COMMUNICATION TO THE EDITOR

An APL function for the approximation of multiplier values in system dynamics models

In *Appl. Math. Modelling* 1979, **3**, 312, a research note by me appeared entitled: 'simulating system dynamics in APL'.

Response by colleagues to the technique described there has centred on the macro-function '*M*' that appears in line 5 of the user-defined *POP* listed on page 313. It is noted in the next to last paragraph on page 313 that:

> 'This function (*M*), once defined, may be utilised reapeatedly in defining system dynamics models such as *POP*'.

The purpose of the paper was to describe how APL could be easily utilized to define models both in terms of a concise notation and the ability to simulate the model directly on a computer supporting an APL interpreter. As such, the APL function *POP* was written in the most straightforward way for comparison to the DYNAMO version of *POP* that also appears on page 313. As with the DYNAMO version, macro routines such as *TABLE*, *LENGTH*, *PLTPER* were not described. However, given that there is only one macro function required to execute the model written in APL (*M*), and in the knowledge that it was not described in the original paper, it should be made available to those researchers wishing to carry out the simulation. The following then illustrates the APL Code that composes *M*:

```
[0]          ∇R←V  M  W
[1]     →((ρV[2;])<CK←V[2;]⍳W)/4
[2]     R←V[1;CK]
[3]     →0
[4]     S←(Z←Z[⍋Z←V[2;],W])⍳W
[5]     XY←V[;V[2;]⍳Z[(S-1),S+1]]
[6]     MI←(XY[1;2]-XY[1;1])÷XY[2;2]-XY[2;1]
[7]     R←((MI×W)-MI×XY[2;1])+XY[1;1]
[8]     ∇
```

For the purposes of clarity, all executable expressions are presented individually rather than in the stacking mode of the earlier paper (multiple individual expressions situated on one line separated by the diamond).

As can be seen in the above listing, in comparison with its use in *POP*, the function *M* is dyadic; that is, it executes with two arguments on either side of the function name. *V* and *W* correspond to *BRFM*/*DRFM* and *F* in *POP*. The function also returns an explicit result in the form of *R* that corresponds to *BR* and *DR* respectively multiplied by .25 and .2 and *P*. With this brief explanation and a copy of the function in his active APL workspace, the researcher is now equipped to define and simulate system dynamics models as presented in the original paper.

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BOOK REVIEW

Finite element approximations of the Navier–Stokes equations
V. Girault and P.-A. Raviart
Springer Verlag, Berlin – Heidelberg – New York 1979, 200 pp., DM25.00, US $13.80

This book includes a presentation of a mathematical foundation and mixed finite element methods for numerical solution of the Stokes problem and of the stationary and time dependent Navier–Stokes equations. The intent of the authors was to provide adequate treatment of recent developments in the rapidly advancing field of mathematical theory and analysis with finite element methods. The rigorous treatment of basic mathematical aspects as well as numerical and discretization aspects of the chosen finite element approaches for solving Navier–Stokes differential equations is fairly comprehensive. The attention which was given to methods of the mixed type, in particular, seems to be one of the most valuable features of the book. Responding to the challenge of some pressing needs, to the delight of physicists and engineers, the book emphasizes to some extent the application of this theory to numerical hydrodynamics. With this book the authors have contributed toward bridging the so-called 'gap' between the aspirations of mathematicians in the mathematical theory of finite element methods and the ambitions of practising scientists. The paper is mathematically well presented, and from a stylistic point of view, the English is what one would expect in a good test. The book is recommended to scientists and to universities and research organizations interested in numerical computations.

The book is well organized to offer a good understanding of the material. With noticeable depth and proficiency, the authors have selected and assembled numerous fragments of the mathematical theory. In addition, they made use of a series of carefully selected results, developed by others and by themselves. It is to the authors' credit that they produced a text which is strongly related, at times, to contributions by others without becoming redundant. The nature of the finite elements themselves and mathematical and variational foundations of the method have been presented. The text

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is filled with developed proofs of approximately 45 theorems, 40 lemmas and a dozen corollaries, as well as many more given without proofs, in addition to numerous hypotheses, definitions and remarks. All are directly related to the Stokes problem or to the Navier–Stokes equations. The proofs are clearly presented and could be comprehended with some effort; however, they are still the hardest part to understand because of the very nature of mathematical development: each proof is related to preceding derivations. With the use of a section on bibliographical notes and numerous references to the original contributions, the interested reader is directed to further proofs and results.

In summary, the book is arranged in five chapters. The presentation of mathematical foundation of the Stokes problem in chapter 1 includes essentials of elliptic boundary value problems, function spaces, decomposition of vector fields, as well as discussions of the analysis of an abstract variational problem and theory of the Stokes problem itself. Chapter 2 presents a numerical solution of the Stokes problem. An abstract approximation result and a method for the solution of this problem is given. Chapter 3 presents a mixed finite element method for solving the Stokes problem which involves a mixed approximation of an abstract problem. An application to the homogeneous Stokes problem is included. Chapter 4 presents the stationary Navier–Stokes equations, a discussion of a class of nonlinear problems, the application to the Navier-Stokes equations as well as a method for approximation of these equations. A mixed method for approximating the Navier–Stokes problem is then given. The mathematical presentation in Chapter 5 centres on the incompressible time-dependent Navier–Stokes equations. It includes a discussion of the continuous problem and a presentation of a one-step method of numerical solution by semi-discretization as well as semi-discretization with a multistep method. Among the treated applications to a wide range of problems are incompressible and viscous flow examples such as stream function and stream function- and velocity-pressure formulations. Complimentary details of specific problems and other examples have also been referenced.

Some topics included in the book will now be considered. Discussions are given in the introductory chapter of the Dirichlet and Neumann's problems for the harmonic and biharmonic operators as well as essentials and results on Sobolev spaces, linear partial differential equations of elliptic type, and special Hilbert spaces used in mechanics problems and in incompressible flow. An abstract framework is constructed which is deemed adaptable for solution of linear boundary value problems such as the Stokes problem. To deal with the constraint, two algorithms are proposed which are deemed useful for the continuous problem and for discretization problems in particular. The existence and uniqueness of the solution of the Stokes system is also established here. In Chapter 2, attention is given to the approximation of an abstract variational problem. The results are applicable to a variety of situations such as the study of mixed and hybrid equilibrium finite element methods for second-order elliptic problems. As an application of this theory, a finite element method based on the formulation of the Stokes problem is constructed here. The difficulties encountered in the practical construction of simple and continuous divergence-free velocity are elucidated. A second-order method is derived by modification of spaces to improve on loss of precision. A regularization algorithm is developed to allow for a reduction of large systems of linear equation to smaller systems. The examples given made use of continuous velocities with a small but non-vanishing divergence.

Chapter 3 includes the development of a method based upon discontinuous but exactly divergence-free functions. The abstract problem studied in the introductory chapter is put now in a weaker setting which leads to a mixed formulation. A mixed approximation is then derived from this formulation. The developed theory is thus used to formulate and approximate the Stokes problem for the two-dimensional case. To allow judging matters in their proper relationship, a nonlinear generalization of the abstract variational problem is put forward in chapter 4. This nonlinear theory is then applied to the solution of stationary Navier–Stokes equations with proposed finite element approximation of the problem in two dimensions. A discussion is given here on discretization with respect to space variables. Finally, the mixed finite element method, developed in the preceding chapter, is adapted to the two-dimensional Navier–Stokes problem with homogeneous boundary conditions. The theoretical background needed for the inclusion of the critical time-variable in a boundary value problem is first discussed in chapter 5. A variational formulation is then derived for the time-dependent Navier–Stokes equations. Proof is given of the existence theorem and a uniqueness result. The concluding two sections of the chapter are devoted to numerical solution of the transient Navier–Stokes equations and to discretization with respect to the time-variable. A first order one step semi-discrete method is analysed for illustration of the type of argument used in dealing with semi-discretization. To overcome such deficiency when higher accuracy is desired with differential equations, semi-discretization with a two-step method has been introduced.

The authors intended to present a self-contained text. However, many a reader will find the material difficult, if he has no previous exposure to Hilbert and Sobolev spaces, for instance, as well as a good knowledge of partial differential equation theory and a familiarity with the Navier–Stokes equations. The book should be well received in academic circles as well as by all who are either interested in advanced aspects of, or work in the field of mathematical theory of finite element methods. Now that this field is sufficiently advanced, the full potential of the method should be realized in fluid mechanics. Because of the sophisticated form and contents of the book, it can be used as a reference on mathematical tools or as a text for an advanced course in applied mathematics.

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