \to 1| = 1$, $|\frac{1}{2} \to 0| = |1 \to \frac{1}{2}| = \frac{1}{2}$.

In his [9], Łukasiewicz reported a truth-functional definition of "possibility" proposed by Tarski for the 3-valued logic $Ł_3$. Tarski defined "p is possible", $\Diamond p$, as follows: $\Diamond p =_{\text{Def}} \neg p \to p$. Hence, for $Ł_3$, Tarski's operator has the following truth table:

p	$\Diamond p$
0	0
$\frac{1}{2}$	1
1	1

We are now going to show that, within the framework of vector logic, these previous results by Łukasiewicz and Tarski provide strong hints about how to build vectorial modal operators using a recursive procedure.

In the basic vector logic, the matrix that executes the implication, $\mathbf{L}$, generates, when the inputs are vectors $\mathbf{u}, \mathbf{v} \in \mathcal{T}$, the following segmental projection (Mizraji [12]): $\mu_L = 1 - \alpha(1 - \beta)$. Note that if $\alpha = \beta = \frac{1}{2}$, we get $\mu_L = \frac{3}{4}$; hence, this scalar implication does not coincide with $Ł_3$ implication ($|\frac{1}{2} \to \frac{1}{2}| = 1$). Note also that in the Kleene 3-valued logic, it is $|\frac{1}{2} \to \frac{1}{2}| = \frac{1}{2}$ (see Rescher [13]). Due to this discrepancy, Tarski's definition of possibility does not work in the basic vector logic. However, Tarski's definition provides the fundamental clue for the design of vector modal operators. Note that in the previously described vector logic, the expression $\neg p \to p$ is represented by the equation $\mathbf{L}(\mathbf{Nu} \otimes \mathbf{u}) = \mathbf{L}(\mathbf{N} \otimes \mathbf{I})(\mathbf{u} \otimes \mathbf{u})$. But, since $\mathbf{L}(\mathbf{N} \otimes \mathbf{I}) = \mathbf{D}$, we get $\mathbf{L}(\mathbf{Nu} \otimes \mathbf{u}) = \mathbf{D}(\mathbf{u} \otimes \mathbf{u})$. This expression is equivalent to the classical binary tautology $p \vee q\ eq\ \neg p \to q$.

Let us construct the following sequence:

$$\mathbf{u}_1 = \mathbf{D}(\mathbf{u} \otimes \mathbf{u})$$

$$\mathbf{u}_2 = \mathbf{D}(\mathbf{u}_1 \otimes \mathbf{u}_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathbf{u}_{r+1} = \mathbf{D}(\mathbf{u}_r \otimes \mathbf{u}_r).$$

We now define each term of this sequence as a "possibility of order r", in brief, r-possibility. We use the notation $\Diamond_r(\mathbf{u}) \equiv \mathbf{u}_r$.

Definition 4.1 We define the "crisp" possibility (c-possibility) operator $\Diamond$ as the limit of this sequence for $r \to \infty$:

$$\Diamond(\mathbf{u}) \equiv \lim_{r \to \infty} \{\mathbf{D}(\mathbf{u}_r \otimes \mathbf{u}_r)\}.$$

The importance of this limit will be shown in Section 6.

The "necessities of order r" (r-necessities) are generated by the successive terms of the following iterative procedure:

$$\mathbf{v}_1 = \mathbf{C}(\mathbf{v} \otimes \mathbf{v})$$

$$\mathbf{v}_2 = \mathbf{C}(\mathbf{v}_1 \otimes \mathbf{v}_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathbf{v}_{r+1} = \mathbf{C}(\mathbf{v}_r \otimes \mathbf{v}_r).$$

The notation for the r-necessity is $\Box_r(\mathbf{v}) \equiv \mathbf{v}_r$. The limit of this sequence allows to define a vectorial "crisp" necessity (c-necessity) operator:

Definition 4.2 $\Box(\mathbf{v}) \equiv \lim_{r \to \infty} \{\mathbf{C}(\mathbf{v}_r \otimes \mathbf{v}_r)\}.$

It can be immediately proved, using the De Morgan relations between **D** and **C**, that the r-possibilities and the r-necessities, are linked by matrix versions of the Aristotelian equivalences.

*5 **Matrix modal operators*** Assume that a pair of orthonormal vectors **s**, **n** has been selected from a vector space V_Q. The recursive construction showed previously allows us to define the modalities of order r by means of the following expressions:

$$\Diamond_r(\mathbf{u}) = \left(\prod_{i=0}^{r-1} \mathbf{D}^{[f(i)]}\right)\mathbf{u}^{[f(r)]}$$

$$\Box_r(\mathbf{v}) = \left(\prod_{i=0}^{r-1} \mathbf{C}^{[f(i)]}\right)\mathbf{v}^{[f(r)]},$$

where $f(i) = 2^i$; the exponents represent the Kronecker powers. For instance, the possibility and the necessity of order 3 are given by the expressions:

$$\Diamond_3(\mathbf{u}) = [\mathbf{D}(\mathbf{D}\otimes\mathbf{D})(\mathbf{D}\otimes\mathbf{D}\otimes\mathbf{D}\otimes\mathbf{D})]\,(\mathbf{u}^{[8]})$$

$$\Box_3(\mathbf{v}) = [\mathbf{C}(\mathbf{C}\otimes\mathbf{C})(\mathbf{C}\otimes\mathbf{C}\otimes\mathbf{C}\otimes\mathbf{C})]\,(\mathbf{v}^{[8]}).$$

Note that **u** and **v** are Q-dimensional column vectors; hence, the Kronecker exponent $f(r)$ generates a $Q^{f(r)}$-dimensional column vector. Consequently, any modality of order r is generated by a matrix of order $Q \times Q^{f(r)}$ acting on a vector of order $Q^{f(r)} \times 1$. The output, naturally, is a Q-dimensional vector.

It is easy to demonstrate, from the properties of the Kronecker product, that the De Morgan relation between **D** and **C** generates Aristotelian relations between r-modalities: $\Box_r(\mathbf{u}) = \mathbf{N}\Diamond_r(\mathbf{Nu})$. We can illustrate the general demonstration procedure for the particular case of modalities of order 2:

$$\begin{aligned}\Box_2(\mathbf{u}) &= [\mathbf{C}(\mathbf{C}\otimes\mathbf{C})]\,(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\\ &= [\mathbf{ND}(\mathbf{N}\otimes\mathbf{N})(\mathbf{C}\otimes\mathbf{C})]\,(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\\ &= [\mathbf{ND}(\mathbf{NC}\otimes\mathbf{NC})]\,\mathbf{N}^{[4]}\mathbf{N}^{[4]}(\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u}\otimes\mathbf{u})\\ &= [\mathbf{ND}(\mathbf{D}\otimes\mathbf{D})]\,(\mathbf{Nu}\otimes\mathbf{Nu}\otimes\mathbf{Nu}\otimes\mathbf{Nu}) = \mathbf{N}\Diamond_2(\mathbf{Nu}).\end{aligned}$$

The same constructive procedure of demonstration can be extended to modalities of any order. Interpreting the modalities as the successive terms of a sequence, we define the crisp modal operators as the limit of these sequences:

$$\Diamond(\mathbf{u}) \equiv \lim_{r\to\infty}\{\Diamond_r(\mathbf{u})\}$$

$$\Box(\mathbf{v}) \equiv \lim_{r\to\infty}\{\Box_r(\mathbf{v})\}.$$

These limits can be interpreted as the product of a rectangular non-finite matrix acting over a non-finite Kronecker power, generating a Q-dimensional vector as output. These limit matrix operations are symbolically representable by the following expressions:

$$\Diamond(\mathbf{u}) = \left(\prod_{i=0}^{\infty} \mathbf{D}^{[f(i)]}\right)\mathbf{u}^{[\infty]}$$

$$\Box(\mathbf{v}) = \left(\prod_{i=0}^{\infty} \mathbf{C}^{[f(i)]}\right)\mathbf{v}^{[\infty]}.$$

6 "Crisp" vectorial modalities on logical subspaces We are going to show in this section that when the crisp vector modal operators act on the logical subspaces defined in Section 3, we obtain vectorial versions of some of the classical results of modal logic. The strategy that we follow here is to analyze, for arbitrary vectors $\mathbf{u},\mathbf{v}$ belonging to the logical subspace $\mathcal{T}$, the scalar projections of the successive r-modalities.

Given $\mathbf{u} = \alpha\mathbf{s} + (1-\alpha)\mathbf{n}$, $\alpha \in [0,1]$, the segmental projection of the possibility of order r is $\alpha_r \equiv \mathbf{s}^{\mathrm{T}}\mathbf{u}_r$. Therefore, $\alpha_r = 1 - (1-\alpha)^t$, $t = 2^r$.

Let $\Diamond\alpha \equiv \mathbf{s}^{\mathrm{T}}(\Diamond\mathbf{u})$; consequently, $\Diamond\alpha = \lim_{t\to\infty}[1 - (1-\alpha)^t]$, with

$$\Diamond\alpha = \begin{cases} 1 & \text{iff } \alpha \neq 0 \\ 0 & \text{iff } \alpha = 0. \end{cases}$$

This last equation is, in fact, a classical definition of "possibility" for scalar truth values that take their values from the interval [0,1] (see Lewis and Langford [8] and Rescher [13]). Hence, due to the fact that the space $\mathcal{T}$ is closed under the propositional operator $\mathbf{D}$, the sequence converges to the following limits:

$$\Diamond\mathbf{u} = \begin{cases} \mathbf{s} & \text{iff } \mathbf{u} \neq \mathbf{n} \\ \mathbf{n} & \text{iff } \mathbf{u} = \mathbf{n}. \end{cases}$$

The equivalent analysis of r-necessities shows that for a basic vector logic, given a vector $\mathbf{v} = \beta\mathbf{s} + (1-\beta)\mathbf{n}$, the sequence corresponding to the segmental projections are given by $\beta_r \equiv \mathbf{s}^{\mathrm{T}}\mathbf{v}_r$. Defining $\Box\beta \equiv \mathbf{s}^{\mathrm{T}}(\Diamond\mathbf{v})$, we obtain $\Box\beta = \lim_{t\to\infty}\beta^t$, $t = 2^r$.

Hence, it corresponds to the classical definition of necessity for scalar variables belonging to [0,1]:

$$\Box\beta = \begin{cases} 1 & \text{iff } \beta = 1 \\ 0 & \text{iff } \beta \neq 1. \end{cases}$$

Consequently, if the argument is an arbitrary vector belonging to the logical subspace $\mathcal{T}$, the operation used to define the vectorial necessity, converges to the following limits:

$$\Box\mathbf{v} = \begin{cases} \mathbf{s} & \text{iff } \mathbf{v} = \mathbf{s} \\ \mathbf{n} & \text{iff } \mathbf{v} \neq \mathbf{s}. \end{cases}$$

In the usual axiomatic presentations of modal logics, for the construction of the two basic modal operators, once defined one of them, the Aristotelian relationship between necessity and possibility is used to define the other. For instance, once the possibility operation $\Diamond p$ is built, the necessity operation, $\Box p$, is defined as follows:

$$\Box p =_{\text{Def}} \neg\Diamond\neg p.$$

In the framework of the basic vector logic, the "crisp" modal operators $\Diamond$ and $\Box$ are constructed from the matrix operators **D** and **C**, respectively, and their link is shown in the following theorem:

Theorem 6.1 $\quad \Box\mathbf{u} = \mathbf{N}\Diamond(\mathbf{Nu}).$

Proof:
$$\Box\mathbf{u} = \lim_{r\to\infty}\{\mathbf{C}(\mathbf{u}_r \otimes \mathbf{u}_r)\}$$
$$= \lim_{r\to\infty}\{\mathbf{ND}(\mathbf{N}\otimes\mathbf{N})(\mathbf{u}_r \otimes \mathbf{u}_r)\}$$
$$= \mathbf{N}[\lim_{r\to\infty}\{\mathbf{D}(\mathbf{Nu}_r \otimes \mathbf{Nu}_r)\}] = \mathbf{N}\Diamond(\mathbf{Nu}).$$

The scalar version of our system of modal logic is connected with the probabilistic logics, but it constitutes a special case in which all the operations are truth-functional. Rescher has demonstrated that a non truth-functional probabilistic logic satisfies the axioms of system S5 (see Rescher [13]).

Given the propositions p and q, consider the following expressions:

1. $p \to \Diamond p$
2. $\Box p \to p$
3. $\Box p \to \Diamond p$
4. $\Box(p\to q) \to (\Box p \to \Box q)$.

For the basic vector logic, the preceding expressions become the following equalities $(\mathbf{u},\mathbf{v} \in \mathcal{T})$:

1′. $\mathbf{L}(\mathbf{u}\otimes\Diamond\mathbf{u}) = \mathbf{s}$
2′. $\mathbf{L}(\Box\mathbf{u}\otimes\mathbf{u}) = \mathbf{s}$
3′. $\mathbf{L}(\Box\mathbf{u}\otimes\Diamond\mathbf{u}) = \mathbf{s}$
4′. $\mathbf{L}[\Box\mathbf{L}(\mathbf{u}\otimes\mathbf{v})\otimes\mathbf{L}(\Box\mathbf{u}\otimes\Box\mathbf{v})] = \mathbf{s}$.

This set of equalities can be easily demonstrated taking into consideration the following properties of the vectorial implication:

$$\mathbf{L}(\mathbf{u}\otimes\mathbf{s}) = \mathbf{L}(\mathbf{n}\otimes\mathbf{u}) = \mathbf{s}$$
$$\mathbf{L}(\mathbf{s}\otimes\mathbf{u}) = \mathbf{u}$$
$$\mathbf{L}(\mathbf{u}\otimes\mathbf{n}) = \mathbf{Nu}.$$

Proof: (of 1′) If $\mathbf{u}=\mathbf{n}$, $\Diamond\mathbf{u}=\mathbf{n}$ and $\mathbf{L}(\mathbf{n}\otimes\mathbf{n}) = \mathbf{Nn} = \mathbf{s}$. If $\mathbf{u}\neq\mathbf{n}$, $\Diamond\mathbf{u} = \mathbf{s}$ and $\mathbf{L}(\mathbf{s}\otimes\mathbf{s}) = \mathbf{s}$. Hence, $\mathbf{L}(\mathbf{u}\otimes\Diamond\mathbf{u}) = \mathbf{s}\,\forall\mathbf{u}\in\mathcal{T}$.

It is immediate to see that the axiom of Brouwer $p\to\Box\Diamond p$ has an equivalent version in the basic vector logic: $\mathbf{L}(\mathbf{u}\otimes\Box\Diamond\mathbf{u}) = \mathbf{s}$. The strict implication defined by Lewis (see Lewis and Langford [8]), $p \prec q \underset{\text{Def}}{=} \neg\Diamond(p\wedge\neg q)$, can be expressed in vector logic by the equation $\mathbf{L}_e(\mathbf{u},\mathbf{v}) \equiv \mathbf{N}\Diamond[\mathbf{C}(\mathbf{u}\otimes\mathbf{Nv})]$. By means of De Morgan's relations, we get $\mathbf{L}_e(\mathbf{u},\mathbf{v}) = \mathbf{N}\Diamond[\mathbf{NL}(\mathbf{u}\otimes\mathbf{v})]$. Hence, by Theorem 6.1 $\mathbf{L}_e(\mathbf{u},\mathbf{v}) = \Box[\mathbf{L}(\mathbf{u}\otimes\mathbf{v})]$.

*7 **Extensional quantification*** The classical Aristotelian relation between modal operators, symbolically expressed "$\Box = \neg\Diamond\neg$", is analogous to the relation between quantifiers in the first order predicate calculus, symbolically "$\forall = \neg\exists\neg$", where $\forall$ represents the universal quantifier, and $\exists$ represents the existential quantifier.

The established connection between the operators $\Diamond$ and $\Box$ and matrices **D** and **C**, respectively, has a counterpart in the theory of vectorial quantifiers. We will now describe an extensional version of the quantifiers in the framework of basic vector logic.

Let us assume a predicate $P(x)$ and a universe defined by a finite set of variables $x_i : U = \{x_1, x_2, \ldots, x_R\}$. Suppose the existence of a valuation Val(P) that assigns to each $P(x_i)$ a vectorial truth value: $\mathrm{Val}[P(x_i)] = \mathbf{u}_i, \mathbf{u}_i \in \{\mathbf{s}, \mathbf{n}\}$.

The extensional definitions for the classical quantifiers are given by

$$\exists x P(x) \; eq \; P(x_1) \vee P(x_2) \vee \ldots \vee P(x_R)$$

$$\forall x P(x) \; eq \; P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_R).$$

The matrix operators disjunction **D** and conjunction **C** allow the construction of vector versions for the quantifiers, using as a guide the previous extensional definitions. Consider the following sequence:

$$\mathbf{d}_1 = \mathbf{u}_1$$

$$\mathbf{d}_2 = \mathbf{D}(\mathbf{u}_2 \otimes \mathbf{d}_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathbf{d}_{n+1} = \mathbf{D}(\mathbf{u}_{n+1} \otimes \mathbf{d}_n),$$

where $\mathbf{u}_i = \mathrm{Val}[P(x_i)]$. We define

$$\exists[\mathbf{u}] \equiv \lim_{n \to R} \{\mathbf{d}_n\}.$$

Consider now the sequence:

$$\mathbf{c}_1 = \mathbf{v}_1$$

$$\mathbf{c}_2 = \mathbf{C}(\mathbf{v}_2 \otimes \mathbf{c}_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathbf{c}_{n+1} = \mathbf{C}(\mathbf{v}_{n+1} \otimes \mathbf{c}_n),$$

with $\mathbf{v}_i = \mathrm{Val}[P(x_i)]$. Hence, we define

$$\forall[\mathbf{u}] \equiv \lim_{n \to R} \{\mathbf{c}_n\}.$$

Note that these constructions are also possible for infinite numerable universes U. With the preceding definitions, the existence of an x_i such that $P(x_i)$ is true, implies $\exists[\mathbf{u}] = \mathbf{s}$. On the other hand, if $P(x_i)$ is true for any $x_i \in U$, we have $\forall[\mathbf{u}] = \mathbf{s}$. These results can be immediately proved by analyzing the segmental projections.

In order to generalize the argument, we assume in what follows that the valuation represented by vector **u** admits uncertainties. Hence we assume that

$\mathbf{u} \in \mathcal{T}$, and a particular $P(x_i)$ has a valuation $\mathbf{u}_i = \alpha_i \mathbf{s} + (1 - \alpha_i)\mathbf{n}$. The corresponding segmental projections are

$$\mathbf{s}^{\mathrm{T}} \exists [\mathbf{u}] = 1 - (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_R)$$

$$\mathbf{s}^{\mathrm{T}} \forall [\mathbf{u}] = \alpha_1 \alpha_2 \cdots \alpha_R.$$

It is interesting to notice that these formulae have been described at the end of the XVII century, in a comment about the confiability of testimonies (see [1] and [7], p. 184).

It can be demonstrated that the vector quantifiers are connected by the following theorem:

Theorem 7.1 $\forall [\mathbf{u}] = \mathbf{N} \exists [\mathbf{N}\mathbf{u}]$.

Proof: Let $\mathbf{c}_3[\mathbf{u}] = \mathbf{C}[\mathbf{u}_3 \otimes \mathbf{C}(\mathbf{u}_2 \otimes \mathbf{u}_1)] = \mathbf{C}(\mathbf{I} \otimes \mathbf{C})(\mathbf{u}_3 \otimes \mathbf{u}_2 \otimes \mathbf{u}_1)$. Using De Morgan's relations, we get:

$$\begin{aligned} &\mathbf{C}(\mathbf{I} \otimes \mathbf{C})(\mathbf{u}_3 \otimes \mathbf{u}_2 \otimes \mathbf{u}_1) \\ &\quad = \mathbf{ND}(\mathbf{N} \otimes \mathbf{N})[\mathbf{I} \otimes \mathbf{ND}(\mathbf{N} \otimes \mathbf{N})](\mathbf{u}_3 \otimes \mathbf{u}_2 \otimes \mathbf{u}_1) \\ &\quad = \mathbf{ND}[\mathbf{NI} \otimes \mathbf{NND}(\mathbf{N} \otimes \mathbf{N})](\mathbf{u}_3 \otimes \mathbf{u}_2 \otimes \mathbf{u}_1) \\ &\quad = \mathbf{ND}(\mathbf{I} \otimes \mathbf{D})(\mathbf{Nu}_3 \otimes \mathbf{Nu}_2 \otimes \mathbf{Nu}_1). \end{aligned}$$

Hence

$$\mathbf{c}_3[\mathbf{u}] = \mathbf{Nd}_3[\mathbf{Nu}].$$

The extension by induction of this procedure completes the proof.

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