Quantum error correction in the presence of spontaneous emission

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We present a quantum error correcting code that is invariant under the conditional time evolution between spontaneous emissions and which can correct for one general error. The code presented here generalizes previous error correction codes in that not all errors lead to different error syndromes. This idea may lead to shorter codes than previously expected. [S1050-2947(97)00101-7]

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I. INTRODUCTION

With the discovery of an algorithm to factorize a large number on a quantum computer in polynomial time instead of exponential time as required by a classical computer [1], the question of how to implement such a quantum computer has received considerable attention [2]. However, realistic estimates soon showed that decoherence processes and spontaneous emission severely limit the bit size of the number that can be factorized [3,4]. It has become clear that the solution to the problem does not lie in an increase in the lifetime of the transitions used in the computation. Attention has now shifted towards the investigation of methods to encode qubits such that the correction of errors due to interaction with the environment becomes possible. In a number of recent publications, possible encoding schemes have been considered and theoretical work has been undertaken to elucidate the structure of quantum error correction codes [5–22]. However, we show that these codes do not perfectly correct errors due to the conditional time evolution [23] between spontaneous emissions . This has the effect that, for example, the encoded lower state of a qubit, which, if unencoded, is not influenced by the conditional time evolution, acquires an error due to the conditional time evolution. We then proceed to construct a code that is able to correct *one* general error and is able to correct to *all orders* the errors due to the conditional time evolution between spontaneous emissions. By one general error we mean an arbitrary one bit operation acting on a single bit of the code. The conditional time evolution, however, contains terms that act on many qubits. Our code proposed in this paper has the ability to correct a special kind of error (here due to the conditional time evolution) to all orders. This is an interesting feature, as one would be interested to correct those errors which frequently occur to higher order than rare errors. The code presented here is optimal in the sense that it uses the smallest possible number of qubits required to perform its task (correcting one general error and all errors due to the conditional time evolution).

II. SINGLE ERROR CORRECTING CODES

Several codes have been proposed to encode one qubit which can correct one general error, i.e., amplitude and phase error or a combination of both applied to the same qubit. An example [10] of such a code is one where state $|0\rangle$ is represented by

$$|0_L\rangle=|00000\rangle+|11100\rangle-|10011\rangle-|01111\rangle+|11010\rangle +|00110\rangle+|01001\rangle+|10101\rangle \quad (1)$$

and the state $|1\rangle$ by

$$|1_L\rangle=|11111\rangle-|00011\rangle+|01100\rangle-|10000\rangle-|00101\rangle +|11001\rangle+|10110\rangle-|01010\rangle, \quad (2)$$

where the subscript L indicates that the encoded state $|i_L\rangle$ differs from the initial state $|i\rangle$. We omit the obvious normalization factor in the states $|0_L\rangle$ and $|1_L\rangle$ throughout this letter as they are irrelevant for the present analysis. Starting with a state $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$, this is encoded as $|\psi_L\rangle=\alpha|0_L\rangle+\beta|1_L\rangle$. If the state suffers an amplitude error A_i (which acts as a NOT operation on qubit i) or a phase error P_i (which gives the upper state of qubit i a minus sign) or the combination A_iP_i of both to the ith qubit of $|\psi_L\rangle$ it is possible to reconstruct the initial state $|\psi\rangle$. The code given in Eqs. (1) and (2) has the attractive feature that it is optimal in the sense that it only requires five qubits which can be shown to be the minimal possible number [13]. Using ideas similar to classical error correcting codes one can estimate that if one wants to encode l qubits in terms of n qubits in such a way that one can reconstruct the state after t general errors, then the inequality

$$2^l\sum_{i=0}^{t} 3^i\binom{n}{i}\leqslant 2^n \quad (3)$$

has to be satisfied [11]. The bound Eq. (3) is related to the sphere packing bound in classical coding theory [24]. The reason for that is that Eq. (3) was obtained using the assumption that different errors lead to different mutually orthogonal error syndromes. However, we will later see that the code presented in this paper (like the one presented in [5]) in fact violates this assumption which shows that it may be possible to find codes that go beyond Eq. (3).

The code given in Eqs. (1) and (2) does not correct for multiple errors. Especially, it is not able to correct to all orders for errors that arise due to the conditional time evolution between spontaneous emissions. The conditional time evolution between spontaneous emissions is unavoidable and it differs from the unit operation because the fact that no spontaneous emission has taken place provides information about the state of the system and therefore changes its wave function. The conditional time evolution of the system under the assumption that no spontaneous emission has taken place is given by the nonunitary time evolution operator $\exp\{-iH_{\rm eff}t/\hbar\}$ [23]. For the case that the qubits are not driven by external fields we obtain for the code given in Eqs. (1) and (2) the effective Hamilton operator

$$H_{\rm eff}=\sum_{i=1}^{5} -i\hbar\Gamma\sigma_{11}^{(i)}, \tag{4}$$

where $\sigma_{11}^{(i)}$ is the projector $|1\rangle\langle 1|$ onto the excited state of the ith qubit leaving all other qubits unaffected. 2Γ is the Einstein coefficient of the upper level 1 of the qubits. If we apply the conditional time evolution $\exp(-iH_{\rm eff}t/\hbar)$ to the encoded state

$$|\psi_L\rangle=\alpha|0_L\rangle+\beta|1_L\rangle \tag{5}$$

and subsequently apply the appropriate error correction procedure for this five-bit code [10] we do *not* recover the original state. This becomes obvious in the special case $\Gamma t\gg 1$ in which one obtains

$$|\psi_C\rangle=|00000\rangle+|00010\rangle+|01000\rangle-|01110\rangle+|10000\rangle +|10010\rangle+|11000\rangle+|11110\rangle. \tag{6}$$

This shows that this five-bit code is not able to correct errors due to the conditional time evolution exactly. Especially striking is the effect when we assume that $\beta=0$, i.e., we encode the (stable) ground state. The conditional time evolution then leads to no errors in the unencoded state while it changes the encoded state such that it cannot be corrected perfectly anymore. Note, however, that the error introduced by the conditional time evolution is, for short times, of fourth order. If, however, a spontaneous emission (or any other kind of error) occurs then a subsequent conditional time evolution induces contributions which after error correction lead to second-order errors in the state. Our code presented later in this paper preserves the encoded state in both cases perfectly, i.e., to all orders.

The reason that the code [Eqs. (1) and (2)] cannot perfectly correct errors due to the conditional time evolution derives from the fact that the words (product states) of which the code consists do not all have the same number of excited states. This leads to a difference in the rate at which the amplitude of these states decays. The amplitude of $|00000\rangle$ remains unchanged under the conditional time evolution while the amplitude of $|11100\rangle$, for example, decreases at a rate $\exp(-3\Gamma t)$. This can be seen as a multiple amplitude error with which the code cannot cope. This problem is not restricted to the five-bit code given in [10] but is present in all other previously proposed codes. It should be noted that it is not necessary to observe the system for these conclusions to hold. If we do not observe the system, it then has to be described by a density operator, whose time evolution follows the appropriate Bloch equations. This time evolution can in principle be decomposed into individual trajectories each of which consists of no-jump evolutions interrupted by spontaneous emissions [23]. For each of these trajectories our considerations above hold and, therefore, also hold for the incoherent sum of these trajectories which make up the ensemble. Therefore our error correction code is not restricted to a particular measurement scheme such as, for example, the detection and reconstruction scheme discussed in [25], where it is necessary to detect individual quantum jumps. Nevertheless, such a detection of individual jumps would improve the performance of our code, as that would exclude the contribution of multiple quantum jumps with which our code cannot cope. This would enhance the importance of the conditional time evolution as a error source compared to other sources and it is here where our code is superior to previous codes.

III. CORRECTING SPONTANEOUS EMISSION

The discussion of Sec. II shows that it is of some interest to construct a quantum error correcting code that corrects errors due to the conditional time evolution to all orders. This is possible, and in the following we present such a quantum error correcting code.

The following code was constructed starting from the code (1) and (2). State $|0\rangle$ is encoded as

$$|0_L\rangle=|00001111\rangle+|11101000\rangle-|10010110\rangle-|01110001\rangle +|11010100\rangle+|00110011\rangle+|01001101\rangle +|10101010\rangle, \tag{7}$$

while state $|1\rangle$ is encoded as

$$|1_L\rangle=|11110000\rangle-|00010111\rangle+|01101001\rangle-|10001110\rangle -|00101011\rangle+|11001100\rangle+|10110010\rangle -|01010101\rangle. \tag{8}$$

The state Eq. (7) encoding the logical 0 was obtained in the following way. We started with state Eq. (1) and for each word, e.g., $|11100\rangle$ we constructed the bitwise inverse, i.e., $|00011\rangle$. We concatenated the two words where the second one is taken in reverse bit order to obtain $|1110011000\rangle$. This method, applied to all words in Eq. (1), already yields a possible code. However, it is possible to shorten the code by removing bits 5 and 6 from every word. This then yields Eq. (7) and analogously Eq. (8). Subsequently a computer search was made for potentially shorter codes; this revealed no such codes, so we conclude that $n=8$ qubits is the minimum number required for the task of correcting one general error while errors due to the conditional time evolution are corrected perfectly. In the following we present some interesting properties of the code and demonstrate that it indeed has the claimed error correction properties. However, this code differs in many ways from previously proposed codes. First of all, it violates the conditions given for quantum error correcting codes in [11] thereby showing that these conditions are overly restrictive. As these conditions were used to derive the inequality Eq. (3), their violation indicates that there might exist codes that require less qubits than expected from Eq. (3). However, we did not yet succeed to construct a code that violates Eq. (3). One should also realize that the code-

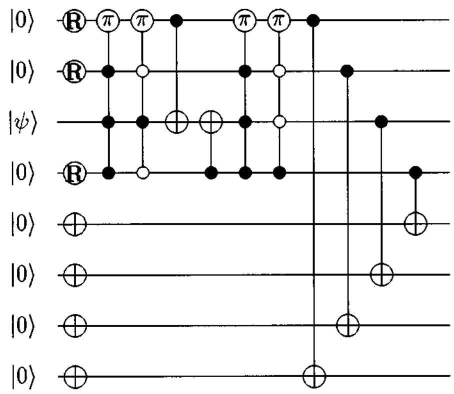

FIG. 1. The encoding network: **R** describes a one bit rotation which takes $|0\rangle \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle \rightarrow (|0\rangle - |1\rangle)/\sqrt{2}$. An encircled cross denotes a NOT operation while a dot denotes a control bit. For a filled circle the operation is carried out if the control bit is 1; for an empty circle the operation is carried out if the control bit is 0. A circle with a π represents multiplication with phase $\exp(i\pi)$. Qubit 3 is in the state $|\psi\rangle$ that we wish to encode, while all other qubits are initially in their ground state $|0\rangle$.

words in the code Eqs. (7) and (8) do *not* form a linear code as this would imply that $|00000000\rangle$ is a codeword which in turn would render impossible the task of constructing a code with codewords of equal excitation. Nevertheless, the codewords of $|0_L\rangle$ form a coset of a linear code. The coset leader is $|00001111\rangle$. This contrasts slightly with other codes such as those presented in [5,8–10]. The codewords of the code (1) and (2), for example form a linear code. Given the initial state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we obtain the code Eqs. (7) and (8) using the network given in Fig. 1. To correct the error that may have appeared we first apply the encoder in the reverse direction (right to left). After the application of the decoder, the third qubit contains information about the encoded state while the remaining seven qubits contain the error syndrome, from which one can infer the type and location of the error. We measure the qubits of the error syndrome and apply, according to the result of our measurement, a suitable unitary operation on qubit 3. We assume that after the measurement all the other qubits are reset to their ground state $|0\rangle$ so that, in principle, we can reencode the state again using the same qubits.

In Table I we give all possible outcomes of the measurement and the corresponding state of the third qubit. The necessary unitary transformation that has to be applied onto the third qubit is then obvious. Careful inspection of Table I reveals that this error correction scheme has, for some errors, a slightly different effect than expected. Take, for example, a phase errors P_1 on bit 1 and compare with the effect of a phase error P_8 on bit 8. We observe that they both lead to the same error syndrome but that the resulting state differs by a global phase -1. Therefore it is not possible to correct both states in such a way that they go over to the initial state. After the correction they differ by a global phase -1. But this also shows that the dimension of the space $\mathcal{H}_{code}$ spanned by the code together with all states that result from it by single errors is 2×21 and not as expected from Eq. (3) 2×25. The latter number results from the considerations of Ekert and Macchiavello [11] who have presented a set of

TABLE I. One obtains an error syndrome, i.e., the state of all qubits except qubit 3, depending on the error that occurred and the place in which it occurred. P_i indicates a sign change of the upper level of qubit i, A_i an amplitude error which is given by the transformation $|0\rangle \leftrightarrow |1\rangle$. The product of both applied to the same qubit gives the third kind of error. Note that the error syndrome is not able to distinguish between P_i and P_{9-i} which leads to global phases in some of the corrected states. This table does not take into account that before and after the error a conditional time evolution takes place.

Error	Error syndrome	State of qubit 3
None	0000000	$\alpha\|0\rangle+\beta\|1\rangle$
P_1	1000000	$\alpha\|0\rangle+\beta\|1\rangle$
P_2	0100000	$\alpha\|0\rangle+\beta\|1\rangle$
P_4	0010000	$\alpha\|0\rangle+\beta\|1\rangle$
A_5	0001000	$\alpha\|0\rangle+\beta\|1\rangle$
A_6	0000100	$\alpha\|0\rangle+\beta\|1\rangle$
A_7	0000010	$\alpha\|0\rangle+\beta\|1\rangle$
A_8	0000001	$\alpha\|0\rangle+\beta\|1\rangle$
P_3	1010000	$\alpha\|0\rangle-\beta\|1\rangle$
A_2	0010010	$\alpha\|0\rangle-\beta\|1\rangle$
P_6	1010000	$-\alpha\|0\rangle+\beta\|1\rangle$
A_2P_2	0110010	$-\alpha\|0\rangle+\beta\|1\rangle$
A_6P_6	1010100	$-\alpha\|0\rangle+\beta\|1\rangle$
P_5	0010000	$-\alpha\|0\rangle-\beta\|1\rangle$
P_7	0100000	$-\alpha\|0\rangle-\beta\|1\rangle$
P_8	1000000	$-\alpha\|0\rangle-\beta\|1\rangle$
A_5P_5	0011000	$-\alpha\|0\rangle-\beta\|1\rangle$
A_7P_7	0100010	$-\alpha\|0\rangle-\beta\|1\rangle$
A_8P_8	1000001	$-\alpha\|0\rangle-\beta\|1\rangle$
A_1P_1	1110001	$\beta\|0\rangle+\alpha\|1\rangle$
A_4P_4	1011000	$\beta\|0\rangle+\alpha\|1\rangle$
A_3P_3	1110100	$\beta\|0\rangle-\alpha\|1\rangle$
A_1	0110001	$-\beta\|0\rangle-\alpha\|1\rangle$
A_3	0100100	$-\beta\|0\rangle-\alpha\|1\rangle$
A_4	1001000	$-\beta\|0\rangle-\alpha\|1\rangle$

conditions that have to be satisfied by any quantum error correction code. The violation of these conditions by the code Eq. (7) and (8) leads to these different predictions for the dimension of $\mathcal{H}_{code}$. More general conditions can be derived and it can be checked easily that our code satisfies these conditions [13,26] while it violates the conditions given in [11].

So far we have shown that our code can indeed correct a general single error without taking into account the conditional time evolution due to spontaneous emission. Now we show that our code is able to correct errors due to the conditional time evolution perfectly, i.e., to all orders. For our code given in Eqs. (7) and (8) the conditional time evolution under the assumption that no spontaneous emission has taken place is generated by the effective Hamilton operator

$$H_{\rm eff} = \sum_{i=1}^{8} -i\hbar\Gamma\sigma_{11}^{(i)}. \tag{9}$$

If the code undergoes a conditional time evolution before it experiences an error like, e.g., a spontaneous emission, it is obvious that the code Eqs. (7)-(8) will work properly, as it is invariant under the conditional time evolution $\exp(-iH_{\rm eff}t/\hbar)$. However, it is not so obvious that the code corrects general single errors that occur *before* or in between the conditional time evolution. As we do not know the time at which the general error occurs, this situation will almost certainly occur and has to be examined. If the error was a phase error, then no problem will occur, as this error does not change the excitation of the state. However, for amplitude errors or a combination of amplitude and phase errors we have to investigate the code more closely. The problem is that, for example, after an amplitude error in the first qubit, we obtain

$$A_1|0_L\rangle=|10001111\rangle-|11110001\rangle+|11001101\rangle +|10110011\rangle+|0110100\rangle-|00010110\rangle +|01010100\rangle+|00101010\rangle. \quad (10)$$

Now the code words have a different degree of excitation so that their relative weights will change during the subsequent conditional time evolution. However, for $|\psi_L\rangle=\alpha|0_L\rangle+\beta|1_L\rangle$ we have the relations $|\psi_L\rangle=\alpha|0_L\rangle+\beta|1_L\rangle$

$$e-iH_{\rm eff}t/\hbar A_i|\psi_L\rangle=\tfrac{1}{2}e^{-3\Gamma t}\{(1+e^{-2\Gamma t})A_i-(1-e^{-2\Gamma t})A_iP_i\}|\psi_L\rangle \quad (11)$$

and

$$e-iH_{\rm eff}t/\hbar A_iP_i|\psi_L\rangle=\tfrac{1}{2}e^{-3\Gamma t}\{-(1-e^{-2\Gamma t})A_i+(1+e^{-2\Gamma t})A_iP_i\}|\psi_L\rangle. \quad (12)$$

Equation (11) shows that after an amplitude error A_i on the ith qubit, the conditional time evolution transforms the state into a superposition of a state without conditional time evolution after this amplitude error, and a state without conditional time evolution obtained after a combined amplitude and phase error A_iP_i on the ith qubit. Inspecting Table I we see that both errors A_i and A_iP_i lead to a different error syndrome. A measurement of the syndrome will then indicate one or the other error, A_i or A_iP_i, which can then be corrected. Therefore the code (7) and (8) corrects properly even if the error is followed by a conditional time evolution.

IV. CONCLUSIONS

We conclude that the code presented here is able to correct a single general error *and*, in addition, errors due to the conditional time evolution to *arbitrary* order. Our code can correct a general error to first order *and* a special kind of error to all orders. This is an interesting result as it shows that it is possible to correct special kinds of errors to all orders. As some errors are more frequent than others it would be in our interest to correct those errors to higher order than less frequently occurring errors. We have adapted our code to errors due to the conditional time evolution between spontaneous emissions. Other applications will require different adaptions. The code presented here (similar to the one given in [5]) violates the conditions for quantum codes given in [11] which shows that these conditions are overly restrictive, as they exclude codes like the one presented here that map different errors onto the same error syndromes. This can lead to the construction of shorter quantum error correction codes than expected from the quantum sphere packing bound as derived in [11]. These results may become important in different fields such as quantum computation, the distribution of entangled particles, and in quantum cryptography [27–30].

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