Continuous-Time Modeling of Random Searches: Statistical Properties and Inference

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Abstract

Mathematical modeling of random searches is of great relevance in the field of physics, chemistry, biology or modern ecology. A large number of existing studies record the search movement at equidistant time intervals and model such time series data directly with discrete-time random walks, such as Lévy flights and correlated random walks. Given the increasing availability of high resolution observation data, statistical inference for search paths based on such high resolution data has recently become one of the major interests and has raised an important issue of robustness of random walk models to the sampling rate, as estimation results for the discrete observation data are found to be largely different at different sampling rates even when the underlying movement is supposedly independent of scale. To address this issue, in this paper, we propose to model the continuous-time search paths directly with a continuous-time stochastic process for which the observer makes statistical inference based on its discrete observation. As continuous-time counterparts of Lévy flights, we consider two-dimensional Lévy processes and discuss the relevance of those models based upon advantages and limitations in terms of statistical properties and inference. Among the proposed models, the Brownian motion is most tractable in various ways while its Gaussianity and infinite variation of sample paths do not well describe the reality. Such drawbacks in statistical properties may be remedied by employing the stable and tempered stable Lévy motions while those models are less tractable and cause an issue in statistical inference.

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1 Introduction

The random search problem has long attracted continuing attention due to its board interdisciplinary range of applications. In particular, animal foraging movements do not seem to be simply deterministic and thus such characteristics are often modelled with two-dimensional spatial stochastic processes, in particular, random walk models. Examples include biased random walks and correlated random walks (Okubo and Levin [33], Turchin [53]). Those models have been analyzed by many authors with respect to various aspects, such as modeling, statistical methodology, simulation, and empirical data analysis. Search movement is made on the continuous-time basis and such a movement path is usually recorded at equidistant time intervals, that is to say, a path is mapped into a broken line where the nodes correspond to animal position at certain observation times. The movement along the broken line, or the random walk in other words, can then be quantified by the distribution of probabilities for the stepsize or ”jump” and the turning angle. In contemporary practice of animal movement studies, researches have indeed worked with such discrete-time random walks in order to reach a good understanding of the underlying continuous-time movement path (see, for example, Codling et al. [9]).

Given the availability of high resolution data of animal movement paths, statistical analysis and modeling can provide a way of closer look at animal movement at a fundamental level, while the approach based on random walks has recently brought up an important issue on robustness of statistical analysis to the sampling rate, that is to say, whether or not these paths remain invariant to the sampling scale used by the observer. The effect of sub-sampling

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and rediscretization on such underlying models has been investigated by many authors. (See, for example, Plank and Codling [38], Kawai and Petrovskii [17] and Kawai [18].) A Lévy movement pattern is in general assumed to be fractal and thus the sampling scale used by the observer should not affect the observed properties (Reynolds and Frye [42]). Recently, however, Plank and Codling [38] demonstrated that the observed properties of simulated Lévy walks are dependent largely on the sampling rate and the value of the exponent in the underlying walk and also how a simulated intermittent random walk can be misidentified as a Lévy path even when rigorous statistical methodologies are employed.

In principle, animal movements involve finite distances moved in finite times and cannot be truly scale invariant. Hence, movement paths are, by nature, not supposed to be robust to the sampling rate. For example, a correlated random walk acts as a good model in this context; the sampling rate is known to have a significant effect on the apparent properties of the movement pattern (Bovet and Benhamou [6]). Alternatively, Viswanathan et al. [55] assumed that movements constitute a truncated Lévy walk, where steps are truncated at points where food is found, so that unrealistically large jumps do not occur and thus the movements are never scale invariant. Moreover, the movement process is modelled with a composite random walk with intermittent phases of extensive and intensive movements [5].

All those arguments are in agreement with the evident fact that any non-fractal discrete-time stochastic processes cannot be robust to the sampling rate. As mentioned earlier, however, animals move on the continuous-time basis and, in principle, such a movement path should not be interpreted in more than one way. To address this robustness issue, it is certainly desirable to model the continuous-time dynamics of random searches directly with a continuous-time stochastic process. In this framework, the observer is required to make statistical inference for the underlying continuous-time dynamics based on its discrete observation. In this paper, as continuous-time counterparts of Lévy flights [12, 41, 25, 50, 54, 55], we consider two-dimensional Lévy processes to model spatial movement paths and discuss the relevance of those models based upon advantages and limitations from several different standpoints, in particular, in terms of statistical properties and inference. We also raise a variety of challenges caused by continuous-time modeling and its statistical analysis to address in the future.

2 Models

Animals make a movement on the continuous-time basis, as a matter of course, whether its movement path is purely continuous or contains some discontinuities (for instance, kangaroos make nearly instantaneous jumps). Such continuous-time movement paths are often observed at equidistant time points and are mapped into a broken line where the nodes indicate animal position at observation times. In the literature, such nodes and lines are mathematically modelled with discrete-time random walks, which we begin our discussion with. Let $\{X_n\}_{n=0,1,2,\dots}$ be a discrete-time stochastic process in $\mathbb{R}^2$, where the integer n indicates the discrete time steps. Of particular practical interest are the cases where the magnitude (that is, jump or step size) $\{\|Z_n\|\}_{n\in\mathbb{N}}$ of the increments $\{Z_n\}_{n\in\mathbb{N}}(=: \{X_n - X_{n-1}\}_{n\in\mathbb{N}})$ are independent and identically distributed (iid) with Gaussian distributed length with $\mathscr{N}(0,\sigma^2)$ or Pareto distributed length with probability density

$$f(z;\alpha,0,\tau) := \frac{\alpha\tau^{\alpha}}{z^{\alpha+1}}, \quad z \in (\tau,+\infty), \tag{2.1}$$

where $\alpha > 0$ and $\tau > 0$ and with the tail probability

$$\mathbb{P}(\|Z_1\| > z) = (\tau/z)^{\alpha}, \quad z \in (\tau,+\infty). \tag{2.2}$$

Here, τ indicates the minimum step size being considered, while α is the exponent (usually written with μ related through $\alpha = \mu - 1$). Moreover, In applications to animal movement, this Pareto length is usually referred to as a ”power law distribution” (for instance, of the jump size z), the resultant random walk is called ”Lévy flight”, and such movement pattern is called ”Lévy movement pattern” (Bartumeus et al.[1], Sims et al. [51], Viswanathan et al. [55]). See Kawai and Petrovskii [17] and Kawai [18] for the issue of data interpretation from a statistical inference point of view. A Lévy movement pattern is generally assumed to be scale-invariant, that is to say, it has the fractal property that the scaling used by the observer should not affect the observed properties (Boyer et al. [7], Reynolds and Frye [42]). In particular, the Gaussian distribution is closed under convolution; in other words, the sum of independent Gaussian random variables is again Gaussian. In turn, the Pareto distribution belongs to a class of the subexponential distribution, namely, if $\{\|Z_n\|\}_{n\in\mathbb{N}}$ is a sequence of iid Pareto random variables with the above density function, then it holds that for each $n \in \mathbb{N}$, as $z \uparrow +\infty$,

$$\mathbb{P}\left(\|Z_1\| + \cdots + \|Z_n\| > z\right) \sim n\mathbb{P}\left(\|Z_1\| > z\right) = n\left(\tau/z\right)^{\alpha}, \tag{2.3}$$

where the equality holds by (2.2) and where $\sim$ indicates the asymptotic equivalence, that is, "$f(z) \sim g(z)$ as $z \uparrow +\infty$" means that "$f(z)/g(z) \to 1$ as $z \uparrow +\infty$." This implies that the density $f^{(*n)}(z;\alpha,\tau)$ of the n-fold convolution satisfies the same order of decay

$$f^{(*n)}(z;\alpha,0,\tau) \sim nf(z;\alpha,0,\tau),$$

as $z \uparrow +\infty$. This implies that at least the parameter α might be asymptotically robust to the sampling rate.

Before proceeding to the next model, let us remark that there is a different class of random walks with the term "*Lévy*" for modeling animal movement path, that is, Lévy walks, which are also characterized by the existence of rare but extremely large steps (Raposo et al. [41]). In the Lévy walks, however, the constant velocity implies the time of travel proportional to the total path length, causing the mean-square displacement to grow with time superlinearly. In contrast, the total path length of Lévy flights (Shlesinger et al. [50]) corresponds to the total time of travel in Lévy walks.

A serious drawback of the Pareto distribution is attributed to its infinite variance. In statistical physics, random walks with heavy probability tails and still with a finite variance have been developed through various different truncations of the Pareto distribution. The pioneering work of Mantegan and Stanley [27] is the constitution for such *truncated* Lévy flights. In Koponen [23], the analytic expression for characteristic function of truncated Lévy flights was derived. In mathematical ecology, there exist more than one construction of the truncated Lévy flight, one of which is defined through the density function

$$f(z;\alpha,\kappa,\tau) := \frac{1}{\kappa^{\alpha}\Gamma(-\alpha,\kappa\tau)}\frac{e^{-\kappa z}}{z^{\alpha+1}} = \frac{e^{-\kappa z}}{\alpha(\kappa\tau)^{\alpha}\Gamma(-\alpha,\kappa\tau)}f(z;\alpha,0,\tau),$$

which is indeed exponential tempering, rather than truncation (for example, Kawai and Petrovskii [17], Mashanova, Oliver and Jansen [29]). Another definition is made through a literal truncation of the Pareto density, such as

$$p(z;\alpha,\tau_{\min},\tau_{\max}) = \frac{\alpha}{\tau_{\min}^{-\alpha} - \tau_{\max}^{-\alpha}} z^{-\alpha-1}, \quad z \in (\tau_{\min},\tau_{\max}),$$

where $0 < \tau_{\min} < \tau_{\max} < +\infty$ (for example, Plank and Codling [38]). Either definition improves the ordinary Pareto distribution by capturing various statistical properties of the movement step length, such as a finite variance, non-fractality and aggregational Gaussianity.

As discussed earlier, such random walk models are purely of discrete-time type and are known to be not robust to the sampling rate. This is because the observer does not (or pretends not to) look at the underlying continuous-time movement path anymore, once such movement paths are discretely observed. With this in mind, we consider three continuous-time spatial stochastic processes in the class of Lévy processes, which may serve as the first step towards realistic modeling of movement paths.

2.1 Brownian Motion

The Brownian motion (or the Wiener process) has been most widely used in modeling continuous-time probabilistic phenomena. Consider the Brownian motion $\{t\gamma + B_t : t \geq 0\}$, where $\gamma := [\gamma_1, \gamma_2]^{\top}$ and $\{B_t : t \geq 0\}$ is a centered Brownian motion in $\mathbb{R}^2$ with variance-covariance matrix

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} =: \Sigma, \tag{2.4}$$

with $\sigma_1 > 0$, $\sigma_2 > 0$ and $\rho \in [-1,+1]$. Here, the parameter γ is the constant drift, which essentially governs advection, while σ_1^2 and σ_2^2 are variance of the movement and ρ is the correlation between two components. In particular, when $\rho = 0$, the two components are independent. The Brownian motion (without drift) enjoys the self-similarity property, that is, for any $h > 0$,

$$\left\{h^{-1/2}B_{ht} : t \geq 0\right\} \stackrel{\mathscr{L}}{=} \{B_t : t \geq 0\},$$

which is essentially equivalent to the scale-free property. Here, we denote by $\mathscr{L}$ the *l*aw and by $\stackrel{\mathscr{L}}{=}$ the identity in law (or distribution).

2.2 Stable Lévy Motion

The stable Lévy motion has been used in several fields of application, such as statistical physics, queueing theory, mathematical finance, to mention just a few. In this paper, we call the stochastic process $\{X_t : t \geq 0\}$ a stable Lévy motion if it is a Lévy process in $\mathbb{R}^2$ without Gaussian component and with characteristic function

$$\begin{aligned}\mathbb{E}\left[e^{i\langle y,X_t\rangle}\right] &= \exp\left[t\left(i\langle y,\gamma\rangle + \int_S\int_0^{+\infty}\left(e^{i\langle y,r\xi\rangle}-1\right)\frac{dr}{r^{\alpha+1}}\lambda(d\xi)\right)\right] \\ &= \exp\left[t\left(i\langle y,\gamma\rangle + \Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\int_S|\langle y,\xi\rangle|^\alpha\left(1-i\tan\frac{\pi\alpha}{2}\mathrm{sgn}(\langle y,\xi\rangle)\right)\lambda(d\xi)\right)\right],\end{aligned} \tag{2.5}$$

where $\alpha \in (0,2)$ is the so-called stability index, $\lambda(d\xi)$ is a finite positive symmetric measure on S (the unit sphere, or circle, in $\mathbb{R}^2$), the drift $\gamma \in \mathbb{R}^2$, and $\Gamma(-\alpha)\cos(\pi\alpha/2)$ is negative over $\alpha \in (0,2)$ and is continuous at $\alpha = 1$ with value $-\pi/2$. (Note that any probability distribution is uniquely characterized by its characteristic function, or equivalently its Fourier transform.) The parameter α corresponds to the exponent of power law and the measure λ controls the direction of jumps. On the one hand, the stable Lévy motion with $\alpha \approx 0$ generates nearly ballistic trajectories, which have been thought of as the most efficient search strategy in destructive foraging, in which the forager may feed at a given target site only once. It is known, on the other hand, that the stable Lévy motion with $\alpha = 2$ corresponds to the Brownian motion, which we exclude from consideration in the name of stable motion. (See, for example, Sato [48] and Samorodnitsky and Taqqu [47] for various basic facts on the stable law and process.)

If the drift γ is zero, then the stable Lévy motion enjoys the self-similarity property, like the Brownian motion but with a different exponent, that is, for any $h > 0$,

$$\left\{h^{-1/\alpha}X_{ht} : t \geq 0\right\} \stackrel{\mathscr{L}}{=} \{X_t : t \geq 0\}.$$

By the generalized central limit theorem due to Gnedenko and Kolmogorov, the scaled sum of iid Pareto random variables tend to a stable distribution. Let $\{\|Z_n\|\}_{n\in\mathbb{N}}$ denote a sequence of iid Pareto random variables with density function (2.1). Then, when $\alpha \in (1,2)$, the scaled sum

$$\left(\frac{2\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi\tau^\alpha}\right)^{1/\alpha} n^{-1/\alpha}\left(\|Z_1\| + \cdots + \|Z_n\| - n\frac{\alpha\tau}{\alpha-1}\right)$$

converges in law to the stable law (2.5) with $\lambda(d\xi) = -(\Gamma(-\alpha)\cos(\pi\alpha/2))^{-1}\delta_{\{[1,0]^\top\}}(d\xi)$. (This explains the asymptotic equivalence (2.3).) This fact indicates that the Lévy flight can be thought of as a random walk formed through equidistant discrete observation of a stable motion. In this paper, we consider the following two subclasses of the stable Lévy motion.

2.2.1 Rotation-Invariant Case

We refer the case where the control measure $\lambda(d\xi)$ is moreover uniform on the unit circle to *rotation-invariant*, indicating that each independent jump has *no preference* in the direction (Reynolds et al. [43]). If the total mass of the uniform control measure is $\lambda(S) = -\sigma^\alpha/(\Gamma(-\alpha)\cos(\pi\alpha/2))$ with a suitable positive constant σ, the characteristic function (2.5) reduces to the simpler form

$$\mathbb{E}\left[e^{i\langle y,X_t\rangle}\right] = e^{t(i\langle y,\gamma\rangle - \sigma^\alpha\|y\|^\alpha)}. \tag{2.6}$$

Multi-dimensionality of the model poses a serious challenge. For example, the probability density $f_\alpha(x)$ of the marginal X_1 with $\gamma = 0$ and $\sigma = 1$ is given by

$$f_\alpha(x) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\langle x,y\rangle - \|y\|^\alpha} dy, \quad x \in \mathbb{R}^2, \tag{2.7}$$

which does not reduce to any practically useful form. Numerical treatment of this density function is thus prohibitively expensive.

2.2.2 Independent Components

Such computational difficulty in the rotation-invariant model may be circumvented by employing the stable Lévy motion with independent components. Let the control measure $\lambda(d\xi)$ be a sum of delta measures as

$$\lambda(d\xi) = -\frac{\sigma^\alpha}{2\Gamma(-\alpha)\cos(\pi\alpha/2)}\left[\delta_{\{[1,0]^\top\}}(d\xi) + \delta_{\{[-1,0]^\top\}}(d\xi) + \delta_{\{[0,1]^\top\}}(d\xi) + \delta_{\{[0,-1]^\top\}}(d\xi)\right], \tag{2.8}$$

with some $\sigma > 0$. (A similar argument holds for the case when the control measure $\lambda(d\xi)$ is supported homogeneously only on the orthonormal basis of $\mathbb{R}^2$. In this paper, we focus on the setting (2.8), without loss of generality.) Then, the characteristic function (2.5) reduces to

$$\mathbb{E}\left[e^{i\langle y,X_t\rangle}\right] = e^{t(iy_1\gamma_1 - \sigma^\alpha|y_1|^\alpha)} e^{t(iy_2\gamma_2 - \sigma^\alpha|y_2|^\alpha)}, \tag{2.9}$$

where $y = [y_1, y_2]^\top \in \mathbb{R}^2$. In contrast to the characteristic function (2.6) of the rotation invariant model, independence of the two components is evident. Each component has the so-called power-law tail, that is to say, a random variable Z with $\mathbb{E}[e^{iyZ}] = \exp[iy\gamma - \sigma^\alpha|y|^\alpha]$, $y \in \mathbb{R}$, has the asymptotic power-law tail behavior

$$\mathbb{P}(Z > z) \sim \frac{\sigma^\alpha}{2\Gamma(1-\alpha)\cos(\pi\alpha/2)} z^{-\alpha}, \quad z \uparrow +\infty, \tag{2.10}$$

in a similar manner to the exact Pareto tail (2.2).

2.3 Tempered Stable Lévy Motion

Finally, we introduce the tempered stable Lévy motion, which corresponds to the smoothly truncated Lévy flights in physical sciences (Koponen [23]). Its featuring properties were rigorously discussed by Rosiński [46], such as a stable-like behavior over short intervals and an aggregational Gaussianity. We call the stochastic process $\{X_t^{(\mathrm{ts})} : t \geq 0\}$ a tempered stable Lévy motion if it is a Lévy process in $\mathbb{R}^2$ without Gaussian component and with characteristic function

$$\begin{aligned}\mathbb{E}\left[e^{i\langle y,X_t^{(\mathrm{ts})}\rangle}\right] &= \exp\left[t\left(i\langle y,\gamma\rangle + \int_S\int_0^{+\infty}\left(e^{i\langle y,r\xi\rangle} - 1\right)\frac{e^{-\kappa(\xi)r}}{r^{\alpha+1}} dr\,\lambda(d\xi)\right)\right] \\ &= \exp\left[t\left(i\langle y,\gamma\rangle + \Gamma(-\alpha)\int_S\left((1 - i\langle y,\xi/\kappa(\xi)\rangle)^\alpha - 1 + i\alpha\langle y,\xi/\kappa(\xi)\rangle\right)\lambda(d\xi)\right)\right],\end{aligned} \tag{2.11}$$

where $\alpha \in (0,2)$, $\gamma \in \mathbb{R}^2$, $\kappa(\xi)$ is a measurable mapping from the unit circle S to $\mathbb{R}_+$ satisfying the integrability condition $\int_S \kappa(\xi)^{\alpha-2}\lambda(d\xi) < +\infty$, and $\lambda(d\xi)$ is the symmetric control measure on the unit circle as in (2.5). Obviously, the term "tempered" comes from the exponential tempering $e^{-\kappa(\xi)r}$ in the characteristic function (2.11). The motion is nearly the (non-tempered) stable motion for a very small $\kappa(\xi) > 0$, while its increments are essentially exponential if $\kappa(\xi)$ is very large. Here, we do not go into the specification of the control measure, unlike in the case of the stable Lévy motion.

3 Technical Results

In this section, we prepare technical results of the proposed models and describe statistical inference for later discussions. We begin with the basics of discrete sampling of the underlying continuous-time stochastic process, which our statistical experiment is based on.

3.1 Discrete Observations and Sampling Frequency

Hereafter, we let n denote the number of observations and suppose that the sample $(X_{t_{n,1}}, X_{t_{n,2}}, \ldots, X_{t_{n,n}})$ is available at equidistant observation points $t_{n,k} := k\Delta$, $k = 1,\ldots,n$, where Δ indicates the equidistant time stepsize. If Δ is very small, then sampling is *high frequency*, while a larger Δ indicates *low frequency* sampling. We write $Z_{n,k} := X_{t_{n,k}} - X_{t_{n,k-1}}$, $k = 1,\ldots,n$, for increments observed at equidistant time stepsize Δ, whether high or low frequency. Moreover, we often use the terminology "ultra high frequency" for the case where the stepsize Δ is *extremely* close to zero. (In general, the distinction among low, high and ultra high is rather qualitative than quantitative.) High frequency sampling has recently attracted much attention due to increasing availability of high resolution data of individual animal movement in ecology (Edwards et al. [12], Mashanova et al. [29], Plank and Codling [38]). Ultra high frequency sampling reflects the best possible experiment environment; in other words, strictly speaking, observation over a whole interval is never possible even with recent high technology; for example, although high resolution video recording apparently provides a continuous movement, it is still discrete even at a ultra high frequency. (It is noteworthy that there however exists some doubt of relevance of ultra high frequency sampling since in practice, the signal tends to decreases while the observation error might not be negligible and may dominate the measurements at some point.)

3.2 Simulation Methods

The increments $\{Z_{n,k}\}_{k=1,\ldots,n}$ of the Brownian motion of Section 2.1 are iid Gaussian with $\mathcal{N}(\Delta\gamma, \Delta\Sigma)$, where Σ is the variance-covariance matrix defined by (2.4). Therefore, generation of sample paths (by increments) is straightforward.

The rotation-invariant stable motion of Section 2.2.1 can be simulated through the infinite shot noise series representation on a fixed finite horizon $[0,T]$;

$$\{X_t : t \in [0,T]\} \leftarrow \left\{ t\gamma + \sum_{n=1}^{+\infty} \left(\frac{\Gamma(1-\alpha)\cos(\pi\alpha/2)}{\sigma^\alpha T} \Gamma_n \right)^{-1/\alpha} \xi_n \, \mathbb{1}(T_n \leq t) : t \in [0,T] \right\}, \tag{3.1}$$

where $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of iid uniform random variables on $[0,T]$, $\{\Gamma_n\}_{n\in\mathbb{N}}$ are arrival times of a standard Poisson process, and $\{\xi_n\}$ is a sequence of iid random vectors with $\xi_n := [\cos\eta_n, \sin\eta_n]^\top$ and $\{\eta_n\}_{n\in\mathbb{N}}$ is a sequence of uniform random variables on $[0,2\pi)$. (See [47] for details.) (Computationally realistic) sample path simulation of the rotation-invariant stable Lévy motion can be performed only through a finite truncation of the infinite series (3.1). In Figure 1, we draw typical sample paths of the Brownian motion with $\rho = 0$ (left) and the rotation-invariant stable Lévy motions with $\alpha = 1.8$ (center) and with $\alpha = 0.9$ (right) on the unit time interval $[0,1]$ with 5000 increments. From the Brownian motion to the stable Lévy motion with $\alpha = 1.8$, we observe more large movements. Then, towards a smaller α of the stable Lévy motions, the sample path looks more ballistic.

Figure 1: Typical spatial sample paths of a Brownian motion with $\rho = 0$ (leftmost) and rotation-invariant stable Lévy motions with $\alpha = 1.8$ (center) and $\alpha = 0.9$ (rightmost).

The stable Lévy motion with independent components of Section 2.2.2 can be simulated through the following infinite shot noise series representation (3.1) where $\{\eta_n\}_{n\in\mathbb{N}}$ here is a sequence of iid uniform random variables on $\{0, \pi/2, \pi, 3\pi/2\}$ instead. Furthermore, independence of two components provides an easier simulation method for *one-dimensional increments*; for each k, $Z_{n,k} \stackrel{\mathscr{L}}{=} \Delta\gamma + [Z_x(\Delta), Z_y(\Delta)]^\top$, where Z_x and Z_y are iid random variables as

$$Z_x(\Delta) \stackrel{\mathscr{L}}{=} Z_y(\Delta) \stackrel{\mathscr{L}}{=} \left(\frac{\Delta\sigma^\alpha}{\cos(\pi\alpha/2)\cos(V)}\right)^{1/\alpha} \sin\left(\alpha(V+\pi/2)\right) \left(\frac{\cos(V-\alpha(V+\pi/2))}{E}\right)^{(1-\alpha)/\alpha},$$

with E being a standard exponential random variable and V a uniform random variable on $(-\pi/2, \pi/2)$. Simulation based on the above expression is significantly much simpler than that using the infinite series (3.1). (We defer illustration of typical sample paths of this model to Section 4.)

Sample path simulation of the tempered stable Lévy motion of Section 2.3 can also be performed through a finite truncation of the infinite series representation (Rosiński [46]) as follows. Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of iid standard exponential random variables and let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of iid uniform random variables on $(0,1)$. Then, the tempered stable Lévy motion can be generated through

$$\left\{X_t^{(\mathrm{ts})} : t \in [0,T]\right\} \leftarrow \left\{t\gamma + \sum_{n=1}^{+\infty} \min\left[\left(\frac{\Gamma(1-\alpha)\cos(\pi\alpha/2)}{\sigma^\alpha T}\Gamma_n\right)^{-1/\alpha}, \frac{E_n U_n^{1/\alpha}}{\kappa(\xi_n)}\right] \xi_n \mathbb{1}(T_n \le t) : t \in [0,T]\right\}, \tag{3.2}$$

where $\sigma^\alpha := \lambda(S)$ here and where $\{T_n\}_{n\in\mathbb{N}}$, $\{\Gamma_n\}_{n\in\mathbb{N}}$, and $\{\xi_n\}_{n\in\mathbb{N}}$ are the ones appeared in (3.1). (A finite truncation of the infinite series representation is studied in Imai and Kawai [15] from a computational point of view. Similarly to the stable motion, some simpler simulation methods are discussed in Kawai and Masuda [20].) In Figure 2, we draw typical sample paths of the rotation-invariant stable Lévy motion and the rotation-invariant tempered stable Lévy motions with $\kappa(\xi) \equiv 1.0$ and $\kappa(\xi) \equiv 2.0$. The stability index is fixed at $\alpha = 0.9$ in common. Thanks to the minimum in the series representation (3.2), extremely large jumps tend to be (randomly) truncated. As the figures illustrate, sample paths behave smoother with larger tempering $k(\xi)$.

3.3 Variation of Movement Paths

In this section, we discuss the proposed models in terms of variation of movement paths, which indicates the amount of movement undergone by the sample path over a finite time interval $[0,T]$. To define this, fix $T > 0$ and let $\{s_k\}_{k=0,\ldots,n}$ denote the time partition, that is, non-decreasing positive constants satisfying $0 =: s_0 \le s_1 \le \cdots \le s_n := T$, not necessarily equidistant. Also, the length of the longest subinterval is assumed to go to zero; $\max_{k=0,\ldots,n-1}(s_{k+1} - s_k) \to 0$. Then, the variation of a trajectory $\{X_t : t \in [0,T]\}$ on a fixed finite time interval $[0,T]$

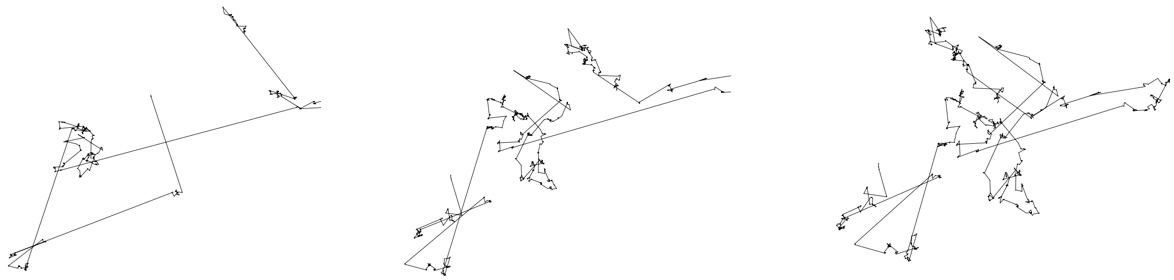

Figure 2: Typical spatial sample paths of the rotation-invariant stable Lévy motion (corresponding to $\kappa(\xi) \equiv 0$, left) and the rotation-invariant tempered stable Lévy motions with $\kappa(\xi) \equiv 1.0$ (center) and $\kappa(\xi) \equiv 2.0$ (right). The stability index is $\alpha = 0.9$ in common.

is defined by

$$\lim_{n \uparrow +\infty} \sum_{k=0}^{n-1} \left\| X_{s_{k+1}} - X_{s_k} \right\| .$$

We present the variation property of the proposed models below, all of which hold regardless of directional preference of the movement. (For proof, see, for example, Sato [48].)

Theorem 3.1. *The following statements hold with probability one;*

(i) *the variation of the Brownian motion of Section 2.1 is infinite,*
(ii) *the variation of the stable Lévy motion of Section 2.2 with* $\alpha \in [1,2)$ *is infinite,*
(iii) *the variation of the stable Lévy motion of Section 2.2 with* $\alpha \in (0,1)$ *is finite,*
(iii) *the variation of the tempered stable Lévy motion of Section 2.3 with* $\alpha \in [1,2)$ *is infinite,*
(iv) *the variation of the tempered stable Lévy motion of Section 2.3 with* $\alpha \in (0,1)$ *is finite.*

In principle, no animal can physically make a movement of infinite length over a finite time interval. The above results suggest that in terms of variation, the stable and tempered stable Lévy motions with $\alpha \in (0,1)$ well describe movement paths in the sense that the amount of its movement undergone by the sample path over a finite time interval is finite. In contrast, the Brownian motion, the stable and tempered stable Lévy motions with $\alpha \in [1,2)$ fail to do so, as the sample paths make a movement of infinite length.

3.4 Moments of Increments

In the case of the Brownian motion, the increments are iid Gaussian with $\mathcal{N}(\Delta\gamma, \Delta\Sigma)$, where Σ is the variance-covariance matrix defined by (2.4). Therefore, each increment has finite moment of every polynomial order, that is, $\mathbb{E}[\|Z_{n,1}\|^p] < +\infty$ for $p > 0$.

Turing to the stable motions of Section 2.2, the increments are iid stable random vectors. Irrespective of the stepsize Δ and the control measure $\lambda(d\xi)$, its increments have finite moment of polynomial order only strictly less than α, that is,

$$\mathbb{E}\left[\|Z_{n,1}\|^p\right] \begin{cases} < +\infty, & \text{if } p < \alpha, \\ = +\infty, & \text{if } p \geq \alpha, \end{cases}$$

identical to the moment property of the Pareto distribution (2.1).

In the case of the tempered stable motion of Section 2.3, the increments are iid tempered stable random vectors. The condition on the tempering exponent $\kappa(\xi)$ ensures the square integrability $\int_{\mathbb{R}_0^2} \|z\|^2 \nu(dz) = \Gamma(2 - \alpha) \int_S \kappa(\xi)^{\alpha-2} \lambda(d\xi) < +\infty$ and thus a finite variance is guaranteed, unlike the (non-tempered) stable motion of Section 2.2. (See Theorem 25.3 of Sato [48] for technicality.)

3.5 Short- and Long-Time Behaviors of the Tempered Stable Lévy Motion

The tempered stable Lévy motion exhibits desirable short- and long-time behaviors; over short intervals, it behaves like a non-tempered stable Lévy motion, while it approximates a Brownian motion in the long run. We state below the results without proof as they are special cases of [46] and without specifying constants to omit non-essential details.

Theorem 3.2. *Let $\{X_t^{(\mathrm{ts})} : t \geq 0\}$ be the tempered stable motion satisfying (2.11) with $\gamma = 0$.*
(i) Short time behavior: It holds that as $h \downarrow 0$,

$$\left\{h^{-1/\alpha} X_{ht}^{(\mathrm{ts})} : t \geq 0\right\} \stackrel{\mathscr{L}}{\to} \{X_t : t \geq 0\}, \tag{3.3}$$

where $\{X_t : t \geq 0\}$ is a suitable stable Lévy motion of Section 2.2 with the same stability index α.
(ii) Long time behavior: It holds that as $h \uparrow +\infty$,

$$\left\{h^{-1/2} X_{ht}^{(\mathrm{ts})} : t \geq 0\right\} \stackrel{\mathscr{L}}{\to} \{B_t : t \geq 0\}, \tag{3.4}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion in $\mathbb{R}^2$ with mean zero and with variance-covariance matrix $\Gamma(2-\alpha)\int_S \kappa(\xi)^{\alpha-2}\xi\xi^\top \lambda(d\xi)$.

Above, the convergence $\stackrel{\mathscr{L}}{\to}$ conserves all sample path properties (for example, continuity and differentiability of the sample path), not only the distribution at some fixed point in time.

3.6 Fisher Information and Local Asymptotic Normality

In this section, we discuss a statistical issue of the proposed models on the basis of the Fisher information and the local asymptotic normality property, both of which serve as vital concepts in asymptotic statistical analysis. The local asymptotic normality property for a differentiable statistical model with the parameter $\theta \in \mathbb{R}^d$ is defined through the weak convergence of the likelihood ratio to the Gaussian shift experiment; for each $h \in \mathbb{R}^d$, as the number of observations tend to increase ($n \uparrow +\infty$),

$$\frac{d\mathbb{P}_{\theta+R_n(\theta)h}}{d\mathbb{P}_\theta}\Big|_{\mathscr{F}_n} \stackrel{\mathscr{L}}{\to} \exp\left[\langle h, Z(\theta)\rangle - \frac{1}{2}\langle h, \mathscr{I}(\theta)h\rangle\right], \tag{3.5}$$

under $\mathbb{P}_\theta$, where $\mathbb{P}_\theta|_{\mathscr{F}_n}$ is a probability measure associated with θ restricted to the filtration $\mathscr{F}_n$, which is the σ-field generated by the discrete observation $(X_{t_{n,1}}, X_{t_{n,2}}, \ldots, X_{t_{n,n}})$. Here, the equidistant time stepsize Δ may be either constant or decreasing in the number n of observations. In this paper, we are particularly interested in the latter case and rewrite it as Δ_n so that $\Delta_n \downarrow 0$, corresponding to (relatively) high frequency sampling and its limit. Moreover, $\{R_n(\theta)\}_{n\in\mathbb{N}}$ is a sequence of diagonal matrices in $\mathbb{R}^{d\times d}$ whose diagonal entries tend to zero, $\mathscr{I}(\theta)$ is a non-negative definite deterministic matrix in $\mathbb{R}^{d\times d}$, called the Fisher information matrix, and $Z(\theta) \sim \mathscr{N}(0, \mathscr{I}(\theta))$ under $\mathbb{P}_\theta$. If the above weak convergence holds, then we say that *the local asymptotic normality property holds at point θ with the rate $R_n(\theta)$ and the Fisher information matrix $\mathscr{I}(\theta)$*. If the local asymptotic normality property holds with non-singular $\mathscr{I}(\theta)$, then a unbiased estimator $\{\widehat{\theta}_n\}_{n\in\mathbb{N}}$ of θ is asymptotically optimal in a neighborhood of θ if

$$R_n(\theta)^{-1}\left(\widehat{\theta}_n - \theta\right) \stackrel{\mathscr{L}}{\to} \mathscr{N}\left(0, \mathscr{I}(\theta)^{-1}\right),$$

under $\mathbb{P}_\theta$, that is, such estimators achieve asymptotically the Cramer-Rao lower bound $\mathscr{I}(\theta)^{-1}$ for the estimation variance, provided that the inverse $\mathscr{I}(\theta)^{-1}$ is well defined. Below, we give results for the Brownian motion and the stable Lévy motion. In order to avoid overloading with unnecessary lengthy details, we do not specify the constants $H_1(\alpha)$, $M_1(\alpha)$, $H_2(\alpha)$ and $M_2(\alpha)$.

Theorem 3.3. *Let Θ_1 and Θ_2 be bounded convex domains, whose closures satisfy*

$$\overline{\Theta}_1 \subset \left\{ [\gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho]^\top \in \mathbb{R}^5 \middle| \gamma_1 \in \mathbb{R}, \gamma_2 \in \mathbb{R}, \sigma_1 \in \mathbb{R}_+, \sigma_2 \in \mathbb{R}_+, \rho \in [-1,+1] \right\},$$
$$\overline{\Theta}_2 \subset \left\{ [\gamma_1, \gamma_2, \alpha, \sigma]^\top \in \mathbb{R}^4 \middle| \gamma_1 \in \mathbb{R}, \gamma_2 \in \mathbb{R}, \alpha \in (0,2), \sigma \in \mathbb{R}_+ \right\}.$$

(i) Consider the Brownian motion of Section 2.1. Suppose $\Delta_n \downarrow 0$ satisfies $\sqrt{n\Delta_n} \uparrow +\infty$. The local asymptotic normality property holds at the point $\theta := [\gamma_1, \gamma_2, \sigma_1, \sigma_2, \rho]^\top \in \Theta_1$ with

$$R_n(\theta) = \mathrm{diag}\left(\frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n\Delta_n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right), \tag{3.6}$$
$$\mathscr{I}(\theta) = \frac{1}{1-\rho^2} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} & 0 & 0 & 0 \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{2-\rho^2}{\sigma_1^2} & -\frac{\rho^2}{\sigma_1\sigma_2} & -\frac{\rho}{\sigma_1} \\ 0 & 0 & -\frac{\rho^2}{\sigma_1\sigma_2} & \frac{2-\rho^2}{\sigma_2^2} & -\frac{\rho}{\sigma_2} \\ 0 & 0 & -\frac{\rho}{\sigma_1} & -\frac{\rho}{\sigma_2} & \frac{1+\rho^2}{1-\rho^2} \end{bmatrix}.$$

In particular, the matrix $\mathscr{I}(\theta)$ is singular if and only if $\rho = \pm 1$.
(ii) Consider the rotation-invariant stable Lévy motion of Section 2.2.1. Suppose $\Delta_n \downarrow 0$ satisfies $\sqrt{n}\Delta_n^{1-1/\alpha} \uparrow +\infty$. The local asymptotic normality property holds at the point $\theta := [\gamma_1, \gamma_2, \alpha, \sigma]^\top \in \Theta_2$ with

$$R_n(\theta) = \mathrm{diag}\left(\frac{1}{\sqrt{n}\Delta_n^{1-1/\alpha}}, \frac{1}{\sqrt{n}\Delta_n^{1-1/\alpha}}, \frac{1}{\sqrt{n}|\ln\Delta_n|}, \frac{1}{\sqrt{n}} \right), \quad \mathscr{I}(\theta) = \begin{bmatrix} \frac{M_1(\alpha)}{\sigma^2} & 0_{2\times1} & 0_{2\times1} \\ 0_{1\times2} & \frac{H_1(\alpha)}{\alpha^4} & \frac{H_1(\alpha)}{\sigma\alpha^2} \\ 0_{1\times2} & \frac{H_1(\alpha)}{\sigma\alpha^2} & \frac{H_1(\alpha)}{\sigma^2} \end{bmatrix},$$

with $H_1(\alpha) > 0$ and $M_1(\alpha)$ is a positive definite matrix in $\mathbb{R}^{2\times2}$, both of which depend only on α. In particular, the matrix $\mathscr{I}(\theta)$ is singular.
(iii) Consider the stable Lévy motion with independent components of Section 2.2.2. Suppose $\Delta_n \downarrow 0$ satisfies $\sqrt{n}\Delta_n^{1-1/\alpha} \uparrow +\infty$. The local asymptotic normality property holds at the point $\theta := [\gamma_1, \gamma_2, \alpha, \sigma]^\top \in \Theta_2$ with

$$R_n(\theta) = \mathrm{diag}\left(\frac{1}{\sqrt{n}\Delta_n^{1-1/\alpha}}, \frac{1}{\sqrt{n}\Delta_n^{1-1/\alpha}}, \frac{1}{\sqrt{n}|\ln\Delta_n|}, \frac{1}{\sqrt{n}} \right), \quad \mathscr{I}(\theta) = \begin{bmatrix} \frac{M_2(\alpha)}{\sigma^2} & 0 & 0 & 0 \\ 0 & \frac{M_2(\alpha)}{\sigma^2} & 0 & 0 \\ 0 & 0 & \frac{H_2(\alpha)}{\alpha^4} & \frac{H_2(\alpha)}{\sigma\alpha^2} \\ 0 & 0 & \frac{H_2(\alpha)}{\sigma\alpha^2} & \frac{H_2(\alpha)}{\sigma^2} \end{bmatrix},$$

with $H_2(\alpha) > 0$ and $M_2(\alpha) > 0$, both of which depend only on α. In particular, the matrix $\mathscr{I}(\theta)$ is singular.

4 Discussion

Given the increasing availability of high resolution observation data of random search paths, it is a natural interest to investigate movement paths at different sampling rates. It has been argued in the recent literature that statistical inference for the time series observation itself tend to give different results at different sampling rates. To address this issue, we have considered three continuous-time spatial stochastic processes when directly describing random searches. We now discuss the relevance of those models on the basis of their advantages and limitations in terms of statistical properties and inference.

4.1 Statistical Properties

We have presented various relevant statistical properties of the proposed models. In particular, all the proposed models provide simulation methods in an implementable form (Section 3.2), which enables us to conduct simulation-based study.

First, increments of the Brownian motion of Section 2.1 with equidistant time stepsize are iid Gaussian with no directional preference. It has long been a widely accepted consensus (Viswanathan [54], for example) and is observed in Figure 1 that the resulting Gaussian step length is too regular to properly describe typical individual movement paths. Even in the continuous-time modeling framework, the Brownian motion fails to capture key statistical properties of movement paths; for example, the variation of its sample path is infinite over a finite time interval (Theorem 3.1 (i)), which is physically impossible for any animal to achieve.

The stable motion of Section 2.2.1 achieves a better description of movement paths. As illustrated in Figure 1, the Brownian walker frequently returns to previously visited locations. By contrast, the rotation-invariant stable Lévy motion revisits sites far less often. This is in agreement with a conclusion often made in the literature that Lévy walkers can outperform Brownian walkers in places where prey is scarce (Viswanathan [54]). In this respect, the sample path with smaller α looks more realistic as a foraging movement. On the additional basis of the variation (Theorem 3.1 (ii) and (iii)), the stable Lévy motion with $\alpha \in (0,1)$ seems to act as a good candidate.

Still, there exist at least two remaining issues with the stable motion. One is its fractal property, which is too beautiful to be realistic, while the other is infinite variance (and even infinite mean for $\alpha \in (0,1)$) which is simply against reality. Those issues may be addressed by employing the tempered stable motion of Section 2.3. In particular, we have seen that the tempered stable Lévy motion with $\alpha \in (0,1)$ has a finite variation and a finite variance, with which its sample paths describe the movement in the continuous-time modeling framework sufficiently well, not only by eye based on Figure 2. Also, the short-time stable and long-time Gaussian behaviors account for the recent statistical observation (for example, Plank and Codling [38]) through the effect of sampling frequency; if frequency is sufficiently high, the observed lengths essentially follow Pareto, while they tend to a distribution with a lighter tail after subsampling, which is obvious in some sense due to the central limit theorem, as the observed lengths have a finite variance in practice.

It is important to discuss here that the stable (and tempered stable) motion with independent components might not be really appropriate for modeling movement paths. We draw various sample paths in Figure 3 to demonstrate this point. In the case of the stable Lévy motion, the trajectory with independent components looks unrealistically orthogonal. Such orthogonality occurs when either one of two independent components jumps with one very large length and the other with no such long jumps during the observation interval. In contrast, the Brownian motion with independent components ($\rho = 0$) is indeed rotationally invariant.

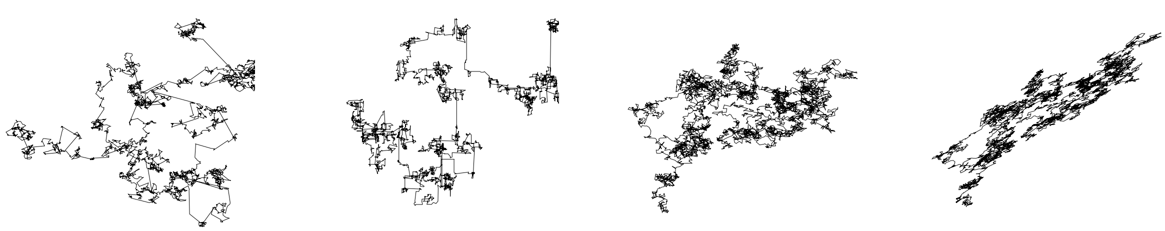

Figure 3: Typical sample paths of the rotation-invariant stable Lévy motion with $\alpha = 1.8$ (leftmost), the stable Lévy motion with independent components with $\alpha = 1.8$ (the second from the left), the Brownian motions with $\rho = 0$ (the second from the right) and $\rho = 0.7$ (rightmost),

4.2 Statistical Inference

For the Brownian motion, the Fisher information matrix is singular (Theorem 3.3 (i)) if the two dimensions are completely correlated, that is, $\rho = \pm 1$; such unrealistic settings can be excluded from consideration right away. It is known that the Fisher information matrix remains the same even under low frequency sampling. Regardless of sampling frequency, thus, asymptotic normality of efficient joint estimation is readily guaranteed. Also, the maximum likelihood estimator, which is easily computable, is known to achieve this asymptotic efficiency. As a consequence, one may choose the time stepsize Δ, without worrying about normality of the joint maximum likelihood estimator.

The optimal rate in (3.6) provides an essential idea about estimation under high frequency sampling, in agreement with the discussion made in Codling et al. [8]. Namely, on the one hand, as the drift parameter γ describes a global behavior of sample paths, the drift parameter cannot be correctly estimated only based on a fixed time interval. For example, if one might mistakenly focus on a fixed interval with very negatively skewed path when the true drift is positive, which is occasionally possible, then estimation would indicate a negative drift. Hence, the total observation length $n\Delta$ is required to be very large. On the other hand, the variances and correlation $[\sigma_1, \sigma_2, \rho]^\top$ describe a local behavior of sample paths, estimatable from a finite interval. In other words, we only need a short interval (the total observation window $n\Delta$ may even be decreasing) without looking at the global tendency of sample paths.

Just as is often the case, getting beyond the Gaussian regime is a typical example of "more realistic model with less mathematical tractability." The rotation-invariant stable motion of Section 2.2.1 exhibits a desired power-law type movement (Dziubanski [11]), while its marginal probability density function (2.7) is numerically prohibitive. Due to the singular Fisher information (Theorem 3.3 (ii)), either the stability index α or the scale σ is desired to be *a priori* known to attain asymptotic optimal joint normality of unbiased efficient estimators, in particular, when the observation is made at ultra high frequency. Even when one parameter is given, the efficient maximum likelihood estimator (that is, root finding of the likelihood equation) is numerically intractable as computation of the multidimensional stable density and its derivatives is computationally prohibitive.

In addition, we would like to emphasize that *a priori* knowledge of a parameter is technically different from leaving it as a nuisance parameter. For example, in the case of the Pareto random walk model, its jump-size distribution requires a (strictly positive) threshold, which is often set in a somewhat heuristic manner by discarding tiny jumps or noises (Sims, Righton and Pitchford [52], Edwards [13], White, Enquist and Green [56]). Such an arbitrarily set threshold is still nothing but a nuisance parameter and may have a significant impact on estimation results (Kawai and Petrovskii [17]).

By employing the stable motion with independent components of Section 2.2.2, we can apply estimation procedures for the one-dimensional stable motion separately to two coordinates due to their independence. Although statistical methods for the one-dimensional stable Lévy motion are already computationally demanding, they are relatively well known (Masuda [28], Nolan [31]), together with a computer code for the maximum likelihood estimation (Nolan [32]). Again, due to the singularity of the Fisher information in the limit, a great care is required for estimation based on ultra high resolution data. Despite a slight advantage in statistical inference, as discussed earlier with Figure 3, the stable motion with independent components may not be a realistic candidate due to its orthogonal-looking movement paths.

Turing to the tempered stable motion of Section 2.3, which appears to be the most realistic among the proposed models, little is known about its statistical inference. For example, the probability density function of tempered stable laws is unavailable in any useful form even in the one-dimension setting, and thus so are all essential quantities, such as the likelihood ratio function, the the likelihood equation and its root. In addition, unlike the well known power-law decay (2.10) of the stable motion, asymptotic behavior of the probability density of the symmetric tempered stable laws is still an open problem, irrespective of the directional preference. We conjecture that the two parameter (α, σ) end up causing the singularity of the Fisher information, just as in the stable motion (Theorem 3.3 (ii) and (iii)). This is so because, as seen in the short time behavior (3.3), its increments behave like the (non-tempered) stable motion under ultra high frequency sampling.

4.3 Perspectives and Challenges

Existing studies have only focused on time series modeling of discrete observations of a random search path, while such discrete-time models are not robust to the sampling rate, as often argued in the literature. In other words, statistical inference at different sampling rates tend to misidentify the underlying continuous-time movement path. Meanwhile, continuous-time modeling is designed to directly describe the underlying continuous-time movement path and thus make it possible to perform inference for the underlying continuous-time path given its discrete observation data, not for the time series formed by discrete sample itself. Hence, by nature, continuous-time models are fully robust to the sampling frequency in statistical inference, as long as the underlying path is the interest for estimation.

In this paper, we have taken the first step towards continuous-time modeling of movement paths, by considering two-dimensional Lévy processes as realistic and/or mathematically tractable candidates; the Brownian motion, the stable motion and the tempered stable motion. Relatively straightforward simulation methods are available for all those models. This is a key feature when using such models in applications.

We have discussed further advantages and limitations of those continuous-time models on the basis of statistical properties and inference. In short, as is often the case, there is an inevitable tradeoff; analytically tractable models are not statistically realistic, while models incorporating more statistical features are too complex in various ways; analytically, numerically and statistically, etc.

The Brownian motion allows for analytical computations of many quantities without recourse to simulation and provides optimal parameter estimation methods regardless of the sampling rate, while its sample path is not considered as a good description of movement paths. Second, the stable motion of finite variation captures many statistical properties except for infinite variance. Nevertheless, statistical inference for the model is significantly much more complex than the case of the Brownian motion. In particular, under ultra high frequency sampling, its Fisher information matrix is singular. Third, the tempered stable motion captures even more statistical properties than the (non-tempered) stable motion, including a finite variance, short-time Pareto and long-time Gaussian behaviors, while almost none is known about its statistical inference as its probability density function and its tail behaviors are still unknown. The range of sampling periods for which short-time Pareto and long-time Gaussian behaviors occur depends largely on the parameters of the model and is an interesting topic to investigate.

Still, the proposed models are relatively mathematically tractable and way too simple to describe the reality. Much more remains to be done to bridge the gap between such stochastic models and the reality. In particular, the proposed models are all in the class of Lévy processes, which satisfy the unrealistic assumption of independent and stationary increments. Thus, they neither cover a multi-state movement based on a combination of intensive and extensive search modes (Bartumeus [2], Benhamou [3]) nor describe non-destructive search strategies, in where the forager may visit and feed at the same target site many times (Morales et al. [30], Patterson et al. [36], Pedersen et al. [37]).

Lévy flights are not optimal search strategies in the case of nonrevisitable targets [41], where intermittent strategies come into effect. (See [5] for a nice review on intermittent search.) Intermittent search processes switch between local extensive search phases and ballistic relocation phases, for example, the leftmost and rightmost figures in Figure 1, respectively. The question of determining optimal search strategies (or equivalently, minimizing the relocation time) has been given growing attention, for example, [34, 35, 45]. In particular, it is known [26] that once optimised, a scale free system based on Pareto jumps is adaptable to changing parameters and is more robust than intermittent systems with fixed scales [4]. Classical statistical methods are directly applicable to a partial trajectory within each phase, while evidently not when two or more phases are under consideration. It is certainly an interesting research direction to develop a quantitative statistical methods for such change-point detection, such as [30], rather than on the basis of visual detection.

Although we have focused on Markovian Lévy process models as continuous-time counterparts of Lévy walks, there are non-Markovian features to be taken into account in modeling, such as directional preference and autocorrelation (Bovet and Benhamou [6], Plank and Codling [38], Shlesinger [49]). For example, fractional Brownian and Lévy motions are continuous-time stochastic processes with autocorrelations and are examined by Reynolds [44] in their discrete-time form as alternatives to Lévy walks. In general, such non-Markovian properties tend to clearly show up when observing movement paths in detail, that is, under higher frequency sampling. On the contrary, as discussed in the literature (Raposo et al. [41], Viswanathan [55]), even a correlated random walk resembles Brownian motion at a lower sampling rate, due to the tangling impact of the turning angles. Our proposed models thus have the potential to be very useful for modeling random search paths at a relatively low observation rate when such non-Morkovian properties become negligible. Moreover, in relation to sampling rate, observation error is another largely open question in particular under high frequency sampling where the signal-to-noise ratio tends to be small.

Finally, singularity of the Fisher information that we have encountered in this paper is not really a rare phenomenon (Kawai [19]); at least two such concrete examples are known. It is known (Kawai [16], Kawai and Petrovskii [17]) that in the fractional Brownian motion model (Reynolds [44]), the joint presence of the self-similar index H and the volatility σ causes singularity. The other is the Meixner Lévy process, suffering from a somewhat different type of singularity between two scale parameters (Kawai and Masuda [21]). Note that like the tempered stable motion of Section 2.3, a Meixner Lévy process behaves like a stable Lévy process with stability index 1 (a Cauchy process, in other words). Interestingly, the normal inverse Gaussian Lévy process exhibits a similar short-range Cauchy behavior, while it possesses the invertible Fisher information matrix (Kawai and Masuda [22]). Given the increasing availability of high resolution observation data, the issue of singular Fisher information is likely to appear more frequently and is a subject of future research.

A Proof

Proof of Theorem 3.3. The complete proof entails rather lengthy arguments of somewhat routine nature. To avoid overloading the paper, we omit the proof of (i) and only give a short sketch of the proof of (ii) without non-essential details. For the proof of (i), see, for example, Ferguson [14]. The results (ii) and (iii) can be shown in a similar manner to Masuda [28]. Define for each $n \in \mathbb{N}$,

$$\varepsilon_{n,k} := \varepsilon_{n,k}(\gamma, \alpha, \sigma, \Delta_n) := \frac{X_{t_{n,k}} - X_{t_{n,k-1}} - \Delta_n \gamma}{\Delta_n^{1/\alpha} \sigma}, \quad k = 1, \ldots, n,$$

where $\varepsilon_{n,1}$ has the probability density function $f_\alpha(x)$ given by (2.7). Thanks to the stationarity and independence of increments of Lévy processes, the log-likelihood function to be maximized with discrete observations $\{X_{t_{n,k}}\}_{k=1,\ldots,n}$ is as simple as $\ell_n(\theta) = \sum_{k=1}^n (-\ln \sigma - \alpha^{-1} \ln \Delta_n + \ln f_\alpha(\varepsilon_{n,k}))$. We can show that

$$\nabla_\gamma(\ell_n(\theta)) = -\frac{1}{\Delta^{1/\alpha-1}\sigma} \sum_{k=1}^n \frac{\nabla f_\alpha(\varepsilon_{n,k})}{f_\alpha(\varepsilon_{n,k})},$$
$$\frac{\partial}{\partial \alpha}(\ell_n(\theta)) = \frac{n \ln \Delta_n}{\alpha^2} + \sum_{k=1}^n \left[\frac{\ln \Delta_n}{\alpha^2} \frac{\langle \varepsilon_{n,k}, \nabla f_\alpha(\varepsilon_{n,k})\rangle}{f_\alpha(\varepsilon_{n,k})} - \frac{\int_{\mathbb{R}^2} \|y\|^\alpha \ln \|y\| e^{-i\langle \varepsilon_{n,k}, y\rangle - \|y\|^\alpha} dy}{(2\pi)^2 f_\alpha(\varepsilon_{n,k})} \right],$$
$$\frac{\partial}{\partial \sigma}(\ell_n(\theta)) = -\frac{n}{\sigma} - \frac{1}{\sigma} \sum_{k=1}^n \frac{\langle \varepsilon_{n,k}, \nabla f_\alpha(\varepsilon_{n,k})\rangle}{f_\alpha(\varepsilon_{n,k})},$$

where $\nabla f_\alpha(x) := \nabla_x(f_\alpha(x))$. For the rest, we refer to [28]. (In particular, for (iii), each entry of the matrix $\mathscr{I}(\theta)$ is well defined since the tail behavior of the density function is given by $g(r\xi) = O(r^{-2-\alpha})$ as $r \uparrow +\infty$ for each ξ in the unit circle S. See [11] for details.) □

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