The Discrete Diffraction Transform

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Abstract

In this paper, we define a discrete analogue of the continuous diffracted projection. We define the discrete diffracted transform (DDT) as a collection of the discrete diffracted projections taken at specific set of angles along specific set of lines. The 'discrete diffracted projection' is defined to be a discrete transform that is similar in its properties to the continuous diffracted projection. We prove that when the DDT is applied to a set of samples of a continuous object, it approximates a set of continuous vertical diffracted projections of a horizontally sheared object and a set of continuous horizontal diffracted projections of a vertically sheared object. A similar statement, where diffracted projections are replaced by the X-ray projections, holds for the discrete 2D Radon transform (DRT), is also proved. We prove that the discrete diffraction transform is rapidly computable and invertible. Some of the underlying ideas came from the definition of DRT. Unlike the DRT, though, this transform cannot be used for reconstruction of the object from the set of rotated projections.

Key word: diffraction tomography, discrete diffraction transform, Radon transform.
AMS Subject Classfication: 44A12, 78A45

1 Introduction

Ultrasound imaging is an example of diffracted tomography (see [4]). X-ray tomography, mathematically described by the continuous Radon transform, is an example of non-diffracted tomography imaging. In both cases, the transforms act on a physical body that result in projection, which is continuous in theory, but in practice it is always discrete, because the number and the size of the receivers that collect the energies, be it X-ray or ultrasound, which are used to illuminate the body, are finite .

In practice, the fact that projections are always discrete, it gives rise to the question whether there exists a transform that accepts a discrete set of samples of a continuous object and produces a set of discrete projections that approximates the actual projections of the object. If such transform exists and it is invertible, we can use the inverse transform to reconstruct the samples of the original object from its projections.

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The notion of a 2D Radon transform, which acts on discrete 2D objects, is defined in [6]. Actually, the 2D Radon transform of a discrete object along a line can be viewed as a 'discrete projection' of the object. The discrete Radon transform (DRT) is a collection of these projections along a specific set of lines. This transform is invertible and rapidly computable. Its complexity is $O(N \log N)$, where $N = n^2$ is the number of pixels in the image. The DRT is used to approximate the X-ray projections of the object. It is based on a discrete set of samples of the continuous object. The inverse DRT is used to reconstruct the object from the set of rotated projections.

In this paper, we use the ideas, which underly the definition of the 2D Radon transform, to define a 'discrete diffracted projection', which is a discrete transform similar in its properties to the continuous diffracted projection. We also define a discrete diffraction transform (DDT) as a collection of discrete diffracted projections along specific set of lines. We explain how the DDT is related to the continuous diffracted projections. We prove that the discrete diffraction transform is rapidly computable and invertible. Unlike the discrete Radon transform, this transform cannot be used to reconstruct the object from the set of rotated projections.

In this paper, we consider two-dimensional objects only. A two-dimensional physical object in the continuous case is represented by a real-valued 'object function' $f(x, y)$ of two real arguments, which describes some physical characteristic of the object. The object function represents the density of the object in X-ray tomography and the refractive index of the object in diffraction tomography. Suppose $f(x, y)$ represents some two-dimensional physical object bounded in space. Since the object is bounded, there exists a constant D such that $f(x, y) = 0$ outside the square $[-D, D] \times [-D, D]$. In the discrete setting, the object is described by a discrete set of values where the Cartesian set of samples is $o[u, v] = f\left(\frac{2D}{M}u, \frac{2D}{M}v\right)$, $u, v \in [-N : N]$, $M = 2N + 1$, for some positive integer N. The discrete object of size $M \times M$ is the square matrix $\left\{o[u, v] \in \mathbb{R} \,\middle|\, u, v \in [-N : N]\right\}$.

In the rest of the paper, we assume that N is a positive integer, $M = 2N + 1$ and $o[u, v]$ is a discrete object of size $M \times M$. The notation $o[u, v]$ will denote either the value of the matrix (the discrete object) at indices u and v or the discrete object (the matrix) itself.

The discrete Fourier transform (DFT) of a discrete object means either the trigonometric polynomial defined by

$$\widehat{o}(\omega_1, \omega_2) \triangleq \sum_{u=-N}^{N} \sum_{v=-N}^{N} o[u, v]\, e^{-i\frac{2\pi}{M}(\omega_1 u + \omega_2 v)} \quad \omega_1, \omega_2 \in \mathbb{R}, \tag{1.1}$$

or the set of samples of this trigonometric polynomial on the discrete set $\omega_1, \omega_2 \in [-N : N]$.

The paper is organized as follows. Section 2 describes the trigonometric interpolation and the shear transformation, which play a major role in the derivation of the discrete diffraction transform. Section 3 reviews the discrete Radon transform [6], which establishes the framework for the proposed discretization. Section 4 reviews the continuous diffraction tomography, including the physical background and the continuous diffraction theorem. Section 5 describes a discretization of a continuous diffracted projection along the y-axis. Definition of the discrete diffracted projection in section 6 is based on this discretization. Section 6 presents the definition of the discrete diffraction projections, which forms a basis for a definition of the discrete diffraction transform in section 7. Section 8 analyzes the relation between the discrete diffracted projections and the continuous diffracted projections of a sheared object. Section 9 describes the implementation of the discrete diffraction transform and its numerical results. Section 10 contains extensive proof of the theorem from section 5.

2 Trigonometric interpolation and shear transformation

We mentioned in the introduction that both the 2D Radon Transform and the discrete diffracted projections, which we define in Section 6, act on a discrete object $o[u, v]$, where N is some positive integer and $u, v \in [-N : N]$. In both cases, we first define a discrete transform for the case when the projection is taken along the x-axis or along the y-axis, and then expand the definition for projections at the other directions. The expansion to non-vertical and non-horizontal directions requires interpolation of a discrete object $o[u, v]$ (for the reasons that will be explained later).

In this section, we define the scaled trigonometric interpolation which has certain properties that make it well-suited for use with DFT, define the shear transformation of discrete objects using this interpolation and show that the DFT of a sheared discrete object is the shear of the object's DFT just like the continuous Fourier transform of a sheared object is a shear of the object's Fourier transform.

2.1 Trigonometric interpolation

Definition 2.1. *Trigonometric polynomial of order N is an expression of the form*

$$T(x) = \sum_{n=-N}^{N} c_n e^{inx}, \tag{2.1}$$

where c_n are complex numbers.

Theorem 2.2. *([7](pg.1) Uniqueness of a trigonometric interpolating polynomial)*

Given $2N + 1$ points $x_{-N}, \dots, x_0, \dots, x_N$, which are distinct modulo 2π, and arbitrary numbers $y_{-N}, \dots, y_0, \dots, y_N$, there always exists a unique polynomial (2.1) such that $T(x_k) = y_k \quad k = -N, \dots, N$.

The polynomial $T(x)$ is called the **(trigonometric) interpolating polynomial** corresponding to points x_k and values y_k. The points $x_{-N}, \dots, x_0, \dots, x_N$ are often called *fundamental* or *nodal* points of the interpolation, or "interpolation nodes".

The trigonometric interpolating polynomial, which corresponds to $\{x_n\}_{n=-N}^{N}$ and nodes $\{\frac{2\pi}{M}n\}_{n=-N}^{N}$ is, explicitly given by

$$x(t) = \frac{1}{M} \sum_{k=-N}^{N} \widehat{x}_k e^{ikt} = \sum_{n=-N}^{N} x_n D_M \left(\frac{2\pi}{M} n - t \right), \tag{2.2}$$

where $M = 2N + 1$, $\{\widehat{x}_k\}_{k=-N}^{N}$ is the 1D DFT of $\{x_n\}_{n=-N}^{N}$ and $D_M(t) \triangleq \frac{1}{M} \frac{sin(\frac{M}{2}t)}{sin(\frac{1}{2}t)} = \frac{1}{M} \sum_{k=-N}^{N} e^{ikt}$ is the Dirichlet kernel of order N.

Definition 2.3. ***Scaled trigonometric polynomial*** *of order* N *with a* ***scaling factor*** α *is an expression of the form* $T_\alpha(x) \triangleq \sum_{n=-N}^{N} c_n e^{i\alpha n x}$ *where* c_n *are complex numbers and* α *is a positive real number.*

Theorem 2.4. *(Uniqueness of a scaled trigonometric interpolating polynomial)*

Let N *be a positive integer,* α *be a positive real number. Given* $2N+1$ *points* $x_{-N}, \ldots, x_0, \ldots, x_N$*, which are distinct modulo* $\frac{2\pi}{\alpha}$*, and arbitrary numbers* $y_{-N}, \ldots, y_0, \ldots, y_N$*, there always exists a unique scaled trigonometric polynomial of order* N *with a scaling factor* α *such that* $T_\alpha(x_k) = y_k \quad k = -N, \ldots, N$.

Proof. Follows from Theorem 2.2. □

The polynomial $T_\alpha(x)$ is called the **scaled trigonometric interpolating polynomial** of order N with a scaling factor α that corresponds to points x_k and values y_k.

One way to interpolate the sequence of values $\{x_n\}_{n=-N}^{N}$ at equidistant nodes $\{nT\}_{n=-N}^{N}$, where $T \in \mathbb{R}^+$, is to use the $x_T(t) \triangleq x\left(\frac{2\pi}{MT}t\right)$, where $x(t)$ is given by Eq. 2.2. $x_T(t)$ is a scaled trigonometric interpolating polynomial with the scaling factor $\frac{2\pi}{MT}$ that corresponds to points $\{nT\}_{n=-N}^{N}$ and values $\{x_n\}_{n=-N}^{N}$.

When $T = 1$, we denote the corresponding scaled trigonometric polynomial by by $\widetilde{x}(t)$. Thus,

$$\widetilde{x}(t) \triangleq x\left(\frac{2\pi}{M}t\right) = \frac{1}{M}\sum_{k=-N}^{N} \widehat{x}_k e^{i\frac{2\pi}{M}kt} = \sum_{n=-N}^{N} x_n \widetilde{D}_M(n-t), \tag{2.3}$$

where $\widetilde{D}_M(t) \triangleq D_M\left(\frac{2\pi}{M}t\right)$ is called a scaled Dirichlet kernel.

Definition 2.5. *A two-dimensional trigonometric polynomial of order* N *is an expression of the form*

$$T(x,y) = \sum_{k=-N}^{N}\sum_{l=-N}^{N} c_{k,l} e^{\imath[kx+ly]},$$

where $c_{k,l}$ *are complex numbers.*

A two-dimensional trigonometric polynomial of degree N that interpolates an arbitrary set of values $\{x_{u,v}\}_{u,v=-N}^{N}$ at nodes $\left\{\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right)\right\}_{u,v=-N}^{N}$ is explicitly given by

$$x(t,s) = \frac{1}{M^2}\sum_{k=-N}^{N}\sum_{l=-N}^{N} \widehat{x}_{k,l} e^{\imath[kt+ls]} = \sum_{u=-N}^{N}\sum_{v=-N}^{N} x_{u,v} D_M\left(\frac{2\pi}{M}u - t, \frac{2\pi}{M}v - s\right),$$

where $M = 2N+1$, $\{\widehat{x}_{k,l}\}_{u,v=-N}^{N}$ is the 2D DFT of $\{x_{u,v}\}_{u,v=-N}^{N}$ and

$$D_M(t,s) \triangleq D_M(t)D_M(s) = \frac{1}{M^2}\sum_{u=-N}^{N}\sum_{v=-N}^{N} e^{\imath[ut+vs]} \tag{2.4}$$

is the two-dimensional Dirichlet kernel of order N. Such an interpolating polynomial is unique.

2.2 Shear transformation and its properties

Definition 2.6. *(Continuous shear) Let $f(x, y)$ be a real-valued function. For a fixed $s \in \mathbb{R}$, the real-valued function $f_s^{\mathrm{h}}(x, y) \triangleq f(x + sy, y)$, is called a 'horizontal shear' of $f(x, y)$. Similarly, the real-valued function $f_s^{\mathrm{v}}(x, y) \triangleq f(x, y + sx)$, is called 'vertical shear' of $f(x, y)$. The parameter s describes the 'shear' size applied to the object.*

Theorem 2.7 establishes the relation between the Fourier transform of a sheared object and the Fourier transform of the original object.

Theorem 2.7. *(2D continuous Fourier transform of a sheared object)*

Let $f(x, y)$ be a real-valued function of two real variables. Let s be a real number. Then,

$$\widehat{f_s^{\mathrm{h}}}(\omega_x, \omega_y) = \widehat{f}(\omega_x, \omega_y - s\omega_x) \text{ and } \widehat{f_s^{\mathrm{v}}}(\omega_x, \omega_y) = \widehat{f}(\omega_x - s\omega_y, \omega_y)$$

where $\widehat{f}(\omega_x, \omega_y)$ is the 2D Fourier transform of $f(x, y)$.

This theorem states that the 2D Fourier transform of a horizontally sheared object is a vertical shear of the object's 2D Fourier transform, and 2D Fourier transform of a vertically sheared object is a horizontal shear of the object's 2D Fourier transform.

A similar result to Theorem 2.7 holds for discrete objects. Let N be a positive integer and let $o[u, v]$ be a discrete object defined on $u, v \in [-N : N]$. We cannot directly apply the same formula as in the continuous case in order to define horizontal shear of $o[u, v]$, since $u + sv$ is, in general, not an integer. We therefore begin by defining horizontal and vertical interpolation of a discrete object $o[u, v]$.

Definition 2.8. *(Horizontal and vertical interpolation of a discrete object) Let $o[u, v]$, $u, v \in [-N : N]$, be a discrete object. Then, $\widetilde{o}_{\mathrm{h}}(x, v) \triangleq \sum_{u=-N}^{N} o[u, v]\widetilde{D}_M(u - x)$, $x \in \mathbb{R}$, $v \in [-N : N]$ is called a* horizontal interpolation *of the discrete object $o[u, v]$, and $\widetilde{o}_{\mathrm{v}}(u, y) \triangleq \sum_{v=-N}^{N} o[u, v]\widetilde{D}_M(v - y)$, $u \in [-N : N]$, $y \in \mathbb{R}$, is called a* vertical interpolation *of the discrete object $o[u, v]$. The subscripts h and v mean* horizontal *and* vertical*, respectively. $\widetilde{D}_M(t)$ is a scaled Dirichlet kernel defined by Eq. (2.3).*

For a fixed v, $\widetilde{o}_{\mathrm{h}}(x, v)$ is the scaled trigonometric interpolation of the sequence $\{o[u, v] \mid u \in [-N : N]\}$. Since the first argument of $\widetilde{o}_{\mathrm{h}}(x, v)$ is continuous, we can shear it horizontally using the continuous shear transformation. We define the *horizontal shear of a discrete object* $o[u, v]$ by resampling $\widetilde{o}_{\mathrm{h}}(x, v)$ on the set $\{(u, v) \mid u, v \in [-N : N]\}$.

Definition 2.9. *(Horizontal and vertical shear of a discrete object) Let $o[u, v]$, $u, v \in [-N : N]$, be a discrete object. Let s be a real number. The discrete object $o_s^{\mathrm{h}}[u, v] = \tilde{o}_{\mathrm{h}}(u + sv, v)$, $u, v \in [-N : N]$, where $\tilde{o}_{\mathrm{h}}(x, v)$ is given by Definition 2.8, is called a* horizontal shear *of $o[u, v]$. Similarly, the discrete object $o_s^{\mathrm{v}}[u, v] = \tilde{o}_{\mathrm{h}}(u, v + su)$, $u, v \in [-N : N]$, is called a* vertical shear *of $o[u, v]$. The superscripts h and v mean* horizontal *and* vertical*, respectively.*

The following theorem relates the 2D DFT of a sheared discrete object to the 2D DFT of the original object.

Theorem 2.10. *(Discrete 2D Fourier transform of a sheared discrete object)*
Let $o[u,v]$, $u,v \in [-N:N]$, be a discrete object. Let $s \in \mathbb{R}$. Then, $\widehat{o_s^{\mathrm{h}}}(k,\omega) = \widehat{o}(k,\omega - sk)$ and $\widehat{o_s^{\mathrm{v}}}(\omega,k) = \widehat{o}(\omega - sk, k)$, where $\omega \in \mathbb{R}$, $k \in [-N:N]$ and $\widehat{o}(\omega_1,\omega_1)$ is given by Eq. (1.1).

Proof.

$$\begin{aligned}\widehat{o_s^{\mathrm{h}}}(k,\omega) &= \sum_{u=-N}^{N}\sum_{v=-N}^{N} o_s^{\mathrm{h}}[u,v]e^{-i\frac{2\pi}{M}[uk+v\omega]} = \sum_{u=-N}^{N}\sum_{v=-N}^{N} \widetilde{o_h}(u+sv,v)e^{-i\frac{2\pi}{M}[uk+v\omega]} \\ &= \sum_{v=-N}^{N} e^{-i\frac{2\pi}{M}v\omega}\left(\sum_{u=-N}^{N} \widetilde{o_h}(u+sv,v)\,e^{-i\frac{2\pi}{M}uk}\right).\end{aligned} \tag{2.5}$$

By definition 2.8, $\widetilde{o}_{\mathrm{h}}(t,v)$ is a scaled trigonometric interpolation of $\{o[u,v] \big| u \in [-N:N]\}$ at integer nodes $[-N:N]$. The bracketed expression in Eq. (2.5) is the DFT of the sequence $\{\widetilde{o}_{\mathrm{h}}(u+sv,v) \,\big|\, u \in [-N:N]\}$ for k. In [1] we show that when we use scaled trigonometric interpolation, the DFT shift property can be generalized for the case of a non-integer shift, i.e.

$$\sum_{u=-N}^{N} \widetilde{o}_{\mathrm{h}}(u+sv,v)\,e^{-i\frac{2\pi}{M}uk} = \sum_{u=-N}^{N} o[u,v]\,e^{-i\frac{2\pi}{M}uk}\,e^{-i\frac{2\pi}{M}(-sv)k}. \tag{2.6}$$

From Eqs.(2.5) and (2.6) we conclude that

$$\begin{aligned}\widehat{o_s^{\mathrm{h}}}(k,\omega) &= \sum_{v=-N}^{N} e^{-i\frac{2\pi}{M}v\omega}\sum_{u=-N}^{N} o[u,v]\,e^{-i\frac{2\pi}{M}uk}\,e^{-i\frac{2\pi}{M}(-sv)k} = \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]\,e^{-i\frac{2\pi}{M}(v\omega+uk-svk)} \\ &= \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]\,e^{-i\frac{2\pi}{M}(uk+v(\omega-sk))} = \widehat{o}(k,\omega-sk).\end{aligned}$$

The proof of the second statement is similar. □

3 2D discrete Radon transform

In this section, we briefly review the 2D discrete Radon transform that is introduced in [6]. We keep our current notation for the discrete object, but in addition, we assume that N is an even positive integer and $o[u,v] = 0$ whenever $(u,v) \notin \left[-\frac{N}{2} : \frac{N}{2}-1\right] \times \left[-\frac{N}{2} : \frac{N}{2}-1\right]$. This assumption is introduced in order to comply to the definition of the discrete Radon transform from [6], which was defined for discrete objects of size $N \times N$.

The continuous Radon transform is defined by the set of all line integrals of the object. Loosely speaking, the 2D discrete Radon transform is defined by summing the values of a discrete object $o[u,v]$ along a discrete set of lines. Discrete Radon transform along vertical lines is defined by

Definition 3.1. *(2D Radon transform along vertical lines) Let N be an even positive integer, $o[u,v]$ be a discrete object. Then, $Radon(\{x=t\},o) \triangleq \sum_{v=-N}^{N} o[t,v]$, $t \in [-N:N]$.*

Similarly, we define

Definition 3.2. *(2D Radon transform for horizontal lines) Let N be an even positive integer, $o[u,v]$ be a discrete object. Then, $Radon(\{y=t\},o) \triangleq \sum_{u=-N}^{N} o[u,t]$, $t \in [-N:N]$.*

The key question is how to process lines of the discrete transform that do not pass through grid points. In [6], all the lines in $\mathbb{R}^2$ are partitioned into two families, namely, *basically vertical lines* and *basically horizontal lines*. A basically vertical line is a line of the form $x = sy + t$ where the slope $|s| \leq 1$. A basically horizontal line is a line of the form $y = sx + t$ where the slope $|s| \leq 1$.

Definition 3.3. *[6](2D Radon Transform along basically vertical lines) Let N be an even positive integer, $M = 2N + 1$ and $s \in [-1, 1]\backslash 0$. Then, for all $t \in [-N : N]$ $Radon(\{x = sy + t\}, o) \triangleq \sum_{v=-N}^{N} \tilde{o}_{\mathrm{h}}(sv + t, v)$, where $\tilde{o}_{\mathrm{h}}(x, v)$ is given by Definition 2.8.*

Definition 3.4. *[6](2D Radon transform along basically horizontal lines) Let N be an even positive integer, $M = 2N + 1$ and $s \in [-1, 1]\backslash 0$. Then, for all $t \in [-N : N]$ $Radon(\{y = sx + t\}, o) \triangleq \sum_{u=-N}^{N} \tilde{o}_{\mathrm{v}}(u, su + t)$, where $\tilde{o}_{\mathrm{v}}(u, y)$ is given by Definition 2.8.*

Equivalently, we can define the 2D discrete Radon transform for basically vertical lines as

Definition 3.5. *Let $o[u, v]$ be a discrete object, $s \in [-1, 1]\backslash 0$. Then, for all $t \in [-N : N]$, $Radon(\{x = sy + t\}, o) = Radon(\{x = t\}, o^{\mathrm{h}}_s)$, where $o^{\mathrm{h}}_s[u, v]$ is given by Definition 2.9.*

To verify that Definition 3.5 is indeed equivalent to Definition 3.3, we fix $s \in [-1, 1]$ and $t \in [-N : N]$. Then,

$$Radon(\{x = t\}, o^{\mathrm{h}}_s) = \sum_{v=-N}^{N} o^{\mathrm{h}}_s(t, v) = \sum_{v=-N}^{N} \tilde{o}_{\mathrm{h}}(t + sv, v) = Radon(\{x = sy + t\}, o).$$

Thus, a basically vertical 2D Radon of a discrete object $o[u, v]$ along the line $x = sy + t$ is equivalent to a vertical 2D Radon of a horizontally sheared object $o^{\mathrm{h}}_s[u, v]$. Similarly, for a basically horizontal lines we have

Definition 3.6. *Let $o[u, v]$ be a discrete object and $s \in [-1, 1]\backslash\{0\}$. Then, for all $t \in [-N : N]$, $Radon(\{y = sx + t\}, o) = Radon(\{y = t\}, o^{\mathrm{v}}_s)$ where $o^{\mathrm{v}}_s[u, v]$ is given by Definition 2.9.*

Thus, a basically horizontal 2D Radon of a discrete object $o[u, v]$ along the line $y = sx + t$ is equivalent to a horizontal 2D Radon of a the vertically sheared object $o^{\mathrm{v}}_s[u, v]$.

4 Continuous diffraction tomography

In this section, we briefly review the continuous theory of diffraction tomography. We describe the physical settings of diffraction tomography, give an expression for the scattered field, and state the Fourier diffraction theorem. The material in this section is borrowed from [4](chapter 6).

4.1 Typical diffraction tomography experiment

In a typical diffraction tomography experiment, a physical body, suspended in a homogeneous medium, is illuminated by a plane wave and the scattered field is measured by detectors located on a line normal

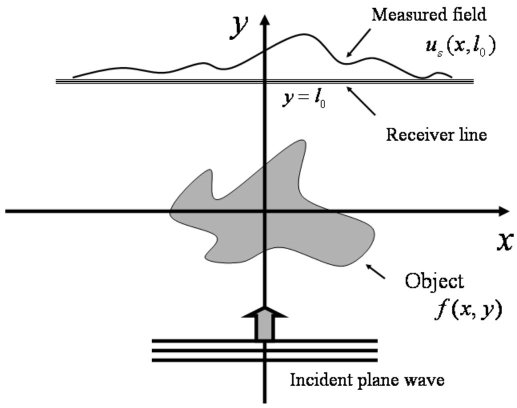

Figure 4.1: Typical diffraction tomography experiment

to the direction of the wave propagation. This line is called the **receiver line**. In transmission tomography, this line is located at the far side of the object (see Fig. 4.1).

In this paper, we consider only the 2D case, although the theory can be readily extended to 3D. In cases where a 3D varies slowly along one of the dimensions, the 2D theory can be applied. This assumption is often made in conventional computerized tomography where 3D models are generated using 2D slices of the object.

We describe next the conventional mathematical model for computing a projection of an object for a given plane wave. The object is described by an 'object function' $f(x, y)$, which is a linear function of the refractive index of the object at location (x, y). A two-dimensional plane wave in homogeneous medium is described by

$$u_o(x, y) = e^{i(\omega_x x + \omega_y y)}. \tag{4.1}$$

This expression is completely specified by the vector (ω_x, ω_y) that is called a propagation vector or a wave vector. The length $\omega_0 = \sqrt{\omega_x^2 + \omega_y^2}$ of this vector is called **wave number** of the plane wave. The **wavelength** of the plane wave is given by $\lambda = \frac{2\pi}{\omega_0}$. The wave propagates in the direction given by (ω_x, ω_y). The orientation of the receiver line depends on the direction of the wave propagation. However, all receivers are assumed to be located at the same distance from the origin.

The **total field** is the field that results from illuminating the measured body with a plane wave. A projection is generated by measuring the total field on the receiver line. To compute a projection we need an expression that describes the total field. Since the measured body usually contains inhomogeneities, Eq. (4.1) is not applicable.

We consider the total field $u(x, y)$ as the sum of two components $u_o(x, y)$ and $u_s(x, y)$. $u_o(x, y)$, known as the **incident field**, is the field present without any inhomogeneities, as given by Eq. (4.1). $u_s(x, y)$, known as the **scattered field**, is the part of the total field that can be attributed solely to the inhomogeneities. We use an approximated expression for $u_s(x, y)$, called the first Born approximation.

The Born approximation for the incident field in Eq. (4.1) is given by

$$u_s(x',y') = \frac{i}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i[\omega_x x+\omega_y y]}\int_{-\infty}^{\infty}\frac{1}{\beta}e^{i[\alpha(x'-x)+\beta|y'-y|]}\,d\alpha\,dy\,dx \quad \beta=\sqrt{\omega_0^2-\alpha^2}. \quad (4.2)$$

The inner integral in Eq. (4.2) represents a cylindrical wave that is centered at (x,y) as a superposition of plane waves. For points with $y' > y$, the plane waves propagate upward, while for $y' < y$ the plane waves propagate downward. In addition, for $|\alpha| \le \omega_0$, the plane waves are of the ordinary type, propagating in the direction given by $\tan^{-1}(\beta/\alpha)$. However, for $|\alpha| > \omega_0$, β becomes imaginary, the waves decay exponentially and they are called evanescent waves. Evanescent waves are usually of no significance beyond about 10 wavelengths from the source, so in the subsequent discussion they will not be taken into consideration.

4.2 The continuous Fourier diffraction theorem

The Fourier diffraction theorem relates the Fourier transform of a diffracted projection to the Fourier transform of the object. It will be established for the case where the direction of the incident plane wave is along the positive y-axis. In this case, the incident field is given by $u_o(x,y) = e^{i\omega_0 y}$, and the scattered field is measured by a linear array of receivers located at $y = l_0$, where l_0 is greater than any y-coordinate within the object (see Fig. 4.1). The term $|y'-y|$ in Eq. (4.2) can be replaced by $l_0 - y$ and the resulting formula is rewritten as

$$u_s(x',l_0) = \frac{i}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i\omega_0 y}\int_{-\infty}^{\infty}\frac{1}{\beta}e^{i[\alpha(x'-x)+\beta(l_0-y)]}\,d\alpha\,dy\,dx, \quad \beta=\sqrt{\omega_0^2-\alpha^2}. \quad (4.3)$$

Let $\widehat{u}_s(\omega,l_0)$ denote the Fourier transform of $u_s(x,l_0)$ with respect to x. The physics of wave propagation dictates that the highest angular frequency in the measured scattered field on the line $y = l_0$ is unlikely to exceed ω_0. Therefore, in almost all practical situations, $\widehat{u}_s(\omega,l_0) = 0$ for $\left|\omega\right| > \omega_0$. This is consistent with neglecting the evanescent waves as was described earlier. By taking the Fourier transform of the scattered field (Eq. (4.3)), we get

$$\widehat{u}_s(\omega,l_0) = \frac{i}{\sqrt{\omega_0^2-\omega^2}}e^{i\sqrt{\omega_0^2-\omega^2}l_0}\widehat{f}(\omega,\sqrt{\omega_0^2-\omega^2}-\omega_0) \quad \text{for } |\omega|<\omega_0, \quad (4.4)$$

where $\widehat{f}(\omega_1,\omega_2)$, which is a two-dimensional Fourier transform of $f(x,y)$, given by

$$\widehat{f}(\omega_x,\omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(\omega_x x+\omega_y y)}dx\,dy.$$

The proof of Eq. (4.4) can be found in [4], chapter 6.

Assume that $-\omega_0 \le \omega \le \omega_0$. The points $(\omega, \sqrt{\omega_0^2-\omega^2}-\omega_0)$ form a semicircular arc in the frequency plane. Equation (4.4) is a particular case of the Fourier diffraction theorem for the case of a plane wave directed along the positive y-axis. In the general case, the continuous Fourier diffraction theorem is:

Theorem 4.1. *(The continuous Fourier diffraction theorem) Given an object $f(x, y)$, the continuous Fourier transform of the forward scattered field u_s, measured on the receiver line $y = l_0$, is equal to the 2D Fourier transform $\widehat{f}(\omega_1, \omega_2)$ of the object along a semicircular arc. That is,*

$$\widehat{u}_s(\omega, l_0) = \frac{i}{\sqrt{\omega_0^2 - \omega^2}} e^{i\sqrt{\omega_0^2-\omega^2}l_0} \widehat{f}(\omega, \sqrt{\omega_0^2 - \omega^2} - \omega_0), \quad |\omega| < \omega_0.$$

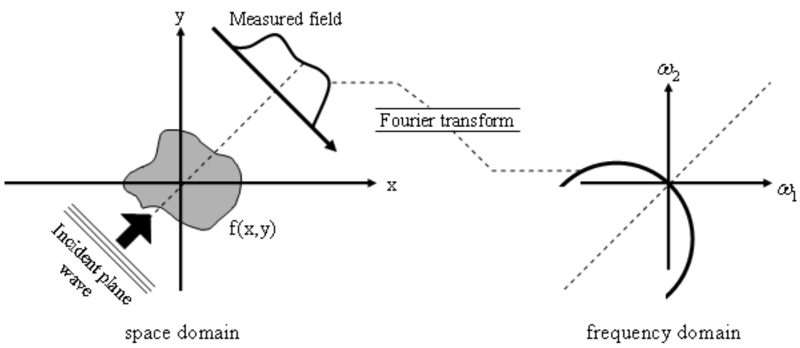

Figure 4.2: Visualization of the continuous Fourier diffraction theorem

In the general case, when the direction of the plane wave is different from the direction of the y-axis, the Fourier transform of the diffracted projection is a slice of the 2D Fourier transform of the object along a semicircular arc rotated in the direction of the plane wave, as shown in Fig. 4.2.

Depending on the context, the terms 'diffracted field' and 'diffracted projection' are used for describing either the physical measurements performed during the diffraction tomography experiment or the Born approximation of the scattered field with discarded evanescent waves. The Born approximation with discarded evanescent waves is given by

$$u_d(x', y') \triangleq \frac{i}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i[\omega_x x + \omega_y y]} \int_{-\omega_0}^{\omega_0} \frac{1}{\beta} e^{i[\alpha(x'-x)+\beta|y'-y|]} \, d\alpha \, dy \, dx, \quad \beta = \sqrt{\omega_0{}^2 - \alpha^2}.$$

5 Discretization of a vertical diffracted projection

In order to define a discrete counterpart of the continuous diffracted projection, we take a closer look at the definition of the 2D Radon transform. It was shown in [6] that the discrete 2D Radon transform along the vertical lines approximates the continuous vertical projection of the object when it applied to samples of a continuous object on a Cartesian grid, .

In this section, we propose a discretization that approximates the vertical diffracted projection. The definition of a discrete diffracted projections in section 6 is based on this proposed discretization. Consider the object function $f(x, y)$. If we ignore the evanescent waves, then the Born approximation of the vertical diffracted field is given by

$$u_d(x', y') = \frac{\imath}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{\imath\omega_0 y} \int_{-\omega_0}^{\omega_0} \frac{1}{\sqrt{\omega_0{}^2 - \alpha^2}} e^{\imath[\alpha(x'-x)+\sqrt{\omega_0{}^2-\alpha^2}|y'-y|]} \, d\alpha \, dy \, dx. \tag{5.1}$$

In section 5.1, we introduce a discretization of the inner integral in Eq. (5.1). Using this discretization, we introduce in section 5.2 a discretization of the triple integral in Eq. (5.1). In section 8.2, we show that if $o[u, v]$ is a discrete object that was obtained by sampling $f(x, y)$ on a Cartesian grid, then, the proposed discretization approximates the samples of the continuous vertical diffracted projection of $f(x, y)$.

This approximation is valid for a specific choice of the wavenumber of the plane wave that is used for illumination. In section 8.3, we show that this choice of the wavenumber is in some sense optimal.

5.1 The Inner integral in Eq. (5.1)

In this section, we prove that the inner integral in Eq. (5.1) is a Lipschitz function, propose for it a discretization and prove the convergence of its discretization.

We introduce a special notation for the inner integral in Eq. (5.1).

Definition 5.1. *Let $\omega_0 \in \mathbb{R}^+$. For arbitrary $x, y, x', y' \in \mathbb{R}$ we define*

$$K(x, y, x', y') \triangleq \int_{-\omega_0}^{\omega_0} \frac{1}{\sqrt{\omega_0^2 - \alpha^2}} e^{\imath[\alpha(x'-x)+\sqrt{\omega_0^2-\alpha^2}|y'-y|]}\, d\alpha. \tag{5.2}$$

Definition 5.2. *(Lipschitz class $Lip_C(\alpha, \Omega)$) Let $\Omega \subseteq \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{C}$ satisfies the condition*

$$|f(x) - f(y)| \leq C\|x - y\|^{\alpha}, \quad 0 < \alpha \leq 1$$

for all $x, y \in \Omega$, then, we say that f belongs to the class $Lip_C(\alpha, \Omega)$. When the value of the constant C is not important, we say that f is Lipschitz α on Ω.

Theorem 5.3. *$K(x, y, x', y')$ in Definition 5.1 is a continuous function. Moreover, for any $D \in \mathbb{R}^+$ there exists $C \in \mathbb{R}^+$ such that for any fixed $x' \in [-D, D]$ the expression $K(x, y, x', D)$, as a function of x and y, belongs to $Lip_C(1; \Omega)$, where $\Omega = \{(x, y) \mid x, y \in [-D, D]\}$.*

Proof.

Consider the integrand from the definition of $K(x, y, x', y')$ given by Eq. (5.2)

$$f_\alpha(x, y, x', y') \triangleq \frac{1}{\sqrt{\omega_0^2 - \alpha^2}} e^{\imath[\alpha(x'-x)+\sqrt{\omega_0^2-\alpha^2}|y'-y|]}. \tag{5.3}$$

This is a complex-valued function of the real variable α. Therefore,

$$\Re e K(x, y, x', y') = \Re e \int_{-\omega_0}^{\omega_0} f_\alpha(x, y, x', y') d\alpha = \int_{-\omega_0}^{\omega_0} \Re e f_\alpha(x, y, x', y') d\alpha. \tag{5.4}$$

Similar equality holds for the imaginary part of $K(x, y, x', y')$.

The absolute value of $\Re e f_\alpha(x, y, x', y')$ is dominated by the function $\frac{1}{\sqrt{{\omega_0}^2 - \alpha^2}}$, which is integrable on $[-\omega_0, \omega_0]$. Consequently, the integral in Eq. (5.4) converges uniformly in x, y, x', y'. Also, $\Re e f_\alpha(x, y, x', y')$ is continuous for $\alpha \in (-\omega_0, \omega_0)$ and $x, y, x', y' \in \mathbb{R}$. Consequently, $\Re e K(x, y, x', y')$ is a continuous function of x,y,x' and y' ([2] p.465). The same is true for $\Im m K(x, y, x', y')$, therefore,

$K(x,y,x',y')$ is a continuous function of x,y,x' and y'. The integrand has a continuous derivative with respect to x: $\frac{\partial f_\alpha}{\partial x}(x,y,x',y') = \frac{-\imath\alpha}{\sqrt{\omega_0^2-\alpha^2}} e^{\imath[\alpha(x'-x)+\sqrt{\omega_0^2-\alpha^2}|y'-y|]}$. Since f_α is a complex-valued function of a real variable α, we have

$$\frac{\partial \Re e f_\alpha(x,y,x',y')}{\partial x} = \Re e \frac{\partial f_\alpha(x,y,x',y')}{\partial x} \tag{5.5}$$

and similar equality holds for the imaginary part.

The absolute value of $\Re e \frac{\partial f_\alpha}{\partial x}(x,y,x',y')$ is dominated by the function $\frac{|\alpha|}{\sqrt{{\omega_0}^2-\alpha^2}}$, which is integrable on $[-\omega_0,\omega_0]$. Therefore, the integral $\int_{-\omega_0}^{\omega_0} \Re e \frac{\partial f_\alpha}{\partial x}(x,y,x',y')d\alpha$ converges uniformly in x,y,x',y'. Hence, the expression in Eq. (5.4) can be differentiated with respect to x under the integral ([2] p.467), and $\left|\frac{\partial \Re e K(x,y,x',y')}{\partial x}\right| \le \int_{-\omega_0}^{\omega_0} \frac{|\alpha|}{\sqrt{{\omega_0}^2-\alpha^2}} d\alpha = 2\omega_0$. A similar inequality holds for the $\Im m K(x,y,x',y')$. Therefore, $\left|\frac{\partial K(x,y,x',y')}{\partial x}\right| \le 2\sqrt{2}\omega_0$ that is, $\frac{\partial K}{\partial x}(x,y,x',y')$ is uniformly bounded in x,y,x' and y'. In particular, for any $y,x' \in [-D,D]$, the function $K(x,y,x',D)$, as a function of x, belongs to $Lip_{2\sqrt{2}\,\omega_0}(1,[-D,D])$.

Now, we fix $y' = D$. For any small positive real number ε, $f_\alpha(x,y,x',D)$ has a continuous derivative with respect to y on any interval $[-D,D-\varepsilon]$. The derivative is given by $\frac{\partial f_\alpha(x,y,x',D)}{\partial y} = -ie^{\imath[\alpha(x'-x)+\sqrt{\omega_0^2-\alpha^2}|y'-y|]}$. The absolute value of $\frac{\partial f_\alpha}{\partial y}(x,y,x',D)$ is less or equal to 1. Using the same reasoning as in the case of the x-derivative above, we conclude that the expression at the right-hand side of Eq. (5.2) can be differentiated under the integral with respect to y and $\left|\frac{\partial K(x,y,x',D)}{\partial y}\right| \le 2\sqrt{2}\omega_0$, that is, $K_y(x,y,x',D)$ is bounded uniformly in x,y and x'. In particular, for any $x,x' \in [-D,D]$, the function $K(x,y,x',D)$, as a function of y, belongs to $Lip_{2\sqrt{2}\omega_0}(1,[-D,D))$. Since $K(x,y,x',D)$ is continuous as a function of y, it belongs to $Lip_{2\sqrt{2}\omega_0}(1,[-D,D])$.

We conclude that for $x,y,x' \in [-D,D]$, the function $K(x,y,x',D)$ is uniformly Lipschitz in both x and y coordinates. The Lipschitz constant in both cases is $C \triangleq 2\sqrt{2}\omega_0$. Consider an arbitrary $x' \in [-D,D]$. For any $x_1,y_1,x_2,y_2 \in [-D,D]$ we have

$$|K(x_1,y_1,x',D) - K(x_2,y_2,x',D)| \le |K(x_1,y_1,x',D) - K(x_2,y_1,x',D)| + |K(x_2,y_1,x',D) - K(x_2,y_2,x',D)| \le C|x_1-x_2| + C|y_1-y_2| \le \sqrt{2}C\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Therefore, for any $x' \in [-D,D]$, the expression $K(x,y,x',D)$, as a function of x and y, belongs to $Lip_{4\omega_0}(1;\Omega)$. □

We approximate the integral in Eq.(5.2) by means of its Riemann sum with equispaced nodes. The choice of equispaced nodes is not arbitrary. In fact, it is this choice of nodes that allows for an efficient computation of the discrete diffraction transform defined in section 7 to take place.

Definition 5.4. *Let $\omega_0 \in \mathbb{R}^+$, $N \in \mathbb{N}$, $M = 2N+1$. Let $f:\mathbb{R}^2 \to \mathbb{R}$. For arbitrary $x,y,x',y' \in \mathbb{R}$, we define*

$$K_N(x,y,x',y') \triangleq \frac{2\omega_0}{M} \sum_{k=-N}^{N} \frac{1}{\sqrt{\omega_0^2 - \left(\frac{2\omega_0 k}{M}\right)^2}} e^{\imath\left[\frac{2\omega_0 k}{M}(x'-x)+\sqrt{\omega_0^2-\left(\frac{2\omega_0 k}{M}\right)^2}|y'-y|\right]}. \tag{5.6}$$

Theorem 5.5. *Let $\Omega \subseteq \mathbb{R}^4$ be a bounded set, $\omega_0 \in \mathbb{R}^+$,$N \in \mathbb{N}$. Then, $K_N(x, y, x', y')$ converges to $K(x, y, x', y')$ uniformly in $(x, y, x', y') \in \Omega$.*

The proof of theorem 5.5 is given in section 10

5.2 Discretization of a vertical diffracted projection

In this section, we discretize the integral given in Eq. (5.1). We denote by $\mathbb{C}^0(\mathbb{R}^2)$ the set of all the continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Given a function $f(x, y) \in \mathbb{C}^0(\mathbb{R}^2)$, we assume that there exists D such that$f(x, y) = 0$ whenever $|x| > D$ or $|y| > D$. In what follows, we assume that the constant D and the wavenumber ω_0 in Eq. (5.1) are known and fixed.

Definition 5.6. *Let $D, \omega_0 \in \mathbb{R}^+$. Let $f \in \mathbb{C}^0(\mathbb{R}^2)$. For arbitrary $x', y' \in \mathbb{R}$, we define*

$$T[f](x', y') \triangleq \int_{-D}^{D} \int_{-D}^{D} f(x, y) e^{\imath\omega_0 y}\, K(x, y, x', y')\, dy\, dx. \tag{5.7}$$

Note that the function $T[f]$ in Eq. (5.7) depends on both D and ω_0 though they do not explicitly appear in the notation.

Definition 5.7. *Let $D, \omega_0 \in \mathbb{R}^+$, $N \in \mathbb{N}$, $M = 2N + 1$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. For arbitrary $x', y' \in \mathbb{R}$ we define*

$$T_N[f](x', y') \triangleq \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\omega_0 \frac{2D}{M} v}\, K_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v, x', y'\right). \tag{5.8}$$

Lemma 5.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a closed bounded set. Let $0 < \alpha \le \beta \le 1$, $f \in Lip_{C_1}(\alpha, \Omega)$, and $g \in Lip_{C_2}(\beta, \Omega)$. Then, there exists a positive constant C such that $fg \in Lip_C(\alpha, \Omega)$.*

Proof. The proof is straightforward. □

Theorem 5.9. *(Approximation of a vertical diffracted projection)*

Let $C \in \mathbb{R}^+, \alpha \in (0, 1]$. Let $N \in \mathbb{N}, M = 2N + 1$. Denote $\Omega = \{(x, y) \,|\, x, y \in [-D, D]\}$. Let $f \in \mathbb{C}^0(\mathbb{R}^2) \cap Lip_C(\alpha, \Omega)$. Then, $T_N[f]\,(x', D)$ converges to $T[f](x', D)$ uniformly in $x' \in [-D, D]$.

Proof.

By the triangle inequality

$$\begin{aligned}
&|T[f](x', D) - T_N[f](x', D)| \le \Big| \textstyle\int_{-D}^{D} \int_{-D}^{D} f(x, y) e^{\imath\omega_0 y} K(x, y, x', D) dy\, dx - \\
&\tfrac{4D^2}{M^2} \textstyle\sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\omega_0 \frac{2D}{M} v} K\left(\frac{2D}{M}u, \frac{2D}{M}v, x', D\right) \Big| + \\
&\Big| \tfrac{4D^2}{M^2} \textstyle\sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\omega_0 \frac{2D}{M} v} \left(K\left(\frac{2D}{M}u, \frac{2D}{M}v, x', D\right) - K_N\left(\frac{2D}{M}u, \frac{2D}{M}v, x', D\right)\right) \Big|,
\end{aligned} \tag{5.9}$$

where $T[f]$ is given by Eq. (5.7) and $T_N[f]$ is given by Eq. (5.8).

It follows from the continuity of $f(x, y)$ that there exists a positive constant A such that $\Big|f(x, y)\Big| \le A$ for all $(x, y) \in \Omega$. Consequently, $\Big|f(x, y)e^{\imath\omega_0 y}\Big| \le A$ on Ω.

Assume $\varepsilon > 0$ is arbitrary. From Theorem 5.5, there exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$ and $x, y, x' \in [-D, D]$ we have $|K(x, y, x', D) - K_N(x, y, x', D)| < \frac{\varepsilon}{4D^2A}$. Then, for any $N > N_1$ and any $x' \in [-D, D]$, the last term on the right-hand side of Eq. (5.9) is less than ε.

The first term on the right-hand side of Eq. (5.9) is the absolute value of the difference between the definite integral of $f(x, y)e^{\imath\omega_0 y}K(x, y, x', D)$ and its corresponding Riemann sum. From Theorem 5.3, there exists $C_0 \in \mathbb{R}^+$ such that for any fixed $x' \in [-D, D]$, the expression $K(x, y, x', D)$ belongs to $Lip_{C_0}(1, \Omega)$ as a function of x and y. Also, $e^{\imath\omega_0 y} \in Lip_{\omega_0\sqrt{2}}(1, \Omega)$. Then, from Lemma 5.8, there exists $C_1 \in \mathbb{R}^+$ such that for any $x' \in [-D, D]$, the expression $f(x, y)e^{\imath\omega_0 y}K(x, y, x', D)$, as a function of x and y, belongs to $Lip_{C_1}(\alpha, \Omega)$. Hence, the absolute value of the first term on the right-hand side of Eq. (5.9) is bounded by

$$\frac{4D^2}{M^2}\sum_{u=-N}^{N}\sum_{v=-N}^{N} C_1\left(\frac{2\sqrt{2}D}{M}\right)^{\alpha} = 4D^2C_1\left(\frac{2\sqrt{2}D}{M}\right)^{\alpha}.$$

This expression tends to zero as N grows, and therefore, for any $\varepsilon > 0$ there exists N_2 such that for any $N > N_2$ and any $x' \in [-D, D]$ the absolute value of the first term on the right-hand side of Eq. (5.9) is less than ε. Therefore, if we take $N_0 = \max(N_1, N_2)$ then, for any N greater than N_0 and any $x' \in [-D, D]$, we have $|T[f](x', D) - T_N[f](x', D)| \le 2\varepsilon$, which completes the proof of the theorem. □

Corollary 5.10.

Let $C \in \mathbb{R}^+, \alpha \in (0, 1]$. Denote $\Omega = \{(x, y) \,|\, x, y \in [-D, D]\}$. Let $A \in \mathbb{R}^+$. Denote $S = \left\{f \in \mathbb{C}^0(\mathbb{R}^2) \cap Lip_C(\alpha, \Omega) \mid |f(x, y)| \le A \text{ on } \Omega\right\}$. Then, the convergence of $T_N[f](x', D)$ to $T[f](x', D)$ is uniform in both $x' \in [-D, D]$ and $f \in S$.

Proof. The class S is uniquely defined by C,α and A. To prove the corollary, it is sufficient to show that N_1 and N_2, from the proof of Theorem 5.9, depend on S but not on a specific $f \in S$.

The number N_1 depends only on A since the convergence of $K_N(x, y, x', y')$ to $K(x, y, x', y')$ is independent of f. The number N_2 depends on C_1 and α. From the proof of Lemma 5.8 we see that C_1 depends on the Lipschitz constant C of $f(x, y)$ and on the maximal value A of $f(x, y)$ on Ω. Therefore, N_2 depends on C, α and A but not on a specific $f \in S$. □

5.3 Vertical discrete diffracted projection

Let $f(x, y)$ be an object function. Consider the discrete object that is obtained from sampling $f(x, y)$ on a Cartesian grid:

$$o[u, v] = f\left(\frac{2D}{M}u, \frac{2D}{M}v\right), \quad u, v \in [-N : N]. \tag{5.10}$$

Equation 5.8 defines an approximation of a vertical diffracted projection along y-axis. We define the vertical discrete diffracted projection of $o[u, v]$ as samples of $T_N[f](x', y')$ on the receiver line $y' = D$ at points $x' = \frac{2D}{M}u'$ for a specific wavenumber $\omega_0 = \frac{\pi M}{2D}$. The reason for this specific choice of the wavenumber is given in section 8.3.

We substitute $\omega_0 = \frac{\pi M}{2D}$ into the definition of $T_N[f](x', y')$, to obtain

$$T_N[f](x', y') = \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\pi v} K_N\left(\frac{2D}{M}u, \frac{2D}{M}v, x', y'\right). \tag{5.11}$$

Then, we expand $K_N\left(\frac{2D}{M}u, \frac{2D}{M}v, x', y'\right)$ using its definition (Eq. (5.11)) with $\omega_0 = \frac{\pi M}{2D}$.

$$T_N[f](x', y') = \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\pi v} \sum_{k=-N}^{N} \frac{e^{\imath\frac{\pi}{D}\left[k(x'-\frac{2D}{M}u)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}|y'-\frac{2D}{M}v|\right]}}{\sqrt{\left(\frac{M}{2}\right)^2-k^2}}. \tag{5.12}$$

We substitute $x' = \frac{2D}{M}u'$ and $y' = D$ in Eq. (5.12) and we get for the left side of Eq. (8.6)

$$T_N[f]\left(\frac{2D}{M}u', D\right) = \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) e^{\imath\pi v} \sum_{k=-N}^{N} \frac{e^{\imath\frac{2\pi}{M}\left[k(u'-u)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}|\frac{M}{2}-v|\right]}}{\sqrt{\left(\frac{M}{2}\right)^2-k^2}}. \tag{5.13}$$

The definition of the vertical discrete diffracted projection of a discrete object $o[u, v]$ in Section 6 is based on a modification of this expression. We replace $f\left(\frac{2D}{M}u, \frac{2D}{M}v\right)$ by $o[u, v]$. To eliminate the constant D, which is related to physical dimensions of the object that makes no sense in the discrete setting, we multiply Eq. (5.13) by factor $\frac{D^2}{\pi^2}$. This results in

$$\left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} o[u, v] e^{\imath\pi v} \sum_{k=-N}^{N} \frac{e^{\imath\frac{2\pi}{M}\left[k(u'-u)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}|\frac{M}{2}-v|\right]}}{\sqrt{\left(\frac{M}{2}\right)^2-k^2}}. \tag{5.14}$$

6 Discrete diffracted projections (DDP)

The 2D Radon transform along basically vertical lines wass defined in section 3 in two steps: first, a vertical projection is defined, and then, general basically vertical projections are defined as vertical projections of a horizontally sheared object. Discretization of the 2D diffracted transform follows the same lines.

First, we define the vertical discrete diffracted projection of a discrete object based on Eq. (5.14). This definition, being applied to samples of a continuous object on a Cartesian grid, approximates continuous vertical diffracted projection of the object. Then, we define the basically vertical discrete diffracted projection as a vertical discrete diffracted projection of a horizontally sheared discrete object. The same principle, where the words 'vertical' and 'horizontal' being swapped, is used to define a discrete diffracted projections along basically horizontal lines.

The rest of the section is organized as follows. We formally define the vertical/horizontal diffracted projections. Next, we define the basically vertical/horizontal discrete diffracted projections. We conclude by formulating and proving the *discrete Fourier diffraction theorem*, which relates the 1D DFT of the discrete diffracted projection to the 2D DFT of the object.

6.1 Projection types

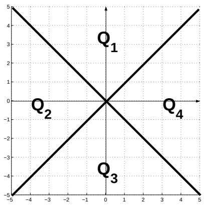

Figure 6.1: Division of all the directions to quarters

Each direction vector in $\mathbb{R}^2$ can be specified by the angle it creates with the x-axis. We divide the set of all possible directions to four quarters

$$\begin{aligned} Q_1 &\triangleq \{\theta \mid \theta \in [\pi/4, 3\pi/4]\}, & Q_2 &\triangleq \{\theta \mid \theta \in [3\pi/4, 5\pi/4]\} \\ Q_3 &\triangleq \{\theta \mid \theta \in [5\pi/4, 7\pi/4]\}, & Q_4 &\triangleq \{\theta \mid \theta \in [-\pi/4, \pi/4]\}. \end{aligned}$$

Q_1–Q_4 are illustared in Fig. 6.1. Quarters Q_1 and Q_3 together form the set of all the “basically vertical” directions. The projections along the directions from Q_1 are called the “basically vertical up-going” projections. Projections along the directions from Q_3 are called “basically vertical down-going” projections. A projection in a “basically vertical” direction is specified as a pair (i, s) where $i = 1, 3$ is the number of the quarter and $s \in [-1, 1]$ is the slope between the y-axis and the line $x = sy$ in the direction where the projection is taken.

The quarters Q_2 and Q_4 together form the set of all “basically horizontal” directions. Projections along the directions from Q_2 are called “basically horizontal left-to-right” projections. Projections along the directions from Q_4 are called “basically vertical right-to-left” projections. A projection in a “basically horizontal” direction is specified by the pair (i, s) where $i = 2, 4$ is the number of the quarter and $s \in [-1, 1]$ is the slope between the x-axis and the line $y = sx$ along which the projection is taken.

6.2 Definition of the discrete diffracted projections

The definition of the discrete diffracted projections is based on Eq. (5.14). We denote by $p^{i,s}_{[o]}(u)$ the discrete diffracted projection of an object $o[u, v]$ in the direction specified by quarter i and slope s.

Definition 6.1. *(Discrete diffracted projection along a vertical line) Let $o[u, v]$ be a discrete object.*

- *A vertical up-going discrete diffracted projection of $o[u, v]$ is defined by*

$$p^{1,0}_{[o]}(u') \triangleq \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} o[u, v] e^{i\pi v} \sum_{k=-N}^{N} \frac{1}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}} e^{i\frac{2\pi}{M}\left(k(u'-u) + \sqrt{\left(\frac{M}{2}\right)^2 - k^2}\,|\frac{M}{2} - v|\right)}.$$

- *A vertical down-going discrete diffracted projection of* $o[u,v]$ *is defined by*

$$p^{3,0}_{[o]}(u') \triangleq \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]e^{-i\pi v} \sum_{k=-N}^{N} \frac{1}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}} e^{i\frac{2\pi}{M}\left(k(u'-u)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}\,|-\frac{M}{2}-v|\right)}.$$

Definition 6.2. *(Discrete diffracted projection along a horizontal line) Let* $o[u,v]$ *be a discrete object.*

- *A horizontal left-to-right discrete diffracted projection of* $o[u,v]$ *is defined by*

$$p^{4,0}_{[o]}(v') \triangleq \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]\,e^{i\pi u} \sum_{k=-N}^{N} \frac{1}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}} e^{i\frac{2\pi}{M}\left(k(v'-v)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}\,|\frac{M}{2}-u|\right)}.$$

- *A horizontal right-to-left discrete diffraction projection of* $o[u,v]$ *is defined by*

$$p^{2,0}_{[o]}(v') \triangleq \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]\,e^{-i\pi u} \sum_{k=-N}^{N} \frac{1}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}} e^{i\frac{2\pi}{M}\left(k(v'-v)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}\,|-\frac{M}{2}-u|\right)}.$$

We define the DDP of a "basically vertical" line as a vertical projection of a horizontally sheared object.

Definition 6.3. *(Discrete diffracted projection along basically vertical lines) Let* $o[u,v]$ *be a discrete object,* $s \in [-1,1]\backslash\{0\}$*. Let* $o^{\mathrm{h}}_s[u,v]$ *be the horizontal shear of* $o[u,v]$ *given by Definition 2.9. Then,*

- *The basically vertical up-going DDP along the line* $x = sy$ *is defined as*

$$p^{1,s}_{[o]}(u) \triangleq p^{1,0}_{[o^{\mathrm{h}}_s]}(u), \quad u \in \mathbb{Z}. \tag{6.1}$$

- *The basically vertical down-going DDP along the line* $x = sy$ *is defined as*

$$p^{3,s}_{[o]}(u) \triangleq p^{3,0}_{[o^{\mathrm{h}}_s]}(u), \quad u \in \mathbb{Z}. \tag{6.2}$$

We define the DDP of a "basically horizontal" line as a horizontal projection of a vertically sheared object.

Definition 6.4. *(Discrete diffracted projection along basically horizontal lines) Let* $o[u,v]$ *be a discrete object,* $s \in [-1,1]\backslash\{0\}$*. Let* $o^{\mathrm{v}}_s[u,v]$ *be a vertical shear of* $o[u,v]$ *given by Definition 2.9. Then,*

- *The basically horizontal left-to-right DDP along the line* $y = sx$ *is defined as*

$$p^{4,s}_{[o]}(v) \triangleq p^{4,0}_{[o^{\mathrm{v}}_s]}(v), \quad v \in \mathbb{Z}. \tag{6.3}$$

- *The basically horizontal right-to-left DDP along the line* $y = sx$ *is defined as*

$$p^{2,s}_{[o]}(v) \triangleq p^{2,0}_{[o^{\mathrm{v}}_s]}(v), \quad v \in \mathbb{Z}. \tag{6.4}$$

Having defined the discrete diffracted projections, we now provide an alternative definition that is based on the two-dimensional Fourier transform of the object.

Theorem 6.5. *(Fourier representation of DDP)*

Let N be a positive integer, $M = 2N+1$. Let $o[u,v]$ be a discrete object and let

$$w(k) = \frac{4\pi^2}{M} \frac{1}{\sqrt{(M/2)^2 - k^2}} e^{i\pi\sqrt{(M/2)^2-k^2}}. \tag{6.5}$$

Then, for any $s \in [-1,1]$ and $t \in \mathbb{N}$

$$p^{1,s}_{[o]}(t) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{h}}_s}\left(k, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right), \tag{6.6}$$

$$p^{3,s}_{[o]}(t) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{h}}_s}\left(k, \left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right), \tag{6.7}$$

$$p^{4,s}_{[o]}(t) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{v}}_s}\left(-\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right), k\right), \tag{6.8}$$

$$p^{2,s}_{[o]}(t) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{v}}_s}\left(\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right), k\right). \tag{6.9}$$

Proof. We prove Eq. (6.6). The proofs of Eqs. (6.7), (6.8) and (6.9) are similar.

From Definition 6.3, $p^{1,s}_{[o]}(u') = p^{1,0}_{[o^{\mathrm{h}}_s]}(u')$. Expanding the right-hand side using Definition 6.1 we get

$$p^{1,s}_{[o]}(u') = \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N}\sum_{v=-N}^{N} o^{\mathrm{h}}_s[u,v] e^{i\pi v} \sum_{k=-N}^{N} \frac{1}{\sqrt{(M/2)^2 - k^2}} \cdot e^{i\frac{2\pi}{M}\left(k(u'-u) + \sqrt{(M/2)^2-k^2}\,|\frac{M}{2} - v|\right)}. \tag{6.10}$$

Since $v \in [-N:N]$ we can replace $|\frac{M}{2} - v|$ by $(\frac{M}{2} - v)$. Equation 6.10 can then be rewritten as:

$$p^{1,s}_{[o]}(u') = \sum_{k=-N}^{N} \left(\frac{2\pi}{M}\right)^2 \frac{1}{\sqrt{(M/2)^2 - k^2}} e^{i\pi\sqrt{(M/2)^2-k^2}} e^{i\frac{2\pi}{M}ku'} \cdot \left[\sum_{u=-N}^{N}\sum_{v=-N}^{N} o^{\mathrm{h}}_s[u,v] e^{-i\frac{2\pi}{M}\left(ku - \left(\frac{M}{2} - \sqrt{(M/2)^2-k^2}\right)v\right)}\right]. \tag{6.11}$$

Using the weight function defined by Eq. (6.5), we can rewrite Eq. (6.11) as

$$p^{1,s}_{[o]}(u') = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}ku'}\, \widehat{o^{\mathrm{h}}_s}\left(k, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right)$$

which completes the proof of Eq. (6.6). □

From Eqs. (6.6)–(6.9), we see that the discrete diffracted projection is periodic with period $M = 2N+1$.

6.3 The discrete Fourier diffraction theorem

The continuous Fourier diffraction theorem, given by Theorem 4.1, relates the one-dimensional Fourier transform of a continuous diffracted projection of an object with the of two-dimensional Fourier transform of the object along a semicircular arc. In this section, we prove the discrete Fourier diffraction theorem, which establishes a similar result for the discrete case. We denote the DFT of the sequence $\left\{p^{i,s}_{[o]}(n)\right\}_{n=-N}^{N}$ by $\widehat{p^{i,s}_{[o]}}(l),\ l \in [-N,\ldots,N]$.

Theorem 6.6. *(Discrete Fourier diffraction theorem)*

Let $o[u,v]$ be a discrete object and s be a real number such that $|s| \le 1$. Let $w(k)$ be a weight function defined by Eq. (6.5). Then, for any $l \in [-N:N]$

$$\widehat{p^{1,s}_{[o]}}(l) = w(l)\cdot\widehat{o}\left(l, -sl - \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right), \tag{6.12}$$

$$\widehat{p^{3,s}_{[o]}}(l) = w(l)\cdot\widehat{o}\left(l, -sl + \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right), \tag{6.13}$$

$$\widehat{p^{4,s}_{[o]}}(l) = w(l)\cdot\widehat{o}\left(-sl - \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right), l\right), \tag{6.14}$$

$$\widehat{p^{2,s}_{[o]}}(l) = w(l)\cdot\widehat{o}\left(-sl + \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right), l\right). \tag{6.15}$$

Proof.

We prove Eq. (6.12). The proofs of Eqs. (6.13),(6.14) and (6.15) are similar. From Theorem 6.5

$$p^{1,s}_{[o]}(t) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{h}}_{s}}\left(k, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right).$$

By taking the 1-D Fourier transform of both sides we get

$$\widehat{p^{1,s}_{[o]}}(l) = \sum_{t=-N}^{N}\frac{1}{M}\sum_{k=-N}^{N} w(k)\, e^{i\frac{2\pi}{M}kt}\, \widehat{o^{\mathrm{h}}_{s}}\left(k, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right) e^{-i\frac{2\pi}{M}tl}.$$

Rearranging the terms at the right-hand side yields

$$\widehat{p^{1,s}_{[o]}}(l) = \frac{1}{M}\sum_{k=-N}^{N} w(k)\, \widehat{o^{\mathrm{h}}_{s}}\left(k, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right)\sum_{t=-N}^{N} e^{i\frac{2\pi}{M}t(k-l)}.$$

Since $\sum_{t=-N}^{N} e^{i\frac{2\pi}{M}t(k-l)} = M\cdot\delta_M(k-l)$, where δ_M is the periodic Kronecker Delta with period M, we obtain

$$\widehat{p^{1,s}_{[o]}}(l) = w(l)\, \widehat{o^{\mathrm{h}}_{s}}\left(l, -\left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right). \tag{6.16}$$

For any $l \in [-N:N]$ and $\omega \in \mathbb{R}$ in Theorem 2.10 we get

$$\widehat{o_s^{\mathrm{h}}}(l,\omega) = \widehat{o}(l,\omega - sl). \tag{6.17}$$

We combine Eqs. (6.16) and (6.17) to get

$$\widehat{p_{[o]}^{1,s}}(l) = w(l)\,\widehat{o}\left(l, -sl - \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right),$$

which completes the proof of Eq. (6.12).

□

6.4 Geometric illustration of the discrete Fourier diffraction theorem

The discrete set of points used by the discrete diffraction theorem has a special structure. According to Theorem 6.6, for a basically vertical up-going projection $p_{[o]}^{1,s}(l)$, we sample the Fourier transform of the object o on the set

$$\left\{\left(l, -sl - \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right) \;\middle|\; l \in [-N:N]\right\}. \tag{6.18}$$

Let $\phi_s^1(x) \triangleq -sx - \left(\frac{M}{2} - \sqrt{(M/2)^2 - x^2}\right)$. The set of points, described by Eq. (6.18), consists of points with integer abscissae that lie on the curve

$$y = \phi_s^1(x)\ , \quad x \in [-M/2, M/2] \tag{6.19}$$

in the Fourier domain. For $s = 0$ and $x \in [-M/2, M/2]$, Eq. (6.19) becomes $y = \phi_0^1(x) = -\left(\frac{M}{2} - \sqrt{(M/2)^2 - x^2}\right)$. This equation describes the upper half-circle of the circle $x^2 + \left(y + \frac{M}{2}\right)^2 = \left(\frac{M}{2}\right)^2$. The corresponding curve for $s \neq 0$ and $x \in [-M/2, M/2]$ is $y = \phi_s^1(x) = -sx + \phi_0^1(x)$. It is the same half circle that is vertically sheared by $(-sx)$. The left part of Fig. 6.2 presents some examples of these curves for different values of s. For each fixed s, points with integer coordinates that lie on each curve correspond to the set defined by Eq. (6.18).

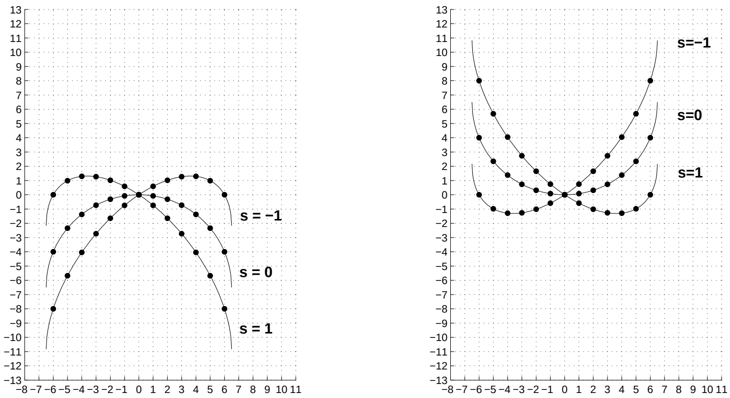

Figure 6.2: Samples in the Fourier domain that correspond to $p_{[o]}^{1,s}$ (left) and $p_{[o]}^{3,s}$ (right)

Similarly, for basically vertical down-going projections $p_{[o]}^{3,s}$, theorem (6.6) states that we sample the Fourier transform of the object o on the set

$$\left\{\left(l, -sl + \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right)\right) \;\middle|\; l \in [-N:N]\right\}. \tag{6.20}$$

The right part of Fig. 6.2 presents an example of these curves for different values of s. For each fixed s, points with integer coordinates, which lie on each curve, correspond to the set defined by Eq. (6.20).

For basically horizontal left-to-right projections, the corresponding set of points $p_{[o]}^{4,s}$ in the Fourier domain is given by $\left\{\left(-sl - \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right), l\right) \;\middle|\; l \in [-N:N]\right\}$. This set is obtained from the set described by Eq.(6.18) by swapping the axes – see Fig. 6.3 for an illustration. Finally, for basically horizontal left-to-right projections, the corresponding set of points $p_{[o]}^{2,s}$ in the Fourier domain is given by $\left\{\left(-sl + \left(\frac{M}{2} - \sqrt{(M/2)^2 - l^2}\right), l\right) \;\middle|\; l \in [-N:N]\right\}$. This set is obtained from the set described by Eq. (6.20) by swapping the axes – see Fig. 6.3 for an illustration.

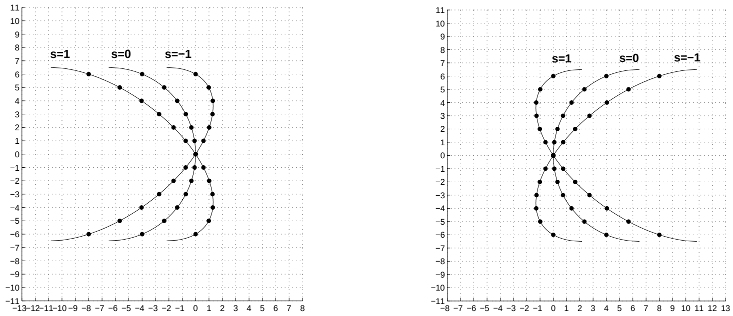

Figure 6.3: Samples in the Fourier domain that correspond to $p_{[o]}^{4,s}$ (left) and $p_{[o]}^{2,s}$ (right)

7 The discrete diffraction transform (DDT)

In this section, we define the DDT as a collection of discrete diffracted projections. We prove that this transform is invertible and rapidly computable.

7.1 Definition of the discrete diffraction transform

Let N be a positive even integer and $M = 2N+1$. Let $o[u,v]$ be a discrete object of size $M \times M$. The DDT is defined as a collection of discrete diffracted projections that correspond to the set of slopes $\left\{s = \frac{l}{N} \mid l \in [-N:N]\right\}$. Formally, we define

Definition 7.1. *(Discrete diffracted transform) Let $o[u,v]$ be a discrete object. For $i = 1, \dots, 4$ and $k, l \in [-N : N]$*

$$\mathcal{D}_{[o]}(i,l,k) \triangleq \begin{cases} p_{[o]}^{1,\frac{l}{N}}(k) & \text{if } \ i=1 \\ p_{[o]}^{2,\frac{l}{N}}(k) & \text{if } \ i=2 \\ p_{[o]}^{3,\frac{l}{N}}(k) & \text{if } \ i=3 \\ p_{[o]}^{4,\frac{l}{N}}(k) & \text{if } \ i=4. \end{cases}$$

Thus, the DDT is a transform that maps a discrete object of size $(2N+1) \times (2N+1)$ into an array of size $4 \times (2N+1) \times (2N+1)$. In the following sections, we show that the DDT is invertible and can be computed in $O(N^2 \log N)$ operations.

The discrete Fourier diffraction theorem maps the discrete diffracted projection into a set of samples of $\widehat{o}(\omega_1, \omega_2)$. We want to find the set of samples $\widehat{o}(\omega_1, \omega_2)$ that corresponds to a collection of projections that form the DDT.

Denote $A(k) \triangleq \frac{M}{2} - \sqrt{(M/2)^2 - k^2}$. The sample points that correspond to projections $p^{i,\frac{l}{N}}$ form, for $l, k \in [-N : N]$, the sets $S_1 = \left\{\left(k, -\frac{l}{N}k - A(k)\right)\right\}$, $S_2 = \left\{\left(-\frac{l}{N}k + A(k), k\right)\right\}$, $S_3 = \left\{\left(k, -\frac{l}{N}k + A(k)\right)\right\}$ and $S_4 = \left\{\left(-\frac{l}{N}k - A(k), k\right)\right\}$.

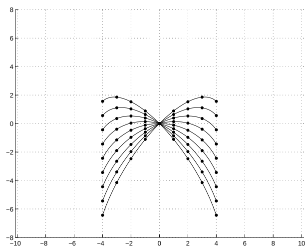

Figure 7.1: Sample set S_1

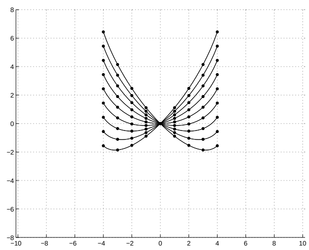

Figure 7.2: Sample set S_3

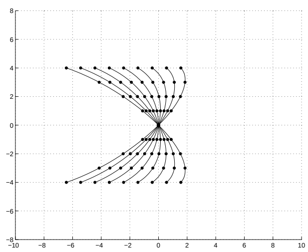

Figure 7.3: Sample set S_4

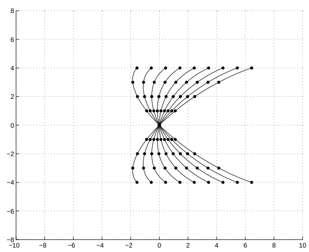

Figure 7.4: Sample set S_2

Figures 7.1 through 7.4 provide geometrical illustration of these sets when $N = 4$. We denote the set of points that correspond to the DDT in the Fourier domain by

$$S_{\mathcal{D}} \triangleq S_1 \cup S_2 \cup S_3 \cup S_4. \tag{7.1}$$

7.2 Efficient computation of the DFT on non-Cartesian grids

Consider the set $G = G_1 \cup G_2$ where

$$G_1 = \left\{(l \cdot f(k) + g(k), \alpha k) \,\middle|\, l, k \in [-N : N]\right\}, G_2 = \left\{(\alpha k,\, l \cdot f(k) + g(k)) \,\middle|\, l, k \in [-N : N]\right\}, \quad (7.2)$$

$\alpha \in \mathbb{R}^+$, $f(k)$ and $g(k)$ are arbitrary real-valued functions. The Cartesian grid is a special case of the grid G for $\alpha = 1$, $f(k) = 1$ and $g(k) = 0$. The pseudo-polar grid, given in [5], is a special case of the grid G when $\alpha = 1$, $f(k) = -2k/N$ and $g(k) = 0$. We present an algorithm that for a given discrete object $o[u, v]$, samples its Fourier transform $\widehat{o}(\omega_1, \omega_2)$ on G in $O(N^2 \log N)$ operations.

The algorithm for computing the Fourier transform of an object $o[u, v]$ in the grid G is based on the *fractional Fourier transform* (FrFT).

Definition 7.2. *[3](Fractional Fourier transform) Let $\{x_n\}_{n=-N}^{N}$ be a sequence of complex numbers. Then, for an arbitrary $\alpha \in \mathbb{R}$ and $k \in [-N : N]$, the fractional Fourier transform is defined as $\left[\hat{f}^{\alpha} X\right](k) \triangleq \sum_{n=-N}^{N} x_n e^{-i\frac{2\pi}{M}\alpha n k}$.*

Lemma 7.3. *[3](Efficient computation of the Fractional Fourier Transform)*

Let $\{x_n\}_{n=-N}^{N}$ be a sequence of complex numbers. For an arbitrary $\alpha \in \mathbb{R}$, the fractional Fourier transform $\left\{\left[\hat{f}^{\alpha}\{x_n\}\right](k) \mid k \in [-N : N]\right\}$ can be computed in $O(N \log N)$ operations assuming that the exponential factors are precomputed.

Fractional Fourier transform ([3]) is based on the same idea (originally by Bluestein [9]) as the chirp z-transform [10].

There is a number of techniques, commonly known as unequally spaced FFT (USFFT) that allow for an efficient evaluation of the DFT at arbitrary set of points within a prescribed precision [8]. Following [5], we use the Fractional Fourier transform rather than USFFT in our algorithm, since it does not include interpolation step, unlike USFFT methods, therefore,it is theoretically exact. However, in practical computations with a prescribed accuracy, there are situations when USFFT is more effective than FrFT [12] (pp778-779). Therefore, either FrFT or USFFT can be used in the implementation of the algorithm.

Next, we use the fractional Fourier transform to derive an efficient algorithm for sampling $\widehat{o}(\omega_1, \omega_2)$ on G_1 given by Eq. 7.2.

Lemma 7.4. *(Efficient computation of $\widehat{o}(\omega_1, \omega_2)$ on G_1)*
Let $f(k)$ and $g(k)$ be two real-valued functions that are defined for $k \in [-N : N]$. Let $\alpha \in \mathbb{R}^+$. Then, the values $\{\widehat{o}(lf(k) + g(k), \alpha k) |\ l, k \in [-N : N]\}$ can be computed in $O(N^2 \log N)$ operations, assuming that the exponential factors $e^{-i\frac{2\pi}{M}\alpha k u}$, $e^{-i\frac{2\pi}{M}f(k)lu}$ and $e^{-i\frac{2\pi}{M}g(k)u}$ are precomputed for all values $k, u, l \in [-N : N]$.

Proof.

$$\widehat{o}(lf(k)+g(k),\alpha k) = \sum_{u=-N}^{N}\sum_{v=-N}^{N} o[u,v]e^{-i\frac{2\pi}{M}([lf(k)+g(k)]u+\alpha kv)} = \sum_{u=-N}^{N}\underbrace{\left(\sum_{v=-N}^{N} o[u,v]e^{-i\frac{2\pi}{M}\alpha kv}\right)}_{A(u,k)}$$

$$\cdot e^{-i\frac{2\pi}{M}lf(k)u}e^{-i\frac{2\pi}{M}g(k)u} = \sum_{u=-N}^{N}\underbrace{\left(A(u,k)e^{-i\frac{2\pi}{M}g(k)u}\right)}_{B(u,k)}e^{-i\frac{2\pi}{M}lf(k)u} = \sum_{u=-N}^{N} B(u,k)e^{-i\frac{2\pi}{M}lf(k)u}. \tag{7.3}$$

For a fixed u, the set $\{A(u,k) \mid k \in [-N:N]\}$ can be computed in $O(N\log N)$ operations, because it is the fractional Fourier transform of $o[u,v]$ on the second variable, and the exponential factors were precomputed. Thus, the set $\{A(u,k) \mid u,k \in [-N:N]\}$ can be computed in $O(N^2\log N)$ operations. Based on $\{A(u,k) \mid u,k \in [-N:N]\}$, we can compute $\{B(u,k) \mid u,k \in [-N:N]\}$ in $O(N^2)$ operations, because $B(u,k)$ was obtained from $A(u,k)$ using multiplication by a precomputed value. Remains to estimate the complexity to compute Eq. (7.3). For a fixed k, $x_u \triangleq B(u,k)$ and $\beta \triangleq f(k)$, this expression is the fractional Fourier transform $\sum_{u=-N}^{N} x_u e^{-i\frac{2\pi}{M}\beta ul}$.

From Lemma 7.3, this expression can be computed in $O(N\log N)$ operations for $l \in [-N:N]$. Thus, for $k,l \in [-N:N]$, the expression in Eq. (7.3) can be computed in $O(N^2\log N)$ operations. Consequently, $\{\widehat{o}(lf(k)+g(k),\alpha k) | \; l,k \in [-N:N]\}$ can be computed in $O(N^2\log N)$ operations. □

The algorithm, which computes $\widehat{o}(\omega_1,\omega_2)$ on G_2, is similar.

7.3 Efficient computation of the DDT

Theorem 7.5. *(Efficient computation of the discrete diffraction transform)*

Let N be a positive even integer and $M = 2N+1$. Let $o[u,v]$ be a discrete object of size $M \times M$. Then, the set $\left\{ \mathcal{D}_{[o]}(i,l,k) \;\middle|\; i \in \{1,2,3,4\},\; l,k \in [-N:N]\right\}$, given by Definition 7.1, can be computed in $O(N^2\log N)$ operations.

Proof. From the definition of the DDT, the set described by $\mathcal{D}_{[o]}(i,l,k)$ is the union of the sets $S_i = \left\{ p_{[o]}^{i,\frac{l}{N}}(k) \;\middle|\; l,k \in [-N:N]\right\}$, $i = 1,\ldots,4$. We show that S_1 can be computed in $O(N^2\log N)$ operations. The proofs for S_2, S_3, and S_4 are similar. From the Fourier diffraction theorem for $k \in [-N:N]$, we have $\widehat{p}_{[o]}^{1,\frac{l}{N}}(k) = w(k)\cdot\widehat{o}\left(k, -\frac{l}{N}k - \left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right)$. By choosing $\alpha = 1$, $f(k) = -\frac{k}{N}$, $g(k) = -\left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)$ and by the application of Lemma 7.4, we get that

$$\left\{\widehat{o}\left(k, -\frac{l}{N}k - \left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2}\right)\right) \;\middle|\; k,l \in [-N:N]\right\} \tag{7.4}$$

can be computed in $O(N^2\log N)$ operations. If $w(k)$ are precomputed, we get that the set

$$\left\{\widehat{p_{[o]}^{1,\frac{l}{N}}}(k) \;\middle|\; k,l \in [-N:N]\right\} \tag{7.5}$$

can be computed in $O(N^2 \log N)$ operations using the values given by Eq. (7.4). For a fixed $l \in [-N : N]$, the values $\left\{ p_{[o]}^{1,\frac{l}{N}}(k) \mid k \in [-N : N] \right\}$ can be computed from the set of frequency samples $\left\{ \widehat{p_{[o]}^{1,\frac{l}{N}}}(k) \mid k \in [-N : N] \right\}$ in $O(N \log N)$ operations by using the 1D inverse FFT. Repeating this for all $l \in [-N : N]$, gives a total of $O(N^2 \log N)$ operations. Hence, the set S_1 can be computed in $O(N^2 \log N)$ operations. The proofs for S_2, S_3, and S_4 are similar. $\square$

7.4 Invertibility of the DDT

Theorem 7.6. *(Invertibility of the discrete diffraction transform)*

The discrete diffraction transform of any discrete object $o[u, v]$ is invertible.

Proof.

Let $\mathcal{D}_{[o]}(i, l, k)$, $i = 1, \ldots, 4$, $\quad l, k \in [-N : N]$. We want to reconstruct the original object $o[u, v]$. We prove a stronger result, namely, that $o[u, v]$ can be reconstructed from $\mathcal{D}_{[o]}(i, l, k)$ for $i = 1, 4$, $\quad l, k \in [-N : N]$.

Assume that $\mathcal{D}_{[o]}(i, l, k)$ is known for $i = 1, 4$, $\quad l, k \in [-N : N]$. From Definition 7.1, this means that $p_{[o]}^{1,\frac{l}{N}}(k)$ and $p_{[o]}^{4,\frac{l}{N}}(k)$ are known for any $k, l \in [-N : N]$. Denote $A \triangleq \left(\frac{M}{2} - \sqrt{(M/2)^2 - k^2} \right)$. By using the discrete Fourier diffraction theorem (Theorem 6.6), we can compute the values of $\widehat{o}(\omega_1, \omega_2)$ on the sets $S_1 = \left\{ \left(k, -\frac{l}{N}k - A\right) \right\}$ and $S_4 = \left\{ \left(-\frac{l}{N}k - A, k\right) \right\}$ for $l, k \in [-N : N]$. We show that the values of $\widehat{o}(\omega_1, \omega_2)$ on the Cartesian grid $\left\{ (k_x, k_y) \mid k_x, k_y \in [-N : N] \right\}$ can be found from the values of $\widehat{o}(\omega_1, \omega_2)$ on the sets S_1 and S_4. Let choose $0 \neq k_x \in [-N : N]$.

$$\widehat{o}(k_x, \omega) = \sum_{u=-N}^{N} \sum_{v=-N}^{N} o[u, v] e^{-i\frac{2\pi}{M}(k_x u + \omega v)} = \sum_{v=-N}^{N} \left(\sum_{u=-N}^{N} o[u, v] e^{-i\frac{2\pi}{M} k_x u} \right) e^{-i\frac{2\pi}{M}\omega v}.$$

Denote $y_v \triangleq \sum_{u=-N}^{N} o[u, v] e^{-i\frac{2\pi}{M} k_x u}$. Let $\widehat{y}(\omega) = \sum_{v=-N}^{N} y_v e^{-i\frac{2\pi}{M} v\omega}$ be the discrete Fourier transform of $\{y_n\}$. Then, $\widehat{y}(\omega)$ is a scaled trigonometric polynomial of order N with scaling factor $\frac{2\pi}{M}$. We can find the values of $\widehat{y}(\omega)$ on the set $S = \left\{ -\frac{k_x}{N} l - \left(\frac{M}{2} - \sqrt{(M/2)^2 - k_x{}^2} \right) \;\middle|\; l \in [-N : N] \right\}$ from the values of $\widehat{o}(\omega_1, \omega_2)$ on the set S_1 from $\widehat{o}(k_x, \omega) = \widehat{y}(\omega)$. The set S consists of M distinct points since $k_x \neq 0$ by choice. The maximal distance between two points in the set S is $2|k_x| \leq 2N < M$. Consequently, the points in S are distinct modulo M. We get that $\widehat{y}(\omega)$ is uniquely defined by its samples on the set S. Sampling $\widehat{y}(\omega)$ at $k_y \in [-N : N]$, gives us the values of $\widehat{o}(\omega_1, \omega_2)$ on the set $\{(k_x, k_y) \mid k_y \in [-N : N]\}$. Since k_x was chosen as an arbitrary non-zero element of $[-N : N]$, we can find the values of $\widehat{o}(\omega_1, \omega_2)$ on the set $\left\{ (k_x, k_y) \mid k_x \in [-N : N] \setminus \{0\},\ k_y \in [-N : N] \right\}$. The set $\left\{ (0, k_y) \mid k_y \in [-N : N] \right\}$ is a subset of S_4 and we get the values of $\widehat{o}(\omega_1, \omega_2)$ on this set as well.

Thus, we found the values of $\widehat{o}(\omega_1, \omega_2)$ on the set $\left\{ (k_x, k_y) \mid k_x, k_y \in [-N : N] \right\}$. This set forms the 2D DFT of $o[u, v]$. By applying the inverse 2D DFT, we find $o[u, v]$ for all $u, v \in [-N : N]$.

$\square$

In the proof of Theorem 7.6, we reconstructed the original object from a subset of the projections that forms the DDT of the object, namely, projections that belong to quarters 1 and 4. In the same way, we can reconstruct the object from the projections that belong to any pair of quarters, where one of the quarters consists of basically vertical directions and the other for basically horizontal directions.

8 DDP as an approximation of diffracted projection of a sheared object

DDP was defined in section 6. It is based on the discretization of a continuous diffracted projection along the y-axis. In section 8.1, we describe an expression that approximates the vertical diffracted projection of a sheared object $f(x,y)$ based on samples of $f(x,y)$ on the set $\left\{\left(\frac{2D}{M}u, \frac{2D}{M}v\right) \mid u,v \in [-N:N]\right\}$. In section 8.2, we show that DDP approximates a diffracted projection of a sheared object for a specific wavenumber choice. In section 8.3, we show that this wavenumber choice is, in some sense, optimal.

8.1 Discretziation of a vertical diffracted projection of a sheared object

Definition 8.1. *Let $f:\mathbb{R}\to\mathbb{R}$. We denote by $f_N(x)$ the one-dimensional trigonometric interpolating polynomial of degree N corresponding to points $\left\{\frac{2\pi}{M}u\right\}_{u=-N}^{N}$ and values $\left\{f\left(\frac{2\pi}{M}u\right)\right\}_{u=-N}^{N}$.*

Definition 8.2. *Let $f:\mathbb{R}^2\to\mathbb{R}$. We denote by $f_N(x,y)$ the two-dimensional trigonometric interpolating polynomial of degree N corresponding to points $\left\{\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) \mid u,v \in [-N:N]\right\}$ and values $\left\{f\left(\frac{2\pi}{M}u, \frac{2\pi}{M}u\right) \mid u,v \in [-N:N]\right\}$.*

Definition 8.3. *Let $D\in\mathbb{R}^+$,$f:\mathbb{R}^2\to\mathbb{R}$. We define*

$$f_N^D(x,y) \triangleq \sum_{u=-N}^{N}\sum_{v=-N}^{N} f\left(\frac{2D}{M}u, \frac{2D}{M}v\right) D_M\left(\frac{2\pi}{M}u - \frac{\pi}{D}x, \frac{2\pi}{M}v - \frac{\pi}{D}y\right).$$

This is a two-dimensional scaled trigonometric interpolating polynomial that corresponds to the samples of $f(x,y)$ on the set $\left\{\left(\frac{2D}{M}u, \frac{2D}{M}v\right) \mid u,v \in [-N:N]\right\}$.

Note that when $D=\pi$, we have $f_N^D(x,y) = f_N(x,y)$.

Lemma 8.4. *Let $f:\mathbb{R}^2\to\mathbb{R}$, $N\in\mathbb{N}$, $v\in[-N:N]$. Let $g(x) \triangleq f\left(x, \frac{2\pi}{M}v\right)$. Then $g_N(x) = f_N\left(x, \frac{2\pi}{M}v\right)$.*

Proof. The expression $f_N\left(x, \frac{2\pi}{M}v\right)$, which is considered as a function of x, is a trigonometric polynomial of degree N. For any $u\in[-N:N]$, we have $f_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) = f\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v\right) = g\left(\frac{2\pi}{M}u\right)$. The claim of the lemma follows from Theorem 2.2 on the uniqueness of the one-dimensional trigonometric interpolating polynomial. □

Definition 8.5. *Let $s\in[-1,1]$, $f:\mathbb{R}^2\to\mathbb{R}$. The horizontal shear of f is defined as $f_s(x,y) \triangleq f(x+sy,y)$.*

Definition 8.6. *Let* $s \in [-1,1]$ *and* $f \in \mathbb{C}^0(\mathbb{R}^2)$*. For arbitrary* $x',y' \in \mathbb{R}$, $T^s[f](x',y') \triangleq \int_{-D}^{D}\int_{-D}^{D} f(x+sy,y)e^{\imath\omega_0 y} K(x,y,x',y')\,dy\,dx$, *where* $T[f]$ *is given by Eq. (5.7).*

Note that $T^s[f](x',y') = T[f_s](x',y')$.

Definition 8.7. *Let* $s \in [-1,1]$, $N \in \mathbb{N}$, $M = 2N+1$*. Let* $f : \mathbb{R}^2 \to \mathbb{R}$*. For arbitrary* $x',y' \in \mathbb{R}$

$$T_N^s[f](x',y') \triangleq \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N}\sum_{v=-N}^{N} f\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) e^{\imath\omega_0\frac{2D}{M}v} K_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v, x', y'\right).$$

Note that $T_N^s[f](x',y') = T_N[f_s](x',y')$.

Lemma 8.8. *[1](p.104)(Uniform convergence of a shifted interpolation)*

Let $A \in [0,\pi]$, $C \in \mathbb{R}^+$, $\alpha \in (0,1]$, $N \in \mathbb{N}$*. Let* $f(x) \in Lip_C(\alpha,\mathbb{R})$ *such that* $f(x) = 0$ *whenever* $|x| \geq A$*. Then, for any* $|\delta| \leq \pi - A$*, we have* $|f_N(x-\delta) - f(x-\delta)| \leq \Phi(C,\alpha,N)$, $x \in [-\pi,\pi]$, *where* $f_N(x)$ *is the one-dimensional trigonometric interpolating ploynomial of degree* N *corresponding to points* $\{\frac{2\pi}{M}u\}_{u=-N}^{N}$ *and values* $\{f\left(\frac{2\pi}{M}u\right)\}_{u=-N}^{N}$ *and* $\Phi(C,\alpha,N)$ *is a function independent of both* f *and* A*, such that* $\lim_{N\to\infty}\Phi(C,\alpha,N) = 0$.

Theorem 8.9. *(Approximation of diffracted projections of a sheared object) Let* $D,\omega_0 \in R^+$*. Let* $f \in \mathbb{C}^0(\mathbb{R}^2) \cap Lip_C(\alpha,\mathbb{R}^2)$*, such that* $f(x,y) = 0$ *whenever* $|x|+|y| \geq D$*. Then,* $T_N^s[f_N^D](x',D)$ *converges to* $T^s[f](x',D)$ *uniformly in* $x' \in [-D,D]$ *and* $s \in [-1,1]$.

Proof. It is sufficient to prove the theorem for $D = \pi$ since we can always scale the variables x and y. This affects the constant C in $Lip_C(\alpha,\mathbb{R}^2)$ but it does not affect α. In diffraction tomography context, $f(x,y)$ describes a physical object, therefore, scaling of x and y corresponds to a change in the metric units. From the definition, $T^s[f](x',\pi) = T[f_s](x',\pi)$. By the note from Definition 8.3, for any function $g : \mathbb{R}^2 \to \mathbb{R}$, we have $g_N^D = g_N$ when $D = \pi$. Therefore,

$$\left|T^s[f](x',\pi) - T_N^s[f_N^D](x',\pi)\right| = \left|T[f_s](x',\pi) - T_N^s[f_N](x',\pi)\right| = \\ \left|T[f_s](x',\pi) - T_N[f_{s_N}](x',\pi) + T_N[f_{s_N}](x',\pi) - T_N^s[f_N](x',\pi)\right|. \quad (8.1)$$

In Eq. (8.1) we added and subtracted $T_N[f_{s_N}]$, where f_{s_N} is a 2D trigonometric interpolation of f_s. By the fact that $T_N[f_{s_N}] = T_N[f_s]$ (see the note to Definition 8.7) and by the application of the triangle inequality, we get

$$\left|T^s[f](x',\pi) - T_N^s[f_N](x',\pi)\right| \leq \left|T[f_s](x',\pi) - T_N[f_{s_N}](x',\pi)\right| + \left|T_N[f_s](x',\pi) - T_N^s[f_N](x',\pi)\right|. \quad (8.2)$$

Since $f \in Lip_C(\alpha,\mathbb{R}^2)$, then, the function $f_s \in Lip_{(\sqrt{3})^\alpha C}(\alpha,\mathbb{R}^2)$ for $s \in [-1,1]$. Using Theorem 5.9, we conclude that the difference $\left|T[f_s](x',\pi) - T_N[f_{s_N}](x',\pi)\right|$ tends to zero uniformly in $x' \in [-\pi,\pi]$. f is continuous on the bounded set $|x|+|y| \leq \pi$ and equals zero outside this set. Hence, f is bounded. Denote $A = \sup_{(x,y)\in\mathbb{R}^2}|f(x,y)|$. We have $\sup_{(x,y)\in\mathbb{R}^2}|f_s(x,y)| = A$ for any $s \in [-1,1]$.

From Corollary 5.10, it follows that the convergence of the first term in Eq. (8.2) to zero is uniform not only in $x' \in [-\pi, \pi]$ but also in $s \in [-1, 1]$. We denote the second term of Eq. (8.2) by

$$R_N(x') \triangleq \left| T_N[f_s](x', \pi) - T_N^s[f_N](x', \pi) \right|. \tag{8.3}$$

Expanding the right-hand side of Eq. 8.3 using Definitions 5.7 and 8.7, we get

$$R_N(x') = \Bigg| \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} \left(f\left(\frac{2\pi}{M}u + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) - f_N\left(\frac{2\pi}{M}u + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right) \cdot e^{\imath\omega_0 v\frac{2\pi}{M}} K_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v, x', \pi\right) \Bigg|.$$

By applying the triangle inequality to the definition of $K_N(x, y, x', y')$, given by Eq. 5.6, we get $|K_N(x, y, x', y')| \leq \frac{2\omega_0}{M} \sum_{k=-N}^{N} \left(\omega_0^2 - \left(\frac{2\omega_0 k}{M}\right)^2\right)^{-\frac{1}{2}}$. The expression on the right-hand side is bounded. Therefore, there exists a constant C_0 such that for any N we have $\left|e^{\imath\omega_0 v\frac{2\pi}{M}} K_N(x, y, x', y')\right| < C_0$. Therefore, for any $N \in \mathbb{N}$ and $x' \in [-\pi, \pi]$,

$$R_N(x') < C_0 \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} \left| f\left(\frac{2\pi}{M}u + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) - f_N\left(\frac{2\pi}{M}u + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) \right|. \tag{8.4}$$

Denote $g_v(x) \triangleq f\left(x, \frac{2\pi}{M}v\right)$ for a fixed $v \in [-N : N]$. This function belongs to $Lip_C(\alpha, \mathbb{R})$. From Lemma 8.4, we have $g_{v_N}(x) = f_N\left(x, \frac{2\pi}{M}v\right)$. Since $f(x, y) = 0$ whenever $|x| + |y| \geq \pi$, then, $g_v(x) = 0$ for $|x| \geq \pi - \frac{2\pi}{M}v$. From Lemma 8.8, there exists a function $\Phi(C, \alpha, N)$ that tends to zero as N grows such that for any $x \in [-\pi, \pi]$ and any $v \in [-N : N]$ we have $\left|g_v\left(x + s\frac{2\pi}{M}v\right) - g_{v_N}\left(x + s\frac{2\pi}{M}v\right)\right| \leq \Phi(C, \alpha, N)$. It is important to note that $\Phi(C, \alpha, N)$ is the same for all $v \in [-N : N]$. By using the definition of $g_v(x)$, we see that for any $v \in [-N, N]$ and $x \in [-\pi, \pi]$ $\left|f\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right) - f_N\left(x + s\frac{2\pi}{M}v, \frac{2\pi}{M}v\right)\right| \leq \Phi(C, \alpha, N)$ holds. For any $\varepsilon > 0$, there exists N_0 such that for any $N > N_0$ we have $|\Phi(C, \alpha, N)| < \frac{\varepsilon}{4C_0\pi^2}$. By using Eq. (8.4), we conclude that when $N > N_0$, $|R_N(x')| \leq \varepsilon$ holds for any $x' \in [-\pi, \pi]$ and $s \in [-1, 1]$. □

The condition "$f(x, y) = 0$ whenever $|x| + |y| \geq D$" is imposed on f in the statement of Theorem 8.9 in order to avoid the "wraparound" effect resulting from the use of trigonometric interpolation.

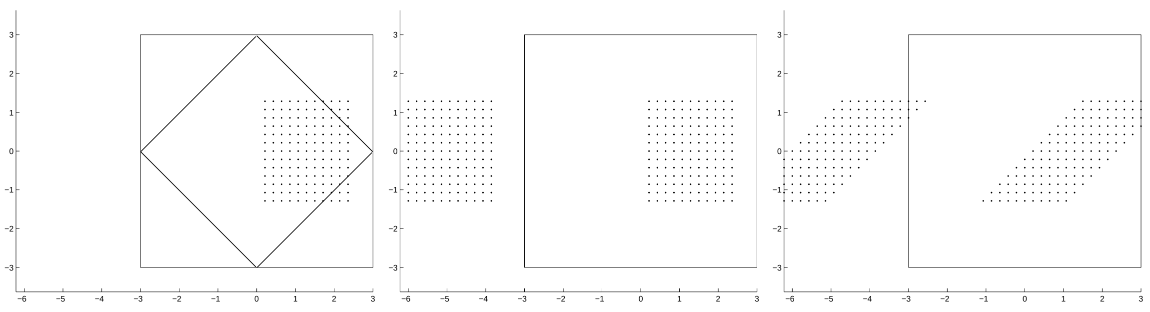

Figure 8.1: Left: Sampled object. Middle: After interpolation. Right: Some points of the left replica fall within the $[-D : D] \times [-D : D]$ square after shear

Figure 8.1 illustrates a typical wraparound effect. Figure 8.1 (right) represents the horizontal shear of the interpolated object that corresponds to slope $s = -1$. We see that some points of the replica of the object are now within the square $[-D, D] \times [-D, D]$. Therefore, the discretization $T_N^s[f_N^D]$ of the vertical diffracted projection of a sheared object is affected by these points, which are not present in the original object.

8.2 DDP as an approximation of a diffracted projection

Theorem 8.10. *Let $f(x, y)$ be an object function. Consider the discrete object that is obtained from sampling $f(x, y)$ on a Cartesian grid:*

$$o[u, v] = f\left(\frac{2D}{M}u, \frac{2D}{M}v\right), \quad u, v \in [-N : N]. \tag{8.5}$$

Let $p_{[o]}^{1,0}$ (Definition 6.1) and $p_{[o]}^{1,s}$ (Eq. 6.1) be the discrete diffracted projections of $o[u, v]$ defined in section 6.2. Assume that the wavenumber is $\omega_0 = \frac{\pi M}{2D}$. Then, for any $u' \in [-N : N]$ and $s \in [-1, 1]$ we have

$$T_N[f]\left(\frac{2D}{M}u', D\right) = \frac{D^2}{\pi^2}\, p_{[o]}^{1,0}(u'), \tag{8.6}$$

$$T_N^s[f_N^D]\left(\frac{2D}{M}u', D\right) = \frac{D^2}{\pi^2}\, p_{[o]}^{1,s}(u'). \tag{8.7}$$

Proof. To see that Eq.(8.6) is correct, we compare Eq. (5.13) and the definition of $p_{[o]}^{1,0}(u')$. The only difference between these two expressions is the constant factor $\frac{D^2}{\pi^2}$.

In order to prove Eq.(8.7), we substitute $\omega_0 = \frac{\pi M}{2D}$ into the definition of $T_N^s[f](x', y')$.

$$T_N^s[f_N^D](x', y') = \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) e^{\imath\pi v}\, K_N\left(\frac{2\pi}{M}u, \frac{2\pi}{M}v, x', y'\right).$$

Then, we expand $K_N\left(\frac{2D}{M}u, \frac{2D}{M}v, x', y'\right)$ by using its definition (Eq. (5.11)) with $\omega_0 = \frac{\pi M}{2D}$ and substitute $x' = \frac{2D}{M}u'$ and $y' = D$:

$$T_N^s[f_N^D]\left(\frac{2D}{M}u', D\right) = \left(\frac{2D}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) e^{\imath\pi v} \sum_{k=-N}^{N} \frac{e^{\imath\frac{2\pi}{M}\left[k(u'-u)+\sqrt{\left(\frac{M}{2}\right)^2-k^2}\,|\frac{M}{2}-v|\right]}}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}}. \tag{8.8}$$

From the definition of $p_{[o]}^{1,s}$ we have

$$p_{[o]}^{1,s}(u') = \left(\frac{2\pi}{M}\right)^2 \sum_{u=-N}^{N} \sum_{v=-N}^{N} o_s^{\mathrm{h}}[u, v] e^{\imath\pi v} \sum_{k=-N}^{N} \frac{e^{i\frac{2\pi}{M}\left[k(u'-u)+\sqrt{(M/2)^2-k^2}\,|\frac{M}{2}-v|\right]}}{\sqrt{\left(\frac{M}{2}\right)^2 - k^2}}.$$

Thus, to prove Eq. (8.7), it remains to show that $f_N^D\left(\frac{2D}{M}u + s\frac{2D}{M}v, \frac{2D}{M}v\right) = o_s^{\mathrm{h}}[u,v]$. Indeed, from the definition of $f_N^D(x,y)$ (Definition 8.3) we get

$$\begin{aligned} & f_N^D\left(\tfrac{2D}{M}u + s\tfrac{2D}{M}v, \tfrac{2D}{M}v\right) = \textstyle\sum_{n=-N}^{N}\sum_{k=-N}^{N} f\left(\tfrac{2D}{M}n, \tfrac{2D}{M}k\right) D_M\left(\tfrac{2\pi}{M}(n-u-sv), \tfrac{2\pi}{M}(k-v)\right) \\ & = \textstyle\sum_{n=-N}^{N} D_M\left(\tfrac{2\pi}{M}(n-u-sv)\right)\sum_{k=-N}^{N} f\left(\tfrac{2D}{M}n, \tfrac{2D}{M}k\right) D_M\left(\tfrac{2\pi}{M}(k-v)\right) \\ & = \textstyle\sum_{n=-N}^{N} f\left(\tfrac{2D}{M}n, \tfrac{2D}{M}v\right) D_M\left(\tfrac{2\pi}{M}(n-u-sv)\right) \end{aligned} \tag{8.9}$$

since $D_M\left(\frac{2\pi}{M}n\right)$ equals one for $n = Mk$, $k \in \mathbb{Z}$, and zero otherwise. On the other hand, from the definition of o_s^{h}, we have

$$o_s^{\mathrm{h}}[u,v] = \sum_{n=-N}^{N} o[n,v]\widetilde{D}_M(n-u-sv) = \sum_{n=-N}^{N} o[n,v] D_M\left(\frac{2\pi}{M}(n-u-sv)\right). \tag{8.10}$$

By comparing Eqs.(8.9) and (8.10) and using Eq.(8.5), we see that the left-hand sides of both equations are equal, which completes the proof.

□

8.3 Optimality of the wavenumber choice

Theorem 8.10 states that for a fixed N and a fixed wavenumber $\omega_0 = \frac{\pi M}{2D}$, we can approximate the basically vertical diffracted projection sampled at M equidistant points $\frac{2D}{M}u$, $u \in [-N : N]$ on the receiver line $y = D$ by using the basically vertical DDP. $\omega_0 = \frac{\pi M}{2D}$ binds the wavenumber with size N of the grid. This means, for example, that for a fixed ω_0, we cannot take N to infinity, therefore, we cannot reach an arbitrary degree of precision when approximating the vertical diffracted projection by means of the vertical DDP. Nevertheless, it turns out that this constraint agrees very well with the existing practical constraints. The following discussion is based on the study of the experimental limitations of the diffracted tomography given in [4].

Let T denote the sampling interval (the distance between detectors on the receiver line). From the Nyquist theorem, the effect of a nonzero sampling interval can be modeled by a lowpass filtering, where the highest measured frequency ω_{meas} is given by $\omega_{meas} = \frac{\pi}{T}$. If we discard the evanescent waves (which are of no significance beyond about 10 wavelengths from the source), then, the highest received wavenumber is $\omega_{max} = \omega_0$. By equating the highest measured frequency to the highest received wavenumber, we get that the highest wavenumber that can be used for a given sampling interval is $\omega_0 = \frac{\pi}{T}$.

Consider the sampling interval $T = \frac{2D}{M}$. The highest wavenumber, which can be used for this sampling interval, is $\omega_0 = \frac{\pi}{T} = \frac{\pi M}{2D}$. Note that this is exactly the wavenumber for which we can compute the approximation of the diffracted projection by using the DDP. We see that this wavenumber is optimal in the following sense: if the wavenumber is bigger than $\frac{\pi M}{2D}$, then, due to aliasing, the measured data may not be a good estimate for the received waveform. If the wavenumber is smaller, the sampling interval can be increased without loss of information.

Note that in the above discussion, we did not consider the effect of having a finite number of receivers or the fact that the receiver line is of finite length.

8.4 Reconstruction of a sheared object by using the DDT

Theorems 5.9 and 8.9 were stated for basically vertical up-going projections. Similar results can be proved by minor modifications of the proofs found in previous sections for the other three types of projections (basically vertical down-going, basically horizontal left-to-right and basically horizontal right-to-left).

The discrete diffraction transform (DDT) was defined in section 7 as a set of discrete diffracted projections along lines with slopes from the set $\{\frac{l}{N} \mid l \in [-N : N]\}$. Therefore, the DDT can be thought as an approximation of a set of continuous diffracted projections of a sheared object, namely, vertical projections of $f(x + \frac{l}{N}y, y)$ and horizontal projections of $f(x, y + \frac{l}{N}x)$, $l \in [-N : N]$.

Conversely, the inverse DDT reconstructs the object from the set of continuous projections described above. Basically vertical up-going projections are measured on the receiver line $y = D$ at points whose x-coordinates are $\frac{2D}{M}n$, $n \in [-N : N]$. The rest of the projections are sampled in a similar way, then, the inverse DDT is applied, which results in a discrete object that approximates the samples of the original object on the set $\left\{(\frac{2D}{M}u, \frac{2D}{M}v) \mid u, v \in [-N : N]\right\}$. In this paper, we do not analyze the quality of the reconstruction or its dependence on N.

8.5 The difference between the DRT and the DDT for image reconstruction

Results similar to Theorems 5.9 and 8.9 hold for the 2D Radon transform. We showed in [1, 6] that a basically vertical DRT approximates the vertical continuous Radon transform of a sheared object. Therefore, the DRT approximates the reconstruction of an object from the set of continuous projections of a sheared object, namely, vertical projections of $f(x+\frac{l}{N}y, y)$ and horizontal projections of $f(x, y + \frac{l}{N}x)$, $l \in [-N : N]$. The phrase "continuous projections" in the context of the DRT refers to the continuous Radon transform.

In this section, we show that there is an important difference between the Radon transform and diffracted projections: in the case of the Radon transform, any rotated projection of the object can be obtained from a single vertical projection of a horizontally sheared object (or a single horizontal projection of a vertically sheared object) by means of multiplication by a constant. This is not the case in diffracted tomography.

Consider the continuous Radon transform of the object $f(x, y)$ along a basically vertical line $x = sy$, $s \in \mathbb{R}$ and $|s| \leq 1$. This projection is formed by a set of line integrals along the parallel lines $\{x = sy + t \mid t \in \mathbb{R}\}$ as illustrated on the left side of Fig. 8.2.

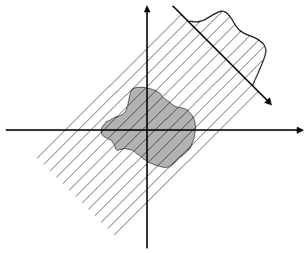

Figure 8.2: The Rotated projection (left) and the vertical projection of a sheared object (right)

The integral of $f(x, y)$ along the line $x = sy + t$ is given by

$$\int_{-\infty}^{\infty} f(sy+t, y)\sqrt{\left(\frac{\partial(sy+t))}{\partial y}\right)^2 + \left(\frac{\partial y}{\partial y}\right)^2} dy = \sqrt{1+s^2}\int_{-\infty}^{\infty} f(sy+t, y)dy. \tag{8.11}$$

The vertical continuous Radon transform of a sheared object $f(x + sy, y)$ is formed by the set of line integrals along the parallel lines $\{x = t \mid t \in \mathbb{R}\}$, as illustrated in the right side of Fig. 8.2. The integral of $f(x + sy, y)$ along the line $x = t$ is given by

$$\int_{-\infty}^{\infty} f(t+sy, y)dy. \tag{8.12}$$

By comparing Eqs.(8.11) and (8.12), we see that the rotated projection of $f(x, y)$ along an arbitrary basically vertical line can be obtained by taking a vertical projection of a sheared object and multiplying it by a constant factor. Conversely, any vertical projection of a horizontally sheared object $o(x + sy, y)$ can be obtained by taking a single rotated projection of the original object $o(x, y)$ and multiplying it by a constant factor. Similar relationships exist between the set of rotated projections along basically horizontal lines and the set of horizontal projections of a vertically sheared object.

This property of the continuous Radon transform makes the inverse DRT appropriate for the reconstruction of an object from a specific set of rotated projections. Indeed, given a set of rotated projections, we can compute a set of sheared projections by a multiplication of each projection by an appropriate constant. Now we can apply the inverse DRT to the set of projections of the sheared object, getting as a result some discrete object that approximates the continuous object $f(x, y)$.

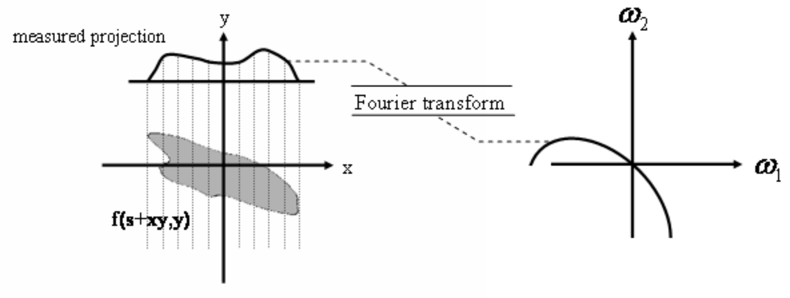

Figure 8.3: A vertical diffracted projection of an horizontally sheared object

Let see whether a similar result holds for the DDT. From the continuous Fourier diffraction theorem (Theorem 4.1), the set of points in the Fourier domain, which corresponds to an up-going projection along the positive y-axis, forms a half-circle of radius ω_0 (the wavenumber of the plane wave used for the illumination), that is centered at the point $(0, -\omega_0)$. In section 2.2, we showed that the continuous 2D Fourier of a horizontally sheared object is a vertical shear of the object's 2D Fourier transform. Consequently, the set of points in the Fourier domain, which corresponds to the vertical diffracted projection of a horizontally sheared object, forms a sheared half-circle (see Fig.8.3). However, by the rotational property of the 2D Fourier transform, the set of points in the Fourier domain, which corresponds to the vertical diffracted projection of a rotated object, forms a rotated half-circle.

In general, it is impossible to cover a fixed sheared half-circle with a single rotated half-circle, which means that no single projection of the rotated object provides us with the set of values in the Fourier domain that corresponds to a vertical projection of the sheared object. Consequently, there is no one-to-one correspondence between the set of rotated projections of a object and the set of vertical projections of a sheared object. This is the reason why the DDT cannot approximate a set of rotated projections of the object and the inverse DDT cannot reconstruct the object from a set of rotated projections.

9 Implementation and numerical results

We use the operator notation to describe the implementation. Let N be a positive integer, $M = 2N+1$, and $o[u, v]$ be a discrete object of size $M \times M$. We denote by $\mathcal{D} : \mathbb{R}^{M\times M} \longrightarrow \mathbb{R}^{4\times M\times M}$ the DDT from definition 7.1. We denote by $\mathcal{F}_{\mathcal{D}} : \mathbb{R}^{M\times M} \longrightarrow \mathbb{R}^{4\times M\times M}$ the operator that maps a discrete object $o[u, v]$ to the set of samples of $\widehat{o}(\omega_1, \omega_2)$ on the set $S_{\mathcal{D}}$ from Eq. (7.1). We denote by $\mathcal{D}_{\mathcal{F}} : \mathbb{R}^{4\times M\times M} \longrightarrow \mathbb{R}^{4\times M\times M}$ the operator that satisfies $\mathcal{D} = \mathcal{D}_{\mathcal{F}} \circ \mathcal{F}_{\mathcal{D}}$. The existence of this operator is a direct corollary of the discrete Fourier diffraction theorem. Both $\mathcal{F}_{\mathcal{D}}$ and $\mathcal{D}_{\mathcal{F}}$ can be applied in $O(N^2 \log N)$ operations as we can see from the proof of Theorem 7.5.

9.1 The forward transform

Operators $\mathcal{F}_{\mathcal{D}}$ and $\mathcal{D}_{\mathcal{F}}$ were implemented in Matlab. The fast forward transform is computed by successive application of these two operators. To verify the correctness of this implementation, we compare the results to the reference implementation of the DDT, which is based on Definition 7.1.

For the input matrix $o[u, v]$, $u, v \in [-N : N]$, we denote by $\mathcal{D}_{[o]}(i, l, k)$ the output of the reference algorithm and by $\mathcal{D}'_{[o]}(i, l, k)$ the output of the fast algorithm, $i \in \{1, 2, 3, 4\}$, $l, k \in [-N : N]$. For a fixed i and l, the vector $\mathcal{D}_{[o]}(i, l, k)$, $k \in [-N : N]$, is a projection of a sheared object. For each projection, we compute the relative l_2 error between the outputs of the fast and the reference implementation and then we take the maximum over i and l. Both algorithms were executed on $(2N + 1) \times (2N + 1)$ random matrices for different values of N with entries uniformly distributed between 0 and 1. We compare the errors for moderate-sized matrices, since direct implementation of the DDT is extremely slow. Comparison results are given in table 1.

N	2	4	8	16	24
Error	6.95e-16	1.73e-15	1.39e-15	3.14e-15	5.85e-15

Table 1: The error between the outputs of the direct and fast implementations of the DDT. Input image size is $(2N+1) \times (2N+1)$.

Next, we estimate the number of floating point operations (flops) required for the application of the operators $\mathcal{F}_\mathcal{D}$ and $\mathcal{D}_\mathcal{F}$. We assume that $5N \log_2 N$ flops are needed for N-point FFT. Evaluation of an N-point fractional Fourier transform requires $20N \log_2 N + 44N$ flops [3].

From the definition, $\mathcal{F}_\mathcal{D}$ evaluates $\widehat{o}(\omega_1, \omega_2)$ on the set $S_\mathcal{D}$ from Eq. (7.1). $S_\mathcal{D}$ is the union of four sets, wheere each is of the form G_1 or G_2 that were specified in Eq. (7.2) for $\alpha = 1$. In Lemma 7.4, we showed that $\widehat{o}(\omega_1, \omega_2)$ can be evaluated on the set G_1 or G_2 in $O(N^2 \log(N))$ operations. We now derive a more exact estimate. Using the notation in Lemma 7.4, we notice that the evaluation of G_1 for $\alpha = 1$ requires M evaluations of M-point FFT to compute $\{A(u,k) | u,k \in [-N:N]\}$, then, M^2 multiplications by precomputed exponential factors are needed to compute $\{B(u,k) | u,k \in [-N:N]\}$ and then M evaluations of an M-point fractional Fourier transform. The total is $M(5M \log_2 M) + M^2 + M(20M \log_2 M + 44M) = 25M^2 \log_2 M + 45M^2$ flops. Since $S_\mathcal{D}$ is a union of four sets, evaluation of $\mathcal{F}_\mathcal{D}$ requires $100M^2 \log_2 M + 180M^2$ flops.

To estimate the number of flops, it is required to apply the operator $\mathcal{D}_\mathcal{F}$ (see the proof of Theorem 7.5). Using the notation from this proof, we notice that, given samples of $\widehat{o}(\omega_1, \omega_2)$ on the set $S_\mathcal{D}$, the set in Eq. 7.5 can be computed using M^2 multiplications by the precomputed factors $\omega(k), k \in [-N:N]$. Then, the set $\left\{ p_{[o]}^{1,\frac{l}{N}}(k) \mid l,k \in [-N:N] \right\}$ is computed by M applications of the inverse M-point FFT. Application of $\mathcal{D}_\mathcal{F}$ requires the computation of $\left\{ p_{[o]}^{i,\frac{l}{N}}(k) \mid l,k \in [-N:N] \right\}$, $i \in \{1,2,3,4\}$. The number of flops, which are required for the application of $\mathcal{D}_\mathcal{F}$, is $4(5M^2 \log_2 M + M^2) = 20M^2 \log_2 M + 4M^2$.

Since $\mathcal{D} = \mathcal{D}_\mathcal{F} \circ \mathcal{F}_\mathcal{D}$, then, the computation of the DDT on an $M \times M$ input matrix requires $120M^2 \log_2 M + 184M^2$ flops. For comparison, the $2D$-FFT computation of the $M \times M$ matrix requires about $10M^2 \log_2 M$ operations.

For completeness, we present in Table 2 the execution times of the non-optimized Matlab implementation of the DDT. The code was executed on Pentium 4 3.0GHz machine running Windows XP.

N	16	32	64	128	256	512	1024
T_{fwd}	0.10	0.28	0.84	2.90	10.20	39.21	270.98

Table 2: The CPU time (in seconds) required for the computation of the DDT on a $(2N+1)\times(2N+1)$ input matrix.

9.2 The inverse transform

The inverse DDT algorithm is a modification of the iterative inverse DRT algorithm from [5]. Consider the DDT operator $\mathcal{D} = \mathcal{D}_{\mathcal{F}} \circ \mathcal{F}_{\mathcal{D}}$. The inverse transform amounts to the solution of $(\mathcal{D}_{\mathcal{F}} \circ \mathcal{F}_{\mathcal{D}})x = y$ for x. $\mathcal{D}_{\mathcal{F}}$ can be inverted in $O(N^2 \log(N))$ operations, so remains to solve $\mathcal{F}_{\mathcal{D}}\, x = z$ where $z = \mathcal{D}_{\mathcal{F}}{}^{-1} y$.

If, for example, y is not necessarily in the range of the DDT due to noise or measurement errors, we want to solve $\min_x \|\mathcal{F}_{\mathcal{D}}\, x - z\|_2$. Solving this minimization problem is equivalent to solving the normal equations

$$\mathcal{F}_{\mathcal{D}}^{*}\, \mathcal{F}_{\mathcal{D}}\, x = \mathcal{F}_{\mathcal{D}}^{*}\, z, \tag{9.1}$$

where $\mathcal{F}_{\mathcal{D}}^{*}$ is the adjoint of $\mathcal{F}_{\mathcal{D}}$. This operator, like $\mathcal{F}_{\mathcal{D}}$, can be applied in $O(N^2 \log(N))$ operations. Since $\mathcal{F}_{\mathcal{D}}^{*}$ is symmetric and positive definite, we can use the conjugate-gradient method [11] to solve Eq. 9.1. We use the same preconditioner as in [5], to improve the convergence rate of the conjugate gradient algorithm.

Table 3 shows the performance of the iterative inversion algorithm for random images of size $(2N + 1) \times (2N + 1)$ for different values of N. The entries in each image are uniformly distributed between 0 and 1. Given the image, its forward DDT was computed, then, the iterative inversion algorithm was applied to recover the image. The error tolerance of the conjugate gradient method was set to $\varepsilon = 10^{-6}$. Notice that the error tolerance of the conjugate gradient algorithm is specified in terms of the A-norm [11](p. 294). This is not the reconstruction error.

We evaluate the quality of the reconstruction by computing the relative error in the Frobenius norm:

$$E_2 = \frac{\sqrt{\sum_u \sum_v |o[u,v] - \tilde{o}[u,v]|^2}}{\sqrt{\sum_u \sum_v |o[u,v]|^2}} \quad . \tag{9.2}$$

As we see from Table 3, very few iterations are required to invert the DDT with high accuracy. The total complexity of the inversion algorithm is $O(\rho(\varepsilon) N^2 \log N)$, where $\rho(\varepsilon)$ is the number of iterations of the conjugate gradient that are required to achieve accuracy ε. As we can see from the table 3, the value of $\rho(\varepsilon)$ depends very weakly on the size of the reconstructed image.

The execution time of the conjugate gradient algorithm is dominated by the application of operators $\mathcal{F}_{\mathcal{D}}$ and $\mathcal{F}_{\mathcal{D}}^{*}$. Application of $\mathcal{F}_{\mathcal{D}}$ dominates the execution time of the forward DDT. Application of $\mathcal{F}_{\mathcal{D}}^{*}$ takes about the same time as the $\mathcal{F}_{\mathcal{D}}$. Therefore, the execution time $T_{inv}(N)$ of the inverse DDT can be approximately estimated to be $2 \cdot T_{fwd}(N) \cdot \rho(\varepsilon)$. The experimental results, which are given in the column T_{inv}/T_{fwd} in Table 3, confirm this estimate.

N	r	*iterations*	E_2	T_{inv}/T_{fwd}
16	4.380218e-007	8	7.286196e-005	14.00
32	9.812216e-008	10	2.139350e-005	20.05
64	3.635840e-007	10	2.081473e-005	20.98
128	1.749592e-007	11	6.922044e-006	22.10
256	4.632606e-007	11	5.561925e-006	24.01
512	1.176530e-007	12	1.338199e-006	26.85
1024	2.764835e-007	12	1.039228e-006	24.65

Table 3: Inverse DDT: $(2N+1) \times (2N+1)$ is the input image size, r is the residual error of the conjugate gradient algorithm upon termination, *iterations* is a number of iterations required for the convergence of the conjugate gradient algorithm, E_2 is the images reconstruction error, T_{inv}/T_{fwd} is the ratio of the execution times between the inverse and the forward DDT.

10 Convergence of the discretization of the inner integral (Theorem 5.5)

$K(x,y,x',y')$ is an improper integral of $f_\alpha(x,y,x',y')$ that is seen as a function of α on the interval $[-\omega_0,\omega_0]$. $K_N(x,y,x',y')$ is a Riemann sum of the same function. Therefore, Theorem 5.5 states that a sequence of certain Riemann sums of $f_\alpha(x,y,x',y')$ converges uniformly in $(x,y,x',y') \in \Omega$ to the improper integral of this function on $[-\omega_0,\omega_0]$.

In section 10.1, we prove the convergence of certain Riemann sums to the corresponding improper integrals. This result is valid for any set of functions that belong to class $\mathbb{B}_{[0,1]}$ (to be defined later). The convergence is uniform within such a set. In section 10.2, we prove that the set of functions $\{f_\alpha(x,y,x',y') \mid (x,y,x',y') \in \Omega\}$ is in class $\mathbb{B}_{[-1,1]}$, which allows the application of the results from section 10.1 to the proof of Theorem 5.5.

10.1 Convergence lemma

Lemma 10.1. *Let $f : [0,1] \to \mathbb{R}$ be a function that is differentiable on $[0,1)$ such that $|f(\alpha)| \le 1$ for $\alpha \in [0,1]$. Then,*

$$S(f) \triangleq \int_0^1 \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha \tag{10.1}$$

converges.

Proof.

$\frac{1}{\sqrt{1-\alpha^2}}$ is integrable on any interval $[0,1-\varepsilon]$, $0 < \varepsilon < 1$, and $\int_0^{1-\varepsilon} \frac{d\alpha}{\sqrt{1-\alpha^2}} = \arcsin(1-\varepsilon)$. $\lim_{\varepsilon\to 0} \arcsin(1-\varepsilon) = \arcsin(1) = \frac{\pi}{2}$. Consequently, $\int_0^1 \frac{d\alpha}{\sqrt{1-\alpha^2}} = \frac{\pi}{2}$. Since $|f(\alpha)| \le 1$, for any $\alpha \in [0,1]$, we have $\left|\frac{f(\alpha)}{\sqrt{1-\alpha^2}}\right| \le \frac{1}{\sqrt{1-\alpha^2}}$, $\alpha \in [0,1]$. Since f is differentiable on $[0,1)$, it is continuous on $[0,1-\varepsilon]$ for any $\varepsilon \in (0,1)$. Consequently, $\left|\frac{f(\alpha)}{\sqrt{1-\alpha^2}}\right|$ is integrable on any interval $[0,1-\varepsilon]$, $\varepsilon \in (0,1)$. From this fact and the above equations we conclude that for any $\varepsilon \in (0,1)$ $\int_0^{1-\varepsilon}\left|\frac{f(\alpha)}{\sqrt{1-\alpha^2}}\right|d\alpha \leq \int_0^{1-\varepsilon}\frac{d\alpha}{\sqrt{1-\alpha^2}} \leq \int_0^1 \frac{d\alpha}{\sqrt{1-\alpha^2}} = \frac{\pi}{2}$. Then, $\int_0^1\left|\frac{f(\alpha)}{\sqrt{1-\alpha^2}}\right|d\alpha = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\left|\frac{f(\alpha)}{\sqrt{1-\alpha^2}}\right|d\alpha \leq \int_0^1\frac{d\alpha}{\sqrt{1-\alpha^2}} = \frac{\pi}{2}$, which means that $S(f)$ converges absolutely and therefore converges. □

Lemma 10.2.

Let $f : [0,1] \to \mathbb{R}$. Then,

$$S_N(f) \triangleq \frac{2}{2N+1}\sum_{n=0}^{N}\frac{f\left(\frac{2n}{2N+1}\right)}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} - \frac{f(0)}{2N+1}. \tag{10.2}$$

is a Riemann sum of $\frac{f(\alpha)}{\sqrt{1-\alpha^2}}$ on the interval $[0,1]$.

Proof.

We consider a partition of $[0,1]$ with the subdivision points

$$\alpha_0 = 0, \quad \alpha_n = \frac{2n-1}{2N+1}, \quad n \in [1,\ldots,N+1], \tag{10.3}$$

and sampling points

$$\xi_n = \frac{2n}{2N+1}, \quad n \in [0,\ldots,N]. \tag{10.4}$$

Then,

$$\begin{aligned} S_N(f) = \; & \tfrac{2}{2N+1}\textstyle\sum_{n=0}^{N}\frac{f(\frac{2n}{2N+1})}{\sqrt{1-(\frac{2n}{2N+1})^2}} - f(0)\cdot\frac{1}{2N+1} = f(0)\cdot\frac{1}{2N+1} + \frac{2}{2N+1}\sum_{n=1}^{N}\frac{f(\frac{2n}{2N+1})}{\sqrt{1-(\frac{2n}{2N+1})^2}} \\ = \; & f(\xi_0)\cdot(\alpha_1-\alpha_0) + \textstyle\sum_{n=1}^{N}\frac{f(\frac{2n}{2N+1})}{\sqrt{1-(\frac{2n}{2N+1})^2}}(\alpha_{n+1}-\alpha_n) = \sum_{n=0}^{N}\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}}\cdot(\alpha_{n+1}-\alpha_n). \end{aligned}$$

Thus, $S_N(f)$ is a Riemann sum of $\frac{f(\alpha)}{\sqrt{1-\alpha^2}}$.

□

Definition 10.3. *(Function class $\mathbb{B}_{[0,1]}$)*

Let $\mathbb{F} = \{f \mid f : [0,1] \to \mathbb{R}\}$. We say that $\mathbb{B} \subseteq \mathbb{F}$ is of class $\mathbb{B}_{[0,1]}$ if the following claims hold: any $f \in \mathbb{B}$ is differentiable on $[0,1)$, for any $f \in \mathbb{B}$ and $\alpha \in [0,1]$, $|f(\alpha)| \leq 1$ holds, and for any $\varepsilon \in (0,1)$ the set $\{f' \mid f \in \mathbb{B}\}$ is uniformly bounded on $[0,1-\varepsilon]$.

The rest of this section is devoted to the proof of following claim: given $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$, then, the sequence of Riemann sums $S_N(f)$ of the function $\frac{f(\alpha)}{\sqrt{1-\alpha^2}}$ converges to the integral $S(f)$ on the interval $[0,1]$ and the convergence is uniform for $f \in \mathbb{B}$.

The following notation will be used in subsequent lemmas. For a fixed $\varepsilon \in (0,1)$, we denote

$$R^{\varepsilon}(f) \triangleq \int_0^{1-\varepsilon}\frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha \tag{10.5}$$

and

$$T^{\varepsilon}(f) \triangleq \int_{1-\varepsilon}^{1} \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha \ . \tag{10.6}$$

From Eqs.(10.1), (10.5) and (10.6) we get

$$S(f) = R^{\varepsilon}(f) + T^{\varepsilon}(f) \ . \tag{10.7}$$

Equations (10.5)–(10.7) define a split of $S(f)$ around the point $1-\varepsilon$. In order to define a split of $S_N(f)$ around the point $1-\varepsilon$, we use the following definition.

For a fixed $N \in \mathbb{N}^+$ and $\varepsilon \in (0,1)$, we define N_ε to be an integer from $[1,\ldots,N+1]$ such that $\alpha_{N_\varepsilon - 1} < 1-\varepsilon \leq \alpha_{N_\varepsilon}$, where α_n are the subdivision points defined by Eq. (10.3). Note that N_ε depends on both N and ε.

For a fixed $N \in \mathbb{N}^+$ and $\varepsilon \in (0,1)$, we define

$$R_N^{\varepsilon}(f) \triangleq \frac{2}{2N+1} \sum_{n=0}^{N_\varepsilon - 1} \frac{f(\frac{2n}{2N+1})}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} - \frac{f(0)}{2N+1}. \tag{10.8}$$

This is the part of $S_N(f)$ that corresponds to the interval $[0, \alpha_{N_\varepsilon}]$. For a fixed $N \in \mathbb{N}^+$ and $\varepsilon \in (0,1)$, we define

$$T_N^{\varepsilon}(f) \triangleq \frac{2}{2N+1} \sum_{n=N_\varepsilon}^{N} \frac{f(\frac{2n}{2N+1})}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}}. \tag{10.9}$$

This is the part of $S_N(f)$ that corresponds to the interval $[\alpha_{N_\varepsilon}, 1]$. From Eqs.(10.1) ($S_N(f)$), (10.8) and (10.9) we have

$$S_N(f) = R_N^{\varepsilon}(f) + T_N^{\varepsilon}(f). \tag{10.10}$$

See Fig.10.1 for an illustration of $R_N^{\varepsilon}(f)$ and $T_N^{\varepsilon}(f)$, .

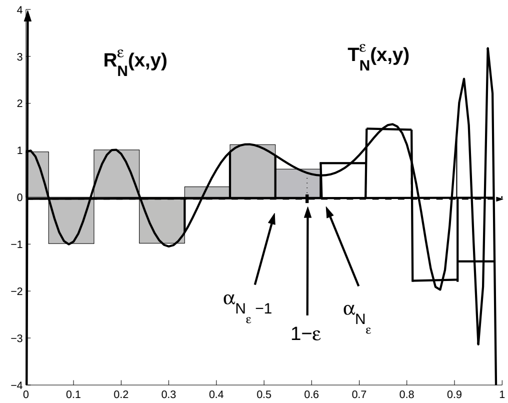

Figure 10.1: Illustration of $R_N^{\varepsilon}(f)$ (colored gray) and $T_N^{\varepsilon}(f)$ (colored white)

In order to prove that when $\mathbb{B}$ is of class $\mathbb{B}_{[0,1]}$, the convergence of $S_N(f)$ to $S(f)$ is uniform for $f \in \mathbb{B}$, we show that for any $\Delta > 0$ the following claims hold (independently of $f \in \mathbb{B}$): there exists ε

such that $|T^{\varepsilon}(f)| < \Delta/4$, there exists N_1 such that $|T_N^{\varepsilon}(f)| < \Delta/2$ for any $N > N_1$, and there exists N_2 such that $|R_N^{\varepsilon}(f) - R^{\varepsilon}(f)| < \Delta/4$ for any $N > N_2$. Lemmas 10.4 though 10.8 contain the proofs.

Lemma 10.4.

Let $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$. For any $\varepsilon \in (0,1)$ and any $f \in \mathbb{B}$, we have $|T^{\varepsilon}(f)| \leq \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{1-\alpha^2}} d\alpha$.

Proof.

By Definition 10.3, for any $f \in \mathbb{B}$ and any $\alpha \in [0,1]$, we have $f|(\alpha)| \leq 1$. Using this fact together with Eq. (10.6) we get

$$|T^{\varepsilon}(f)| = \left| \int_{1-\varepsilon}^{1} \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha \right| \leq \int_{1-\varepsilon}^{1} \left| \frac{f(\alpha)}{\sqrt{1-\alpha^2}} \right| d\alpha \leq \int_{1-\varepsilon}^{1} \frac{1}{\sqrt{1-\alpha^2}} d\alpha.$$

□

Lemma 10.5. *For any $\Delta > 0$ there exists $\varepsilon \in (0,1)$ such that $\int_{1-\varepsilon}^{1} \frac{1}{\sqrt{1-\alpha^2}} d\alpha \leq \Delta$.*

Proof.

We fix $\Delta > 0$. For any $\varepsilon \in (0,1)$, we get from the proof of Lemma 10.1 that $\int_{1-\varepsilon}^{1} \frac{1}{\sqrt{1-\alpha^2}} d\alpha = \frac{\pi}{2} - \arcsin(1-\varepsilon)$. For any $\varepsilon \in (0,1)$, we have $0 \leq \frac{\pi}{2} - \arcsin(1-\varepsilon) \leq 2\sqrt{\varepsilon}$. We choose $\varepsilon = \left(\frac{\Delta}{2}\right)^2$. Then, for any $f \in \mathbb{B}_{[0,1]}$, $\int_{1-\varepsilon}^{1} \frac{1}{\sqrt{1-\alpha^2}} d\alpha \leq 2\sqrt{\varepsilon} = \Delta$. □

Lemma 10.6.

Let $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$. Let $\varepsilon \in (0,1)$. Then, for any $\delta > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ and any $f \in \mathbb{B}$ we have $\left| T_N^{\varepsilon}(f) \right| \leq \int_{1-\varepsilon}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}} + \delta$.

Proof.

Let $f \in \mathbb{B}$. By Definition 10.3, we have $|f(\alpha)| \leq 1$ for any $\alpha \in [0,1]$. From Eq. (10.9) and the triangle inequality we have that for any $N \in \mathbb{N}$

$$|T_N^{\varepsilon}(f)| \leq \frac{2}{2N+1} \sum_{n=N_\varepsilon}^{N} \left| \frac{f(\frac{2n}{2N+1})}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} \right| \leq \frac{2}{2N+1} \sum_{n=N_\varepsilon}^{N} \frac{1}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}}. \tag{10.11}$$

There exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$ we have $\frac{2}{2N+1} < \varepsilon$. Then, $N_\varepsilon < N+1$, which means that ξ_{N_ε} is defined. From Eqs.(10.3) and (10.4) we have $\alpha_{N_\varepsilon} < \xi_{N_\varepsilon}$. Consider a partition of the interval $[\xi_{N_\varepsilon}, 1]$, where the subdivision points are $\{\xi_n \mid n \in \{N_\varepsilon, \ldots, N\}\} \bigcup \{1\}$ and the intermediate points are $\{\xi_n \mid n \in \{N_\varepsilon, \ldots, N\}\}$.

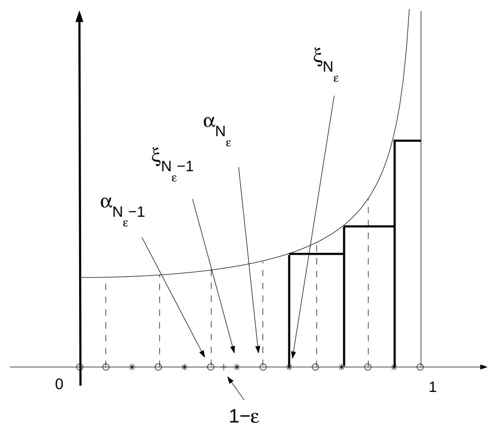

Figure 10.2: The area under the "steps" equals D_N

The Riemann sum of $\frac{1}{\sqrt{1-\alpha^2}}$, which corresponds to this partition, is

$$D_N = \frac{2}{2N+1} \sum_{n=N_\varepsilon}^{N} \frac{1}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{2N}{2N+1}\right)^2}} \cdot \frac{1}{2N+1} \tag{10.12}$$

(see Fig. 10.2 for illustration). From Eqs.(10.11) and (10.12) we get

$$|T_N^\varepsilon(f)| \leq D_N + \frac{1}{\sqrt{1-\left(\frac{2N}{2N+1}\right)^2}} \cdot \frac{1}{2N+1}. \tag{10.13}$$

Since $\frac{1}{\sqrt{1-\alpha^2}}$ monotonically increases on $[0,1]$, D_N is a lower Darboux sum of $\frac{1}{\sqrt{1-\alpha^2}}$ on $[\xi_{N_\varepsilon}, 1]$. From the definition of N_ε, we have $1-\varepsilon \leq \alpha_{N_\varepsilon} < \xi_{N_\varepsilon}$. Consequently,

$$D_N \leq \int_{\xi_{N_\varepsilon}}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}} \leq \int_{1-\varepsilon}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}}. \tag{10.14}$$

From Eqs.(10.13) and (10.14), it follows that $|T_N^\varepsilon(f)| \leq \int_{1-\varepsilon}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}} + \frac{1}{\sqrt{1-\left(\frac{2N}{2N+1}\right)^2}} \cdot \frac{1}{2N+1}$. By simplifying the last term we have

$$\left|T_N^\varepsilon(f)\right| \leq \int_{1-\varepsilon}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}} + \frac{1}{\sqrt{4N+1}}. \tag{10.15}$$

Since the last term in Eq. (10.15) tends to 0 as N grows to infinity, there exists $N_0 > N_1$ such that for any $N > N_0$ we have $\frac{1}{\sqrt{4N+1}} < \delta$. Thus, we found $N_0 \in \mathbb{N}$ such that for any $f \in \mathbb{B}$ we have $|T_N^\varepsilon(f)| \leq \int_{1-\varepsilon}^{1} \frac{d\alpha}{\sqrt{1-\alpha^2}} + \delta$. □

Lemma 10.7.

Let $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$. For any $\varepsilon \in (0,1)$ there exists $C_\varepsilon \in \mathbb{R}^+$ such that for any $f \in \mathbb{B}$ and any $\alpha \in [0, 1-\varepsilon]$ $\left|\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha)\right| \leq C_\varepsilon$.

Proof.

We fix $\varepsilon > 0$. Let $f \in \mathbb{B}$. This function is differentiable on $[0, 1-\varepsilon]$. For any $\alpha \in [0, 1-\varepsilon]$ we have

$$\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha) = \frac{f'(\alpha)}{\sqrt{1-\alpha^2}} + \frac{\alpha f(\alpha)}{(1-\alpha^2)^{3/2}}. \tag{10.16}$$

By Definition 10.3, $|f(\alpha)| \leq 1$ for any $\alpha \in [0,1]$. From the application of the triangle inequality to Eq. (10.16), we get

$$\left|\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha)\right| \leq \frac{|f'(\alpha)|}{\sqrt{1-\alpha^2}} + \frac{1}{(1-\alpha^2)^{3/2}}. \tag{10.17}$$

Since $\mathbb{B}$ is of class $\mathbb{B}_{[0,1]}$, the set of functions $\{f' \mid f \in \mathbb{B}\}$ is uniformly bounded on $[0, 1-\varepsilon]$. Therefore, there exists K_ε such that for any $f \in \mathbb{B}$ and any $\alpha \in [0, 1-\varepsilon]$ we have $|f'(\alpha)| \leq K_\varepsilon$. Then, for any $f \in \mathbb{B}$ and any $\alpha \in [0, 1-\varepsilon]$ we get from Eq. (10.17) $\left|\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha)\right| \leq K_\varepsilon \frac{1}{\sqrt{1-\alpha^2}} + \frac{1}{(1-\alpha^2)^{3/2}}$. We define $C_\varepsilon \triangleq K_\varepsilon \frac{1}{\sqrt{1-(1-\varepsilon)^2}} + \frac{1}{(1-(1-\varepsilon)^2)^{3/2}}$. Then, for any $f \in \mathbb{B}$ and any $\alpha \in [0, 1-\varepsilon]$ we have $\left|\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha)\right| \leq C_\varepsilon$. □

Lemma 10.8. *(Convergence of $R_N^\varepsilon(f)$ from Eq. (10.8))*

Let $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$. Let $\varepsilon \in (0,1)$. Then, for any $\Delta > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ and any $f \in \mathbb{B}$ we have $\left|R_N^\varepsilon(f) - R^\varepsilon(f)\right| < \Delta$.

Proof.

We fix $\varepsilon > 0$. From Lemma 10.7, there exists $C_{\frac{\varepsilon}{2}}$ such that for any $\alpha \in [0, 1-\frac{\varepsilon}{2}]$ and any $f \in \mathbb{B}$ we have $\left|\left(\frac{f(\beta)}{\sqrt{1-\beta^2}}\right)'(\alpha)\right| \leq C_{\frac{\varepsilon}{2}}$. By the mean value theorem, this means that for any $\gamma_1, \gamma_2 \in [0, 1-\frac{\varepsilon}{2}]$ we have $\left|\frac{f(\gamma_1)}{\sqrt{1-\gamma_1{}^2}} - \frac{f(\gamma_2)}{\sqrt{1-\gamma_2{}^2}}\right| \leq C_{\frac{\varepsilon}{2}} |\gamma_1 - \gamma_2|$. There exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$ the inequality $\frac{2}{2N+1} < \frac{\varepsilon}{2}$ holds. Then, for any $N > N_1$ we have $\alpha_{N_\varepsilon} < 1-\frac{\varepsilon}{2}$. Consider an arbitrary $N > N_1$. We know that if in general $f : [a,b] \to \mathbb{R}$ is a continuous function, such that for any $x_1, x_2 \in [a,b]$ the inequality $|f(x_1) - f(x_2)| \leq C|x_1 - x_2|$ holds, and if $\xi \in [a,b]$, then, $\left|f(\xi)(b-a) - \int_a^b f(x)dx\right| \leq C(b-a)^2$. Then, if we apply it to $\frac{f(\alpha)}{\sqrt{1-\alpha^2}}$ we have, for any $n \in \{1, \dots, N_\varepsilon - 1\}$

$$\left|\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}}|\alpha_{n+1} - \alpha_n| - \int_{\alpha_n}^{\alpha_{n+1}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha\right| \leq C_{\frac{\varepsilon}{2}} |\alpha_{n+1} - \alpha_n|^2.$$

Since $|\alpha_{n+1} - \alpha_n| = \frac{2}{2N+1}$ for any $n \in \{1, \dots, N_\varepsilon - 1\}$ we get

$$\left|\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}} \frac{2}{2N+1} - \int_{\alpha_n}^{\alpha_{n+1}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha\right| \leq C_{\frac{\varepsilon}{2}} \left(\frac{2}{2N+1}\right)^2. \tag{10.18}$$

Similarly, for $n = 0$ we get

$$\left|\frac{f(\xi_0)}{\sqrt{1-\xi_0{}^2}} \frac{1}{2N+1} - \int_{\alpha_0}^{\alpha_1} \frac{f(\alpha)}{\sqrt{1-\alpha^2}} d\alpha\right| \leq C_{\frac{\varepsilon}{2}} \left(\frac{1}{2N+1}\right)^2 \leq C_{\frac{\varepsilon}{2}} \left(\frac{2}{2N+1}\right)^2. \tag{10.19}$$

From Eqs.(10.8) and (10.4), we have $R^\varepsilon_N(f) = \frac{2}{2N+1}\sum_{n=0}^{N_\varepsilon-1}\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}} - \frac{f(0)}{2N+1}$. From Eq. (10.5)

$$R^\varepsilon(f) = \int_{\alpha_0}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha - \int_{1-\varepsilon}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha.$$

Thus,

$$|R^\varepsilon_N(f) - R^\varepsilon(f)| = \left|\frac{2}{2N+1}\sum_{n=0}^{N_\varepsilon-1}\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}} - \int_{\alpha_0}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha - \frac{f(0)}{2N+1} + \int_{1-\varepsilon}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha\right|.$$

By the application of the triangle inequality, we get

$$|R^\varepsilon_N(f) - R^\varepsilon(f)| \le \sum_{n=0}^{N_\varepsilon-1}\left|\frac{2}{2N+1}\cdot\frac{f(\xi_n)}{\sqrt{1-\xi_n{}^2}} - \int_{\alpha_n}^{\alpha_{n+1}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha\right| + \left|\frac{f(0)}{2N+1}\right| + \left|\int_{1-\varepsilon}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha\right|.$$

By the application of Eqs.(10.18) and (10.19) we get

$$|R^\varepsilon_N(f) - R^\varepsilon(f)| \le N\cdot C_{\frac{\varepsilon}{2}}\cdot\left(\frac{2}{2N+1}\right)^2 + \left|\frac{f(0)}{2N+1}\right| + \left|\int_{1-\varepsilon}^{\alpha_{N_\varepsilon}} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}d\alpha\right|. \tag{10.20}$$

The first two terms on the right-hand side of Eq. (10.20) tend to zero as N grows. From the definition of N_ε we have $|1-\varepsilon-\alpha_{N_\varepsilon}| \le \frac{2}{2N+1}$, therefore, the third term on the right-hand side of Eq. (10.20) tends to zero as well. Consequently, there exists $N_0 > N_1$ such that for any $N > N_0$ we have $|R^\varepsilon_N(f) - R^\varepsilon(f)| < \Delta$ □

Now we prove that when $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$, the convergence of $S_N(f)$ to $S(f)$ is uniform for $f \in \mathbb{B}$.

Lemma 10.9.

Let $\mathbb{B}$ of class $\mathbb{B}_{[0,1]}$. For any $\Delta > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ and any $f \in \mathbb{B}$ we have $|S_N(f) - S(f)| < \Delta$.

Proof. Consider an arbitrary $\Delta > 0$. From Eqs.(10.7) and (10.10) we know that for any $N \in \mathbb{N}$, any $f \in \mathbb{B}$ and any $\varepsilon \in (0,1)$

$$|S_N(f) - S(f)| = |R^\varepsilon_N(f) + T^\varepsilon_N(f) - R^\varepsilon(f) - T^\varepsilon(f)| \le |R^\varepsilon_N(f) - R^\varepsilon(f)| + |T^\varepsilon_N(f)| + |T^\varepsilon(f)|.$$

By Lemma 10.4, for any $\varepsilon \in (0,1)$ and $f \in \mathbb{B}$, we have $|T^\varepsilon(f)| \le \int_{1-\varepsilon}^1 \frac{1}{\sqrt{1-\alpha^2}}d\alpha$ and from Lemma 10.5 there exists $\varepsilon_0 \in (0,1)$ such that $\int_{1-\varepsilon_0}^1 \frac{1}{\sqrt{1-\alpha^2}}d\alpha \le \frac{\Delta}{4}$. From these two equations we conclude that $|T^{\varepsilon_0}(f)| \le \frac{\Delta}{4}$. Then, for any $N \in \mathbb{N}$ and any $f \in \mathbb{B}$

$$|S_N(f) - S(f)| \le \left|R^{\varepsilon_0}_N(f) - R^{\varepsilon_0}(f)\right| + \left|T^{\varepsilon_0}_N(f)\right| + \frac{\Delta}{4}. \tag{10.21}$$

By Lemma 10.6 there exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$ and any $f \in \mathbb{B}$ we have $\left|T^{\varepsilon_0}_N(f)\right| \le \int_{1-\varepsilon_0}^1 \frac{1}{\sqrt{1-\alpha^2}}d\alpha + \frac{\Delta}{4}$. From $\int_{1-\varepsilon_0}^1 \frac{1}{\sqrt{1-\alpha^2}}d\alpha \le \frac{\Delta}{4}$ we get that for any $N > N_1$ and any $f \in \mathbb{B}$ $\left|T^{\varepsilon_0}_N(f)\right| \le \frac{\Delta}{2}$. Thus, for any $N > N_1$ and any $f \in \mathbb{B}$ we have $|S_N(f) - S(f)| \le \left|R^{\varepsilon_0}_N(f) - R^{\varepsilon_0}(f)\right| + \frac{\Delta}{2} + \frac{\Delta}{4}$ and by Lemma 10.8 there exists $N_2 \in \mathbb{N}$ such that for any $N > N_2$ and any $f \in \mathbb{B}$ we have $\left|R^{\varepsilon_0}_N(f) - R^{\varepsilon_0}(f)\right| < \frac{\Delta}{4}$. We define $N_0 = \max(N_1, N_2)$. Then, from these two equations we conclude that for any $N > N_0$ and any $f \in \mathbb{B}$ $|S_N(f) - S(f)| < \Delta$. □

Definition 10.10. *(Function class $\mathbb{B}_{[-1,1]}$)*

Let $\mathbb{F} = \{f \mid f : [-1,1] \to \mathbb{R}\}$. We say that $\mathbb{B} \subseteq \mathbb{F}$ is of class $\mathbb{B}_{[-1,1]}$ if the following claims hold: any $f \in \mathbb{B}$ is differentiable on $(-1,1)$, for any $f \in \mathbb{B}$ and $\alpha \in [-1,1]$, $|f(\alpha)| \leq 1$ holds, and for any $\varepsilon \in (-1,1)$ the set $\{f' \mid f \in \mathbb{B}\}$ is uniformly bounded on $[-1+\varepsilon, 1-\varepsilon]$.

Lemma 10.11. *Consider $\mathbb{B}$ of class $\mathbb{B}_{[-1,1]}$. Then $\{f|_{[0,1]} \mid f \in \mathbb{B}\}$ is of class $\mathbb{B}_{[0,1]}$.*

Lemma 10.12. *Consider $\mathbb{B}$ of class $\mathbb{B}_{[-1,1]}$. Then $\{f(-x) \mid f \in \mathbb{B}\}$ is of class $\mathbb{B}_{[-1,1]}$.*

Lemma 10.13. *(Convergence lemma)*

Let $\mathbb{B}$ of class $\mathbb{B}_{[-1,1]}$. For any $\Delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ and any $f \in \mathbb{B}$, we have

$$\left| \frac{2}{2N+1} \sum_{n=-N}^{N} \frac{f\left(\frac{2n}{2N+1}\right)}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} - \int_{-1}^{1} \frac{f(\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha \right| < \Delta. \tag{10.22}$$

Proof. Denote $A_{i,j} \triangleq \frac{2}{2N+1}\sum_{n=i}^{j} \frac{f\left(\frac{2n}{2N+1}\right)}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}}$ and $B_{c,d} \triangleq \int_c^d \frac{f(\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha$.

From Lemmas 10.11 and 10.9 there exists $N_1 \in \mathbb{N}$ such that for any $N > N_1$ and any $f \in \mathbb{B}$ $\left|A_{0,N} - \frac{f(0)}{2N+1} - B_{0,1}\right| < \frac{\Delta}{2}$. Consider the set of functions $\{f(-x) \mid f(x) \in \mathbb{B}\}$. By Lemmas 10.12, 10.11 and 10.9 there exists $N_2 \in \mathbb{N}$ such that for any $N > N_2$ and any $f \in \mathbb{B}'$ we have $\left|A_{0,N} - \frac{f(0)}{2N+1} - B_{0,1}\right| < \frac{\Delta}{2}$. Therefore, for any $N > N_2$ and any $f \in \mathbb{B}$ we have $\left|\frac{2}{2N+1}\sum_{n=0}^{N} \frac{f\left(-\frac{2n}{2N+1}\right)}{\sqrt{1-\left(\frac{2n}{2N+1}\right)^2}} - \frac{f(0)}{2N+1} - \int_0^1 \frac{f(-\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha\right| < \frac{\Delta}{2}$. Since $\int_0^1 \frac{f(-\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha = \int_{-1}^0 \frac{f(\alpha)}{\sqrt{1-\alpha^2}}\, d\alpha$ we can rewrite this equation as $\left|A_{-N,0} - \frac{f(0)}{2N+1} - B_{-1,0}\right| < \frac{\Delta}{2}$. By the application of the triangle inequality to the left-hand side of Eq. (10.22), we get $|A_{-N,N} - B_{-1,1}| \leq \left|A_{0,N} - \frac{f(0)}{2N+1} - B_{0,1}\right| + \left|A_{-N,0} - \frac{f(0)}{2N+1} - B_{-1,0}\right|$. We choose $N_0 = \max(N_1, N_2)$. Then, from the above equations we get that for any $N > N_0$ and any $f \in \mathbb{B}$, $|A_{-N,N} - B_{-1,1}| < \Delta$.

□

10.2 Convergence of $K_N(x, y, x', y')$ (Theorem 5.5)

In this section, we prove Theorem 5.5. It is sufficient to prove this theorem for $\omega_0 = 1$. The proof for an arbitrary ω_0 can be then obtained by scaling x, y, x' and y' by ω_0.

The function $f_\alpha(x, y, x', y')$ (Eq. (5.3)) is complex valued. To prove Theorem 5.5 it is thus sufficient to show that for any $\varepsilon > 0$ there exists N_0 such that for any $N > N_0$ and any x, y, x' and y' in Ω

$$\left| \frac{2}{M} \sum_{n=-N}^{N} \Re e\left[f\left(\frac{2n}{M}; x, y, x', y'\right)\right] - \int_{-1}^{1} \Re e\left[f_\alpha(x, y, x', y')\right]\, d\alpha \right| < \varepsilon/\sqrt{2} \tag{10.23}$$

and that similar inequality holds for the imaginary part of $f_\alpha(x, y, x', y')$.

Lemma 10.14.

Let $\Omega \subset \mathbb{R}^2$ be a bounded set. Then, for a fixed $\varepsilon \in (0,1)$ there are constants C_1 and C_2 such that

$$\left|\left(\frac{\partial \cos(x\beta + |y|\sqrt{1-\beta^2})}{\partial \beta}\right)(\alpha)\right| < C_1 \text{ and } \left|\left(\frac{\partial \sin(x\beta + |y|\sqrt{1-\beta^2})}{\partial \beta}\right)(\alpha)\right| < C_2 \tag{10.24}$$

for arbitrary $(x,y) \in \Omega$ and $\alpha \in [-1+\varepsilon, 1-\varepsilon]$.

Proof.

Denote $f^{x,y}(\alpha) \triangleq \cos(x\alpha+|y|\sqrt{1-\alpha^2})$. Its derivative is $\left(\frac{\partial f^{x,y}(\beta)}{\partial \beta}\right)(\alpha) = \frac{\sin(x\alpha+|y|\sqrt{1-\alpha^2})\cdot(|y|\alpha - x\sqrt{1-\alpha^2})}{\sqrt{1-\alpha^2}}$. Since $|\sin(x)| \le 1$ for any x, we have $\left|\left(\frac{\partial f^{x,y}(\beta)}{\partial \beta}\right)(\alpha)\right| \le \left|\frac{|y|\alpha - x\sqrt{1-\alpha^2}}{\sqrt{1-\alpha^2}}\right|$. Applying the triangle inequality and using the fact that $|\alpha| \le 1$ we have $\left|\left(\frac{\partial f^{x,y}(\beta)}{\partial \beta}\right)(\alpha)\right| \le \frac{|y|+|x|}{\sqrt{1-\alpha^2}}$. Since Ω is a bounded set there exists $A > 0$ such that $\Omega \subseteq [-A, A] \times [-A, A]$, and since $|\alpha| \le 1-\varepsilon$ we have $\left|\left(\frac{\partial f^{x,y}(\beta)}{\partial \beta}\right)(\alpha)\right| \le \frac{2A}{\sqrt{1-(1-\varepsilon)^2}}$. The constant $C_1 = \frac{2A}{\sqrt{1-(1-\varepsilon)^2}}$ satisfies the left equation in Eq. (10.24) for arbitrary $(x,y) \in \Omega$ and $\alpha \in [-1+\varepsilon, 1-\varepsilon]$. The proof of the second inequality in Eq. (10.24) is similar. □

Lemma 10.15.

Let $\Omega \subseteq \mathbb{R}^4$ be a bounded set. Consider the two sets of functions

$$\mathbb{B}_1 = \{\cos(\alpha(x'-x) + \sqrt{1-\alpha^2}|y'-y|) \mid (x,y,x',y') \in \Omega\} \tag{10.25}$$

$$\mathbb{B}_2 = \{\sin(\alpha(x'-x) + \sqrt{1-\alpha^2}|y'-y|) \mid (x,y,x',y') \in \Omega\}. \tag{10.26}$$

Then, both $\mathbb{B}_1$ and $\mathbb{B}_2$ are of class $\mathbb{B}_{[-1,1]}$.

Proof.

For any $(x,y,x',y') \in \Omega$, $\cos(\alpha(x'-x) + \sqrt{1-\alpha^2}|y'-y|)$ is differentiable on $(-1,1)$ as a function of α. For any $(x,y,x',y') \in \Omega$, $\alpha \in [-1,1]$, the absolute value of the function is bounded by 1. Since Ω is bounded, the set $\{(x'-x, y'-y) \mid x,y,x',y' \in \Omega\}$ is bounded. Therefore, by Lemma 10.14 for any $\varepsilon \in (0,1)$ the set of functions $\left\{\frac{\partial \cos(\alpha(x'-x)+\sqrt{1-\alpha^2}|y'-y|)}{\partial \alpha} \,\middle|\, (x,y,x',y') \in \Omega\right\}$ is uniformly bounded on $[-1+\varepsilon, 1-\varepsilon]$. Therefore, $\mathbb{B}_1$ satisfies the requirements of Lemma 10.13. The proof for $\mathbb{B}_2$ is similar. □

By applying Lemma 10.13 to the set $\mathbb{B}_1$ from Eq. (10.25), we obtain Eq. (10.23). By applying Lemma 10.13 to $\mathbb{B}_2$ we obtain a similar inequality for the imaginary part of $f_\alpha(x,y,x',y')$. This completes the proof of Theorem 5.5.

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