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# An Interesting Feature of the Radon Transform

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**Abstract**—It is shown that the reconstructed image in two-dimensional, parallel beam tomography has an imaginary component which, while zero for perfect data, is of necessity nonzero with real world data. The imaginary component can be used in at least two ways. The first is as an indicator function of the efficacy of corrective measures taken to correct flaws in the original data or in the processing technique. The second is as an essential ingredient in the many complex number operations that are used to process noisy data: crosscorrelations, autocorrelations, spectral analyses, and so forth.

**Keywords**—Radon transform, Tomography, Complex reconstruction.

## SECTION 1

The Radon transform is the primary data set in the field of tomography. When processed by the principles of this discipline, the data lead to a gray level map of the two-dimensional attenuation coefficient $f(x, y)$ of slices of a three-dimensional object. While tomography is of great value and use in medical diagnosis where it has come into great prominence [1] in x-ray and nuclear magnetic resonance scanners, in different variants it is used in geology, plasma studies, radio astronomy [2] and elsewhere.

Tomography, however, is still a young field and there are features about it which remain to be uncovered. One of these we wish to present now which, while simple, appears not to have been noted as of yet. While interesting in its own right from a purely mathematical point of view, it is expected that this property will have numerous applications. In particular, we are going to show that in all cases of practical interest, each tomographic reconstruction carries with it an imaginary component. And this imaginary component will have at least two roles. First, it can serve as an indicator function of flaws in the Radon transform data since if the data are perfect and the processing is ideal, this imaginary component disappears. Inasmuch as this never happens, the imaginary component will always exist. Second from time to time, it may be desirable to crosscorrelate or autocorrelate the derived imagery. Including the imaginary component gives rise to greater accuracy in these procedures.

We will restrict ourselves to parallel beam tomography with the usual restrictions on the finite extent of $f(x, y)$. It will also be assumed that the Fourier transform of $f(x, y)$ exists. In what follows, initially we shall follow the development given by Rowland [3].

In the next section, we present the derivation. Concluding remarks will appear in Section 3.

## SECTION 2

The Radon transform of the two-dimensional attenuation coefficient $f(x, y)$ consists of straight line integrals of $f(x, y)$ along rays or beams in a plane through the three-dimensional object to

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be analyzed. It is given by

$$p(l',\theta) = \iint f(x,y)\delta(l' - x\cos\theta - y\sin\theta)\,dx\,dy, \tag{1}$$

where the integration is taken over the domain of $f(x,y)$, $\delta(x)$ is the delta function, and $l'$ and $\theta$ are parameters of the normal form of the equation of the straight line that is coincident with the ray.

THEOREM. *In all cases of practical interest, every parallel beam tomographic reconstruction of $f(x,y)$ is complex with the real component given by*

$$g_1(x,y) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_{-\infty}^{\infty} \frac{p(l',\theta)}{(l'-l)^2}\bigg|_{l=x\cos\theta+y\,\sin\theta} dl'd\theta, \tag{2}$$

*and the imaginary component given by*

$$g_2(x,y) = \frac{1}{4\pi}\int_0^{2\pi} \frac{dp(l',\theta)}{dl'}\bigg|_{l'=x\cos\theta+y\,\sin\theta} d\theta. \tag{3}$$

PROOF. The projection slice theorem tells us that the Fourier transform of $p(l',\theta)$ is

$$F(R,\theta) = \int_{-\infty}^{\infty} p(l',\theta)\exp(-i2\pi Rl')\,dl', \tag{4}$$

and that $F(R,\theta)$ is the Fourier transform of $f(x,y)$.

The reconstructed version of $f(x,y)$ we denote by $g(x,y)$ and it is given in polar coordinates by the inverse transform of (4).

$$\begin{aligned} &g(r\cos\varphi, r\sin\varphi),\\ &= \int_0^{2\pi}\int_0^{\infty} RF(R,\theta)\exp(i2\pi rR\cos(\theta-\varphi))\,dR\,d\theta,\\ &x = r\cos\varphi,\\ &y = r\sin\varphi. \end{aligned} \tag{5}$$

The use of (4) in (5) yields

$$g(x,y) = \int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} R\,p(l',\theta)\exp(i2\pi R(l-l'))\,dl'\,d\theta\,dR, \tag{6}$$

where

$$l = r\cos(\theta-\varphi).$$

By reversing the order of integration and using the substitution

$$a = l - l',$$

we obtain

$$g(x,y) = \int_0^{2\pi}\int_{-\infty}^{\infty} p(l',\theta)\int_0^{\infty} R\exp(i2\pi aR)\,dR\,dl'\,d\theta. \tag{7}$$

It is at this point that we depart from the standard derivation by evaluating the inner integral in (7) as it stands without the usual alteration in the limits.

This inner integral is equivalent to

$$I = \int_{-\infty}^{\infty} \cup(R)R\exp(i2\pi aR)\,dR, \tag{8}$$

where $U(R)$ is the Heaviside unit step function. The integral in (8) is evaluated as [4].

$$I = -\frac{i}{4\pi}\frac{d\delta(a)}{da} - \frac{1}{(2\pi a)^2}. \tag{9}$$

When (9) is inserted into (7), then (2) and (3) are obtained. This concludes the proof.

In parallel beam tomography, in principle,

$$p(l', \theta + \pi) = p(-l', \theta), \tag{10}$$

and

$$\left.\frac{dp(l', \theta + \pi)}{dl'}\right|_{l' = x\cos(\theta+\pi) + y\sin(\theta+\pi)} = -\left.\frac{dp(l', \theta)}{dl'}\right|_{l' = x\cos\theta + y\sin\theta}. \tag{11}$$

From (11) we can see that $\frac{dp}{dl'}$ is an odd function about $\pi$. Consequently, (3) is zero when the data are perfect which, of course, due to experimental conditions, is never true. Thus $g_2(x, y)$ is almost always nonzero and is required in cross and autocorrelation operations. The size of $g_2(x, y)$ can also be used as an indicator of the efficacy of any methods that may be applied in order to correct for flaws in the original data.

## SECTION 3

The imaginary component, $g_2(x, y)$, in the tomographic reconstruction of the two-dimensional attenuation function, $f(x, y)$, for the parallel beam case has been derived (3). This function may be used to serve two purposes. When the signal-to-noise ratio is high, $g_2$ can be used as an indicator of structural problems (sampling irregularities, undesired nonlinearities, image motion, etc.). When noise is a significant factor, then $g_2$ becomes a distinctly stochastic variable and its identification as the imaginary part of the reconstruction allows it to be used in complex correlation studies. Since a large body of literature exists concerning statistical studies of this sort, it may now be that the statistical study of tomographic systems can be pursued more accurately. The circumstances and conditions under which $g_2$ actually exists in individual cases needs to be carefully examined, however [5].

Because of the symmetry exhibited in (10), it is possible to compute $g_1(x, y)$, the real part of the reconstruction, within a multiplicative constant using only data gathered for $\theta \in [0, \pi)$. In clinical practice, in order to minimize patient x-ray dosage and/or exposure time, parallel beam machines may be operated in this manner. This, of course, will make $g_2$ unavailable because it needs data gathered for $\theta \in [0, 2\pi)$. The possible utility of $g_2$ in furthering deeper analyses of reconstructions, however, may prompt users of these parallel beam machines to halve the strength of the x-rays applied to their patients while gathering data over the full circle.

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