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Asymptotic periodicity of a food-limited diffusive population model with time-delay

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Abstract

In this paper, a general reaction–diffusion food-limited population model with time-delay is proposed. Accordingly, the existence and uniqueness of the periodic solutions for the boundary value problem and the asymptotic periodicity of the initial-boundary value problem are considered. Finally, the effect of the time-delay on the asymptotic behavior of the solutions is given.

Keywords: Time-delay; Asymptotic periodicity; Upper–lower solution method; Existence; Uniqueness; Food-limited population model

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1. Introduction

When the growth limitations are based on the proportion of available resources not utilized, the food-limited model was proposed in [4,7,8] as follows (Model-1):

$$\frac{dN(t)}{dt} = rN(t)\frac{K - N(t)}{K + cN(t)}.$$

Here the population density is denoted by $N(t)$ and the positive constants r and K represent the growth rate of the population and the carrying capacity of the habitat, respectively. The constant $c > 0$ and r/c is the replacement of mass in the population at K. This model also allows incorporation of both environmental and food effects of toxicant stress. Based on the fact that population density may vary spatially as well as temporally, in [2] the authors proposed two types of diffusive models (Model-2 and Model-3):

$$\frac{\partial N(t,x)}{\partial t} - d\Delta N(t,x) = r(x)N(t,x)\frac{K(x) - N(t,x)}{K(x) + c(x)N(t,x)},$$
$$\frac{\partial N(t,x)}{\partial t} - d\Delta N(t,x) = r(x)N(t,x)\frac{K(x) - aN(t,x) - bN(t-\tau,x)}{K(x) + ac(x)N(t,x) + bc(x)N(t-\tau,x)},$$

where Δ denotes the Laplace operator, $d > 0$ denotes the distributive rate. a and b are positive constants, $c(x)$ is positive and Hölder continuous in x. The time-delay term is induced on the assumption that a growing population requires more food (growth and maintenance) than a saturated one (maintenance only). In [2] the existence and uniqueness of a positive steady state solution are established for these two models by using upper–lower methods. It is shown that as long as the magnitude of the instantaneous self-limitation and/or toxicant effects are larger than that of the time delay effects in Model-3, the solutions of both reaction–diffusion systems have the same asymptotic behavior (extinction or converging to the positive steady state solution, depending on the growth rate of the species).

From another direction, the derivation of Model-1 is based on the fact that the population densities rarely converge monotonically to K and usually have a tendency to fluctuate around the equilibrium; Model-1 can be modified by assuming that the average growth rate is a function of the delayed argument $t - \tau$. Detailed arguments are given in [5,8]. Recently, the authors in [1] also take spatial dispersal and environmental heterogeneity into consideration and consider the following model (Model-4):

$$\frac{\partial N(t,x)}{\partial t} - \Delta N(t,x) = r(x)N(t,x)\frac{K(x) - N(t-\tau,x)}{K(x) + c(x)N(t-\tau,x)}.$$

Stability and bifurcations of the steady state solutions to Model-4 are discussed in [1]. Mathematically speaking, Model-4 can be seen as a particular case of Model-3 for the case $a = 0$ and $b = 1$. However, it should be noted that with $a = 0$, the upper and lower solution techniques mentioned above do not apply.

Furthermore, notice that the carrying capacity and the coefficients in the previous models may vary spatially as well as temporally, and may also vary in a seasonal scale with respect to the seasonal variation of the environment, and also take the effect of time-delay into consideration, based on Model-3 in [2], we can get a generalized reaction–diffusion model as follows:

$$\frac{\partial N(t,x)}{\partial t} - d\Delta N(t,x)$$
$$= r(t,x)N(t,x)\frac{K(t,x) - a_0(t,x)N(t,x) - c_0(t,x)N(t-\tau,x)}{K(t,x) + b_0(t,x)N(t,x) + d_0(t,x)N(t-\tau,x)},$$
$$(t,x) \in R^+ \times \Omega, \tag{1.1}$$
$$B[N](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \tag{1.2}$$
$$N(t,x) = N_0(t,x), \quad (t,x) \in [-\tau, 0] \times \bar{\Omega}. \tag{1.3}$$

Here we do not limit b_0, d_0 to be the forms $a_0 c, c_0 c$ as in Model-3, since these limited forms of coefficients are only for convenience of constructing the upper–lower solutions in [2]. Now that the generalized model is constructed, it is very natural to ask some questions about this model. Is the upper–lower method still valid for this initial-boundary value problem? Are there any similar results to those in [2]? It seems plausible that solutions to (1.1)–(1.3) are attracted to a periodic solution. For a smooth function $u(t,x)$, if there exists a smooth periodic function $\theta(t,x)$ such that $\lim_{t\to\infty}[u(t,\cdot) - \theta(t,\cdot)] = 0$ in $C(\Omega)$, then we say the function $u(t,x)$ has asymptotic periodicity. In this paper we will reveal that this property holds for the solution of our problem by using the upper–lower solution method. Here we mention that in [9] the author gives an early introduction to the method of using upper and lower solution to find periodic solution for reaction–diffusion systems, the readers can refer to it.

It is reasonable to choose the distributive rate $d = 1$ due to the transform method of variables in [1]. If we denote $u(t,x) = N(t,x)$, $a(t,x) = a_0(t,x)/K(t,x)$, $b(t,x) = b_0(t,x)/K(t,x)$, $c(t,x) = c_0(t,x)/K(t,x)$, $d(t,x) = d_0(t,x)/K(t,x)$ and $\phi(t,x) = N_0(t,x)$, then problem (1.1)–(1.3) may be rewritten as:

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x)$$
$$= r(t,x)u(t,x)\frac{1 - a(t,x)u(t,x) - c(t,x)u(t-\tau,x)}{1 + b(t,x)u(t,x) + d(t,x)u(t-\tau,x)}, \quad (t,x) \in R^+ \times \Omega, \tag{1.4}$$
$$B[u](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \tag{1.5}$$
$$u(t,x) = \phi(t,x), \quad (t,x) \in [-\tau, 0] \times \bar{\Omega}. \tag{1.6}$$

In this paper we mainly study the initial-boundary value problem (1.4)–(1.6) under the following elementary hypotheses:

(H_1) Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, the boundary condition is given by

$$B[u] = u \quad \text{or} \quad B[u] = \frac{\partial u}{\partial n} + \gamma(x)u.$$

Here, $\gamma(x) \in C^{1+\alpha}(\partial\Omega)$ $(0 < \alpha < 1)$ and $\gamma(x) \geqslant 0$ on $\partial\Omega$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$, $R^+ = (0, \infty)$.

(H_2) The coefficients $r(t,x)$, $a(t,x)$, $b(t,x)$, $c(t,x)$ and $d(t,x)$ are T-periodic in t and Hölder continuous on $R \times \bar{\Omega}$ with $r(t,x) > 0$; $a(t,x) > 0$; $b(t,x) \geqslant 0$, $c(t,x) \geqslant 0$ and $d(t,x) \geqslant 0$. We denote r_1, a_1, b_1, c_1, d_1 and r_2, a_2, b_2, c_2, d_2 to be the minimum and maximum values of r, a, b, c, d on $[0,T] \times \bar{\Omega}$ with $b_2 > 0$, respectively.

(H_3) The initial function $\phi(t,x) \in C^{0,1}([-\tau,0] \times \bar{\Omega})$ is a nonnegative, nontrivial function, which satisfies the compatibility condition, i.e., $B[\phi](0,x) = 0$ on $\partial\Omega$.

To study the initial-boundary value problem (1.4)–(1.6), we first state some results for the following boundary value problem:

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = u(t,x)\frac{m(t,x) - p(t,x)u(t,x)}{n(t,x) + q(t,x)u(t,x)}, \quad (t,x) \in R^+ \times \Omega, \tag{1.7}$$

$$B[u](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \tag{1.8}$$

where $m(t,x), n(t,x), p(t,x)$ and $q(t,x)$ are T-periodic in t and Hölder continuous on $R \times \bar{\Omega}$ with $m(t,x) > 0$, $n(t,x) > 0$, $p(t,x) > 0$ and $q(t,x) \geqslant 0$. We also denote m_1, n_1, p_1, q_1 and m_2, n_2, p_2, q_2 to be the minimum and maximum values of $m(t,x), n(t,x), p(t,x)$ and $q(t,x)$ on $[0,T] \times \bar{\Omega}$. Problem (1.7), (1.8) is the particular case of problem (1.4), (1.5) with $c(t,x) \equiv d(t,x) \equiv 0$, $r(t,x) = m(t,x)/n(t,x)$, $a(t,x) = p(t,x)/m(t,x)$ and $b(t,x) = q(t,x)/n(t,x)$.

Denote

$$f(t,x,u) = \frac{m(t,x) - p(t,x)u(t,x)}{n(t,x) + q(t,x)u(t,x)},$$

then for any $(t,x) \in R^+ \times \bar{\Omega}$, we can check that $f(t,x,u)$ is strictly decreasing in u for $u \geqslant 0$ and $f(t,x,m_2/p_1) \leqslant 0$. Furthermore, as $m(t,x)$, $n(t,x)$, $p(t,x)$, and $q(t,x)$ are all Hölder continuous on $R^+ \times \bar{\Omega}$, we can also check that $f, f_u \in C(R^+ \times \bar{\Omega} \times R^+)$, and there exists a constant α $(0 < \alpha < 1)$ such that $f(\cdot,\cdot,u), f_u(\cdot,\cdot,u) \in C^{\alpha,\alpha/2}(R^+ \times \bar{\Omega})$. So from the results given in [17,18], we get the following lemma.

Lemma 1.1. *The eigenvalue problem*

$$\frac{\partial \varphi(t,x)}{\partial t} - \Delta\varphi(t,x) - \frac{m(t,x)}{n(t,x)}\varphi(t,x) = \sigma(m/n)\varphi(t,x), \quad (t,x) \in R^+ \times \Omega,$$

$$B[\varphi] = 0, \quad (t,x) \in R^+ \times \partial\Omega, \tag{1.9}$$

where φ is T-periodic in t, has a principal eigenvalue $\sigma(m/n)$ with positive eigenfunction.

(1) *If $\sigma(m/n) \geqslant 0$, then the trivial solution* 0 *is globally asymptotically stable in problem* (1.7), (1.8) *with respect to every nonnegative smooth initial function.*
(2) *If $\sigma(m/n) < 0$, then problem* (1.7), (1.8) *admits a positive T-periodic solution $\theta(t,x)$ on Ω which is globally asymptotically stable with respect to every nonnegative, nontrivial smooth initial function.*

We arrange our paper as follows. In Section 2 we will first study the global asymptotic stability of trivial solution $u = 0$ of the initial-boundary value problem (1.4)–(1.6), and then set forth a result about the periodic quasi-solutions of the boundary value problem (1.4), (1.5). The proof of existence and uniqueness of periodic solutions to boundary value problem (1.4), (1.5) will be given in Section 3. The effects of time-delay on the asymptotic behavior will be discussed in Section 4. Some numerical results will be given as applications in Section 5, and some discussion about the problem is given in Section 6.

2. Stability of trivial solution and periodic quasi-solutions

In this section, we mainly discuss the stability of the steady-state solution $u = 0$ and the existence of the periodic quasi-solutions of the initial boundary value problem (1.4)–(1.6). Here we use the upper–lower solution method, and hence the nonlinear term should have some monotone property, which can be ensured by the following lemma.

Lemma 2.1. *For $t \in R^+$, $x \in \Omega$, $0 \leqslant u \leqslant 1/a_1$ and $\xi \geqslant 0$, if*

$$a_2 d_2 - b_1 c_1 \leqslant a_1(c_1 + d_1), \tag{2.1}$$

then for fixed t, x, u, the function $F(t,x,u,\xi) = (1 - au - c\xi)/(1 + bu + d\xi)$ is monotone decreasing in ξ.

Proof. To claim $F(t,x,u,\xi)$ is monotone decreasing in ξ only if $\partial F/\partial\xi \leqslant 0$.

$$\begin{aligned}
\frac{\partial F}{\partial \xi}(t,x,u,\xi) &= \frac{\partial}{\partial \xi}\left(\frac{1 - au - c\xi}{1 + bu + d\xi}\right) \\
&= \frac{1}{(1 + bu + d\xi)^2}\big[-c(1 + bu + d\xi) - d(1 - au - c\xi)\big] \\
&= -\frac{1}{(1 + bu + d\xi)^2}\big[c + d + (bc - ad)u\big].
\end{aligned}$$

If $a_2 d_2 - b_1 c_1 \leqslant 0$, then $\partial F/\partial\xi \leqslant 0$ is trivially satisfied. If $a_2 d_2 - b_1 c_1 > 0$, then the inequalities $0 \leqslant u \leqslant 1/a_1$ and (2.1) imply that $\partial F/\partial\xi \leqslant 0$. Lemma 2.1 is thus proven. □

Remark. Since for fixed $t \in R^+$, $x \in \Omega$ and $\xi \geqslant 0$, it is easy to see $F(t,x,1/a_1,\xi) \leqslant 0$, so $1/a_1$ is an upper bound of u in $R^+ \times \Omega$ according to the following lemma.

In the following we denote $u_\tau = u(t - \tau, x)$ for simplicity.

Lemma 2.2. *Under the condition* (2.1), *if there exist functions $\bar{u}(t,x)$, $\underline{u}(t,x) \in C^{1,2}(R^+ \times \Omega) \cap C^{0,1}([-\tau,+\infty) \times \bar{\Omega})$ (called coupled upper and lower solutions) such that $\bar{u}(t,x) \geqslant \underline{u}(t,x)$ on $[-\tau,+\infty) \times \bar{\Omega}$, and they satisfy the following inequalities*

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial t} - \Delta\bar{u} \geqslant r\bar{u}\frac{1 - a\bar{u} - c\underline{u}_\tau}{1 + b\bar{u} + d\underline{u}_\tau}, \quad (t,x) \in R^+ \times \Omega, \\
&\frac{\partial \underline{u}}{\partial t} - \Delta\underline{u} \leqslant r\underline{u}\frac{1 - a\underline{u} - c\bar{u}_\tau}{1 + b\underline{u} + d\bar{u}_\tau}, \quad (t,x) \in R^+ \times \Omega,
\end{aligned} \tag{2.2}$$

$$B[\bar{u}] \geqslant 0 \geqslant B[\underline{u}], \quad (t,x) \in R^+ \times \partial\Omega, \tag{2.3}$$

$$\bar{u} \geqslant \phi \geqslant \underline{u}, \quad (t,x) \in [-\tau, 0] \times \bar{\Omega}, \tag{2.4}$$

then the initial-boundary value problem (1.4)–(1.6) *has a unique solution $u \in C^{1,2}(R^+ \times \Omega) \cap C^{0,1}([-\tau,+\infty) \times \bar{\Omega})$ with $\bar{u} \geqslant u \geqslant \underline{u}$ on $[-\tau,+\infty) \times \bar{\Omega}$.*

In case $\bar{u}$, $\underline{u}$ satisfy (2.2), (2.3) with $\bar{u} \geqslant \underline{u}$ on $R^+ \times \bar{\Omega}$, we also call $\bar{u}$, $\underline{u}$ upper and lower solutions of problem (1.4), (1.5). For the proof of Lemma 2.2 we refer to the monotone method in [11,16]. Noting the elementary hypotheses (H_2) and (H_3), there always exists a positive number α such that $\alpha \geqslant \phi(t,x)$ on $[-\tau, 0] \times \bar{\Omega}$. Therefore, we can choose α large enough such that α and 0 is a pair of upper and lower solutions of the problem (1.4)–(1.6) according to (2.2)–(2.4). So Lemma 2.2 implies that problem (1.4)–(1.6) has a unique smooth solution $u(t,x)$ on $[-\tau, +\infty) \times \bar{\Omega}$.

Theorem 2.1. *Let the hypotheses (H_1)–(H_3) and the condition* (2.1) *hold. If $\sigma(r) \geqslant 0$, then the trivial solution $u = 0$ is globally asymptotically stable in* (1.4)–(1.6) *with respect to every nonnegative nontrivial initial function $\phi(t,x)$.*

Proof. Let $U(t,x)$ be the solution of the following parabolic problem:

$$\begin{aligned}
&\frac{\partial U(t,x)}{\partial t} - \Delta U(t,x) = r(t,x)U(t,x)\frac{1 - a(t,x)U(t,x)}{1 + b(t,x)U(t,x)}, \quad (t,x) \in R^+ \times \Omega, \\
&B[U](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \\
&U(0,x) = \phi(0,x), \quad x \in \bar{\Omega}. \qquad (2.5)
\end{aligned}$$

As $\phi(0,x) \geqslant 0$ on $\bar{\Omega}$, it is easy to know that $U(t,x) \geqslant 0$ on $R^+ \times \bar{\Omega}$. Define the function $\tilde{U}(t,x)$ as $\tilde{U}(t,x) = \phi(t,x)$ on $[-\tau, 0] \times \bar{\Omega}$ and $\tilde{U}(t,x) = U(t,x)$ on $R^+ \times \bar{\Omega}$. From the comparison results, we know that $\tilde{U}(t,x)$ and 0 is a pair of upper and lower solutions of the initial-boundary value problem (1.4)–(1.6). Therefore, by Lemma 2.2 there exists a unique solution $u(t,x)$ for (1.4)–(1.6) with $0 \leqslant u \leqslant \tilde{U}$ on $[-\tau, +\infty) \times \bar{\Omega}$. According to the case $\sigma(r) \geqslant 0$ in Lemma 1.1, we have

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(\bar{\Omega})} \leqslant \lim_{t\to\infty} \|U(t,\cdot)\|_{C(\bar{\Omega})} = 0.$$

Theorem 2.1 is thus proven. □

For the following boundary value problem:

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = r(t,x)u(t,x)\frac{1 - a(t,x)u(t,x)}{1 + b(t,x)u(t,x)}, \quad (t,x) \in R^+ \times \Omega, \\
&B[u](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \qquad (2.6)
\end{aligned}$$

if the eigenvalue $\sigma(r) < 0$, then Lemma 1.1 ensures the existence of a positive T-periodic solution $\theta_0(t,x)$ on Ω. Moreover, if further

$$\sigma\left(r\frac{1 - c\theta_{0\tau}}{1 + d\theta_{0\tau}}\right) < 0,$$

then the boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u = ru\frac{(1 - c\theta_{0\tau}) - au}{(1 + d\theta_{0\tau}) + bu}, \quad (t,x) \in R^+ \times \Omega, \\
&B[u](t,x) = 0, \quad (t,x) \in R^+ \times \partial\Omega, \qquad (2.7)
\end{aligned}$$

also has a positive T-periodic solution $\Theta(t,x)$ on Ω. It is easy to check that θ_0 and Θ is a pair of upper and lower solutions of the boundary value problem (1.4), (1.5) with $0 \leqslant \Theta \leqslant \theta_0 \leqslant 1/a_1$. For $\Theta \leqslant u \leqslant \theta_0$, if $a_2 d_2 - b_1 c_1 \leqslant a_1(c_1 + d_1)$, then from Lemma 2.1 we know that the monotone method is valid. Moreover, corresponding to the boundary value problem (1.4), (1.5) we have the system below:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta u_1 = r u_1 \frac{1 - a u_1 - c u_{2\tau}}{1 + b u_1 + d u_{2\tau}}, \quad (t,x) \in R^+ \times \Omega, \\ &\frac{\partial u_2}{\partial t} - \Delta u_2 = r u_2 \frac{1 - a u_2 - c u_{1\tau}}{1 + b u_2 + d u_{1\tau}}, \quad (t,x) \in R^+ \times \Omega, \\ &B[u_1] = B[u_2] = 0, \quad (t,x) \in R^+ \times \partial\Omega. \end{aligned} \tag{2.8}$$

If the solution (u_1, u_2) of system (2.8) exists and the components u_1 and u_2 satisfy $u_1 \geqslant u_2$, then we call u_1 and u_2 a pair of quasi-solutions of the boundary value problem (1.4), (1.5). Using the monotone methods and referring to the results in [10,12,19], we get the following results.

Theorem 2.2. *Let hypotheses* (H_1)–(H_3) *and the condition* (2.1) *hold. If* $\sigma(r) < 0$ *and* $\sigma(r(1 - c\theta_{0\tau})/(1 + d\theta_{0\tau})) < 0$, *then the boundary value problem* (1.4), (1.5) *has a pair of ordered T-periodic quasi-solutions* $\bar{\theta}$, $\underline{\theta}$ *that satisfy system* (2.8) *with* $\Theta \leqslant \underline{\theta} \leqslant \bar{\theta} \leqslant \theta_0$ *on* $R^+ \times \bar{\Omega}$. *Moreover, for every nonnegative nontrivial initial function* ϕ, *the time-dependent solution* $u(t,x)$ *of the initial-boundary value problem* (1.4)–(1.6) *satisfies*

$$\liminf_{t \to \infty} \big[u(t,\cdot) - \underline{\theta}(t,\cdot)\big] \geqslant 0 \geqslant \limsup_{t \to \infty} \big[u(t,\cdot) - \bar{\theta}(t,\cdot)\big] \quad \text{in } C(\bar{\Omega}). \tag{2.9}$$

Remark. If the Laplace operator Δ is substituted by a uniformly strong elliptic operator in the initial-boundary value problem (1.4)–(1.6), then Theorems 2.1 and 2.2 also hold true.

3. Existence and uniqueness of periodic solution and the asymptotic behavior

Based on the results in Theorem 2.2, to reveal the asymptotic periodicity of the solution for the initial-boundary value problem (1.4)–(1.6), it needs to be shown that there exists a periodic solution of the boundary value problem (1.4), (1.5), i.e., $\bar{\theta} \equiv \underline{\theta}$. Here we use the methods in [15] to set forth our arguments. From now on, we assume all the conditions of Theorem 2.2 are satisfied and let

$$R = \frac{r}{(1 + b\bar{\theta} + d\underline{\theta}_\tau)(1 + b\underline{\theta} + d\bar{\theta}_\tau)}.$$

According to Theorem 2.2, $\bar{\theta}$ and $\underline{\theta}$ satisfy the system (2.8) and hence

$$\begin{aligned} &\frac{\partial}{\partial t}(\bar{\theta} - \underline{\theta}) - \Delta(\bar{\theta} - \underline{\theta}) \\ &\quad = r\left\{ \frac{\bar{\theta}(1 - a\bar{\theta} - c\underline{\theta}_\tau)}{1 + b\bar{\theta} + d\underline{\theta}_\tau} - \frac{\underline{\theta}(1 - a\underline{\theta} - c\bar{\theta}_\tau)}{1 + b\underline{\theta} + d\bar{\theta}_\tau} \right\} \\ &\quad = R\big\{\bar{\theta}(1 - a\bar{\theta} - c\underline{\theta}_\tau)(1 + b\underline{\theta} + d\bar{\theta}_\tau) - \underline{\theta}(1 - a\underline{\theta} - c\bar{\theta}_\tau)(1 + b\bar{\theta} + d\underline{\theta}_\tau)\big\} \end{aligned}$$

$$= R\big\{\big[(1+d\bar\theta_\tau)\big(1-a(\bar\theta+\underline{\theta})-c\underline{\theta}_\tau\big)-ab\bar\theta\underline{\theta}\big](\bar\theta-\underline{\theta}) + \underline{\theta}(bc\bar\theta-ad\underline{\theta}+c+d)(\bar\theta_\tau-\underline{\theta}_\tau)\big\}. \tag{3.1}$$

Notice that $0 \leqslant \Theta \leqslant \underline{\theta} \leqslant \bar\theta \leqslant \theta_0 \leqslant 1/a_1$ and the condition (2.1) implies $bc\bar\theta - ad\underline{\theta} + c + d \geqslant 0$. We define the numbers $P,\ Q,\ R_1,\ R_2$ by

$$\begin{aligned}
P &= \sup_{(t,x)\in[0,T]\times\bar\Omega} \big\{(1+d\theta_{0\tau})-(2a\Theta+c\Theta_\tau)(1+d\Theta_\tau)-ab\Theta^2\big\};\\
Q &= \sup_{(t,x)\in[0,T]\times\bar\Omega} \big\{\theta_0(bc\theta_0-ad\Theta+c+d)\big\};\\
R_1 &= \inf_{(t,x)\in[0,T]\times\bar\Omega} \left\{\frac{r}{(1+b\theta_0+d\theta_{0\tau})^2}\right\};\\
R_2 &= \sup_{(t,x)\in[0,T]\times\bar\Omega} \left\{\frac{r}{(1+b\Theta+d\Theta_\tau)^2}\right\}.
\end{aligned} \tag{3.2}$$

It is easy to check that $R_1 \leqslant R \leqslant R_2$ and $Q \geqslant 0$. Notice that $\bar\theta \geqslant \underline{\theta}$, the relations in (3.1) imply that

$$\frac{\partial}{\partial t}(\bar\theta-\underline{\theta}) - \Delta(\bar\theta-\underline{\theta}) \leqslant RP(\bar\theta-\underline{\theta}) + RQ(\bar\theta_\tau-\underline{\theta}_\tau). \tag{3.3}$$

In the following, we search for the sufficient conditions associated with different boundary conditions to ensure $\bar\theta \equiv \underline{\theta}$.

Part A. Dirichlet conditions

For the Dirichlet boundary conditions $\bar\theta = \underline{\theta} = 0$ on $\partial\Omega$, since $\bar\theta - \underline{\theta} \geqslant 0$, we multiply (3.3) by $(\bar\theta-\underline{\theta})$, and integrate it over Ω, then the left-hand side and the right-hand side of (3.3) can be defined, respectively,

$$\begin{aligned}
I &= \int_\Omega (\bar\theta-\underline{\theta})\frac{\partial(\bar\theta-\underline{\theta})}{\partial t}\,dx - \int_\Omega (\bar\theta-\underline{\theta})\Delta(\bar\theta-\underline{\theta})\,dx\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega (\bar\theta-\underline{\theta})^2\,dx + \int_\Omega \big|\nabla(\bar\theta-\underline{\theta})\big|^2\,dx,
\end{aligned} \tag{3.4}$$

$$II = P\int_\Omega R(\bar\theta-\underline{\theta})^2\,dx + Q\int_\Omega R(\bar\theta-\underline{\theta})(\bar\theta_\tau-\underline{\theta}_\tau)\,dx. \tag{3.5}$$

From the Poincaré inequality (see [14]),

$$\int_\Omega \big|\nabla(\bar\theta-\underline{\theta})\big|^2\,dx \geqslant \lambda_1 \int_\Omega (\bar\theta-\underline{\theta})^2\,dx,$$

where λ_1 is the principal eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions. Denote by $\|\cdot\|$ the L^2 norm on Ω. Then from (3.4) we get

$$I \geqslant \frac{1}{2}\frac{d}{dt}\|\bar\theta-\underline{\theta}\|^2 + \lambda_1\|\bar\theta-\underline{\theta}\|^2. \tag{3.6}$$

Case 1. If $P \geqslant 0$, noting that $Q \geqslant 0$ and $0 < R_1 \leqslant R \leqslant R_2$, from (3.5) we have

$$\begin{aligned} II &\leqslant PR_2 \int_\Omega (\bar\theta - \underline\theta)^2\,dx + QR_2 \int_\Omega (\bar\theta - \underline\theta)(\bar\theta_\tau - \underline\theta_\tau)\,dx \\ &\leqslant PR_2 \|\bar\theta - \underline\theta\|^2 + QR_2 \|\bar\theta - \underline\theta\| \cdot \|\bar\theta_\tau - \underline\theta_\tau\| \\ &\leqslant \left(PR_2 + \frac{1}{2}QR_2\right)\|\bar\theta - \underline\theta\|^2 + \frac{1}{2}QR_2\|\bar\theta_\tau - \underline\theta_\tau\|^2. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7) leads to

$$\frac{1}{2}\frac{d}{dt}\|\bar\theta - \underline\theta\|^2 \leqslant \left(PR_2 + \frac{1}{2}QR_2 - \lambda_1\right)\|\bar\theta - \underline\theta\|^2 + \frac{1}{2}QR_2\|\bar\theta_\tau - \underline\theta_\tau\|^2. \tag{3.8}$$

Integrating (3.8) with respect to t on $[0, T]$, and noting that $\|\bar\theta - \underline\theta\|^2$ is T-periodic in t, it yields

$$\begin{aligned} 0 &= \int_0^T \left\{\frac{1}{2}\frac{d}{dt}\|\bar\theta - \underline\theta\|^2\right\} dt \\ &\leqslant \left(PR_2 + \frac{1}{2}QR_2 - \lambda_1\right)\int_0^T \|\bar\theta - \underline\theta\|^2\,dt + \frac{1}{2}QR_2 \int_0^T \|\bar\theta_\tau - \underline\theta_\tau\|^2\,dt \\ &= \left[(P+Q)R_2 - \lambda_1\right]\int_0^T \|\bar\theta - \underline\theta\|^2\,dt. \end{aligned} \tag{3.9}$$

If the inequality $(P+Q)R_2 - \lambda_1 < 0$ is satisfied, then from (3.9) we get

$$\int_0^T \|\bar\theta - \underline\theta\|^2\,dt \equiv 0,$$

which implies $\bar\theta \equiv \underline\theta$ on $R^+ \times \bar\Omega$.

Case 2. If $P < 0$, then the same process as in Case 1 reveals that $PR_1 + QR_2 - \lambda_1 < 0$ ensures $\bar\theta \equiv \underline\theta$ on $R^+ \times \bar\Omega$.

From the above arguments we know that the boundary value problem (1.4), (1.5) has periodic solutions under certain conditions. Now if θ^1 is any other solution of the boundary value problem with $\Theta \leqslant \theta^1 \leqslant \theta_0$, then θ^1 and $\underline\theta$ is also a pair of upper and lower solutions of the boundary value problem (1.4), (1.5) and these also satisfy the system (2.8). Applying the same reasoning as previously for $\bar\theta$ and $\underline\theta$, yields $\theta^1 \equiv \underline\theta$ on $R^+ \times \bar\Omega$, provided certain conditions are satisfied. Hence the T-periodic solution of the boundary value problem is also unique.

To ensure the existence of $\bar\theta$ and $\underline{\theta}$, it suffices to check that $\sigma(r) < 0$, $\sigma(r(1-c\theta_{0\tau})/(1+d\theta_{0\tau})) < 0$ and $a_2d_2 - b_1c_1 \leqslant a_1(c_1+d_1)$. As $r \geqslant r(1-c\theta_{0\tau})/(1+d\theta_{0\tau})$, $\sigma(r) \leqslant \sigma(r(1-c\theta_{0\tau})/(1+d\theta_{0\tau}))$, we can get the above conditions provided that

$$r\frac{1-c\theta_{0\tau}}{1+d\theta_{0\tau}} > \lambda_1.$$

We now state some sufficient conditions for the existence and uniqueness of the periodic solution of problem (1.4), (1.5):

$$\begin{aligned}
&(1)\quad P \geqslant 0,\ (P+Q)R_2 < \lambda_1 < r\frac{1-c\theta_{0\tau}}{1+d\theta_{0\tau}}, \qquad a_2d_2 - b_1c_1 \leqslant a_1(c_1+d_1);\\
&\qquad \text{or}\\
&(2)\quad P < 0,\ PR_1 + QR_2 < \lambda_1 < r\frac{1-c\theta_{0\tau}}{1+d\theta_{0\tau}}, \qquad a_2d_2 - b_1c_1 \leqslant a_1(c_1+d_1),
\end{aligned} \tag{3.10}$$

where P, Q, R_1, R_2 are given by (3.2).

Theorem 3.1. *Let hypotheses* (H_1)–(H_3) *hold. If either* (1) *or* (2) *in* (3.10) *is satisfied, then the boundary value problem* (1.4), (1.5) *with zero Dirichlet boundary condition has a unique, smooth, T-periodic solution $\theta(t,x)$ on $R^+\times\Omega$. Moreover, for every nonnegative, nontrivial initial function $\phi(t,x)$, the time-dependent solution $u(t,x)$ of the initial-boundary value problem* (1.4)–(1.6) *has the asymptotic periodicity*:

$$\lim_{t\to\infty}\big[u(t,\cdot)-\theta(t,\cdot)\big] = 0 \quad \text{in } C(\bar\Omega).$$

Part B. Neumann conditions

In case $\gamma(x)\equiv 0$ on $\partial\Omega$, we have the Neumann boundary conditions $\partial\bar\theta/\partial n = \partial\underline{\theta}/\partial n = 0$. In this case, we can search for sufficient conditions which only depend on the coefficients and not on θ_0, Θ. According to (2.2) and (2.3), we can get a pair of coupled upper and lower positive constant solutions k_2, k_1 for the boundary value problem (1.4), (1.5) by solving the following system:

$$\begin{aligned}
&1 - a_1k_2 - c_1k_1 = 0,\\
&1 - a_2k_1 - c_2k_2 = 0.
\end{aligned} \tag{3.11}$$

Thus, if $a_1 > c_2$, we get

$$k_2 = \frac{a_2-c_1}{a_1a_2-c_1c_2}, \qquad k_1 = \frac{a_1-c_2}{a_1a_2-c_1c_2}. \tag{3.12}$$

It is easy to check that $0 < k_1 \leqslant k_2 \leqslant 1/a_1$.

In this case we can set the numbers $P,\ Q,\ R_1,\ R_2$ as:

$$\begin{aligned}
&P = 1 + d_2k_2 - (2a_1+c_1)(1+d_1k_1)k_1 - a_1b_1k_1^2;\\
&Q = k_2(b_2c_2k_2 - a_1d_1k_1 + c_2 + d_2);\\
&R_1 = \frac{r_1}{(1+b_2k_2+d_2k_2)^2}; \qquad R_2 = \frac{r_2}{(1+b_1k_1+d_1k_1)^2}.
\end{aligned} \tag{3.13}$$

As $a_2d_2 - b_1c_1 \leqslant a_1(c_1 + d_1)$, it is easy to check that $Q \geqslant 0$. If $P < 0$, then following the same process as in Part A, we can get

$$\frac{1}{2}\frac{d}{dt}\|\bar{\theta} - \underline{\theta}\|^2 + \big\|\nabla(\bar{\theta} - \underline{\theta})\big\|^2 \leqslant \left(PR_1 + \frac{1}{2}QR_2\right)\|\bar{\theta} - \underline{\theta}\|^2 + \frac{1}{2}QR_2\|\bar{\theta}_\tau - \underline{\theta}_\tau\|^2. \tag{3.14}$$

Integrating (3.14) with respect to t on $[0, T]$, and noting that $\|\bar{\theta} - \underline{\theta}\|^2$ is T-periodic in t, we get

$$\begin{aligned}\int_0^T \big\|\nabla(\bar{\theta} - \underline{\theta})\big\|^2\,dt &\leqslant \left(PR_1 + \frac{1}{2}QR_2\right)\int_0^T \|\bar{\theta} - \underline{\theta}\|^2\,dt + \frac{1}{2}QR_2\int_0^T \|\bar{\theta}_\tau - \underline{\theta}_\tau\|^2\,dt \\ &= (PR_1 + QR_2)\int_0^T \|\bar{\theta} - \underline{\theta}\|^2\,dt.\end{aligned} \tag{3.15}$$

If $PR_1 + QR_2 < 0$, then from (3.15) we get

$$\int_0^T \|\bar{\theta} - \underline{\theta}\|^2\,dt \equiv 0.$$

Hence $\bar{\theta} \equiv \underline{\theta}$ on $R^+ \times \bar{\Omega}$. It is easy to see that this result cannot be acquired for the case $P \geqslant 0$. Similar arguments to those in Part A show that the T-periodic solution is also unique provided $PR_1 + QR_2 < 0$.

As the principal eigenvalue of $-\Delta$ on Ω with zero Neumann boundary condition is $\lambda_1 = 0$, for the existence of $\bar{\theta}$ and $\underline{\theta}$, it suffices to have

$$r_1\frac{1 - c_2k_2}{1 + d_2k_2} > 0, \quad \text{i.e., } c_2k_2 < 1,$$

which can be ensured by insisting $a_1 > c_2$. Similar to above, we get that the sufficient conditions for the existence and uniqueness of the periodic solution of the boundary value problem (1.4), (1.5) are:

$$\begin{aligned}&a_1 > c_2, \qquad a_2d_2 - b_1c_1 \leqslant a_1(c_1 + d_1),\\ &P < 0, \qquad PR_1 + QR_2 < 0,\end{aligned} \tag{3.16}$$

where P, Q R_1 and R_2 are given by (3.13).

Theorem 3.2. *Let hypotheses* (H_1)–(H_3) *hold. If the conditions in* (3.16) *are satisfied, then the boundary value problem* (1.4), (1.5) *with Neumann boundary condition has a unique, smooth, T-periodic solution $\theta(t,x)$. Moreover, for every nonnegative, nontrivial initial function $\phi(t,x)$, the time-dependent solution $u(t,x)$ of the initial-boundary value problem* (1.4)–(1.6) *has the asymptotic periodicity*:

$$\lim_{t\to\infty}\big[u(t,\cdot) - \theta(t,\cdot)\big] = 0 \quad \text{in } C(\bar{\Omega}).$$

Remark. By using this method, we can get similar results for the Robin boundary condition: $\partial u/\partial n + \gamma(x)u = 0$ with $\gamma(x) \neq 0$.

4. Effect of time-delay

In this section, we consider the effect of the time delay on the asymptotic behavior of the solution $u(t,x)$ of the initial-boundary value problem (1.4)–(1.6). We begin our arguments by dealing with the problem with Dirichlet boundary condition. If the eigenvalue $\sigma(r) < 0$, then Lemma 1.1 ensures the existence of the T-periodic solution for the no-delay boundary value problem:

$$\begin{aligned} &\frac{\partial\theta}{\partial t} - \Delta\theta = r\theta\frac{1-(a+c)\theta}{1+(b+d)\theta}, \quad (t,x) \in R^+ \times \Omega, \\ &\theta = 0, \quad (t,x) \in R^+ \times \partial\Omega. \end{aligned} \tag{4.1}$$

For the initial-boundary value problem (1.4)–(1.6), if the initial function satisfies $0 \leqslant \phi(t,x) \leqslant 1/a_1$ on $[-\tau,0] \times \bar{\Omega}$, then the monotone method implies $0 \leqslant u(t,x) \leqslant 1/a_1$ under the condition (2.1). In order to estimate the effect of the time-delay on the asymptotic behavior of the solution, we first make the following definitions:

$$\begin{aligned} &R = \frac{r}{(1+bu+du_\tau)(1+(b+d)\theta)}; \\ &R_1 = \inf_{(t,x)\in D}\left\{\frac{r}{[1+(b+d)/a_1][1+(b+d)\theta]}\right\}; \\ &R_2 = \sup_{(t,x)\in D}\left\{\frac{r}{1+(b+d)\theta}\right\}; \\ &P = \sup_{(t,x)\in D}\bigl\{1 - a\theta(1+d\theta^2) + d\theta(1+c/a_1)\bigr\}; \\ &Q = \max\Bigl\{\sup_{(t,x)\in D}\bigl|\theta(ad\theta^2 - c - d)\bigr|, \sup_{(t,x)\in D}\bigl|\theta(ad\theta^2 - bc/a_1 - c - d)\bigr|\Bigr\}; \\ &S = \max\Bigl\{\sup_{(t,x)\in D}\bigl|\theta(1 - ad\theta^2 + d)\bigr|, \sup_{(t,x)\in D}\bigl|\theta(1 - ad\theta^2 + bc/a_1 + d)\bigr|\Bigr\} \end{aligned} \tag{4.2}$$

with $D = [0,T] \times \bar{\Omega}$. The following relations can be obtained by subtracting the first equation in (4.1) from Eq. (1.4):

$$\frac{\partial}{\partial t}(u-\theta) - \Delta(u-\theta) = r\left\{\frac{u(1-au-cu_\tau)}{1+bu+du_\tau} - \frac{\theta[1-(a+c)\theta]}{1+(b+d)\theta}\right\}. \tag{4.3}$$

So the right hand (denoted by f) of (4.3) can be rewritten as:

$$f = R\bigl\{u(1-au-cu_\tau)\bigl[1+(b+d)\theta\bigr] - \theta\bigl[1-(a+c)\theta\bigr](1+bu+du_\tau)\bigr\},$$

i.e.,

$$
\begin{aligned}
\frac{f}{R} &= (u-\theta) + d\theta(u-u_\tau) - a(u+\theta)(u-\theta) - c(uu_\tau - \theta^2) - abu\theta(u-\theta) \\
&\quad - ad\theta(u^2 - u_\tau\theta) + bcu\theta(\theta - u_\tau) - cdu_\tau\theta(u-\theta) \\
&= \big[1 - a(u+\theta) - abu\theta - cdu_\tau\theta\big](u-\theta) + d\theta(u-u_\tau) \\
&\quad - c(uu_\tau - \theta^2) - ad\theta(u^2 - u_\tau\theta) + bcu\theta(\theta - u_\tau) \\
&= \big[1 - a(u+\theta) - abu\theta - cdu_\tau\theta\big](u-\theta) + d\theta\big[(u-\theta) + (\theta-\theta_\tau) - (u_\tau - \theta_\tau)\big] \\
&\quad - c\big[u_\tau(u-\theta) + \theta(u_\tau - \theta_\tau) - \theta(\theta - \theta_\tau)\big] - ad\theta\big[\theta(u+\theta)(u-\theta) \\
&\quad - \theta^2(u_\tau - \theta_\tau) + \theta^2(\theta - \theta_\tau)\big] + bcu\theta\big[(\theta-\theta_\tau) - (u_\tau - \theta_\tau)\big] \\
&= \big[1 - cu_\tau - a(u+\theta)(1+d\theta^2) - abu\theta + d\theta(1+cu_\tau)\big](u-\theta) \\
&\quad + \theta(ad\theta^2 - bcu - c - d)(u_\tau - \theta_\tau) + \theta(1 - ad\theta^2 + bcu + d)(\theta - \theta_\tau). \qquad (4.4)
\end{aligned}
$$

Keeping in mind the above definitions, if $P < 0$, we can multiply (4.3) by $(u-\theta)$, and integrate it with respect to x on Ω. Then from the left-hand side and the right-hand side of (4.3), respectively, it follows that

$$
I = \int_\Omega \left[\frac{\partial(u-\theta)}{\partial t} - \Delta(u-\theta)\right](u-\theta)\,dx \geqslant \frac{1}{2}\frac{d}{dt}\|u-\theta\|^2 + \lambda_1\|u-\theta\|^2, \qquad (4.5)
$$

$$
\begin{aligned}
II &= \int_\Omega f\cdot(u-\theta)\,dx \\
&= \int_\Omega R\big[1 - cu_\tau - a(u+\theta)(1+d\theta^2) - abu\theta + d\theta(1+cu_\tau)\big](u-\theta)^2\,dx \\
&\quad + \int_\Omega R\theta(ad\theta^2 - bcu - c - d)(u-\theta)(u_\tau - \theta_\tau)\,dx \\
&\quad + \int_\Omega R\theta(1 - ad\theta^2 + bcu + d)(u-\theta)(\theta - \theta_\tau)\,dx \\
&\leqslant \int_\Omega PR(u-\theta)^2 dx + \int_\Omega R\big|\theta(ad\theta^2 - bcu - c - d)\big|\cdot|u-\theta|\cdot|u_\tau - \theta_\tau|\,dx \\
&\quad + \int_\Omega R\big|\theta(1 - ad\theta^2 + bcu + d)\big|\cdot|u-\theta|\cdot|\theta - \theta_\tau|\,dx \\
&\leqslant PR_1\int_\Omega (u-\theta)^2 dx + QR_2\int_\Omega |u-\theta|\cdot|u_\tau - \theta_\tau|\,dx \\
&\quad + SR_2\int_\Omega |u-\theta|\cdot|\theta - \theta_\tau|\,dx \\
&\leqslant PR_1\|u-\theta\|^2 + QR_2\|u-\theta\|\cdot\|u_\tau - \theta_\tau\| + SR_2\|u-\theta\|\cdot\|\theta - \theta_\tau\|
\end{aligned}
$$

$$\leqslant \left(PR_1 + \frac{1}{2}QR_2\delta_1 + \frac{1}{2}SR_2\delta_2\right)\|u-\theta\|^2 + \frac{QR_2}{2\delta_1}\|u_\tau - \theta_\tau\|^2 + \frac{SR_2}{2\delta_2}\|\theta - \theta_\tau t\|^2. \tag{4.6}$$

Here the Hölder and Cauchy inequalities are used and δ_1 and δ_2 are positive numbers which can be chosen freely. Denote $y(t) = \|u(t,\cdot) - \theta(t,\cdot)\|^2$, $\eta(\tau) = \sup_{0\leqslant t\leqslant T}\|\theta(t,\cdot) - \theta(t-\tau,\cdot)\|^2$. Then from (4.5), (4.6) and $P<0$, we have

$$\frac{d}{dt}y(t) \leqslant (2PR_1 - 2\lambda_1 + QR_2\delta_1 + SR_2\delta_2)y(t) + \frac{QR_2}{\delta_1}y(t-\tau) + \frac{SR_2}{\delta_2}\eta(\tau). \tag{4.7}$$

Similarly, for $P \geqslant 0$, we have

$$\frac{d}{dt}y(t) \leqslant (2PR_2 - 2\lambda_1 + QR_2\delta_1 + SR_2\delta_2)y(t) + \frac{QR_2}{\delta_1}y(t-\tau) + \frac{SR_2}{\delta_2}\eta(\tau). \tag{4.8}$$

Let

$$M(\delta_1,\delta_2) = \begin{cases} 2PR_1 - 2\lambda_1 + QR_2\delta_1 + SR_2\delta_2 + \frac{QR_2}{\delta_1}, & P<0; \\ 2PR_2 - 2\lambda_1 + QR_2\delta_1 + SR_2\delta_2 + \frac{QR_2}{\delta_1}, & P\geqslant 0. \end{cases} \tag{4.9}$$

If $M(\delta_1,\delta_2) \neq 0$, then the inequalities (4.7) and (4.8) can be rewritten in the same form, namely,

$$\frac{d}{dt}\xi(t) \leqslant \left(M(\delta_1,\delta_2) - \frac{QR_2}{\delta_1}\right)\xi(t) + \frac{QR_2}{\delta_1}\xi(t-\tau) \tag{4.10}$$

with $\xi(t) = y(t) + \varepsilon(\tau)$ and

$$\varepsilon(\tau) = \frac{SR_2\eta(\tau)}{\delta_2 M(\delta_1,\delta_2)}. \tag{4.11}$$

The differential equation corresponding to (4.10) is

$$\frac{d}{dt}\xi_1(t) = \left(M(\delta_1,\delta_2) - \frac{QR_2}{\delta_1}\right)\xi_1(t) + \frac{QR_2}{\delta_1}\xi_1(t-\tau). \tag{4.12}$$

The characteristic equation associated with (4.12) is

$$\mu = \left(M(\delta_1,\delta_2) - \frac{QR_2}{\delta_1}\right) + \frac{QR_2}{\delta_1}e^{-\tau\mu}. \tag{4.13}$$

Notice that $QR_2/\delta_1 > 0$ (refer to Hayes Theorem in [13] and the results in the appendix of [6]), if there exist some positive numbers δ_1 and δ_2 such that $(M(\delta_1,\delta_2) - QR_2/\delta_1) + QR_2/\delta_1 = M(\delta_1,\delta_2) < 0$, then Eq. (4.13) only has roots with negative real-parts, that is $\operatorname{Re}\mu < 0$, and the steady-state solution 0 of Eq. (4.12) is asymptotic stable, that is $\lim_{t\to\infty}\xi_1(t) = 0$. From the comparison of (4.10) and (4.12) we can get $\xi(t) \leqslant \xi_1(t)$, and so $y(t) \leqslant \xi_1(t) - \varepsilon(\tau)$. From the condition $M(\delta_1,\delta_2) < 0$, we can deduce the following conditions:

$$\begin{aligned} &\text{(i)} \quad P<0, \quad 2(PR_1 - \lambda_1 + QR_2) + SR_2\delta_2 < 0; \quad \text{or} \\ &\text{(ii)} \quad P\geqslant 0, \quad 2(PR_2 - \lambda_1 + QR_2) + SR_2\delta_2 < 0. \end{aligned} \tag{4.14}$$

In fact $QR_2\delta_1 + QR_2/\delta_1 \geqslant 2\sqrt{QR_2\delta_1 \times QR_2/\delta_1} = 2QR_2$ $(\forall \delta_1 > 0)$. From the above arguments, it directly follows that

Theorem 4.1. *Suppose that hypotheses* (H_1)–(H_3) *and the condition* (2.1) *hold. For each fixed* $\tau > 0$, *if* $\sigma(r) < 0$ *and either condition* (i) *or condition* (ii) *in* (4.14) *is satisfied for a suitable positive number* δ_2, *then for every given initial function* ϕ *with* $0 \leqslant \phi \leqslant 1/a_1$, *the solution* $u(t,x)$ *of the delayed problem* (1.4)–(1.6) *and the* T*-periodic solution* $\theta(t,x)$ *of the no-delay problem* (4.1) *have the relations*

$$\overline{\lim_{t\to\infty}} \left\| u(t,\cdot) - \theta(t,\cdot) \right\|^2_{L^2(\Omega)} \leqslant -\frac{SR_2\eta(\tau)}{\delta_2 M(\delta_1,\delta_2)}, \tag{4.15}$$

where $M(\delta_1,\delta_2)$ *is given by* (4.9) *and* $\eta(\tau) = \sup_{0\leqslant t\leqslant T} \|\theta(t,\cdot) - \theta(t-\tau,\cdot)\|^2$.

Remarks. (1) In a particular case of our food-limited model with $b(t,x) \equiv d(t,x) \equiv 0$, the result Theorem 4.1 is coincident with that in [3] for the case $\tau = mT$ $(m \in N)$ corresponding to the Logistic model.

(2) If the condition $\sigma(r(1 - c\theta_{0\tau})/(1 + d\theta_{0\tau})) < 0$ is satisfied in Theorem 4.1, then R_1, R_2, P, Q and S in (4.2) can be redefined by Θ, θ_0 and θ. If the boundary condition is Neumann type, then R_1, R_2, P, Q and S can be redefined by k_1, k_2 as given by (3.12), and the corresponding condition to that in (4.14) has the form: $2(PR_1 + QR_2) + SR_2\delta_2 < 0$ with $P < 0$.

5. Applications

In this section, we give some numerical results for the asymptotic behavior of Eq. (1.4)–(1.6) on the one-dimensional spatial domain $\Omega = (0,1)$.

Example 1. We consider the following problem with Dirichlet boundary condition:

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) \\ &\quad = (6 + 2\sin 2\pi t)u(t,x)\frac{1 - (3 + \sin 2\pi t)u(t,x) - \frac{1}{4}(1 + \cos 2\pi t)u(t - \frac{1}{2}, x)}{1 + u(t,x)}, \end{aligned} \tag{5.1}$$

$$u(t,0) = u(t,1) = 0, \quad t \in [0,+\infty), \tag{5.2}$$

$$u(t,x) = \sin(\pi x), \quad (t,x) \in [-1/2, 0] \times [0,1] \tag{5.3}$$

with $(t,x) \in (0,\infty) \times (0,1)$ for (5.1). As the principal eigenvalue of $-\Delta$ in $\Omega = (0,1)$ with zero Dirichlet boundary conditions is $\lambda_1 = \pi^2$, and $r(t,x) = 6 + 2\sin 2\pi t \leqslant 8 < \pi^2$, which implies $\sigma(r) > 0$. Also $\phi(t,x) = \sin(\pi x) \geqslant 0$ satisfies the compatibility conditions $\phi(0,0) = \phi(0,1) = 0$. Moreover, it is easy to see that $a_1 = 2, a_2 = 4, b_1 = b_2 = 1, c_1 = 0, c_2 = 1/2$ and $d_1 = d_2 = 0$, and hence $a_2 d_2 - b_1 c_1 \leqslant a_1(c_1 + d_1)$ is naturally satisfied.

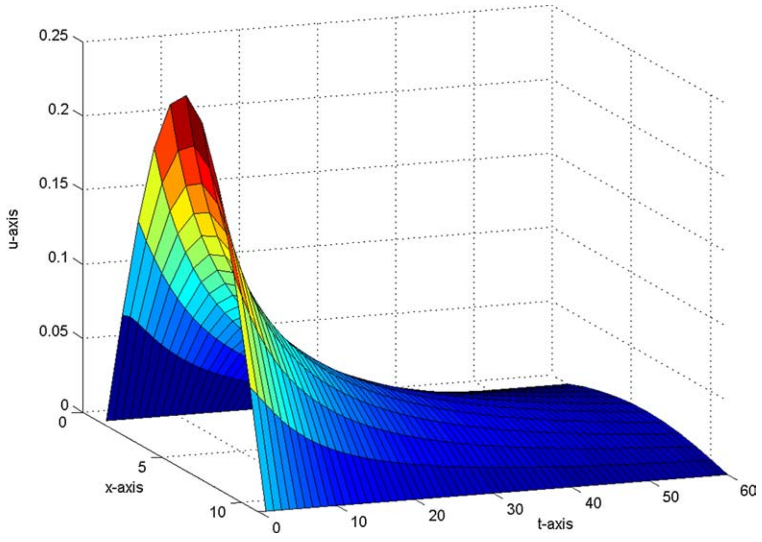

Fig. 1. The trivial solution 0 of problem (5.1)–(5.3) is globally asymptotic stable (see text).

According to Theorem 2.1, the trivial solution $u = 0$ of problem (5.1)–(5.3) is globally asymptotically stable (see Fig. 1), i.e.,

$$\lim_{t\to\infty} u(t,x) = 0, \quad \forall x \in [0,1].$$

Example 2. We consider the problem with Neumann boundary condition:

$$\frac{\partial u(t,x)}{\partial t} - \Delta u(t,x) = r(t,x)u(t,x)\frac{1 - a(t,x)u(t,x) - c(t,x)u(t-1/2,x)}{1 + b(t,x)u(t,x) + d(t,x)u(t-1/2,x)}, \tag{5.4}$$

$$\frac{\partial u(t,0)}{\partial x} = \frac{\partial u(t,1)}{\partial x} = 0, \quad t \in [0,+\infty), \tag{5.5}$$

$$u(t,x) = \phi(t,x), \quad (t,x) \in [-1/2,0] \times [0,1], \tag{5.6}$$

with $(t,x) \in (0,\infty) \times (0,1)$ for (5.4) and

$$\begin{aligned} &r(t,x) = 10; \qquad a(t,x) = 6 + \sin 2\pi t; \qquad b(t,x) = 2; \\ &c(t,x) = \frac{1}{2}(1 + \cos 2\pi t); \qquad d(t,x) = 0; \qquad \phi(t,x) = \frac{1}{8}(1.2 - \cos \pi x). \end{aligned} \tag{5.7}$$

This case $r_1 = r_2 = 10,\ a_1 = 5,\ a_2 = 7,\ b_1 = b_2 = 2,\ c_1 = 0,\ c_2 = 1,\ d_1 = d_2 = 0$. We check the conditions in (3.16) as follows. It is easy to see $a_1 > c_2$ and $a_2 d_2 - b_1 c_1 \leqslant a_1(c_1 + d_1)$. Also, from (3.12),

$$k_2 = \frac{a_2 - c_1}{a_1 a_2 - c_1 c_2} = \frac{7 - 0}{5 \times 7 - 0} = \frac{1}{5}; \qquad k_1 = \frac{a_1 - c_2}{a_1 a_2 - c_1 c_2} = \frac{5 - 1}{5 \times 7 - 0} = \frac{4}{35}. \tag{5.8}$$

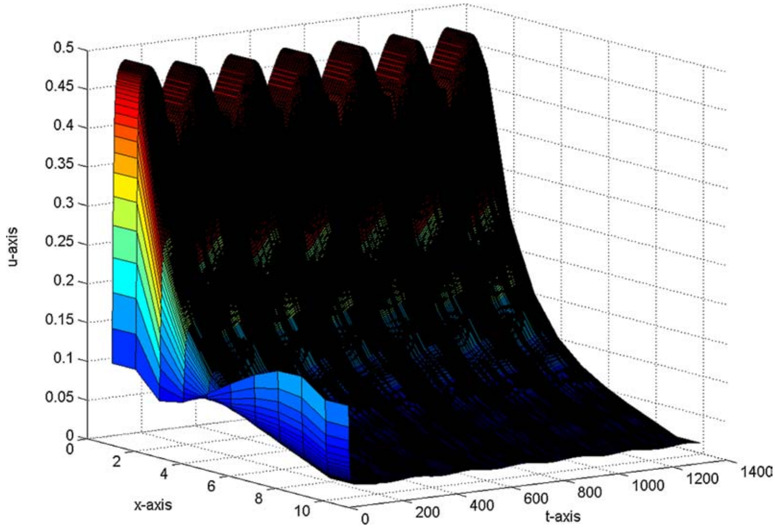

Fig. 2. The asymptotic periodicity of the solution for problem (5.4)–(5.6) (see text).

So from the definitions in (3.13), it follows that

$$\begin{aligned}
P &= 1 + d_2k_2 - (2a_1 + c_1)(1 + d_1k_1)k_1 - a_1b_1k_1^2 \\
&= 1 + 0 - (2 \times 5 + 0)(1 + 0) \times \frac{4}{35} - 5 \times 2 \times \left(\frac{4}{35}\right)^2 = -\frac{67}{245}; \\
Q &= k_2(b_2c_2k_2 - a_1d_1k_1 + c_2 + d_2) \\
&= \frac{1}{5} \times \left(2 \times 1 \times \frac{1}{5} - 0 + 1 + 0\right) = \frac{7}{25}; \\
R_1 &= \frac{r_1}{(1 + b_2k_2 + d_2k_2)^2} = \frac{10}{(1 + 2 \times \frac{1}{5} + 0)^2} = \frac{250}{49}; \\
R_2 &= \frac{r_2}{(1 + b_1k_1 + d_1k_1)^2} = \frac{10}{(1 + 2 \times \frac{4}{35} + 0)^2} = \frac{12250}{1849}.
\end{aligned} \tag{5.9}$$

Clearly $P = -\frac{67}{245} < 0$,

$$PR_1 + QR_2 = -\frac{67}{245} \times \frac{250}{49} + \frac{7}{25} \times \frac{12250}{1849} \approx -1.20979 < 0, \tag{5.10}$$

and it is easy to check that the compatibility conditions for $\phi(t,x)$ are also satisfied. According to Theorem 3.2, the boundary value problem (5.4), (5.5) has a 1-periodic solution $\theta(t,x)$ on $[0,\infty) \times [0,1]$, and the time-dependent solution $u(t,x)$ of the initial boundary value problem (5.4)–(5.6) has the asymptotic periodicity:

$$\lim_{t\to\infty} [u(t,x) - \theta(t,x)] = 0, \quad \forall x \in [0,1].$$

This is in line with our numerical simulation of problem (5.4)–(5.6) as given in Fig. 2.

6. Discussion

We have developed a food-limited model based on the consideration that the carrying capacity and the coefficients may vary spatially as well as temporally, may vary in a seasonal scale with respect to the seasonal variation of the environment, and they may also affected by the time-delay. The main basis of our reaction–diffusion model (1.1)–(1.3) is Model-3 as given in [2].

From the arguments in this paper we see there are some similar results to those in [2], although the system coefficients considered here are allowed to vary temporarily as well as spatially. The results in [2] show that as long as the magnitude of the instantaneous self-limitation and/or toxicant effects are larger than that of the time delay effects in Model-3 (mathematically speaking, the coefficient b is very small, the solution $u(t,x)$ may tend to 0 provided some conditions for the coefficients are satisfied). Ecologically speaking, it means an extinction of the species in the long run as shown in Fig. 1; or the solution $u(t,x)$ converges to a positive steady state solution $U(x)$ depending on the growth rate $r(x)$ of the species. This means that if the species have a suitable growth rate, its population becomes stable, though it is restricted by food supply and/or toxicant level. The theorems given in Sections 2 and 3 show that the solution $u(t,x)$ of problem (1.4)–(1.6) (i.e., $N(t,x)$ for (1.1)–(1.3)) converges to a positive periodic solution $\theta(t,x)$ of the boundary value problem (1.4), (1.5) (see Example 2 in Section 5), provided suitable conditions are imposed on the growth rate, the restriction of the food and/or the effects of the toxicant. This means the population of the species may tend to vary periodically under suitable conditions. Though the conditions for Theorems 3.1 and 3.2 seem very complex, if the coefficients c and d are very small with respect to a and b together with a suitable r, then all the conditions can be met (see Example 2), i.e., if the time-delay effect is very small and the growth rate of the species is suitably chosen, then asymptotic periodicity of the solution may appear.

Theorem 4.1 gives an indication of the effects of the time-delay on the asymptotic behavior of the solution with respect to the periodic solution of the no-delay boundary value problem. Particularly, for the special time delay $\tau = mT$ $(m \in N)$, only if the growth rate is suitable, the population of the species can tend to vary periodically.

Finally, we mention the case:

$$\begin{aligned}
&\frac{\partial\theta}{\partial t} - \Delta\theta = r\theta\frac{1-(a+c)\theta}{1+\rho(a+c)\theta}, \quad (t,x) \in R^+ \times \Omega,\\
&\theta = 0, \quad (t,x) \in R^+ \times \partial\Omega,
\end{aligned} \tag{6.1}$$

where r, a, c are as in (4.1) and ρ is a nonnegative parameter. This case is similar to that of Model-3 with no delay. What we want to illuminate is the effect of ρ on θ.

Assume that $r > \lambda_1$ and hence $\sigma(r) < 0$, so the periodic solution of (6.1) exists. The upper–lower solution method implies that $1/(a_1 + c_1)$ and 0 is a pair of upper and lower solutions of θ and hence $0 \leqslant \theta \leqslant 1/(a_1+c_1)$. Denote θ_0 for the case $\rho = 0$. For a sequence of numbers $0 = \rho_0 < \rho_1 < \cdots < \rho_{n-1} < \rho_n < \cdots$, noting that $0 \leqslant \theta \leqslant 1/(a_1 + c_1)$, it is easy to check

$$r\theta\frac{1-(a+c)\theta}{1+\rho_{n-1}(a+c)\theta} \geqslant r\theta\frac{1-(a+c)\theta}{1+\rho_n(a+c)\theta}, \quad \forall n \in N. \tag{6.2}$$

If θ_n denotes the periodic solution of problem (6.1) with $\rho = \rho_n$ $(n \in N)$, then the comparison results imply that $0 \leqslant \cdots \leqslant \theta_n \leqslant \theta_{n-1} \leqslant \cdots \leqslant \theta_1 \leqslant \theta_0$. So the parameter ρ can affect the size of the periodic solution θ of the Dirichlet problem (6.1). In fact, the periodic solution θ is monotone decreasing with respect to ρ.

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