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1 Selection of optimal artificial boundary condition (ABC) frequencies  
2 for structural damage identification

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9 **Abstract**

10 In this paper, the sensitivities of artificial boundary condition (ABC) frequencies to the  
11 damages are investigated, and the optimal sensors are selected to provide the reliable  
12 structural damage identification. The sensitivity expressions for one-pin and two-pin ABC  
13 frequencies, which are the natural frequencies from structures with one and two additional  
14 constraints to its original boundary condition, respectively, are proposed. Based on the  
15 expressions, the contributions of the underlying mode shapes in the ABC frequencies can be  
16 calculated and used to select more sensitive ABC frequencies. Selection criteria are then  
17 defined for different conditions, and their performance in structural damage identification is  
18 examined with numerical studies. From the findings, conclusions are given.

19 **Key word:** ABC frequency, structural damage identification, numerical simulation, sensitivity  
20 analysis, selection criteria

21 **1. Introduction**

22 Structural Health Monitoring (SHM) is an important subject in today's civil engineering  
23 practice. This is particularly true for large scale structures, for example, bridges and high-rise  
24 buildings, since severe damages or collapse of these structures will cause not only significant  
25 economic loss, but also loss of human lives. Therefore, the interest in the ability to monitor  
26 the structures and detect damage at an early stage becomes pervasive.

27 Among many other approaches to structural health monitoring and damage identification,  
28 model-based methods, in particular finite element model updating, have attracted extensive  
29 attention in the past few decades [1-13]. In such a procedure, the errors in a computational  
30 finite element model are corrected by minimizing the discrepancy between the measured  
31 and simulated response data. The parameters after updating may serve as indicators of the  
32 structural or damage conditions, while the updated FE model as a whole can be used for  
33 current and future performance predictions for the structure in question. The most  
34 commonly used response data for FE model updating are the dynamic modal data, such as  
35 natural frequencies, mode shapes, and to a lesser extent damping. However, in practical  
36 conditions, the noise contained in the response data, especially in mode shapes, dictate that  
37 only a limited amount of such data as acquired from a physical test may be useful in the  
38 actual FE model updating operation [14-16].

39 Comparing to the mode shapes, natural frequencies are generally known to be measurable  
40 with higher accuracy. However, natural frequencies are not sensitive to local damages.  
41 Moreover, since each natural mode has only one frequency, the total number of natural  
42 frequencies that can be measured with high quality is always limited [17-18]. These restrict  
43 the ability of using natural frequency alone in relatively complex problems involving a large  
44 number of variable parameters. It would be highly desirable if additional modal frequencies  
45 can be generated to enhance the frequency dataset in the general damage detection and  
46 structural identification field.

47 The concept of perturbed natural frequencies of a structure under different (perturbed)  
48 boundary conditions opens up a new avenue where more modal frequency data may be  
49 generated for damage identification [19-21]. The development of the theory of the artificial  
50 boundary condition (ABC) methodology, by which the above mentioned perturbed natural  
51 frequencies of the structure with additional pin supports could be derived from the  
52 incomplete FRF matrix measured from the original structure, as well as the subsequent  
53 studies on the effectiveness of such frequencies in structural identification, brings the  
54 incorporation of the perturbed natural frequencies a significant step closer to practical  
55 applications. Several researches have been devoted to the structural damage identification  
56 using ABC frequencies in the last few decades [22-27].

57 Despite the above advancements, since a large variety of perturbed boundary conditions, or  
58 the ABC pin supports, may be configured for the ABC frequencies, appropriate criteria for  
59 the selection of better (or more sensitive) ABC frequencies for inclusion in the structural  
60 identification or FE model updating need be developed.

61 In this paper, the sensitivities of ABC frequencies to the damages are investigated. For  
62 simplicity and without losing generality, only one-pin and two-pin ABC frequencies are  
63 employed. In the study, the approach of deriving anti-resonance (one-pin ABC frequencies)  
64 sensitivity expression for lightly damped structures is adopted and is extended to two-pin  
65 ABC frequencies. On the basis of the expressions of one-pin and two-pin ABC frequency  
66 sensitivities, the contributions of the underlying mode shapes in the ABC frequencies can be  
67 calculated, and this enables the selection of more sensitive ABC frequencies for FE model  
68 updating. Following the basic formulation, numerical studies are then used to examine the  
69 performance of the selected ABC frequencies in damage identification of a lightly damped  
70 structure. Moreover, the size of ABC frequencies for better identification performance is  
71 also studied. Finally, conclusions are given and future work is suggested.

## 72 **2. Theory of ABC frequency sensitivities**

### 73 2.1 Sensitivity of driving point anti-resonance (one-pin ABC frequencies) and mode shape 74 contributions

75 In modal analysis, the frequency response function (FRF) of an un-damped system can  
76 generally be expressed as follows:

77  
78

$$\mathbf{h}_{ij}(\omega) = \sum_{k=1}^n \frac{\Phi_{ik}\Phi_{jk}}{(\omega_k^2 - \omega^2)} \quad (1)$$

79 Where  $i, j$  represent positions of the excitation force and the response, respectively.  $\boldsymbol{\varphi}_{ik}$  and  
 80  $\boldsymbol{\varphi}_{jk}$  are the  $k$ -th order mode shape at points  $i$  and  $j$ , respectively, and  $\omega_k$  is the  $k$ -th order  
 81 natural frequency.

82 When the applied force and measured response are at the same position, the corresponding  
 83 FRF reduces to the driving-point FRF as:

$$84 \quad \mathbf{h}_{ii}(\omega) = \sum_{k=1}^n \frac{\boldsymbol{\varphi}_{ik}^2}{(\omega_k^2 - \omega^2)} \quad (2)$$

85 According to Mottershead [28], Eq.(2) can be rearranged as follows:

$$86 \quad \mathbf{h}_{ii}(\omega) = \sum_{k=1}^n \frac{\boldsymbol{\varphi}_{ik} \det(\Lambda - \omega^2 \mathbf{I})_k \boldsymbol{\varphi}_{ik}}{\det(\Lambda - \omega^2 \mathbf{I})} \quad (3)$$

87 where  $\det(\Lambda - \omega^2 \mathbf{I})_k = (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) \cdots (\omega_{k-1}^2 - \omega^2)(\omega_{k+1}^2 - \omega^2) \cdots (\omega_n^2 - \omega^2)$

88 The driving point anti-resonance frequencies, which are just one-pin ABC frequencies, can be  
 89 obtained by setting Eq.(3) to zero, thus:

$$90 \quad \sum_{k=1}^n \boldsymbol{\varphi}_{ik} \det(\Lambda - \omega^2 \mathbf{I})_k \boldsymbol{\varphi}_{ik} = 0 \quad (4)$$

91 where  $\omega_{1-pin_i}$  denotes anti-resonance from driving-point FRFs measured at point  $i$  (i.e.  
 92 one-pin ABC frequency with pin at point  $i$ ).

93 Differentiating each term in Eq. (4) with respect to a variable parameter  $p$ :

$$94 \quad \frac{\partial}{\partial p} (\boldsymbol{\varphi}_{ik} \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_k \boldsymbol{\varphi}_{ik}) = \frac{\partial \boldsymbol{\varphi}_{ik}}{\partial p} \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_k \boldsymbol{\varphi}_{ik} \\ + \boldsymbol{\varphi}_{ik} \frac{\partial}{\partial p} (\det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_k) \boldsymbol{\varphi}_{ik} + \boldsymbol{\varphi}_{ik} \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_k \frac{\partial \boldsymbol{\varphi}_{ik}}{\partial p} \quad (5)$$

95 Based on Mottershead [28] substituting Eq. (5) into Eq. (4) yields:

$$96 \quad \frac{\partial \omega_{1-pin_i}^2}{\partial p} = 2 \times \frac{\sum_{k=1}^n \frac{\partial \boldsymbol{\varphi}_{ik}}{\partial p} \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_k \boldsymbol{\varphi}_{ik} + \sum_{p=1}^n \frac{\partial \omega_p^2}{\partial p} \left( \sum_{k=p}^n \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_{k,p} \boldsymbol{\varphi}_{ik} \boldsymbol{\varphi}_{ik} \right)}{\sum_{k=1}^n \boldsymbol{\varphi}_{ik} \left( \sum_{p=k}^n \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_{k,p} \right) \boldsymbol{\varphi}_{ik} + \sum_{k=1}^n \boldsymbol{\varphi}_{ik} \left( \sum_{p=k}^n \det(\Lambda - \omega_{1-pin_i}^2 \mathbf{I})_{k,p} \right) \boldsymbol{\varphi}_{ik}} \quad (6)$$

97 From Eq. (6), it is clear that the sensitivity of the one-pin ABC frequencies to a particular  
 98 structural parameter is a combination of the respective sensitivity of mode shapes at the  
 99 same point and the sensitivity of the natural frequencies.

100 Hanson et al. [29] also proposed the expression of anti-resonance sensitivity. A two-DOF  
 101 system was used in their study, leading to a simplified expression of the FRF from Eq. (2)  
 102 with  $n=2$ , and the subsequent sensitivity of the anti-resonance is effectively a special case of  
 103 Eq. (6).

104 Theoretically speaking, based on Eq. (6), it is possible to calculate the contributions  
 105 ("footprint") of the mode shapes in the driving-point anti-resonance (one-pin ABC)

106 frequency sensitivities. Hanson et al. [29] employed a ratio to represent the mode shape  
 107 contributions in the sensitivities of the anti-resonances, with the following expression:

$$108 \quad C = \frac{|\Phi|}{|\Omega| + |\Phi|} \quad (7)$$

109 where C is the relative mode shape contribution ratio,  $\Omega$  denotes the natural frequency  
 110 contribution in the anti-resonance (1-pin ABC) sensitivity, and  $\Phi$  is the mode shape  
 111 contribution in the anti-resonance (1-pin ABC) sensitivity,

$$112 \quad \Omega = \frac{\sum_{p=1}^n \frac{\partial \omega_p^2}{\partial \rho} \left( \sum_{k=1}^n \det(\Lambda - \omega_{r-pin_j}^2 I)_{k,p} \Phi_{ik} \Phi_{ik} \right)}{\sum_{k=1}^n \Phi_{ik} \left( \sum_{p=k}^n \det(\Lambda - \omega_{r-pin_j}^2 I)_{k,p} \right) \Phi_{ik}} \quad (7a)$$

$$113 \quad \Phi = 2 \times \frac{\sum_{k=j}^n \frac{\partial \Phi_{ik}}{\partial \rho} \det(\Lambda - \omega_{r-pin_j}^2 I)_k \Phi_{ik}}{\sum_{k=j}^n \Phi_{ik} \left( \sum_{p=k}^n \det(\Lambda - \omega_{r-pin_j}^2 I)_{k,p} \right) \Phi_{ik}} \quad (7b)$$

114 The effectiveness of using the ratio in a two-DOF system has been verified with both  
 115 numerical and experimental studies [29]. Thus, the (1-pin) ABC frequencies that contain a  
 116 larger mode shape contribution are expected to be relatively more sensitive to damage and  
 117 hence should be selected for the FE model updating.

## 118 2.2 Two-pin ABC frequency sensitivity and mode shape contributions

119 The above method of evaluating the anti-resonance (1-pin ABC) frequency sensitivity can be  
 120 extended to two-pin ABC frequencies, thus Eq. (7) can also be employed to obtain mode  
 121 shape contributions in the two-pin ABC frequency sensitivities.

122 As described in previous studies [20-21], two-pin ABC frequencies can be obtained by  
 123 inverting the  $2 \times 2$  FRF matrix measured at these two pin points, and each element (ABC  
 124 curve) in the inverted matrix can be used to determine the two-pin ABC frequencies by  
 125 identifying the singular (peak) frequencies.

126 For simplicity, consider only the first three natural modes in the structural response, the  
 127  $2 \times 2$  FRF matrix can then be expressed as:

$$128 \quad \mathbf{H} = \begin{bmatrix} \mathbf{h}_{ij} & \mathbf{h}_{jj} \\ \mathbf{h}_{ji} & \mathbf{h}_{jj} \end{bmatrix} \quad (8)$$

129 where  $h_{ii}$  is the FRF containing the first three modes of information,

130

131 
$$\mathbf{h}_{ii} = \frac{\omega_{i1}^2}{\omega_1^2 - \omega^2} + \frac{\omega_{i2}^2}{\omega_2^2 - \omega^2} + \frac{\omega_{i3}^2}{\omega_3^2 - \omega^2} \quad (8a)$$

132 Inverting the above matrix yields:

133 
$$\mathbf{H}^{-1} = \frac{1}{\begin{vmatrix} \mathbf{h}_{ii} & -\mathbf{h}_{ij} \\ \mathbf{h}_{ij} & \mathbf{h}_{jj} \end{vmatrix}} \begin{bmatrix} \mathbf{h}_{jj} & -\mathbf{h}_{ij} \\ -\mathbf{h}_{ij} & \mathbf{h}_{ii} \end{bmatrix}$$

134 (9)

135 From Eq. (9), the singular (peak) frequencies in the inverted matrix, i.e. the two-pin ABC  
136 frequencies, can be calculated by setting  $|\mathbf{h}_{ii}\mathbf{h}_{jj} - \mathbf{h}_{ij}\mathbf{h}_{ji}|$  to zero:

137 
$$\left( \frac{\Phi_{i1}^2}{\omega_1^2 - \omega_{2-pin}^2} + \frac{\Phi_{i2}^2}{\omega_2^2 - \omega_{2-pin}^2} + \frac{\Phi_{i3}^2}{\omega_3^2 - \omega_{2-pin}^2} \right) \left( \frac{\Phi_{j1}^2}{\omega_1^2 - \omega_{2-pin}^2} + \frac{\Phi_{j2}^2}{\omega_2^2 - \omega_{2-pin}^2} + \frac{\Phi_{j3}^2}{\omega_3^2 - \omega_{2-pin}^2} \right) - \left( \frac{\Phi_{i1}\Phi_{j1}}{\omega_1^2 - \omega_{2-pin}^2} + \frac{\Phi_{i2}\Phi_{j2}}{\omega_2^2 - \omega_{2-pin}^2} + \frac{\Phi_{i3}\Phi_{j3}}{\omega_3^2 - \omega_{2-pin}^2} \right)^2 = 0 \quad (10)$$

138 From Eq. (10), the two-pin ABC frequencies with pins at  $i$  and  $j$  can be represented as:

139 
$$\omega_{2-pin}^2 = \frac{\mathbf{A1} \times \omega_3^2 + \mathbf{A2} \times \omega_2^2 + \mathbf{A3} \times \omega_1^2}{\mathbf{A1} + \mathbf{A2} + \mathbf{A3}}$$

140 (11)

141 where  $\mathbf{A1} = (\Phi_{i1}\Phi_{j2} - \Phi_{i2}\Phi_{j1})^2$ ,  $\mathbf{A2} = (\Phi_{i1}\Phi_{j3} - \Phi_{i3}\Phi_{j1})^2$ ,  $\mathbf{A3} = (\Phi_{i2}\Phi_{j3} - \Phi_{i3}\Phi_{j2})^2$

142 The derivative of two-pin ABC frequencies with respect to a variable parameter  $p$  can be  
143 further expressed as follows:

144 
$$\frac{\partial \omega_{2-pin}^2}{\partial p} = \frac{\left( \frac{\partial \mathbf{A1}}{\partial p} \omega_3^2 + \frac{\partial \omega_3^2}{\partial p} \mathbf{A1} + \frac{\partial \mathbf{A2}}{\partial p} \omega_2^2 + \frac{\partial \omega_2^2}{\partial p} \mathbf{A2} + \frac{\partial \mathbf{A3}}{\partial p} \omega_1^2 + \frac{\partial \omega_1^2}{\partial p} \mathbf{A3} \right) (\mathbf{A1} + \mathbf{A2} + \mathbf{A3})}{(\mathbf{A1} + \mathbf{A2} + \mathbf{A3})^2} - \frac{\left( \frac{\partial \mathbf{A1}}{\partial p} + \frac{\partial \mathbf{A2}}{\partial p} + \frac{\partial \mathbf{A3}}{\partial p} \right) (\mathbf{A1} \times \omega_3^2 + \mathbf{A2} \times \omega_2^2 + \mathbf{A3} \times \omega_1^2)}{(\mathbf{A1} + \mathbf{A2} + \mathbf{A3})^2}$$

145 (12)

146 where  $\frac{\partial \mathbf{A1}}{\partial p} = 2(\Phi_{i1}\Phi_{j2} - \Phi_{i2}\Phi_{j1}) \left( \frac{\partial \Phi_{i1}}{\partial p} \Phi_{j2} + \Phi_{i1} \frac{\partial \Phi_{j2}}{\partial p} - \frac{\partial \Phi_{i2}}{\partial p} \Phi_{j1} - \Phi_{i2} \frac{\partial \Phi_{j1}}{\partial p} \right)$

147 
$$\frac{\partial \mathbf{A2}}{\partial p} = 2(\Phi_{i1}\Phi_{j3} - \Phi_{i3}\Phi_{j1}) \left( \frac{\partial \Phi_{i1}}{\partial p} \Phi_{j3} + \Phi_{i1} \frac{\partial \Phi_{j3}}{\partial p} - \frac{\partial \Phi_{i3}}{\partial p} \Phi_{j1} - \Phi_{i3} \frac{\partial \Phi_{j1}}{\partial p} \right) \quad (13)$$

148 
$$\frac{\partial \mathbf{A3}}{\partial p} = 2(\Phi_{i2}\Phi_{j3} - \Phi_{i3}\Phi_{j2}) \left( \frac{\partial \Phi_{i2}}{\partial p} \Phi_{j3} + \Phi_{i2} \frac{\partial \Phi_{j3}}{\partial p} - \frac{\partial \Phi_{i3}}{\partial p} \Phi_{j2} - \Phi_{i3} \frac{\partial \Phi_{j2}}{\partial p} \right)$$

149 According to Eq. (12) and (13), it can be observed clearly that similar to one-pin ABC  
150 frequency sensitivity, the sensitivity of two-pin ABC frequency can also be expressed with a

151 combination of the natural mode shape sensitivities at the same points and the natural  
 152 frequency sensitivities.

153 The mode shape contribution in the two-pin ABC frequency sensitivities can also be  
 154 evaluated using Eq. (7), but the values of the mode shape sensitivity and natural frequency  
 155 sensitivity are now expressed as:

$$\begin{aligned}
 \Phi &= \frac{\left( \frac{\partial A1}{\partial p} \omega_1^2 + \frac{\partial A2}{\partial p} \omega_2^2 + \frac{\partial A3}{\partial p} \omega_3^2 \right) (A1 + A2 + A3)}{(A1 + A2 + A3)^2} - \frac{\left( \frac{\partial A1}{\partial p} + \frac{\partial A2}{\partial p} + \frac{\partial A3}{\partial p} \right) (A1 \times \omega_1^2 + A2 \times \omega_2^2 + A3 \times \omega_3^2)}{(A1 + A2 + A3)^2} \\
 \Omega &= \frac{\left( \frac{\partial \omega_1^2}{\partial p} A1 + \frac{\partial \omega_2^2}{\partial p} A2 + \frac{\partial \omega_3^2}{\partial p} A3 \right) (A1 + A2 + A3)}{(A1 + A2 + A3)^2}
 \end{aligned} \tag{14}$$

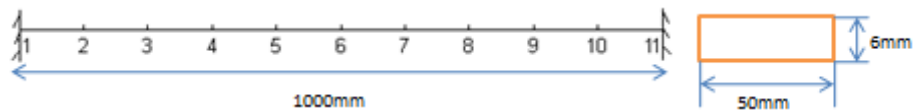
158 In what follows, a numerical case is employed to illustrate the outcome and the soundness  
 159 of the one-pin and two-pin ABC frequency sensitivities with the above equations, before  
 160 utilizing them in the process of selecting ABC frequencies for FE model updating.

### 161 3. Numerical verification of ABC frequency sensitivities

162 A beam model is used for the numerical verification. Both one-pin and two-pin ABC  
 163 frequency sensitivities are calculated using equations in Section 2, and these results are  
 164 compared with those from the direct method, where the ABC frequency sensitivity is  
 165 calculated using the ABC frequencies of the beam before and after the damage.

166 The beam is tested and simulated to validate ABC frequency sensitivities from above  
 167 equations. It is 1m long, and the cross section is 50 × 6 mm, and is made of steel. The beam  
 168 is fully fixed at both ends. The rigidity (EI) of the beam model is slightly tuned from the  
 169 theoretical value to 162 N.m<sup>2</sup> so that its natural frequencies match those from the test  
 170 beam with the same dimension and boundary condition, allowing for easy comparisons in  
 171 other aspects where needed, for example, the one-pin and two-pin ABC frequencies are  
 172 extracted from the test beam and compared to those from the model to verify the accuracy  
 173 of the experimental extracted ABC frequencies, which is beyond the scope of this paper.

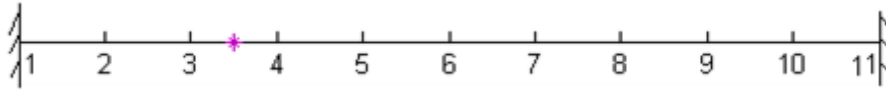
174 The first three natural frequencies from the FE model beam are 29.6Hz, 81.6Hz and 160Hz,  
 175 respectively. The beam is divided into ten elements, thus nine measurement points can be  
 176 employed to obtain ABC frequencies, as depicted in Fig. 1.



177  
 178 **Figure 1 FE model of beam**

179 For convenience, only the first order ABC frequencies are employed in the verification here.  
 180 Similar observations can be extended to higher (2<sup>nd</sup> and 3<sup>rd</sup>) ABC frequencies as well.

181 A damage scenario is simulated with a 1% stiffness reduction at element 3, which is shown  
 182 in Fig. 2.



183

184

**Figure 2 An arbitrary damage scenario in the beam**

185

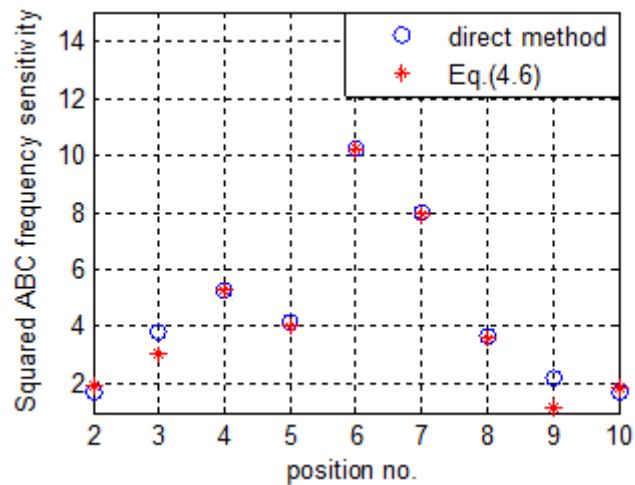
One-pin ABC frequency sensitivity is examined firstly. Given the beam properties, the sensitivity of one-pin ABC frequencies can be obtained using Eq. (6). It is noted that in the present calculation only the first three natural modes are used to form the FRF data.

187

188

Fig.3 compares the results using the sensitivity equations with those calculated directly from the ABC frequencies before and after damage (direct method). The vertical axis is the sensitivity of squared one-pin ABC frequency, which is shown in Eq. (6).

190



191

192

**Figure 3 One-pin ABC frequency sensitivities (to damage in segment 3) obtained using different methods**

193

194

It can be seen that the one-pin ABC frequency sensitivities calculated using Eq. (6) compare well with the direct results. The slight difference may be attributed to the fact that only the first few modes are employed to calculate the one-pin ABC frequency sensitivity using the equations. The variation of the sensitivity indicates that the one-pin ABC frequency (first order herein) shows the highest sensitivity when the pin is positioned at point 6, i.e. the mid-span. This is explicable as the first order ABC frequency when the pin is in the middle has the highest curvature over segment 3 where the particular damage is located.

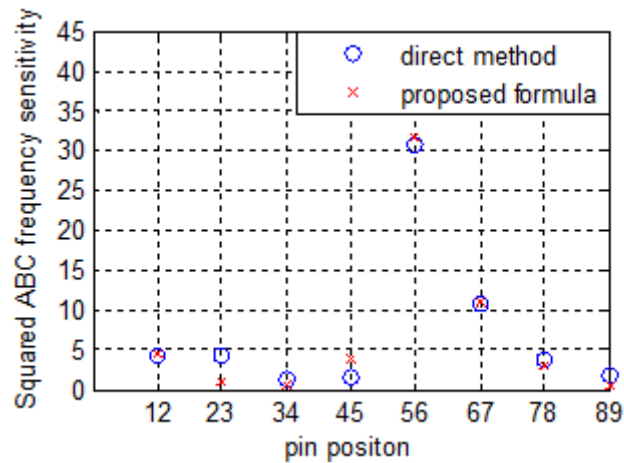
200

201

Next, the sensitivity of two-pin ABC frequencies is examined. The two-pin ABC frequency sensitivities calculated using Eq. (12) and (13) are compared with the direct results of the two-pin ABC frequencies before and after damage. Fig. 4 shows the comparison. Note that numbers in the x-label indicates the two pin positions, for example, "12" means pins located at points 1 and 2. The vertical axis is the sensitivity of squared two-pin ABC frequency shown in Eq. (12). It should be noted that although sensitivities of various two-pin ABC frequencies, including two-pin ABC frequencies with closely located pins and well separated pins, are compared, only sensitivities of adjacent two-pin ABC frequencies are shown in Fig. 4 to cover all the pin locations.

209





210

211

**Figure 4 Two-pin ABC frequency sensitivities using different methods**

212

It can be observed that the sensitivities evaluated using Eq. (12) and (13) agree well with the direct results. Similar to comparison results of one-pin ABC frequency sensitivity, the difference may be attributed to the fact that only the first few modes are used in the calculation with Eq. (12) and (13).

216

The above comparisons confirm that both one-pin and two-pin ABC frequency sensitivities can be satisfactorily evaluated using equations described in Section 2, which also indicates that these sensitivities are closely correlated with the first few (three herein) natural modes. This paves a way for the proposal of a methodology to select the ABC frequencies to be measured / included in a particular damage identification procedure.

221

**4. Selection of optimal ABC frequencies for structural damage identification and numerical validation**

222

223

4.1 Selection methodology

224

The sensitivity of an ABC frequency is defined with respect to a particular structural parameter, as expressed in Eq. (6), (12) and (13). For a set of different structural parameters, a set (vector) of ABC frequency sensitivities can be obtained. Therefore, depending upon whether or not prior knowledge about the damage positions is available, the selection method will differ.

229

4.1.1 Selection of ABC frequencies with prior knowledge of damage positions

230

When prior knowledge about the damage (stiffness reduction in particular) positions is available, the selection process is straightforward. The sensitivity of one-pin and two-pin ABC frequencies with respect to the stiffness of the damaged element can be calculated using Eq. (6), (12) and (13), then the mode shape contributions in the corresponding ABC frequency sensitivities can be obtained using Eq. (7). The ABC frequencies which exhibit higher mode shape contributions are selected, subject to a desirable total number, for FE model updating.

236

237 4.1.2 Selection of ABC frequencies without prior knowledge of damage positions

238 When the damage positions are not known beforehand, as in most practical applications,  
 239 the selection needs to be based on the overall sensitivity of the ABC frequency with respect  
 240 to all possible damage positions.

241 Generally speaking, assuming a structure with  $n$  elements, for a particular ABC frequency, its  
 242 sensitivity to a damage (stiffness reduction) in element  $i$  is defined as  $S_i$ , thus in total  $n$   
 243 sensitivities of the ABC frequency can be obtained, the sensitivity vector  $\mathbf{S}$  containing these  
 244 sensitivities can be written as  $(S_1, S_2, \dots, S_{n-1}, S_n)$ . For each ABC frequency sensitivity, the  
 245 mode shape contribution index  $C$  can be calculated using Eq. (7), thus a vector of index  $\mathbf{C}$   
 246 can be formed as  $\mathbf{C} = (C_1, C_2, \dots, C_{n-1}, C_n)$ . Based on the mode shape contribution vector  $\mathbf{C}$ , the  
 247 overall sensitivity of an ABC frequency may be expressed as:

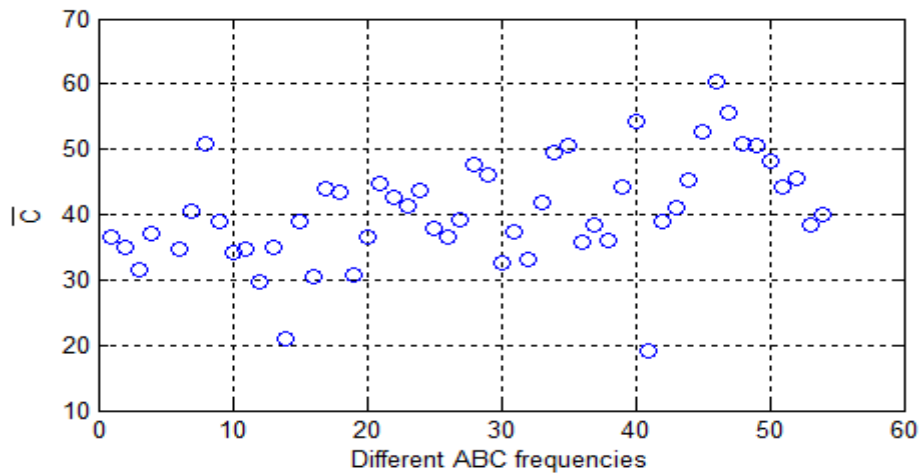
248 
$$\bar{C} = \mu_C + \mu_C / \sigma_C \quad (15)$$

249 where  $\mu_C$  and  $\sigma_C$  are mean value and standard deviation of the vector  $\mathbf{C}$ .

250 With the above index  $\bar{C}$ , the ABC frequencies with higher mean value and smaller standard  
 251 deviation value will be selected, which means these ABC frequency sensitivities have  
 252 collectively higher mode shape contributions to all possible damage scenarios.

253 Take the beam with ten elements shown in Fig. 1 as an example. There could be ten possible  
 254 single damage positions. For each ABC frequency, a sensitivity vector  $\mathbf{S}$  includes ten  
 255 sensitivities with respect to elemental stiffness of ten possible positions, which can give ten  
 256 mode shape contributions in vector  $\mathbf{C}$ . Subsequently the  $\bar{C}$  value in Eq. (15) can be  
 257 calculated for the selection of ABC frequencies.

258 Fig. 5 depicts the index values of different ABC frequencies from the above beam, these  
 259 frequencies include the first two orders of one-pin ABC frequencies and the first order two-  
 260 pin ABC frequencies. From the figure it can be seen that the proposed index values stretch  
 261 over a diverse range, from 20 to 60 herein, and this indicates that separation (selection) of  
 262 the more sensitive ABC frequencies can be made using the index without ambiguity.



263

264

**Figure 5 ABC frequency sensitivity index (mode shape contribution ratio) values**

265 4.2 Numerical verification of the selection criteria

266 4.2.1 Numerical validation with prior knowledge of damage positions

267 The same beam model mentioned above (shown in Fig. 1) is used to verify the performance  
268 of the selection method in FE model updating.

269 For first scenario with prior knowledge of the damage position, an arbitrary single damage is  
270 simulated by a 50% stiffness reduction at the 3<sup>rd</sup> element. The candidate ABC frequencies  
271 include the first two orders of one-pin ABC frequencies and the first order two-pin ABC  
272 frequencies. For comparison, two different sets of ABC frequencies are used for FE model  
273 updating. The first set includes ten ABC frequencies having larger mode shape contributions,  
274 while the second set includes ten ABC frequencies having smaller mode shape contributions.  
275 It should be mentioned that for the present verification, the ABC frequencies are free of  
276 noises (no noises added), and thus the difference in the updating results is only attributable  
277 to the performance of the selected ABC frequencies.

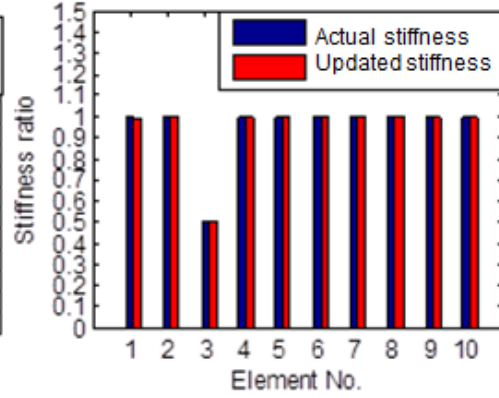
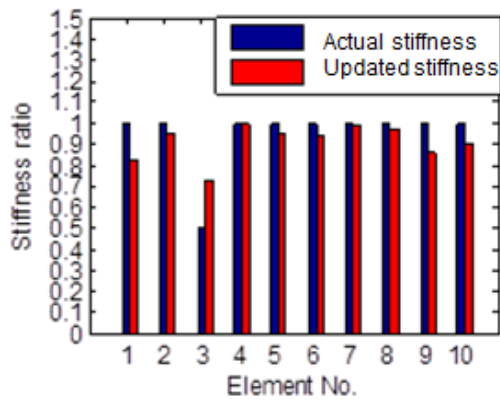
278 The actual structural damage identification is carried using a genetic-algorithm (GA)  
279 powered identification procedure. According to previous studies about GA application [26],  
280 in most cases, the probabilities for crossover and mutation can be selected within 0.6-0.9  
281 and 0.01-0.02 to achieve reasonable solutions, but a lower crossover probability and higher  
282 mutation probability should be used for real coding GA. Therefore, in this study, the  
283 probabilities for crossover and mutation are selected as 0.7 and 0.02, respectively. These  
284 parameters are listed in the Table 1.

285 **Table 1 GA configuration**

Max generation	1,000
Selection method	Ranking selection
Crossover method	Heuristic crossover
Crossover probability	0.7
Mutation method	Uniform mutation
Mutation probability	0.02

286

287 Fig. 6 depicts the updating results using the two different sets of ABC frequencies datasets,  
288 the corresponding root mean square errors and the maximum percentage errors in the  
289 updating results. It is noted that the parameters being updated are represented by a  
290 stiffness ratio, which is defined as the ratio between the variable elemental rigidity and the  
291 original (undamaged) elemental rigidity.

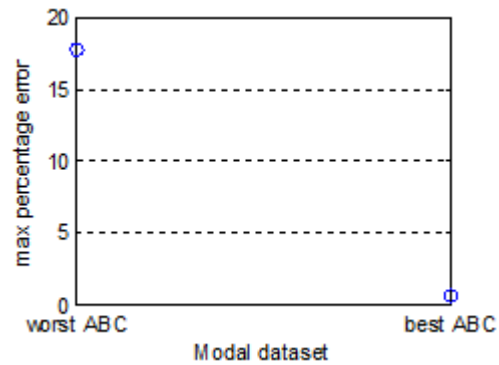
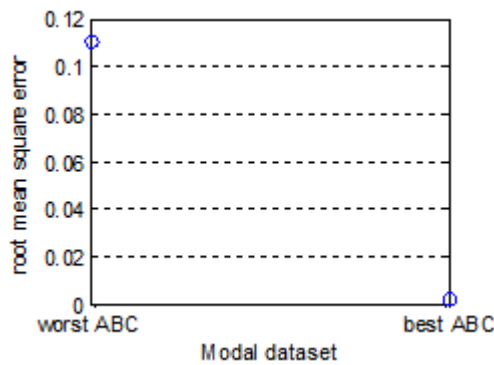


292

293

Using 10 "worst" ABC frequencies

Using 10 "best" ABC frequencies



294

295

Root mean square error

Max percentage error

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Figure 6 Updating results using different ABC frequencies (noise-free) and comparison

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From the updating results shown in Fig. 6, it can be seen that the performance of the "best" 10 ABC frequencies selected by the mode shape contribution criterion is superior over that of the 10 "worst" ABC frequencies. Without the influence of measurement noises, the accuracy of the updating results using the 10 "best" ABC frequencies is almost perfect, with a root mean square error (RMSE) and maximum percentage error (MPE) approaching zero. On the other hand, with the "worst" 10 ABC frequencies, the RMSE and MPE of the updated results are about 11% and 18%, respectively.

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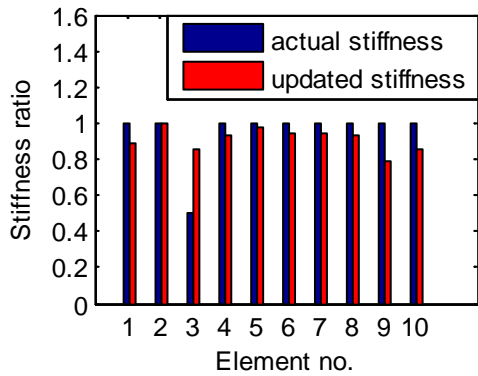
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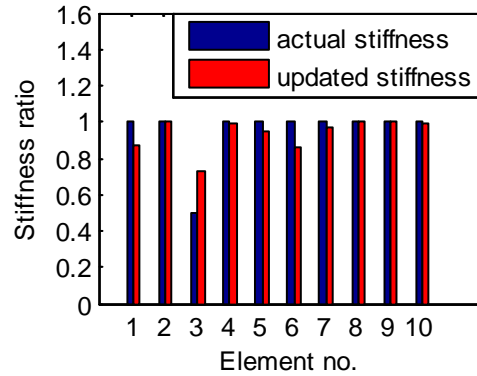
To further evaluate the effect of the selection method under practical conditions, the computed (exact) ABC frequencies are treated by injecting measurement noises before they are employed in the FE updating procedure. For this purpose, a 1% uniformly distributed random error is added to the ABC frequencies to simulate more realistic ABC frequencies as would be extracted from an experiment. Fig. 7 shows the FE model updating results using noise-contaminated ABC frequencies. For comparison, results using 4 sets of ABC frequencies, ranging from collectively the "worst" to the "best" sets according to the different ABC frequency sensitivity index ( $\bar{C}$ ) values calculated using Eq. (15), are presented. It should be mentioned that the same number (ten) ABC frequencies is used in each set.



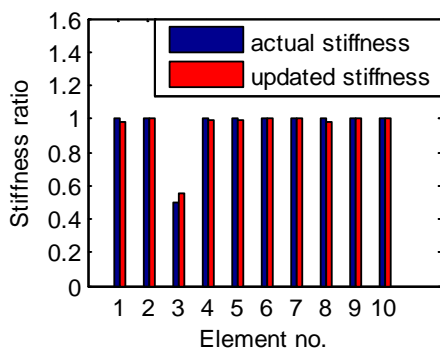
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$\bar{C} < 35$



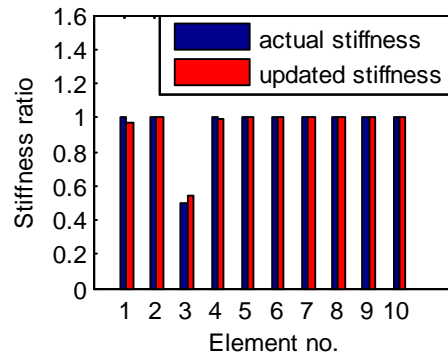
$35 < \bar{C} < 40$



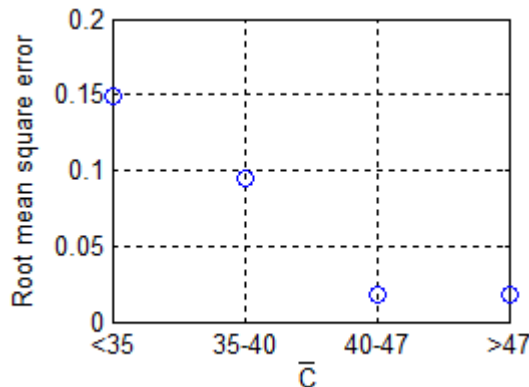
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$40 < \bar{C} < 47$



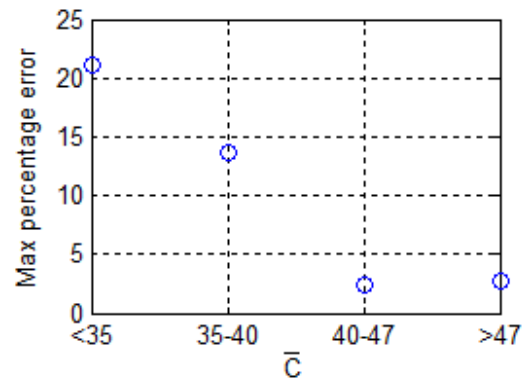
$\bar{C} > 47$



317

318

Root Mean Square Error



Max percentage error

319

**Figure 7 Updating results using different ABC frequencies (noise added) and comparison**

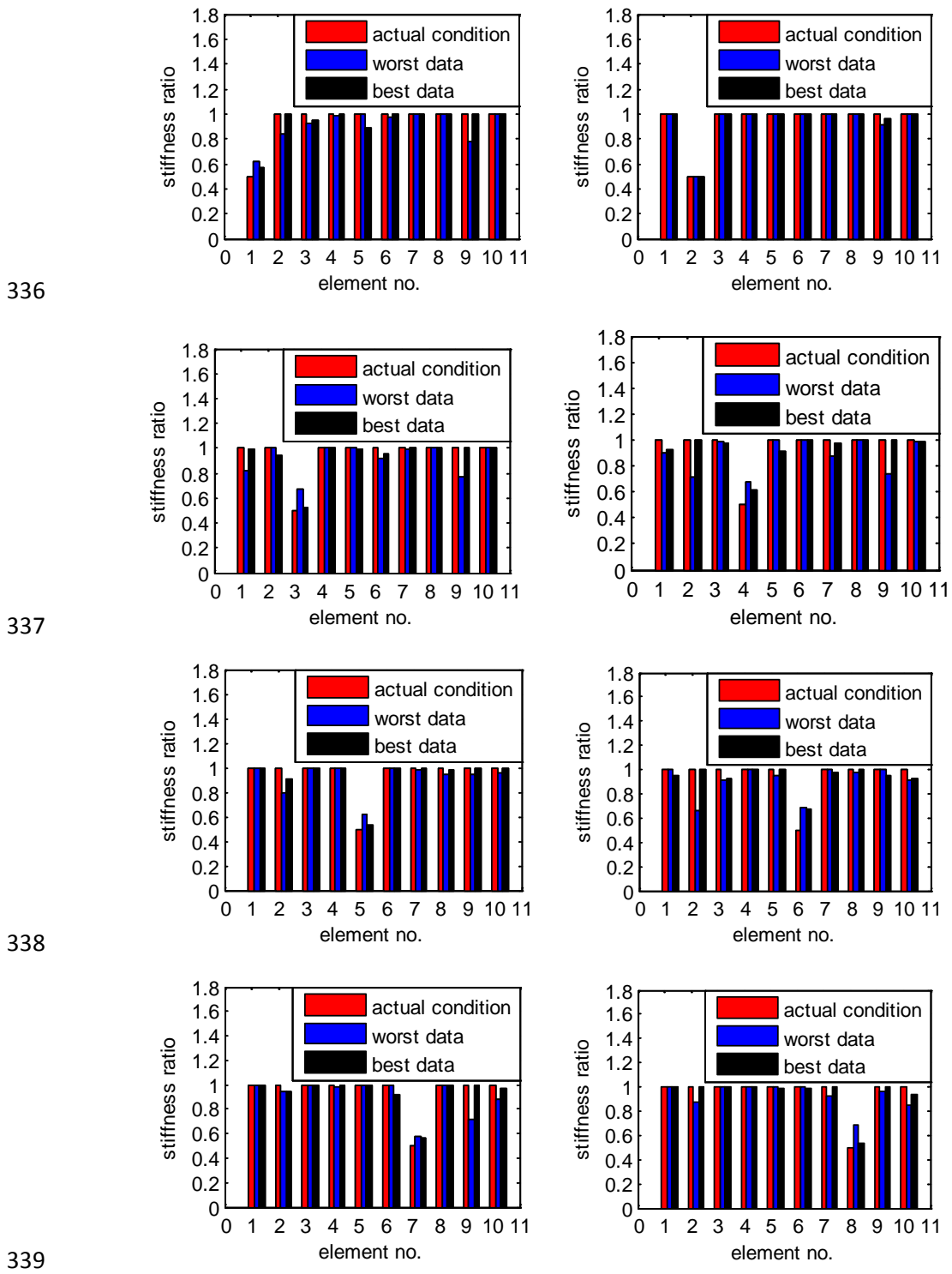
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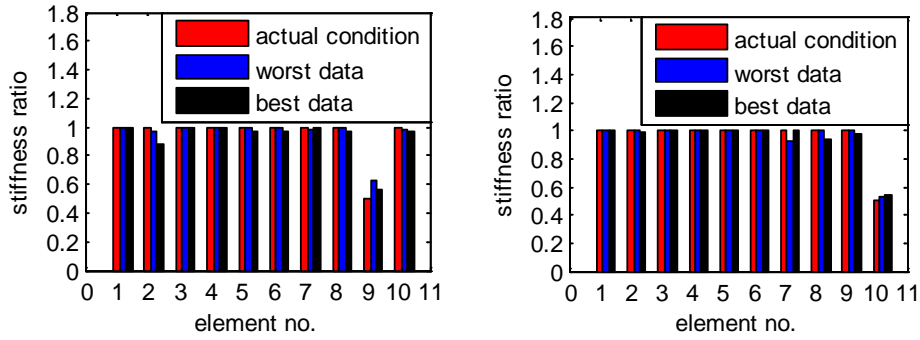
The results and comparison further confirm that, even with the inclusion of the measurement noises in the ABC frequencies, the employment of “better” ABC frequencies that contain higher mode shape contributions tends to yield better updating results. When the “worst” set of ABC frequencies with ABC frequency sensitivity value  $\bar{C}$  less than 35 is used, the updated results exhibit errors as large as 15% in terms of RMSE and 25% in terms of MPE. On the other hand, when the “best” set of ABC frequencies is used, the updating results are very good with only about 2% RMSE and 3% MPE.

327

4.2.2 Numerical validation without prior knowledge of damage positions

328 For this verification, different damage scenarios are considered to examine the performance  
 329 of the proposed index in Eq. (15). Similar to the analysis in the above section, two datasets  
 330 of ABC frequencies are employed in the FE model updating procedure. The first dataset  
 331 includes ten ABC frequencies having the largest sensitivity index values (in terms of the  
 332 mode shape contributions), while ten ABC frequencies with the smallest index values are  
 333 used in the second dataset. 1% uniformly distributed random noises are added to the ABC  
 334 frequencies to simulate measurement errors in all the ABC frequencies. The updating results  
 335 are illustrated in Fig. 8.

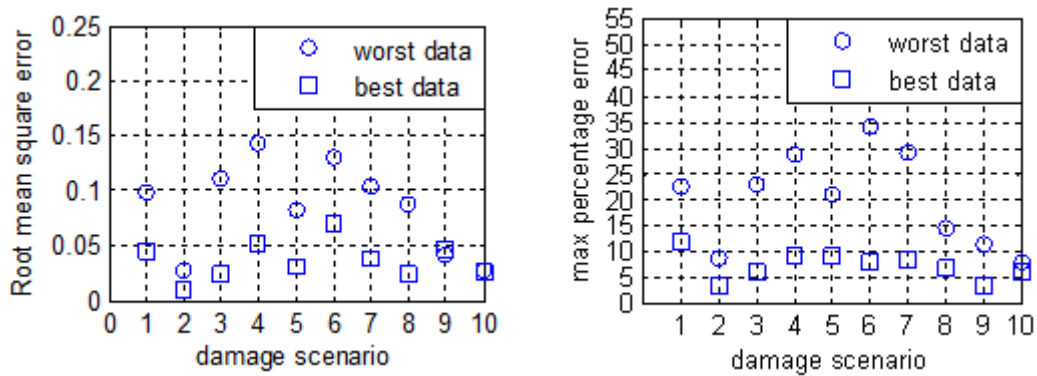




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(a) Updating results for damage in element 1 to 10 consecutively



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343

(b) Updating errors (RMSE and MPE) for each damage scenario

Figure 8 Updating results using the “best” and “worst” sets of ABC frequencies for various damage scenarios

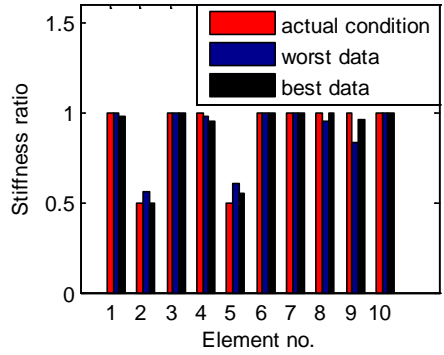
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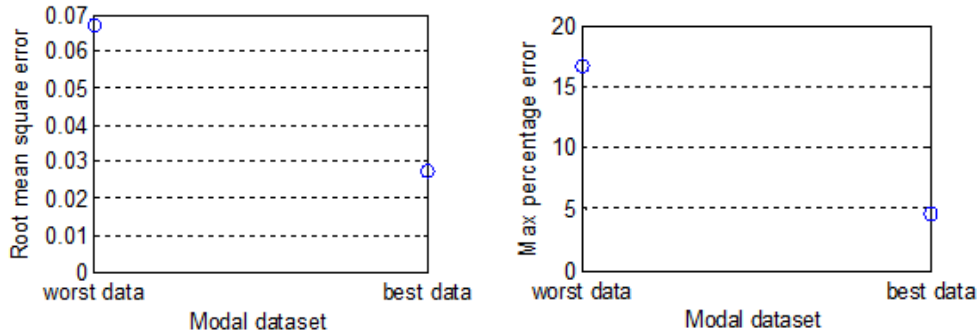
346 From the results shown in Fig. 8, it can be seen that the performance of the “best” ten ABC  
 347 frequencies is clearly and consistently superior over the “worst” ten ABC frequencies. Using  
 348 the ten “best” ABC frequencies, the maximum RMSE and MPE of updated results are about 7%  
 349 and 12%, respectively; whereas with the “worst” ten ABC frequencies the maximum RMSE  
 350 and MPE in the updating results can approach as high as 15% and 35%, respectively.

351 It is worth noting that there exists differences in the updating results for the symmetrical  
 352 damage scenarios, while theoretical speaking, the updating results for symmetrical cases  
 353 should be exactly the same. These differences can be attributed to two factors; one is that  
 354 the added 1% noise in the frequencies may cause variation in the updating results, and  
 355 another is that the searching path in GA may not be the same in each case, which will also  
 356 give different updating results.

357 It should be pointed out that in all the above numerical simulations, only a single damaged  
 358 element has been assumed in each individual case. This is mainly for the convenience of  
 359 presenting and comparing the results across different damage scenarios. In fact, during the  
 360 model updating, all elements have been assumed to have variable damage (stiffness)  
 361 parameters. Therefore, as far as the model updating procedure is concerned, the  
 362 observations are generally applicable, regardless whether the problem involves just one or  
 363 multiple damage locations. For illustration, two multiple-damage cases are considered and  
 364 the updating is performed also using the ‘best’ and ‘worst’ ten ABC frequencies, respectively.  
 365 The results are shown in Fig. 9. It can be observed that for multiple damage scenarios, the  
 366 ‘best’ ten ABC frequencies still perform better than the ‘worst’ ten ABC frequencies.



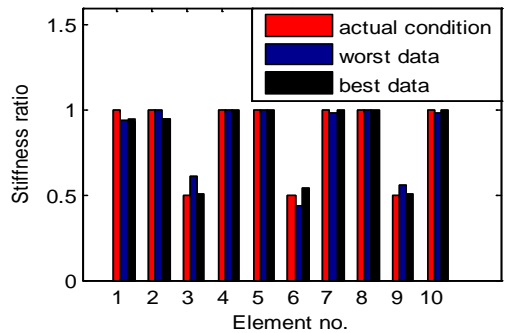
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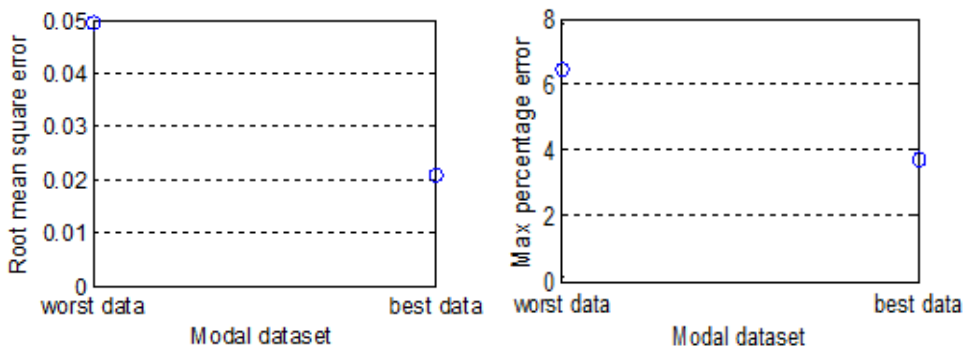
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369

(a) 50% stiffness reductions at elements 2 and 5



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371

372

50% stiffness reductions at elements 3, 6 and 9

373

374

**Figure 9 Updating results using the “best” and “worst” sets of ABC frequencies for multiple damage scenarios**

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It should be pointed out that in the above numerical examples, ten unknown parameters are updated using only ten ABC frequencies, which is deemed adequate for cross-comparison between the use of different sets of ABC frequencies (with different ranges of the sensitivity

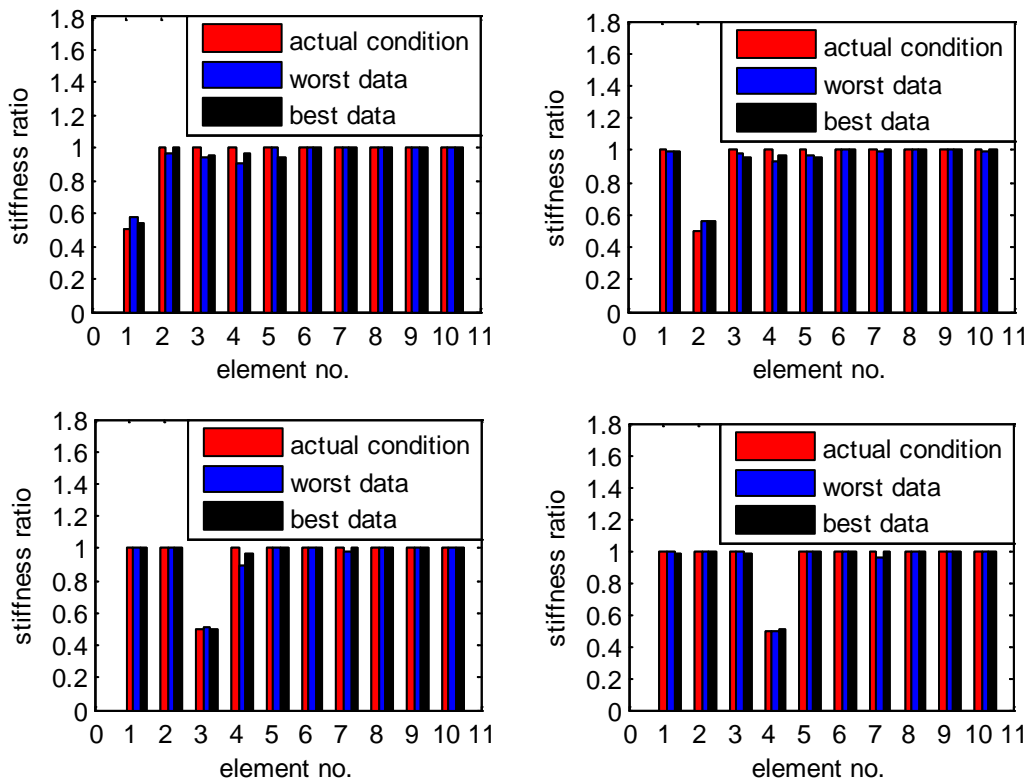


378 index). A general examination with regard to the effect of the ABC data size on the updating  
 379 results will be carried out in the next section.

380 **5. Investigation of size of ABC frequency dataset in structural damage identification**

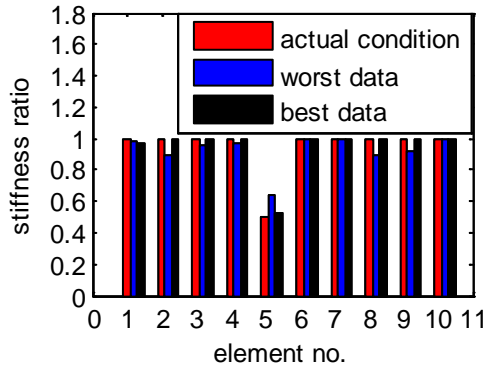
381 As mentioned in previous sections, there is an optimum range of the amount of modal data  
 382 to be included in a FE model updating procedure. It is generally understood that the number  
 383 of modal data should be 2-3 times the number of parameters being updated in order to  
 384 achieve a satisfactory result [30]. Due to inevitable measurement errors, employing too  
 385 many modal data could introduce conflicting tendencies during the updating process and  
 386 thus adversely affect the updating results, and evidences of such phenomenon have been  
 387 reported in a number of occasions [31-32]. The use of the rule of “2-3 times” has been  
 388 adopted in many previous studies [25, 31, 33].

389 In this section, we shall examine whether the above general observations also apply in the  
 390 case of using ABC frequencies. Firstly, two groups of twenty ABC frequencies are employed  
 391 for the model updating; one group includes ABC frequencies with larger sensitivity index ( $\bar{C}$ )  
 392 values, and the other includes ABC frequencies having smaller sensitivity index values. Thus,  
 393 the two groups are effectively the expanded sets from the cases using 10 ABC frequencies in  
 394 Section 4. The updating results and their percentage errors are shown in Fig. 10.

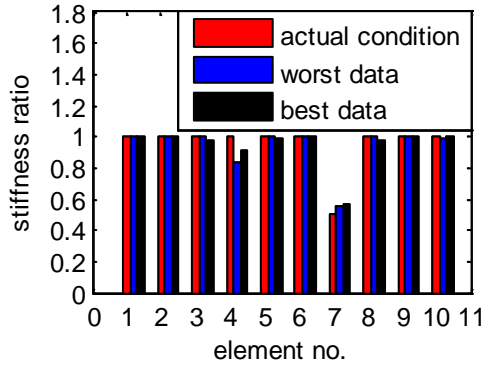
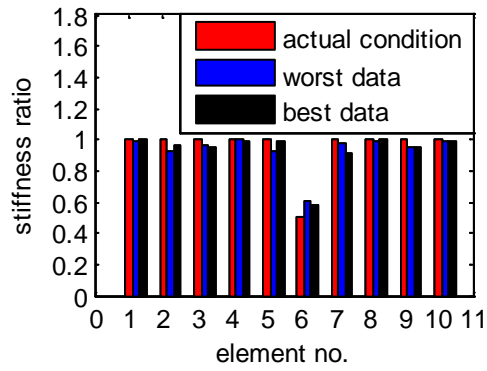


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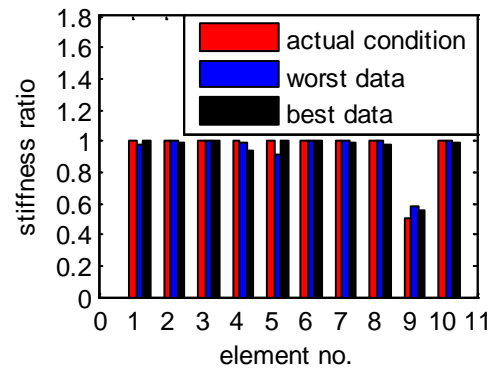
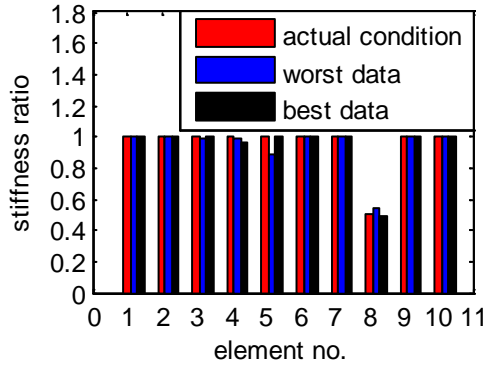
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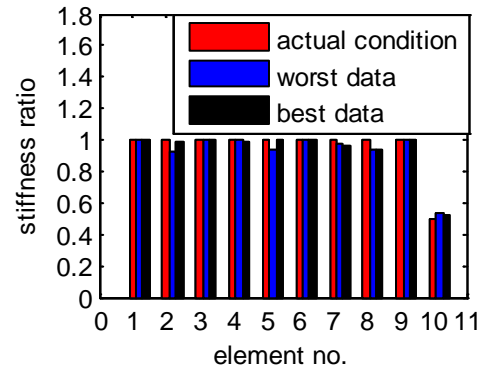
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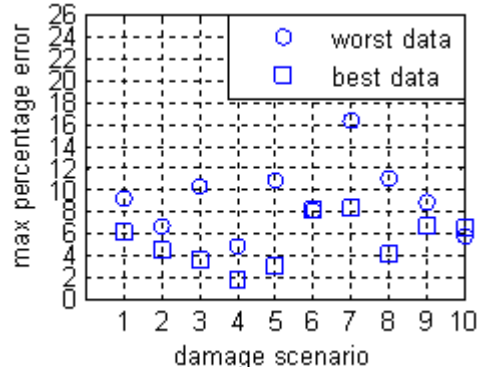
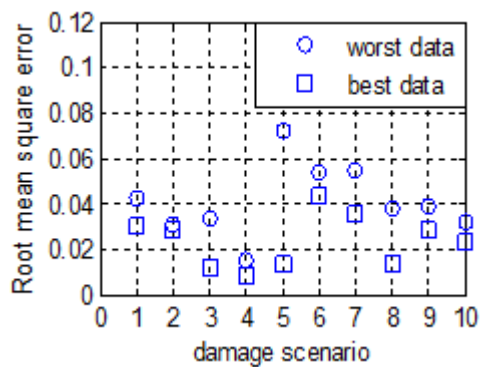


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(a) Updating results for damage in element 1 to 10 consecutively



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(b) Updating errors (RMSE and MPE) for each damage scenario

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Figure 10 Updating results using 20 ABC frequencies for various damage scenarios

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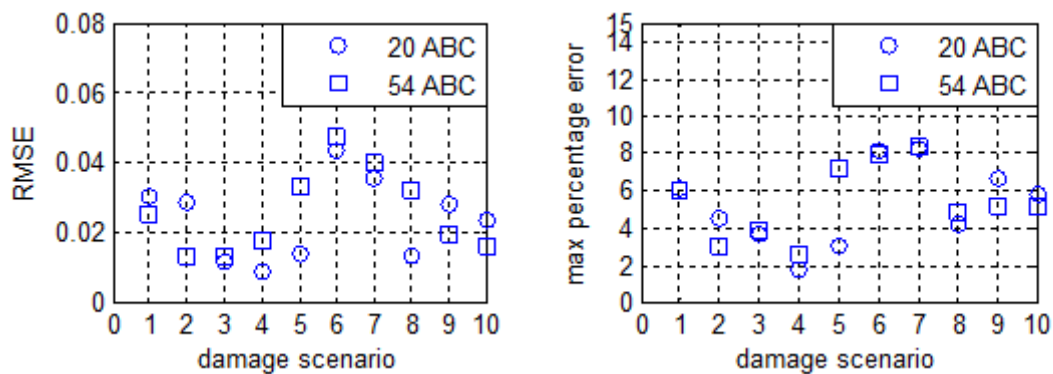
It can be seen that, when 20 ABC frequencies are employed for the updating of 10

405

parameters, the results from the set of ABC frequencies having higher  $\bar{C}$  index values still

406 produce better results. However, comparing with the results using ten ABC frequencies (Fig.  
 407 8(b)), increasing to twenty ABC frequencies does not improve the results much further for  
 408 the case with the “best” ABC frequencies, indicating that the “best” ABC frequencies as  
 409 selected according to the ABC sensitivity index (mode shape contributions) indeed contain  
 410 more significant information. On the other hand, increasing to twenty ABC frequencies in  
 411 the case of the “worst” ABC frequencies does improve the updating results markedly, and  
 412 this is expected according the “2-3 times” rule mentioned before. In the particular case with  
 413 ABC frequencies herein, this may be explained by the fact that, although the ABC  
 414 frequencies with a smaller index contains less (sensitive) mode shape information, by  
 415 increasing the number of ABC frequencies, the overall mode shape information gets  
 416 enhanced.

417 Next, we shall examine the updating results when too many ABC frequencies are involved.  
 418 For this purpose all 54 ABC frequencies, including one-pin ABC frequencies from the first two  
 419 modes and two-pin ABC frequencies from the first mode, are employed in the model  
 420 updating for each damage scenario. Similar to the previous cases, 1% uniformly distributed  
 421 random noises are added to ABC frequencies to simulate measurement errors. Fig. 11 shows  
 422 the results in terms of the RMSE and MPE from using 20 ABC frequencies having larger index  
 423 values and using all 54 ABC frequencies.



424

425

**Figure 11 Updating results using different numbers of ABC frequencies**

426 It can be found clearly that by further increasing ABC frequencies for the model updating,  
 427 results do not show appreciable further improvement. In fact, some results using all 54 ABC  
 428 frequencies exhibit even larger RMSE and MPE errors than those using 20 ABC frequencies,  
 429 for example in damage scenarios 4,5,6,7 and 8.

430 The detailed accuracy profiles will understandably vary as the level of measurement errors  
 431 (herein assumed 1%) in the ABC frequencies and structural setting vary. However, the  
 432 results presented above tend to suggest that with ABC frequencies, the general rule on the  
 433 amount of modal data to be included in the FE model updating still holds. In order words,  
 434 when employing ABC frequencies in practice (containing some normal level of measurement  
 435 noises) for model updating, the number of ABC frequencies should be kept about 2-3 times  
 436 of the number of unknown parameters in order to achieve an effective and efficient FE  
 437 model updating process.

438

## 6. Conclusions

439 In this paper, the sensitivities of ABC frequencies to damage (stiffness reduction) are studied.  
440 On this basis, a methodology for the selection of ABC frequencies in a finite element model  
441 updating / damage identification procedure is proposed to achieve the reliable identification  
442 results.

443 For this purpose, the existing anti-resonance (one-pin ABC) frequency sensitivity expression  
444 is extended to formulate the two-pin ABC frequency sensitivity expression, which includes  
445 the contributions of the natural frequencies and mode shape coordinates at the pin  
446 locations. Numerical examples demonstrate that the ABC sensitivity formulas for both one-  
447 pin and two-pin ABC frequencies match closely the actual sensitivities as calculated directly  
448 from the changes of the respective ABC frequencies corresponding to a particular damage.

449 The mode shape contribution (ratio) in the ABC frequency sensitivity is adopted as a  
450 classifying criterion for the selection of the ABC frequencies in a FE model updating  
451 procedure, such that those with higher mode shape contributions are employed, subjected  
452 to a desirable number limit.

453 Numerical studies on the above proposed selection criteria are carried out for different  
454 damage scenarios, with or without prior knowledge about damage positions. Results  
455 demonstrate that in both situations, the ABC frequencies selected using the proposed  
456 criterion consistently give rise to better updating results. Furthermore, it is verified through  
457 numerical studies that the general rule regarding the number of modal data to be included  
458 in a FE model updating procedure, i.e., being 2-3 times of the number of unknown  
459 parameters, also applies in the case with ABC frequencies.

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