Achievable Rate Regions for Network Coding

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***Abstract*—Determining the achievable rate region for networks using routing, linear coding, or non-linear coding is thought to be a difficult task in general, and few are known. We describe the achievable rate regions for three interesting networks and show that achievable rate regions for linear codes need not be convex.**

I. Introduction

In this paper, a *network* is a directed acyclic multigraph $G = (V, E)$, some of whose nodes are information sources or receivers. Associated with the sources are m generated *messages*, where the i^{th} source message is assumed to be a vector of k_i arbitrary elements of a fixed finite alphabet, $\mathcal{A}$, of size at least 2. At any node in the network, each out-edge carries a vector of n alphabet symbols which is a function (called an *edge function*) of the vectors of symbols carried on the in-edges to the node, and of the node's message vectors if it is a source. Each network edge is allowed to be used at most once (i.e. at most n symbols can travel across each edge). It is assumed that every network edge is reachable by some source message. Associated with each receiver are *demands*, which are subsets of the network messages. Each receiver has *decoding functions* which map the receiver's inputs to vectors of symbols in an attempt to produce the messages demanded at the receiver. The goal is for each receiver to deduce its demanded messages from its in-edges and source messages by having information propagate from the sources through the network.

A $(k_1, \ldots, k_m, n)$ *fractional code* is a collection of edge functions, one for each edge in the network, and decoding functions, one for each demand of each node in the network. A $(k_1, \ldots, k_m, n)$ *fractional solution* is a $(k_1, \ldots, k_m, n)$ fractional code which results in every receiver being able to compute its demands via its decoding functions, for all possible assignments of length-k_i vectors over the alphabet to the i^{th} source message, for all i.

Special codes of interest include *linear codes*, where the edge functions and decoding functions are linear over a finite field, and *routing codes*, where the edge functions and decoding functions simply copy specified input components to output components.[1] Special networks of interest include *multicast* networks, where there is only one source node and every receiver demands all of the source messages, and *multiple-unicast* networks, where each network message is generated by exactly one source node and is demanded by exactly one receiver node.

For each i, the ratio k_i/n can be thought of as the rate at which source i injects data into the network. If a network has a $(k_1, \ldots, k_m, n)$ fractional solution over some alphabet, then we say that $(k_1/n, \ldots, k_m/n)$ is an *achievable rate vector*, and we define the *achievable rate region*[2] of the network as the set

$$S = \overline{\{r \in \mathbf{Q}^m : r \ \text{ is an achievable rate vector}\}}.$$

In this paper, we will sometimes restrict attention to achievable rate regions corresponding to using only linear codes (perhaps over certain finite field alphabets) or only routing codes.

Determining the achievable rate region of an arbitrary network appears to be a formidable task. Alternatively, certain scalar quantities that reveal information about the achievable rates are typically studied. For any $(k_1, \ldots, k_m, n)$ fractional solution, we call the scalar quantity

$$\frac{1}{m}\left(\frac{k_1}{n} + \cdots + \frac{k_m}{n}\right)$$

an *achievable average rate* of the network. We define the *average coding capacity* of a network to be the supremum of all achievable average rates, namely

$$\mathcal{C}^{\text{average}} = \sup\left\{\frac{1}{m}\sum_{i=1}^{m} r_i : (r_1, \ldots, r_m) \in S\right\}.$$

Similarly, for any $(k_1, \ldots, k_m, n)$ fractional solution, we

[1] If an edge function for an out-edge of a node depends only on the symbols of a single in-edge of that node, then, without loss of generality, we assume that the out-edge simply carries the same vector of symbols (i.e. routes the vector) as the in-edge it depends on.

[2] Some authors in the literature refer to this region by other terminology, such as the "capacity region".

call the scalar quantity

$$\min\left(\frac{k_1}{n},\ldots,\frac{k_m}{n}\right)$$

an *achievable uniform rate* of the network. We define the *uniform coding capacity* of a network to be the supremum of all achievable uniform rates, namely

$$\mathcal{C}^{\text{uniform}} = \sup\left\{\min(r_1,\ldots,r_m) : (r_1,\ldots,r_m)\in S\right\}.$$

Note that for any $r \in S$ and $r' \in \mathbf{R}^m$, if each component of r' is nonnegative, rational, and less than or equal to the corresponding component of r, then $r' \in S$. In particular, if $(r_1,\ldots,r_m)\in S$ and $r_i = \min\limits_{1\le j\le m} r_j$, then $(r_i, r_i, \ldots, r_i) \in S$, which implies

$$\mathcal{C}^{\text{uniform}} = \sup\left\{r_i : (r_1,\ldots,r_m)\in S, \quad r_1 = \cdots = r_m\right\}.$$

In other words, all messages can be restricted to having the same dimension $k_1 = \cdots = k_m$ when considering $\mathcal{C}^{\text{uniform}}$. Also, note that

$$\mathcal{C}^{\text{uniform}} \le \mathcal{C}^{\text{average}}.$$

The quantities $\mathcal{C}^{\text{average}}$ and $\mathcal{C}^{\text{uniform}}$ are attained by points on the boundary of the closed set S. It is known that not every network has a capacity which is an achievable rate [3].

If a network's edge functions are restricted to purely routing functions, then we write the capacities as $\mathcal{C}^{\text{average}}_{\text{routing}}$ and $\mathcal{C}^{\text{uniform}}_{\text{routing}}$, and refer to them as the *average routing capacity* and *uniform routing capacity*, respectively. Likewise, for solutions using only linear edge functions, we write $\mathcal{C}^{\text{average}}_{\text{linear}}$ and $\mathcal{C}^{\text{uniform}}_{\text{linear}}$ and refer to them as the *average linear capacity* and *uniform linear capacity*, respectively.

Given random variables $x_1,\ldots x_i$ and $y_1,\ldots,y_j$, we write $x_1,\ldots x_i \longrightarrow y_1,\ldots,y_j$ to mean that $y_1,\ldots,y_j$ are deterministic functions of $x_1,\ldots x_i$.

In this paper, we study three specific networks, namely the Generalized Butterfly network, the Fano network, and the non-Fano network. Various capacities of these networks have been computed in [4], however, the full achievable rate regions of these networks have not been previously determined, to the best of our knowledge.

In this paper, we give the exact achievable rate regions (for routing, linear coding, and non-linear coding) for each of the Generalized Butterfly, Fano, and non-Fano networks. The linear coding achievable rate regions for the Fano and non-Fano networks depend on the characteristic of the finite field alphabet used. Proofs are given for the Generalized Butterfly network, but are omitted due to space for the Fano and non-Fano networks (these proofs will appear in a future publication). Finally, a network is given that demonstrates that the achievable rate region for linear coding need not be convex. This latter result was motivated by a discussion in [6].

The Generalized Butterfly network (studied in Section II and illustrated in Figure 1) has the same topology as the usual Butterfly network, but instead of one source at each of nodes n_1 and n_2, there are two sources at each of these nodes. For each of the source nodes, one of it's source messages is demanded by receiver n_5 and the other by receiver n_6. The usual Butterfly network is the special case when messages a and d do not exist (or are just not demanded by any receiver). A large majority of network coding publications mention in some context the Butterfly network, so it plays an important role in the field.

The Fano network (studied in Section III and illustrated in Figure 2) and the non-Fano network (studied in Section IV and illustrated in Figure 6) were used in [3] as components of a larger network to demonstrate the unachievability of network coding capacity. Specifically, in [3] the Fano network was shown to be solvable if and only if the alphabet size is a power of 2 and the non-Fano network was shown to be solvable if and only if the alphabet size is odd. In [5], the Fano and non-Fano networks were used to build a solvable multicast network whose reverse (i.e. all edge directions change, and sources and receivers exchange roles) was not solvable. In [2], the Fano and non-Fano networks were used to construct a network which disproved a previously published conjecture asserting that all solvable networks are vector linearly solvable over some finite field and some vector dimension.

II. Generalized Butterfly network

Theorem II.1. *The achievable rate regions for either linear (over any finite field alphabet) or non-linear coding are the same for the Generalized Butterfly network and are equal to the closed polytope in $\mathbf{R}^4$ whose faces lie on the 9 planes:*

$$\begin{aligned}
r_a &= 0\\
r_b &= 0\\
r_c &= 0\\
r_d &= 0\\
r_b &= 1\\
r_c &= 1\\
r_a + r_b + r_c &= 2\\
r_b + r_c + r_d &= 2\\
r_a + r_b + r_c + r_d &= 3
\end{aligned}$$

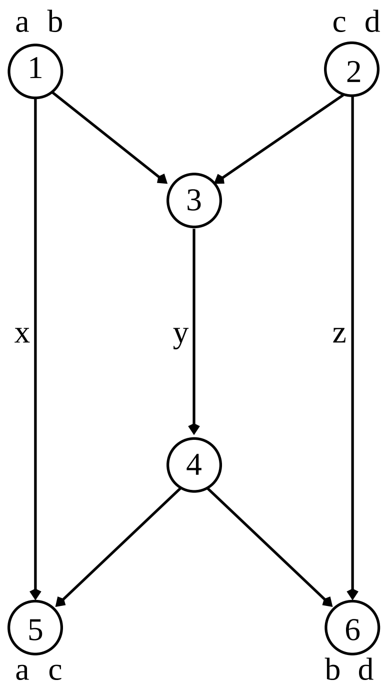

Fig. 1. The Generalized Butterfly network. Source node n_1 generates messages a and b, and source node n_2 generates messages c and d. Receiver node n_5 demands messages a and c, and receiver node n_6 demands messages b and d. The symbol vectors carried on edges $e_{1,5}$, $e_{2,4}$, and $e_{3,6}$ are denoted x, y, and z, respectively.

and whose vertices are the 12 *points:*

$$\begin{array}{llll} (0,0,0,0) & (0,0,0,2) & (2,0,0,0) & (0,1,0,0) \\ (0,0,1,0) & (2,0,0,1) & (1,0,0,2) & (0,0,1,1) \\ (1,1,0,0) & (1,0,1,1) & (1,1,0,1) & (0,1,1,0). \end{array}$$

Furthermore, the coding capacities and linear coding capacities are given by:

$$\mathcal{C}^{\text{uniform}} = \mathcal{C}^{\text{uniform}}_{\text{linear}} = 2/3$$
$$\mathcal{C}^{\text{average}} = \mathcal{C}^{\text{average}}_{\text{linear}} = 3/4.$$

Proof: Consider a network solution over an alphabet $\mathcal{A}$ and denote the source message dimensions by k_a, k_b, k_c, and k_d, and the edge dimensions by n. Let each source be a random variable whose components are independent and uniformly distributed over $\mathcal{A}$. Then the solution must satisfy the following inequalities:

$$k_a \geq 0 \quad (1)$$
$$k_b \geq 0 \quad (2)$$
$$k_c \geq 0 \quad (3)$$
$$k_d \geq 0 \quad (4)$$
$$k_b = H(b) = H(y|a,c,d) \leq n \quad (5)$$
$$k_c = H(c) = H(y|a,b,d) \leq n \quad (6)$$
$$\begin{aligned} k_a + k_b + k_c &= H(a,b,c) = H(x,y|d) \\ &\leq H(x,y) \leq 2n \end{aligned} \quad (7)$$
$$\begin{aligned} k_b + k_c + k_d &= H(b,c,d) = H(y,z|a) \\ &\leq H(y,z) \leq 2n \end{aligned} \quad (8)$$
$$\begin{aligned} k_a + k_b + k_c + k_d &= H(a,b,c,d) = H(x,y,z) \\ &\leq 3n. \end{aligned} \quad (9)$$

(1)–(4) are trivial; (5) follows because $c,d,y \longrightarrow y,z \longrightarrow b,d$ (at node n_6), and therefore $a,c,d,y \longrightarrow a,b,c,d$ and thus $H(a,b,c,d) = H(a,c,d,y)$; similarly for (6); (7) follows because $x,y \longrightarrow a,c$ (at node n_5), $c,d,y \longrightarrow b,d$ (at node n_6), and therefore $d,x,y \longrightarrow a,c,d,y \longrightarrow a,b,c,d$ and thus $H(a,b,c,d) = H(d,x,y)$; similarly for (8); (9) follows because $x,y,z \longrightarrow a,b,c,d$ (at nodes n_5 and n_6). Dividing each inequality in (1)–(9) by n gives the 9 bounding hyperplanes stated in the theorem.

Let $r_a = k_a/n$, $r_b = k_b/n$, $r_c = k_c/n$, and $r_d = k_d/n$, and let $\mathcal{P}$ denote the polytope in $\mathbf{R}^4$ consisting of all 4-tuples (r_a, r_b, r_c, r_d) satisfying (1)–(9). Then $\mathcal{P}$ is bounded by (1)–(4) and (9). One can easily calculate that each point in $\mathbf{R}^4$ that satisfies some set of four of the inequalities (1)–(9) with equality must be one of the 12 points stated in the theorem. Now we show that all 12 such points do indeed lie in $\mathcal{P}$, and therefore their convex hull equals $\mathcal{P}$. The following 5 points lie in $\mathcal{P}$ by taking $n = k_a = k_b = k_c = k_d = 1$ with the following codes over, say, the binary field:

$$\begin{array}{ll} (2,0,0,1): & x = y = a,\ z = d \\ (1,0,0,2): & x = a,\ y = z = d \\ (1,0,1,1): & x = a,\ y = c,\ z = d \\ (1,1,0,1): & x = a,\ y = b,\ z = d \\ (0,1,1,0): & x = b,\ y = b + c,\ z = c \end{array}$$

and the remaining 7 points are achieved by fixing certain messages to be 0.

Since the above codes are all linear, the achievable rate regions for linear and non-linear codes are the same.

By (9), we have $\mathcal{C}^{\text{average}} \leq 3/4$, and this upper bound is achievable by routing using the code given above for the point $(2,0,0,1)$, namely taking $x = y = a$ and

$z = d$. By (8), we have $\mathcal{C}^{\text{uniform}} \le 2/3$; since

$$\begin{aligned}(2/3)(1,1,1,1) = (1/3)&(1,0,1,1)\\ + (1/3)&(1,1,0,1)\\ + (1/3)&(0,1,1,0)\end{aligned}$$

the upper bound of $2/3$ is achievable by a convex combination of the linear codes given above for the points $(1,0,1,1)$, $(1,1,0,1)$, and $(0,1,1,0)$, as follows. Take $k = 2$ and $n = 3$ and use the (linear) code determined by:

$$\begin{aligned}x &= (a_1, a_2, b_2)\\ y &= (c_1, b_1, b_2 + c_2)\\ z &= (d_1, d_2, c_2).\end{aligned}$$

■

Theorem II.2. *The achievable rate region for routing for the Generalized Butterfly network is the closed polytope in* $\mathbf{R}^4$ *bounded by the* 9 *planes in Theorem II.1 together with the plane*

$$r_b + r_c = 1$$

and whose vertices are the 13 *points:*

$(0,0,0,0)$ $(0,0,0,2)$ $(2,0,0,0)$ $(0,1,0,0)$
$(0,1,0,1)$ $(0,0,1,0)$ $(2,0,0,1)$ $(1,0,0,2)$
$(0,0,1,1)$ $(1,0,1,0)$ $(1,1,0,0)$ $(1,0,1,1)$
$(1,1,0,1)$.

Furthermore, the routing capacities are given by:

$$\begin{aligned}\mathcal{C}^{\text{uniform}}_{\text{routing}} &= 1/2\\ \mathcal{C}^{\text{average}}_{\text{routing}} &= 3/4.\end{aligned}$$

Proof: With routing, in addition to the inequalities (1)–(9), a solution must also satisfy

$$k_b + k_c \le n \tag{10}$$

since all of the components of messages b and c must be carried by the edge labeled y. One can show that each point in $\mathbf{R}^4$ that satisfies with equality some set of four of the inequalities (1)–(9) and (10) must be one of the 13 points stated in this theorem (i.e. one of the points $(1,0,1,0)$ and $(0,1,0,1)$, together with 11 of the 12 points stated in Theorem II.1 by excluding the point $(0,1,1,0)$). The proof of Theorem II.1 showed that all vertices of $\mathcal{P}$ except $(0,1,1,0)$ were achievable using routing. The two new points $(1,0,1,0)$ and $(0,1,0,1)$ are achievable using routing by forcing a message to be 0 in the codes for $(1,0,1,1)$ and $(1,1,0,1)$, respectively, which were shown to have routing solutions in the proof of Theorem II.1.

By (10), we have $\mathcal{C}^{\text{uniform}}_{\text{routing}} \le 1/2$, and this upper bound is achievable, for example, by taking a convex combination of codes that achieve $(1,0,1,0)$ and $(0,1,0,1)$, as follows. Take $k = 1$ and $n = 2$ and use the routing code determined by:

$$\begin{aligned}x &= (a, a)\\ y &= (b, d)\\ z &= (d, d).\end{aligned}$$

The capacity $\mathcal{C}^{\text{average}}_{\text{routing}} = 3/4$ follows immediately from the proof of Theorem II.1. ■

III. Fano network

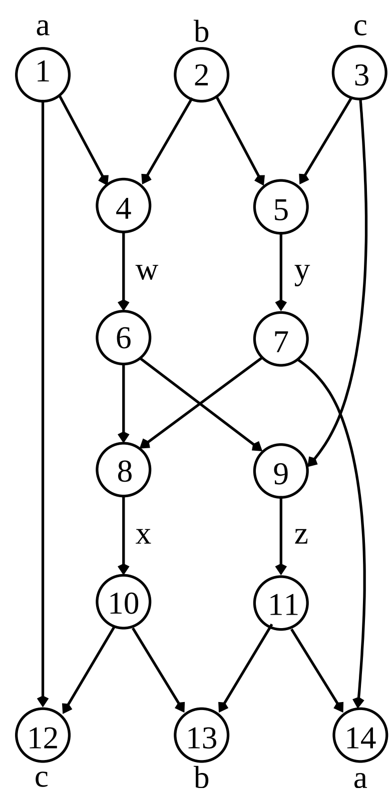

Fig. 2. The Fano network. Source nodes n_1, n_2, and n_3 generate messages a, b, and c, respectively. Receiver nodes n_{12}, n_{13}, and n_{14} demand messages c, b, and a, respectively. The symbol vectors carried on edges $e_{4,6}$, $e_{8,10}$, $e_{5,7}$, $e_{9,11}$ are labeled as w, x, y, and z, respectively.

Theorem III.1. *The achievable rate regions for either linear coding over any finite field alphabet of even characteristic or non-linear coding are the same for the Fano network and are equal to the closed polyhedron in*

$\mathbf{R}^3$ *whose faces lie on the* 7 *planes (see Figure 3):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a = 1$$
$$r_c = 1$$
$$r_b + r_c = 2$$
$$r_a + r_b = 2$$

and whose vertices are the 8 *points:*

$$(0,0,0) \quad (0,0,1) \quad (1,0,0) \quad (0,2,0)$$
$$(0,1,1) \quad (1,0,1) \quad (1,1,0) \quad (1,1,1).$$

It was shown in [2] that for the Fano network, $\mathcal{C}^{\text{average}} = \mathcal{C}^{\text{uniform}} = 1$ and $\mathcal{C}^{\text{uniform}}_{\text{linear}} = 1$ for all even characteristic fields and $\mathcal{C}^{\text{uniform}}_{\text{linear}} = 4/5$ for all odd characteristic fields. The calculation of $\mathcal{C}^{\text{uniform}}_{\text{linear}} = 4/5$ in [2] required a rather involved computation.

Theorem III.2. *The achievable rate region for linear coding over any finite field alphabet of odd characteristic for the Fano network is equal to the closed polyhedron in* $\mathbf{R}^3$ *whose faces lie on the* 8 *planes (see Figure 4):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a = 1$$
$$r_c = 1$$
$$r_a + 2r_b + 2r_c = 4$$
$$2r_a + r_b + 2r_c = 4$$
$$2r_a + 2r_b + r_c = 4$$

and whose vertices are the 10 *points:*

$$(0,0,0) \quad (0,0,1) \quad (1,0,0) \quad (0,2,0)$$
$$(0,1,1) \quad (1,0,1) \quad (1,1,0)$$
$$(2/3,2/3,1) \quad (1,2/3,2/3) \quad (4/5,4/5,4/5).$$

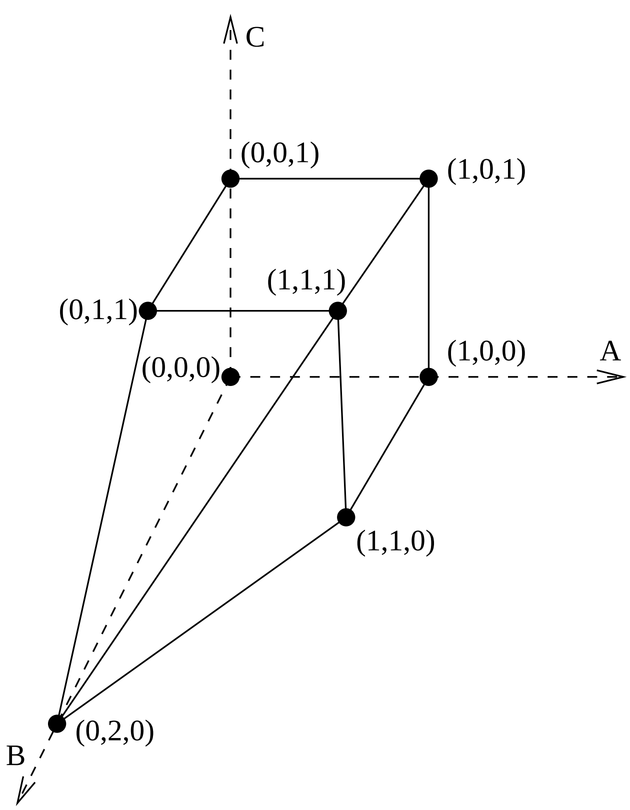

Fig. 3. The achievable coding rate region for the Fano network is a 7-sided polyhedron with 8 vertices.

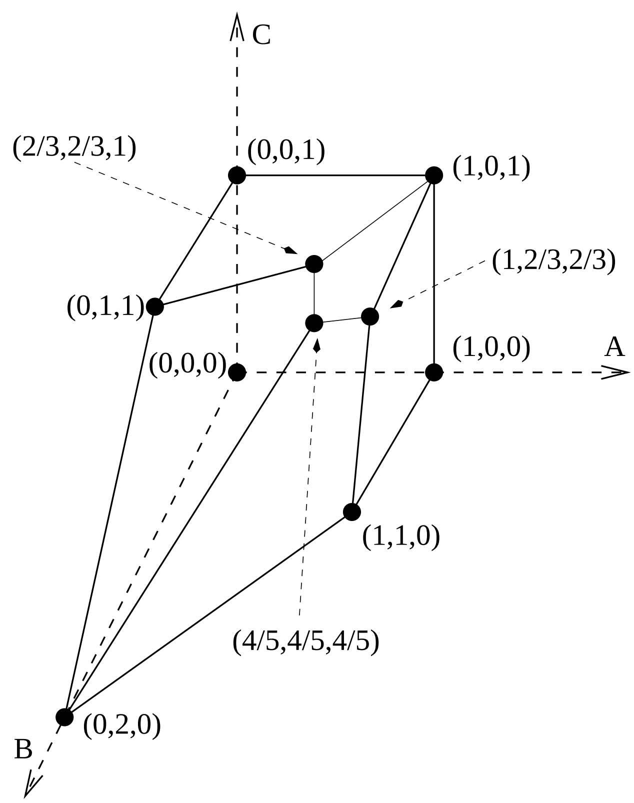

Fig. 4. The achievable linear coding rate region over odd characteristic finite fields for the Fano network is a 8-sided polyhedron with 8 vertices.

Theorem III.3. *The achievable rate region for routing for the Fano network is the closed polyhedron in* $\mathbf{R}^3$ *whose faces lie on the* 6 *planes (see Figure 5):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a = 1$$
$$r_c = 1$$
$$r_a + r_b + r_c = 2$$

and whose vertices are the 7 *points:*

$$(0,0,0) \quad (0,0,1) \quad (1,0,0) \quad (0,2,0)$$
$$(0,1,1) \quad (1,0,1) \quad (1,1,0).$$

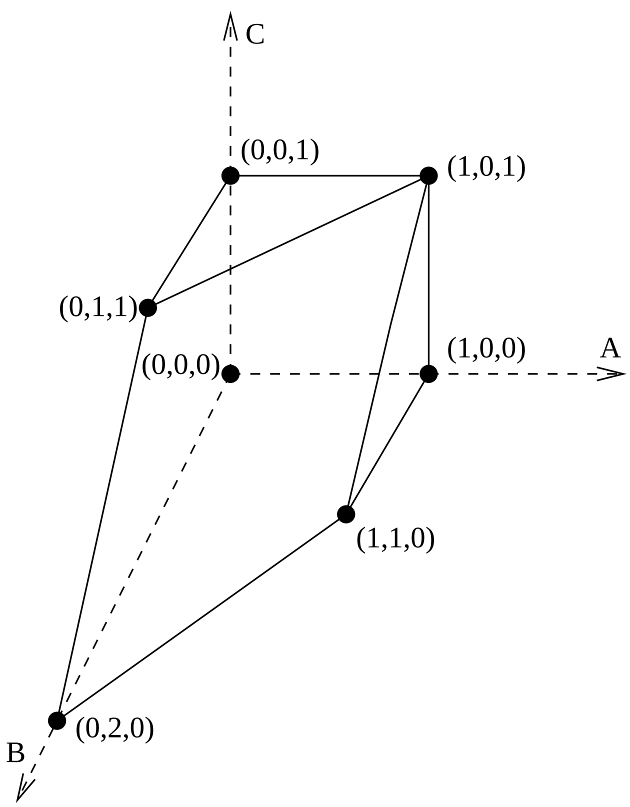

Fig. 5. The achievable routing rate region for the Fano network is a 6-sided polyhedron with 7 vertices.

IV. Non-Fano network

Theorem IV.1. *The achievable rate region for either linear coding over any finite field alphabet of odd characteristic or non-linear coding are the same for the*

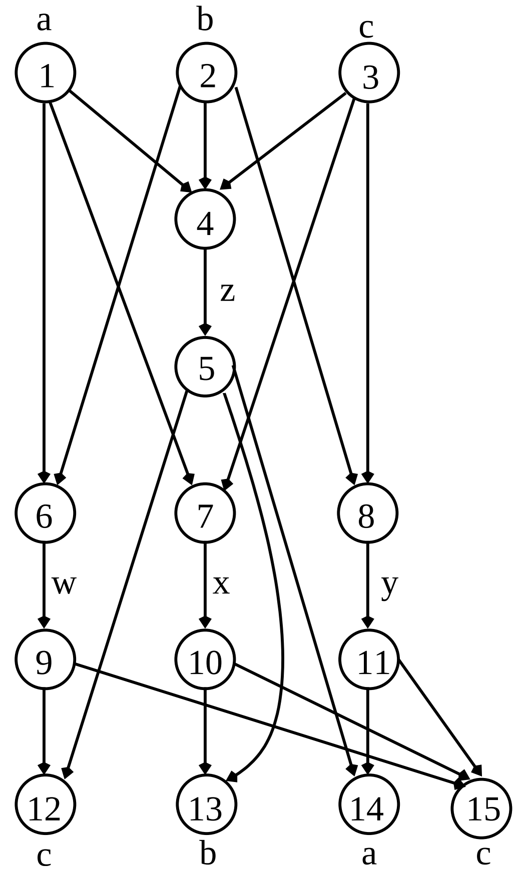

Fig. 6. The non-Fano network. Source nodes n_1, n_2, and n_3 generate messages a, b, and c, respectively. Receiver nodes n_{12}, n_{13}, n_{14}, and n_{15} demand messages c, b, a, and c, respectively. The symbol vectors carried on edges $e_{6,9}$, $e_{7,10}$, $e_{8,11}$, $e_{4,5}$ are labeled as w, x, y, and z, respectively.

non-Fano network and are equal to the closed cube in $\mathbf{R}^3$ *whose faces lie on the* 6 *planes (see Figure 7):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a = 1$$
$$r_b = 1$$
$$r_c = 1$$

and whose vertices are the 8 *points:*

$$(0,0,0) \quad (0,0,1) \quad (1,0,0) \quad (0,1,0)$$
$$(0,1,1) \quad (1,0,1) \quad (1,1,0) \quad (1,1,1).$$

Theorem IV.2. *The achievable rate region for linear coding over any finite field alphabet of even character-*

istic for the non-Fano network is are equal to the closed polyhedron in $\mathbf{R}^3$ *whose faces lie on the* 7 *planes (see Figure 8):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a = 1$$
$$r_b = 1$$
$$r_c = 1$$
$$r_a + r_b + r_c = 5/2$$

and whose vertices are the 10 *points:*

$$\begin{array}{llll} (0,0,0) & (0,0,1) & (1,0,0) & (0,1,0) \\ (0,1,1) & (1,0,1) & (1,1,0) & \\ (1,1,1/2) & (1,1/2,1) & (1/2,1,1). & \end{array}$$

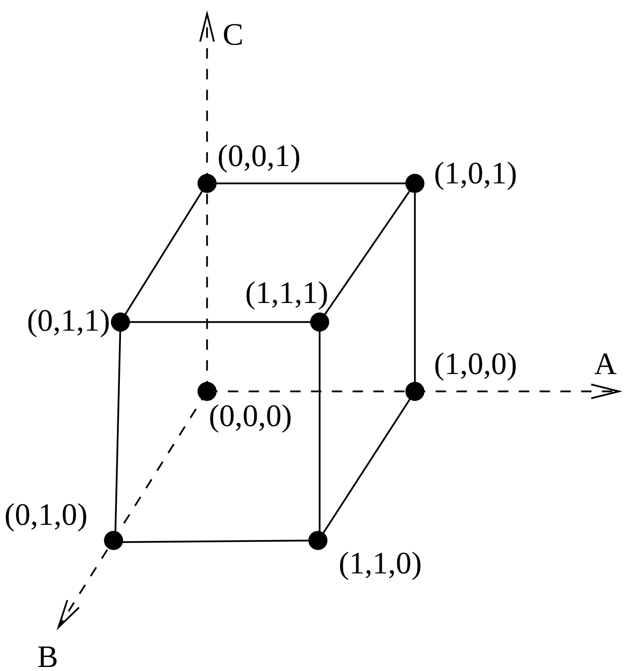

Fig. 7. The achievable coding rate region for the Fano network is a cube in $\mathbf{R}^3$.

Theorem IV.3. *The achievable rate region for routing for the non-Fano network is the closed tetrahedron in*

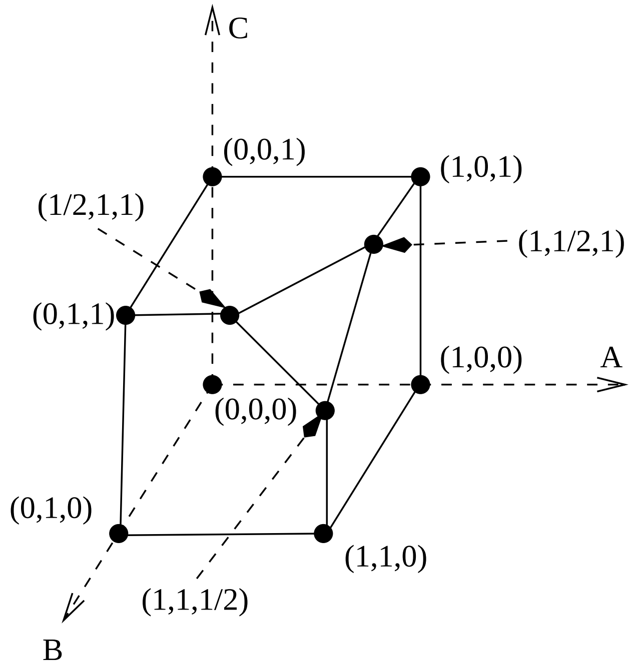

Fig. 8. The achievable linear coding rate region over even characteristic finite fields for the non-Fano network is a 7-sided polyhedron with 10 vertices.

$\mathbf{R}^3$ *whose faces lie on the* 4 *planes (see Figure 9):*

$$r_a = 0$$
$$r_b = 0$$
$$r_c = 0$$
$$r_a + r_b + r_c = 1$$

and whose vertices are the 4 *points:*

$$(0,0,0),\ (0,0,1),\ (1,0,0),\ (0,1,0).$$

V. Non-convex achievable rate region for linear codes

Theorem V.1. *There exists a network whose achievable rate region for linear codes is non-convex.*

(Sketch): Consider the network formed by taking the disjoint union of the Fano and non-Fano networks (with separate messages, so the union is a six-message network). It can be shown that this network's achievable rate region for linear codes contains the points $(4/5, 4/5, 4/5, 1, 1, 1)$ and $(1, 1, 1, 5/6, 5/6, 5/6)$ but not their midpoint $(9/10, 9/10, 9/10, 11/12, 11/12, 11/12)$, and thus is non-convex. ■

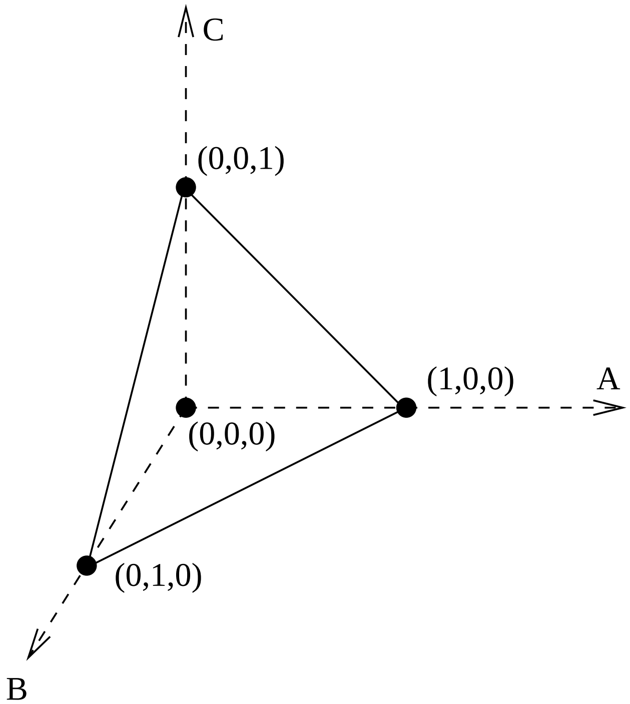

Fig. 9. The achievable routing rate region for the Fano network is a tetrahedron in $\mathbf{R}^3$.

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REFERENCES

[1] A. Blasiak, R. Kleinberg, and E. Lubetzky, "Lexicographic products and the power of non-linear network coding", *arXiv 1108.2489 [cs.IT]*, August 11, 2011.

[2] R. Dougherty, C. Freiling, and K. Zeger, "Insufficiency of linear coding in network information flow", *IEEE Transactions on Information Theory*, vol. 51, no. 8, pp. 2745-2759, August 2005.

[3] R. Dougherty, C. Freiling, and K. Zeger, "Unachievability of network coding capacity", joint issue of *IEEE Transactions on Information Theory & IEEE/ACM Transactions on Networking* vol. 52, no. 6, pp. 2365-2372, June 2006.

[4] R. Dougherty, C. Freiling, and K. Zeger, "Networks, matroids, and non-Shannon information inequalities", *IEEE Transactions on Information Theory*, vol. 53, no. 6, pp. 1949-1969, June 2007.

[5] R. Dougherty and K. Zeger, "Nonreversibility and equivalent constructions of multiple-unicast networks", *IEEE Transactions on Information Theory*, vol. 52, no. 11, pp. 5067-5077, November 2006.

[6] A. Kim and M. Médard, "Computing bounds on network capacity regions as a polytope reconstruction problem", *arXiv 1103.0361 [cs.IT]*, March 2, 2011.