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Broadcasting with linearly bounded transmission faults ☆

L. Gąsieniec [a,1], A. Pelc [b,*]

[a] *Instytut Informatyki, Uniwersytet Warszawski, Banacha 2, 02-097 Warszawa, Poland*
[b] *Département d'Informatique, Université du Québec à Hull, Hull, Québec Canada J8X 3X7*

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Abstract

We consider broadcasting with a linearly bounded number of transmission failures. For a constant parameter $0 < \alpha < 1$ we assume that at most αi faulty transmissions can occur during the first i time units of the communication process, for every natural number i. Every informed node can transmit information to at most one neighbor in a unit of time. Faulty transmissions have no effect. We investigate worst-case optimal non-adaptive broadcasting time under this fault model, for several communication networks. We show, e.g., that for the n-node line network this time is linear in n, if $\alpha < 1/2$, and exponential otherwise. For the hypercube and the complete graph, broadcasting in the linearly bounded fault model can be performed in time logarithmic in the number of nodes. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

In broadcasting, information originally held in one node of the communication network (called the source) has to be transmitted to all other nodes. Two communication models studied in the literature are the *1-port* or *whispering* model and the the *n-port* or *shouting* model (cf. [8, 13]). The first one assumes that every node which already got the source message can transmit it to at most one neighbor in a unit of time and every node can receive information from at most one neighbor in a unit of time. In the shouting model every informed node can inform all its neighbors in a unit of time.

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* Corresponding author. E-mail: pelc@uqah.uquebec.ca.

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Recently, many researchers have studied fault-tolerant broadcasting (and closely related gossiping) [2–7, 9–12, 14, 16–18]. Links or nodes of the network are subject to failures, crash failures being the most frequently considered type: a faulty link or node does not transmit any messages. Two types of restrictions concerning the number of faulty components have been considered. The *bounded* fault model [2, 10–12, 16] assumes an upper bound on the total number of faults and their worst-case location in the network, while in the *probabilistic* model [3–5, 18] faults are supposed random and independent. In the bounded model all nodes need to be informed, as long as the number of faults does not exceed the imposed bound, while in the probabilistic model broadcasting has to be performed with high probability.

Another important characteristic of faults is their duration. Faults may be either *permanent*, i.e., the fault status of a component does not change during the entire communication process [2–5, 10–12, 14, 16–18], or *transient*, i.e., the same component can be faulty in some time units and fault-free in others [6, 7, 9]. Permanent link faults correspond to the situation when the link is physically damaged while transient failures correspond to individual transmission faults. In case of transient faults, a global upper bound on their number during the entire communication process does not seem to be a realistic assumption, as one would expect more transmission faults if the algorithm runs longer. Therefore, two approaches were adopted: in [7] individual transmission faults were assumed random and mutually independent, while in [6, 9] an upper bound was imposed on the number of faulty transmissions (calls) *in every time unit* and the worst-case location of these faults was assumed. However, the latter approach is meaningful only in the shouting model which was adopted in [6, 9]. In the whispering model even one faulty call in each time unit precludes any broadcasting in the worst case, as this may always be the call made by the source. Thus if we want to consider the worst-case location of transmission faults in the whispering model, another type of restriction is needed.

In this paper we adopt the *linearly bounded* fault model. Given a constant $0<\alpha<1$ we assume that at most αi faulty transmissions can occur during the first i time units of the communication process, for every natural number i. This assumption grasps the idea that more faults are possible when the algorithm runs longer and, on the other hand, uses the worst case rather than the random approach. During a fault-free transmission involving a pair of nodes, information can pass in both directions, while faulty transmissions have no effect. Nodes are assumed fault-free. The linearly bounded fault (error) model has been previously used by Pelc [15] and Aslam and Dhagat [1] in the context of searching with errors. The assumption $\alpha<1$ is necessary for broadcasting to be feasible.

In the presence of faults two types of broadcasting algorithms should be distinguished. In *non-adaptive* (also called *oblivious*) algorithms all transmissions are scheduled in advance, while in *adaptive* algorithms nodes can decide which neighbor to call next, based on success or failure of preceding transmissions. Non-adaptive algorithms are usually less efficient, due to lack of flexibility, but are easier to implement. They have been widely studied in the literature (cf. [2, 3, 10–12]). In this paper we restrict attention to non-adaptive algorithms. We also assume that they are *synchronous* (processors use a global clock measuring time units). One step of the algorithm takes one unit of time.

For a fixed parameter $0<\alpha<1$, a given network $\mathcal{N}$ and a source s, a broadcasting algorithm is called *α-safe* if it broadcasts information from the source s to all nodes, whenever the number of faulty transmissions during the algorithm execution satisfies the above linearly bounded assumption with parameter α. By $B(\mathcal{N},\alpha,s)$ we denote the least worst-case running time of non-adaptive α-safe broadcasting from the source s. A non-adaptive α-safe broadcasting algorithm running in worst-case time $B(\mathcal{N},\alpha,s)$, for any source s, is called *optimal*. The maximum of $B(\mathcal{N},\alpha,s)$ over all sources s is denoted by $B(\mathcal{N},\alpha)$ and is called *α-safe broadcasting time* of the network $\mathcal{N}$.

Networks whose α-safe broadcasting time is linear in their fault-free broadcasting time can be considered robust with respect to linearly bounded transmission faults, while those for which α-safe broadcasting time dramatically exceeds broadcasting time without faults, are vulnerable to faulty transmissions. Our goal is to establish α-safe broadcasting time for important communication networks and to find out which of them are robust and which are vulnerable to faults. We first consider very simple networks: the line and the "star" (the tree of diameter 2). The star turns out to be very vulnerable to faulty transmissions: for any positive α, its α-safe broadcasting time is exponential in its size. For the line we prove the surprising result that its α-safe broadcasting time is linear in its length if $\alpha<1/2$ and is exponential otherwise. More generally, for trees of bounded degree d, α-safe broadcasting time is linear in their diameter, for $\alpha<1/d$. It is always exponential in the maximum degree of the tree and exponential in its diameter for $\alpha\geqslant 1/2$. On the other side of the spectrum, we show that hypercubes and complete graphs are robust with respect to transmission faults: their α-safe broadcasting time is logarithmic in their size, for any $\alpha<1$.

The paper is organized as follows. In Sections 2 and 3 we analyze α-safe broadcasting for the line and for the star. In Section 4 we observe that trees are not robust with respect to linearly bounded transmission failures when $\alpha\geqslant 1/2$ and show that rings are robust for any $\alpha<1$. In Section 5 we investigate α-safe broadcasting time for hypercubes and complete graphs. Section 6 contains conclusions and open problems.

2. The line

In this section we consider α-safe broadcasting on the line of length n. We present a non-adaptive linear algorithm for the case $\alpha<1/2$ and show that for $\alpha\geqslant 1/2$ any non-adaptive α-safe broadcasting takes time $\Omega((1/(1-\alpha))^n)$. For this range of the parameter α a matching upper bound $\mathcal{O}((1/(1-\alpha))^n)$ is also obtained. Throughout this section, L_n denotes the line with nodes $v_0,v_1,\ldots,v_n$ and links $l_1,\ldots,l_n$.

2.1. $\alpha < \frac{1}{2}$

Assume that the source is in the leftmost node v_0. Consider the following algorithm.

Algorithm Odd–Even
begin
In odd (even) steps all pairs of nodes joined by odd (even) numbered links communicate.
end.

Lemma 2.1. *Algorithm Odd–Even performs broadcasting in L_n in time at most $n/(1-2\alpha)$, for $\alpha < 1/2$.*

Proof. Assume that before a given step of the algorithm v_k is the rightmost informed node. Node v_{k+1} becomes informed in this step if link l_{k+1} is used in communication in this step and the transmission along this link is fault-free. Thus, any single transmission fault can cause a delay of at most two steps in the broadcasting process. One step of delay is directly caused by the failure and the second one is caused by the fact that link l_{k+1} is not used in communication in the next step. Let T be the worst-case shortest time in which Algorithm Odd–Even broadcasts in L_n in an α-safe way. Since at most $\lfloor \alpha T \rfloor$ transmission failures can occur during the broadcasting process, every failure causes a delay of at most two steps and the message has to traverse n links, the following inequality holds:

$$n + 2\lfloor \alpha T \rfloor \geqslant T.$$

This implies

$$T \leqslant \frac{n}{1-2\alpha}. \qquad \square$$

If the source is in an interior node, the above lemma holds as well. This implies the following result.

Theorem 2.1. *$B(L_n, \alpha) = \mathcal{O}(n)$, for any fixed $0 < \alpha < 1/2$.*

2.2. $\alpha \geqslant \frac{1}{2}$

α-safe broadcasting time of the line dramatically changes in the case $\alpha \geqslant 1/2$. For this range of the parameter α we show an exponential lower bound $\Omega((1/(1-\alpha))^n)$ on the running time of all non-adaptive α-safe broadcasting algorithms on L_n. Then we present such an algorithm running in time $\mathcal{O}((1/(1-\alpha))^n)$.

In order to establish the lower bound we use an adversary argument. Consider any non-adaptive broadcasting algorithm. It can be viewed as a sequence of matchings whose links are used in communication in a given step. Define the *adversary's account* $\mathcal{A}$ which changes during the algorithm execution in the following way. In the begining of the algorithm $\mathscr{A}$ is set to 0. After every step of the algorithm $\mathscr{A}$ is increased by 1. Whenever the adversary uses a transmission fault in a given step, the account $\mathscr{A}$ is decreased by $1/\alpha$. This corresponds to the fact that the adversary has to wait at least $1/\alpha$ time units (i.e. "earn" $1/\alpha$ units on the account) before "spending" one failure. The adversary can place failures in an arbitrary way, as long as the account remains non-negative at all times. Define the *head* to be the rightmost informed node.

Lemma 2.2. *There exists an adversary for which every move of the head after the first step of the algorithm increases $\mathscr{A}$ at least $1/(1-\alpha)$ times.*

Proof. After the first step of the algorithm, $\mathscr{A}$ has value 1. Assume that after a given step t_0 the account $\mathscr{A}$ is positive and v_k is the head. Consider t steps of the algorithm following step t_0. Denote by $w(t)$ the number of steps among those t in which link l_{k+1} is used for communication (call them *white* steps) and let $b(t)=t-w(t)$ (call these remaining steps *black* steps). Consider inequalities

$$w(t)\leqslant\alpha(\mathscr{A}+t) \quad \text{and} \quad b(t)\leqslant\alpha(\mathscr{A}+t). \tag{1}$$

As long as both of them hold, the adversary can put faults always on link l_{k+1} preventing the head from moving right. Thus, any α-safe algorithm has to violate one of those inequalities. Let t_1 be the least positive integer for which one of the inequalities is violated. First assume that $w(t_1)>\alpha(\mathscr{A}+t_1)$. In this case we describe the behavior of the adversary as follows: in steps $t_0+1,\dots,t_0+t_1$ a failure is placed on link l_{k+2} whenever this link is used. In this way, after step t_0+t_1 the head can move by at most one, to node v_{k+1}. Since the second inequality holds in all these steps, this is a legitimate behavior of the adversary. Denote $W=w(t_1)$ and $B=b(t_1)$. We have $W>\alpha(\mathscr{A}+t_1)=\alpha(\mathscr{A}+W+B)$, thus

$$W-\frac{\alpha}{1-\alpha}B>\frac{\alpha}{1-\alpha}\mathscr{A}. \tag{2}$$

Notice that

$$\frac{1-\alpha}{\alpha}\leqslant\frac{\alpha}{1-\alpha} \quad \text{for } \alpha\geqslant\tfrac{1}{2}. \tag{3}$$

During t_1 steps $\mathscr{A}$ is changed to at least $\mathscr{A}+W+B-(B/\alpha)$, because the account $\mathscr{A}$ is increased by $t_1=W+B$ and decreased by using at most B transmission failures.

The final value of the account after step t_0+t_1 is at least

$$\mathscr{A}+W+B-\frac{B}{\alpha}=\mathscr{A}+W-\frac{1-\alpha}{\alpha}B.$$

Applying inequalities (3) and (2) we get

$$\mathscr{A}+W-\frac{1-\alpha}{\alpha}B\geqslant\mathscr{A}+\left(W-\frac{\alpha}{1-\alpha}B\right)\geqslant\mathscr{A}+\frac{\alpha}{1-\alpha}\mathscr{A},$$

thus finally

$$\mathscr{A} + W + B - \frac{B}{\alpha} \geqslant \frac{1}{1-\alpha}\mathscr{A}. \tag{4}$$

In steps $t_0+1,\dots,t_0+t_1$ the head moved by at most 1 and the value of the account increased at least by the factor $1/(1-\alpha)$.

The case when the second of inequalities (1) is violated first, is handled analogously. In this case, the adversary blocks the link l_{k+1} whenever it is used and the head does not move at all, while the value of the account increases as before. □

After the first step of the algorithm the value of $\mathscr{A}$ is 1. By Lemma 2.2, when the head gets to v_n, this value increases to $\Omega((1/(1-\alpha))^n)$. This means that broadcasting time is also $\Omega((1/(1-\alpha))^n)$ because at each step the value of $\mathscr{A}$ can increase by at most 1. This proves the following result.

Theorem 2.2. $B(L_n,\alpha)=\Omega((1/(1-\alpha))^n)$, *for any constant* $\alpha\geqslant 1/2$.

We now describe an α-safe broadcasting algorithm with running time $\mathcal{O}((1/(1-\alpha))^n)$, for any constant $\alpha\geqslant 1/2$. First suppose that the source is in node v_0. Let t_k, for $k=1,\dots,n$, be integers defined as follows:

$$t_0=0,\quad t_1=1,\quad \text{and}\quad t_k=\lfloor\alpha(t_1+\cdots+t_k)\rfloor+1\quad \text{for } k>1. \tag{5}$$

Algorithm Exponential
begin
for $j:=1$ **to** n **do**
  if j is odd
    then in steps $t_{j-1}+1,\dots,t_j$ all pairs of nodes
    joined by odd numbered links communicate
    else in steps $t_{j-1}+1,\dots,t_j$ all pairs of nodes
    joined by even numbered links communicate
end.

First observe that Algorithm Exponential is α-safe. After t_1 steps node v_1 is informed. Assume that after $t_1+\cdots+t_k$ steps node v_k is informed. In steps $t_1+\cdots+t_k+1,\dots,t_1+\cdots+t_k+t_{k+1}$ the link l_{k+1} is used for communication. Since $t_{k+1}>\alpha(t_1+\cdots+t_k+t_{k+1})$ the adversary cannot block this link in all those steps and, consequently, after $t_1+\cdots+t_{k+1}$ steps node v_{k+1} becomes informed.

Since

$$t_{k-1}=\lfloor\alpha(t_1+\cdots+t_{k-1})\rfloor+1\quad\text{and}\quad t_k=\lfloor\alpha(t_1+\cdots+t_k)\rfloor+1,$$

we get

$$t_k=\lfloor\alpha(t_1+\cdots+t_k)\rfloor+1\leqslant\lfloor\alpha(t_1+\cdots+t_{k-1})\rfloor+1+\alpha t_k+1$$

and hence

$$t_k \leqslant t_{k-1} + \alpha t_k + 1. \tag{6}$$

This implies $t_n = \mathcal{O}((1/(1-\alpha))^n)$ and hence $T_n = t_1 + \cdots + t_n = \mathcal{O}((1/(1-\alpha))^n)$. If the source is in an interior node, the same upper bound on time remains valid. Hence, we get the following theorem.

Theorem 2.3. $B(L_n, \alpha) = \mathcal{O}((1/(1-\alpha))^n)$, *for any constant* $\alpha \geqslant 1/2$.

3. The star

Define the star S_n to be the tree with central node v_0 and nodes $v_1, \ldots, v_n$ adjacent to it. Let l_i be the link joining v_0 and v_i. We show that any non-adaptive α-safe broadcasting algorithm for S_n has running time $\Omega((1/(1-\alpha))^n)$, for any constant $0<\alpha<1$. We also show an algorithm running in time $\mathcal{O}((1/(1-\alpha))^n)$.

Without loss of generality, we may assume that the central node v_0 is the source. (If a leaf is the source then after the first step the central node is informed and the situation is the same as if v_0 were the source.) Consider any non-adaptive α-safe broadcasting algorithm $\mathscr{B}$. Let t_i^k be the number of steps in which link l_i is used for communication during the first k steps of algorithm $\mathscr{B}$. Notice that for each $i = 1, \ldots, n$, there must exist a positive integer k such that $t_i^k > \alpha k$. Otherwise the adversary could always preclude informing node v_i. Let $k_i = \min\{k : t_i^k > \alpha k\}$. Notice that in step k_i of algorithm $\mathscr{B}$ the link l_i is used, because of minimality of k_i. This implies $k_i \neq k_j$ whenever $i \neq j$. Thus we may renumber all leaves of S_n in increasing order of k_i and assume, from now on, that $k_i < k_j$ for $1 \leqslant i < j \leqslant n$.

Since $k_i \geqslant k_j$, for all $j = 1, \ldots, i$, we get $k_i \geqslant t_1^{k_1} + \cdots + t_i^{k_i}$. Since $t_i^{k_i} > \alpha k_i$ we get

$$t_i^{k_i} > \alpha(t_1^{k_1} + \cdots + t_i^{k_i}).$$

Since $t_1^{k_1} = 1$ we obtain $t_n^{k_n} = \Omega((1/(1-\alpha))^n)$. There are at least $t_n^{k_n}$ steps of the algorithm in which link l_n is used, hence the running time of algorithm $\mathscr{B}$ must be in $\Omega((1/(1-\alpha))^n)$. We have proved:

Theorem 3.1. $B(S_n, \alpha) = \Omega((1/(1-\alpha))^n)$, *for any constant* $0<\alpha<1$.

We conclude this section by presenting an α-safe broadcasting algorithm for S_n with running time $\mathcal{O}((1/(1-\alpha))^n)$. This is a modification of Algorithm Exponential. Let t_i have the same meaning as in the formulation of this algorithm.

Algorithm Star
begin
for $j := 1$ **to** n **do**
  in steps $t_{j-1} + 1, \ldots, t_j$ node v_0 communicates with v_j
end.

The argument that Algorithm Star is α-safe is similar as for Algorithm Exponential. Its running time is evaluated as before to be in $\mathcal{O}((1/(1-\alpha))^n)$. This proves the following result.

Theorem 3.2. *$B(S_n,\alpha)=\mathcal{O}((1/(1-\alpha))^n)$, for any constant $0<\alpha<1$.*

4. Trees and rings

Techniques and results from the two previous sections can be applied in the context of general trees. First consider trees with maximum degree bounded by a constant d and assume that $\alpha<1/d$. Partition all links of the tree into d disjoint matchings $m_1,\ldots,m_d$ and consider the following generalization of Algorithm Odd–Even on the line: in time unit i all pairs of nodes joined by a link from $m_{i \bmod d}$ communicate. If D is the diameter of the tree, this algorithm performs fault-free broadcasting in time at most dD. Every transmission failure causes delay at most d and in time t there are at most αt failures. Hence, α-safe broadcasting time T for the tree satisfies the inequality $T\leqslant dD+\alpha Td$ which implies $T\leqslant dD/(1-\alpha d)$. This proves the following result.

Theorem 4.1. *For any tree $\mathcal{T}$ with maximum degree bounded by a constant d and any constant $\alpha<1/d$, $B(\mathcal{T},\alpha)=\mathcal{O}(D)$, where D is the diameter of $\mathcal{T}$.*

Thus, for small values of parameter α, bounded degree trees are robust with respect to linearly bounded transmission faults: their α-safe broadcasting time is linear in fault-free broadcasting time. However, for $\alpha\geqslant 1/2$, trees become vulnerable to transmission failures. Theorem 2.2 implies that α-safe broadcasting time is exponential in the diameter of the tree, in this range of parameter values. On the other hand, in view of Theorem 3.2, this time is exponential in the maximum degree of the tree, for any constant $0<\alpha<1$. Hence, if $\alpha\geqslant 1/2$, α-safe broadcasting time of any tree largely exceeds its fault-free broadcasting time.

What are the sparsest networks which are robust with respect to transmission faults for any $0<\alpha<1$? It turns out that rings have this characteristic. Consider an n-node ring R_n with nodes $v_0,\ldots,v_{n-1}$ and links $l_0,\ldots,l_{n-1}$, link l_i joining v_i with $v_{(i+1)\bmod n}$. Let v_0 be the source. First suppose that n is even. Consider the analog of Algorithm Odd–Even on the line: in odd (even) steps all links of the ring with odd (even) indices are used for communication. After step 1 the informed nodes are v_0 and v_n. Fix an even positive integer j and let $v_a,v_{a+1},\ldots,v_0,\ldots,v_b$ be the maximal segment of informed nodes before step j. Let $l'=l_{a-1}$ and $l''=l_b$. Call two consecutive steps, j and $j+1$, a *phase*. (Thus, the first phase consists of steps 2 and 3.) During phase $\{j,j+1\}$ each of links l',l'' is used once for communication. Thus, if x is the number of nodes which got the message in a given phase and f is the number of faults used in this phase then $x+f\geqslant 2$. The maximum number of faults that can occur during k phases is $\alpha(2k+1)$ because one step was done before the first phase and no failure could occur in this step. Hence, the number of nodes informed during k phases is at least $2k - \alpha(2k+1)$. Whenever

$$2k - \alpha(2k+1) \geqslant n-2,$$

broadcasting is completed after time $T = 2k+1$. (Two nodes were informed after the first step, before the first phase.) The latter inequality is satisfied for

$$k = \left\lceil \frac{(n-1)/(1-\alpha)-1}{2} \right\rceil,$$

hence α-safe broadcasting is completed in time at most

$$2\left\lceil \frac{(n-1)/(1-\alpha)-1}{2} \right\rceil + 1.$$

If n is odd, the above algorithm can be slightly modified. In the first step, the link l_0 is used and node v_1 gets the message. Then in even steps we use links with positive even indices and in odd steps links with odd indices, never again using l_0. The analysis can be carried out as before. This proves the following theorem.

Theorem 4.2. $B(R_n, \alpha) = \mathcal{O}(n)$, *for any constant* $0 < \alpha < 1$.

5. The hypercube and the complete graph

Although, as we have shown, linearly bounded transmission faults increase broadcasting time of rings by only a constant factor, for any value of α, rings are not good networks for broadcasting, even without faults: their broadcasting time is linear in their size. This section is devoted to the study of networks for which α-safe broadcasting can be performed fast. We investigate α-safe broadcasting time for hypercubes and complete graphs. We show that in both cases this time is logarithmic in the number of nodes. In the case of the hypercube we obtain the exact value of α-safe broadcasting time and give an optimal algorithm. For the complete graph on n nodes the running time of our broadcasting algorithm is larger than optimal by at most $\mathcal{O}(\log\log n)$.

5.1. The hypercube

We denote by H_r the r-dimensional hypercube and fix a labeling $1, \ldots, r$ of its dimensions. For any natural $t \geqslant r$ consider the following broadcasting algorithm in H_r.

Algorithm Cyclic
begin
In time unit $i := 1, \ldots, t$ every node communicates with its neighbor in dimension $i \bmod r$.
end.

Lemma 5.1. *Algorithm Cyclic achieves broadcasting to all nodes of H_r in the presence of at most k transmission faults in time $t=r+k$.*

Proof. We prove the lemma by induction on r. For $r=1$ and $r=2$ it can be checked directly. Assume $r \geqslant 3$. For $k=1$ the lemma is obvious, so assume $k>1$. Without loss of generality, we may assume that the first transmission from the source (along dimension 1) is fault-free: otherwise the number of faults and time both decrease by 1. Split H_r into two copies of H_{r-1} along dimension 1. Denote by L the copy containing the source and by R the other copy. Thus, after the first time unit, each of L and R contains an informed node. Define a *window* to be a time segment of length r in which transmissions in Algorithm Cyclic are scheduled along consecutive dimensions $2,3,\ldots,r,1$. The number of windows during time $t=r+k$ is $x=\lfloor (r+k-1)/r \rfloor$. Since $r \geqslant 3$ and $k>1$, we have $x \leqslant k/2$.

Let a be the number of transmission faults within L, b the number of transmission faults within R and z the number of transmission faults along dimension 1. First, assume that $a \leqslant k-x$ and $b \leqslant k-x$. Ignore temporarily time units devoted to dimension 1 in the algorithm. By the inductive assumption all nodes of L and all nodes of R are informed after $(r-1)+(k-x)$ time units. At most $x+1$ time units of Algorithm Cyclic are devoted to transmissions along dimension 1, hence both L and R become informed after a total time $(r-1)+(k-x)+(x+1)=r+k$. Next, suppose that one of the numbers a or b exceeds $k-x$. Without loss of generality suppose that $b>k-x$. Hence, $a<x-z$. Consequently, there exists a window in which all transmissions within L are fault-free, followed by a time unit j devoted to dimension 1 with all transmissions fault-free as well. During time units of this window, L becomes informed and in time unit j the information is passed to all nodes in R. By the inductive assumption broadcasting is completed after time at most $(r-1)+a+(x+1) \leqslant r+2x \leqslant r+k$. □

We are now able to establish the exact value of α-safe broadcasting time for the hypercube.

Theorem 5.1. $B(H_r,\alpha)=\lfloor \alpha/(1-\alpha)(r-1) \rfloor + r$, *for any* $0<\alpha<1$.

Proof. We first prove the lower bound. Consider any non-adaptive broadcasting algorithm. After the first $r-1$ time units there remain some uninformed nodes. Pick one such node v. Assume that all transmissions involving node v, during x consecutive time units $r, r+1, \ldots, r+x-1$, are faulty. In order to satisfy the linear bound on the number of faults it suffices to guarantee $x/(r-1+x) \leqslant \alpha$. This is satisfied for $x=\lfloor \alpha/(1-\alpha)(r-1) \rfloor$. At least one more time unit is needed to inform v, thus the total time is at least $\lfloor \alpha/(1-\alpha)(r-1) \rfloor + r$.

In order to prove the upper bound we show that Algorithm Cyclic achieves broadcasting in time at most $\lfloor \alpha/(1-\alpha)(r-1) \rfloor + r$, for any adversary satisfying the linear bound with parameter α. Let $T=r+y$ be the minimum time in which this can be done. It follows that time $T'=r+y-1$ is too short. Hence, in time T' more than $y-1$ faults can occur: otherwise, in view of the previous lemma, the hypercube could be informed in time T' even against a stronger adversary which can place faults arbitrarily. It follows that in time T' at least y faults can occur. This means that $y \leq \alpha T' = \alpha(r+y-1)$ and, consequently, $y \leq \lfloor \alpha/(1-\alpha)(r-1) \rfloor$. Finally, we get $T \leq \lfloor \alpha/(1-\alpha)(r-1) \rfloor + r$, which concludes the proof. □

5.2. The complete graph

We now discuss α-safe broadcasting time for the complete graph on n nodes. If n is a power of 2 this time is equal to that for the hypercube on n nodes and follows from our preceding considerations. For other values of n, however, a modified approach is needed. Fix the parameter $0<\alpha<1$ and let $r = \lfloor \log n \rfloor$. Let b be the number of digits 1 in the binary representation of n and let $q = \lceil \log b \rceil$. Consider the hypercube H_q, called *basic*. Represent n as a sum $2^{x_1} + \cdots + 2^{x_s}$, where $s = 2^q$ and $x_i \leq r$ are natural numbers, for all $i = 1, \ldots, s$. This partition is possible in view of $s \geq b$. At consecutive nodes of H_q attach a copy of H_{x_i}, for $i = 1, \ldots, s$.

We now construct the following broadcasting algorithm. First, apply Algorithm Cyclic to the basic hypercube, running in time

$$T_0 = B(H_q, \alpha) = \left\lfloor \frac{\alpha}{1-\alpha}(q-1) \right\rfloor + q.$$

By Theorem 5.1 this algorithm is α-safe and, hence, all nodes of the basic hypercube become informed. Then, run Algorithm Cyclic in all attached hypercubes in parallel, for T_1 time units, where the value of T_1 is determined below. The aim is to inform all nodes in all attached hypercubes. The total running time of the algorithm is $T_0 + T_1$. The maximum number of faults in the second phase can be $\lfloor \alpha(T_0 + T_1) \rfloor$. Hence, by Lemma 5.1, it suffices to take $T_1 = \lfloor \alpha(T_0 + T_1) \rfloor + r$. Thus, we get $T_1 \leq \alpha(T_0 + T_1) + r$ which implies $T_1 \leq \alpha/(1-\alpha) T_0 + 1/(1-\alpha) r$. It follows that the entire running time of the algorithm is

$$T_0 + T_1 \leq B(H_q, \alpha) + \frac{\alpha}{1-\alpha} B(H_q, \alpha) + \frac{1}{1-\alpha} r.$$

Theorem 5.1 implies that

$$\frac{1}{1-\alpha}(r-1) \leq B(H_r, \alpha) \leq \frac{1}{1-\alpha}(r-1) + 1.$$

Thus, we obtain the following upper bound on the total running time:

$$\begin{aligned} T_0 + T_1 &\leq \frac{1}{1-\alpha}(q-1) + 1 + \frac{\alpha}{1-\alpha}\left(\frac{1}{1-\alpha}(q-1) + 1\right) + \frac{1}{1-\alpha} r \\ &= \frac{1}{1-\alpha} r + \frac{1}{(1-\alpha)^2}(q-1) + \frac{1}{1-\alpha}. \end{aligned}$$

The same argument as in the proof of Theorem 5.1 shows that the number $B(H_r, \alpha)$ is a lower bound on the worst-case running time of any non-adaptive α-safe broadcasting algorithm for n nodes. Thus, for the complete graph K_n we get the following estimates:

$$\frac{1}{1-\alpha}(\lfloor \log n \rfloor - 1) \leqslant B(K_n, \alpha)$$
$$\leqslant \frac{1}{1-\alpha} \log n + \frac{1}{(1-\alpha)^2} \log\log n + \frac{1}{1-\alpha},$$

which prove the following theorem:

Theorem 5.2. $B(K_n, \alpha) = 1/(1-\alpha)\log n + \mathcal{O}(\log\log n)$, *for any constant* $0 < \alpha < 1$.

6. Conclusions

We investigated non-adaptive broadcasting time in various communication networks, under the assumption that at most αi transmission failures can occur in the first i steps of broadcasting, for any natural i and a fixed $0 < \alpha < 1$. It was shown that for bounded degree trees with maximum degree at most d, non-adaptive α-safe broadcasting can be performed in time linear in the diameter of the tree, for any constant $\alpha < 1/d$. Thus, if $\alpha < 1/3$, a complete binary tree is a sparse network in which non-adaptive α-safe broadcasting can be performed in logarithmic time. However, for $\alpha \geqslant 1/2$, trees are not good for non-adaptive α-safe broadcasting: the time required is exponential both in the maximum degree and in the diameter. Rings turned out to be the sparsest networks for which α-safe broadcasting time is linear in fault-free broadcasting time, for any constant $0 < \alpha < 1$. However, in the case of rings, this time is linear in their size. In order to obtain logarithmic α-safe broadcasting time for any constant $0 < \alpha < 1$, we considered hypercubes and complete graphs. This yields an interesting question: what are the sparsest networks for which α-safe broadcasting time is logarithmic for any constant $0 < \alpha < 1$? Is this possible for any networks of bounded degree?

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