Doppler Resilient Golay Complementary Waveforms

Ali Pezeshki, *Member, IEEE*, A. Robert Calderbank, *Fellow, IEEE*, William Moran, *Member, IEEE*, and Stephen D. Howard

***Abstract*—We describe a method of constructing a sequence (pulse train) of phase-coded waveforms, for which the ambiguity function is free of range sidelobes along modest Doppler shifts. The constituent waveforms are Golay complementary waveforms which have ideal ambiguity along the zero Doppler axis but are sensitive to nonzero Doppler shifts. We extend this construction to multiple dimensions, in particular to radar polarimetry, where the two dimensions are realized by orthogonal polarizations. Here we determine a sequence of two-by-two Alamouti matrices where the entries involve Golay pairs and for which the range sidelobes associated with a matrix-valued ambiguity function vanish at modest Doppler shifts. The Prouhet–Thue–Morse sequence plays a key role in the construction of Doppler resilient sequences of Golay complementary waveforms.**

***Index Terms*—Ambiguity function, Doppler resilient waveforms, Golay complementary sequences, Prouhet–Thue–Morse sequence, radar polarimetry, range sidelobe suppression.**

I. Introduction

In sensing and communications it is often required to localize a received signal in time, e.g., to estimate the range of a target from a radar based on the delay in the radar return or to synchronize a mobile handset with a pilot signal sent from a base station. Typically, localization is performed by matched filtering the received signal with the transmitted waveform. The output of the matched filter would ideally be an impulse at the desired delay. Therefore, waveforms with impulse-like autocorrelation functions are of great value in these applications. Phase coding [1] is a common technique in radar for generating waveforms with impulse-like autocorrelation functions. In this technique, a long pulse is phase coded with a unimodular (biphase or polyphase) sequence and the autocorrelation function of the coded waveform is controlled through the autocorrelation function of the unimodular sequence. Examples of sequences that produce good autocorrelation functions are polyphase sequences by Heimiller [2], Frank codes [3], polyphase codes by Chu [4], Barker sequences [5], and generalized Barker sequences by Golomb and Scholtz [6]. It is however impossible to achieve an impulse aperiodic autocorrelation function with a single unimodular sequence. This has led to the idea of using complementary sets of unimodular sequences [7]–[11] for phase coding.

Perhaps the most famous class of complementary sequences are binary complementary sequences introduced by Marcel Golay [7]. Golay complementary sequences (Golay pairs) have the property that the sum of their autocorrelation functions vanishes at all delays other than zero. Thus, if the sequences are transmitted separately and their autocorrelation functions are added together the sum will be an impulse. The concept of complementary sequences was generalized to multiple complementary codes by Tseng and Liu [11], and to multiphase (or polyphase) sequences by Sivaswami [12] and Frank [13]. Properties of complementary sequences, their relationship with other codes, and their applicability in radar have been studied in several articles among which are [7]–[13].

Recently, Howard *et al.* [14], [15] and Calderbank *et al.* [16] combined Golay complementary waveforms with Alamouti signal processing to enable pulse compression for multichannel and fully polarimetric radar systems. In [14]–[16], Alamouti coding is used to coordinate the transmission of Golay complementary waveforms across two orthogonal polarizations in time. Separating Golay complementary waveforms in frequency however is not as straightforward. Frequency separation disturbs the complementary property of the waveforms due to the presence of delay-dependent phase terms. Searle and Howard [17], [18] have recently introduced modified Golay pairs, which are complementary in the squared autocorrelation functions and maintain their complementary property when transmitted over different frequencies. Golay complementary sequences have also been advocated for the next generation guided radar (GUIDAR) systems [19].

The use of complementary sequences have also been explored for data communications. The early work in this context include the introduction of orthogonal complementary codes for synchronous spread spectrum multiuser communications by Suehiro and Hatori [20]. In the 1990s, some researchers including Wilkinson and Jones [21], van Nee [22], and Ochiai and Imai [23] explored the use of Golay complementary sequences as codewords for orthogonal frequency-division multiplexing (OFDM), due to their small peak-to-mean envelope power ratio (PMEPR). However, the major advances in this context are due to Davis and Jedwab [24] and Paterson [25], who derived tight bounds for the PMEPR of Golay complementary sequences

Manuscript received March 1, 2007; revised June 9, 2008. Published August 27, 2008 (projected). This work was supported in part by the National Science Foundation under Grant 0701226, by the Office of Naval Research under Grant N00173-06-1-G006 , and by the Air Force Office of Scientific Research under Grants FA9550-05-1-0443 and FA8750-05-2-0285. The material in this paper was presented in part at the 2007 IEEE Workshop on Statistical Signal Processing, Madison, WI, August 2007.

A. Pezeshki was with the Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544 USA. He is now with the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO 80523 USA.

A. R. Calderbank is with the Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544 USA.

W. Moran is with the Department of Electrical Engineering and Computer Science, The University of Melbourne, Melbourne, Vic. 3010, Australia.

S. D. Howard is with the Defence Science and Technology Organization, P.O. Box 1500, Edinburgh 5111, Australia.

Color versions Figures 2–5 in this paper are available online at http://ieeexplore.ieee.org.

Communicated by G. Gong, Associate Editor for Sequences.

Digital Object Identifier 10.1109/TIT.2008.928292

and related codes from cosets of the generalized first-order Reed–Muller code. Construction of low PMEPR codes from cosets of the generalized first-order Reed–Muller code has also been considered by Schmidt [26] and by Schmidt and Finger [27]. Complementary codes have also been employed as pilot signals for channel estimation in OFDM systems [28]. Orthogonal complementary codes have been advocated by Chen *et al.* [29], [30] and Tseng and Bell [31] for enabling interference-free (both multipath and multiple-access) multicarrier code-division multiple access (CDMA). Other work in this context include the extension of complementary codes using the Zadoff–Chu sequence by Lu and Dubey [32] and cyclic shifted orthogonal complementary codes by Park and Jim [33]. In [34], orthogonal complementary codes have been used in the design of access-request packets for contention resolution in random-access wireless networks.

Despite their many intriguing properties, in practice, a major barrier exists in adoption of Golay complementary sequences for radar and communications; the perfect autocorrelation property of these sequences is extremely sensitive to Doppler shift. Although the effective ambiguity function of complementary sequences is free of delay (range) sidelobes along the zero-Doppler axis, off the zero-Doppler axis it has large-range sidelobes. Most generalizations of Golay complementary sequences, including multiple complementary sequences and polyphase complementary sequences, suffer from the same problem to some degree. Sivaswami [35] has proposed a class of near-complementary codes, called subcomplementary codes, which exhibit some tolerance to Doppler shift. Subcomplementary codes consist of a set of N length-N sequences that are phase-modulated by a binary Hadamard matrix. The necessary and sufficient conditions for a set of phase-modulated sequences to be subcomplementary have been derived by Guey and Bell in [36]. We note that a large body of work exists concerning the design of single polyphase sequences that have Doppler tolerance. A few examples are Frank codes [3], $P1$, $P2$, $P3$, and $P4$ sequences [37], PX sequences [38], and $P(n,k)$ sequences [39], [40]. The design of Doppler tolerant polyphase sequences has also been considered for multiple-input multiple-output (MIMO) radar. In [41], Khan *et al.* have used a harmonic phase structural constraint along with a numerical optimization method to design a set of polyphase sequences with resilience to Doppler shifts for orthogonal netted radar (a special case of MIMO radar). Their design is based on an extension of a work by Deng [42], which utilizes polyphase sequences for orthogonal netted radar.

In this paper, we present a novel and systematic way of designing a Doppler resilient sequence of Golay complementary waveforms, for which the pulse train ambiguity function is free of range sidelobes at modest Doppler shifts. The idea is to determine a sequence of Golay pairs that annihilates the range sidelobes in the low-order terms of the Taylor expansion (around zero Doppler) of the pulse train ambiguity function. It turns out that the Prouhet–Thue–Morse sequence [43]–[46] plays a key role in constructing the Doppler resilient sequence of Golay pairs. We then extend our analysis to the design of a Doppler resilient sequence of Alamouti waveform matrices of Golay pairs, for which the range sidelobes associated with a matrix-valued ambiguity function vanish at modest Doppler shifts. Alamouti matrices of Golay waveforms have recently been shown [14]–[16] to be useful for instantaneous radar polarimetry, which has the potential to improve the performance of fully polarimetric radar systems, without increasing the receiver signal processing complexity beyond that of single-channel matched filtering. Again, the Prouhet–Thue–Morse sequence plays a key role in determining the Doppler resilient sequence of Golay pairs. Finally, numerical examples are presented, demonstrating the resilience of the constructed sequences to modest Doppler shifts.

II. Golay Complementary Waveforms for Radar

A. Golay Complementary Sequences

Definition 1: Two length-L unimodular sequences of complex numbers $x[l]$ and $y[l]$ are Golay complementary if for $k=-(L-1),\ldots,(L-1)$ the sum of their autocorrelation functions satisfies

$$\mathrm{corr}_x[k]+\mathrm{corr}_y[k]=2L\delta_{k,0} \tag{1}$$

where $\mathrm{corr}_x[k]$ is the autocorrelation of $x[l]$ at lag k and $\delta_{k,0}$ is the Kronecker delta function.

Let $X(z)=\mathcal{Z}\{x[l]\}$ and $Y(z)=\mathcal{Z}\{y[l]\}$ be the z-transforms of $x[l]$ and $y[l]$ so that

$$\begin{aligned}X(z)&=x[0]+x[l]z^{-1}+\cdots+x[L-1]z^{-(L-1)}\\ Y(z)&=y[0]+y[l]z^{-1}+\cdots+y[L-1]z^{-(L-1)}.\end{aligned} \tag{2}$$

Then, $x[l]$ and $y[l]$ (or alternatively $X(z)$ and $Y(z)$) are Golay complementary if $X(z)$ and $Y(z)$ satisfy

$$X(z)\widetilde{X}(z)+Y(z)\widetilde{Y}(z)=|X(z)|^2+|Y(z)|^2=2L \tag{3}$$

where $\widetilde{X}(z)=X^*(1/z^*)$ and $\widetilde{Y}(z)=Y^*(1/z^*)$ are the z-transforms of $\widetilde{x}[l]=x^*[-l]$ and $\widetilde{y}[l]=y^*[-l]$, the time reversed complex conjugates of $x[l]$ and $y[l]$.

Henceforth, we drop the discrete time index l from $x[l]$ and $y[l]$ and simply use x and y. We use the notation (x,y) whenever x and y are Golay complementary and call (x,y) a Golay pair. Correspondingly, each member of the pair (x,y) is called a Golay sequence. From (3), it follows that if (x,y) is a Golay pair then $(\pm x,\pm y)$, $(\pm x,\pm\widetilde{y})$, $(\pm\widetilde{x},\pm y)$, $(\pm\widetilde{x},\pm\widetilde{y})$ are also Golay pairs.

B. Golay Pairs for Radar

Suppose N (N even) Golay sequences $x_0,x_1,\ldots,x_{N-1}$ are transmitted from a radar antenna during N pulse repetition intervals (PRIs) to interrogate a radar scene. Assume $(x_0,x_1),(x_2,x_3),\ldots,\ (x_{N-2},x_{N-1})$ are Golay pairs. Consider a point scatterer at delay coordinate d_0. Suppose the scatterer moves at a constant speed, causing a relative Doppler shift of θ_0 [rad] between consecutive PRIs.[1] Assume that the radar PRI is short enough so that during the N PRIs the

[1] We assume that the relative Doppler shift over L *chip* intervals (duration of a single waveform) is negligible. A chip interval is the time interval between two consecutive values in a phase code.

scatterer remains within the same range cell (or delay cell). Let $R_n(z) = \mathcal{Z}\{r_n(k)\}$ denote the z-transform of the radar return associated with the nth PRI where x_n is transmitted. Then, the radar return vector (in z-domain) is given by

$$\underline{\boldsymbol{r}}^T(z) = h_0 z^{-d_0} \underline{\boldsymbol{x}}^T(z) \boldsymbol{D}(\theta_0) + \underline{\boldsymbol{w}}^T(z) \tag{4}$$

where $\underline{\boldsymbol{r}}^T(z) = [R_0(z), \ldots, R_{N-1}(z)]$, h_0 is a scattering coefficient, $\underline{\boldsymbol{x}}^T(z) = [X_0(z), \ldots, X_{N-1}(z)]$ is the transmit signal vector, and $\underline{\boldsymbol{w}}(z)^T$ is a noise vector. The matrix $\boldsymbol{D}(\theta)$ is the following diagonal Doppler modulation matrix:

$$\boldsymbol{D}(\theta) = \mathrm{diag}(1, e^{j\theta}, \ldots, e^{j(N-1)\theta}). \tag{5}$$

If we now process the radar measurement vector $\underline{\boldsymbol{r}}^T(z)$ using the receiver vector $\underline{\widetilde{\boldsymbol{x}}}(z)$ the receiver output will be

$$\underline{\boldsymbol{r}}^T(z)\underline{\widetilde{\boldsymbol{x}}}(z) = h_0 z^{-d_0} G(z, \theta_0) + \underline{\boldsymbol{w}}^T(z)\underline{\widetilde{\boldsymbol{x}}}(z) \tag{6}$$

where $G(z, \theta)$ is given by

$$G(z,\theta) = \underline{\boldsymbol{x}}^T(z)\boldsymbol{D}(\theta)\underline{\widetilde{\boldsymbol{x}}}(z) = \sum_{n=0}^{N-1} e^{jn\theta} |X_n(z)|^2. \tag{7}$$

The function $G(z,\theta)$ is the z-transform of the *ambiguity function* [1] (ignoring the range aliases) of the pulse train $x_0, x_1, \ldots, x_{N-1}$. Along the zero-Doppler axis ($\theta = 0$) $G(z,\theta)$ is given by

$$G(z, 0) = \sum_{n=0}^{N-1} |X_n(z)|^2 = NL \tag{8}$$

where the second equality follows from the fact that $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are Golay pairs. This shows that the ambiguity function of the pulse train $x_0, x_1, \ldots, x_{N-1}$ is an impulse function in delay (constant in z-domain) and is hence free of range sidelobes. Off the zero-Doppler axis, however, this is no longer the case. In fact, even for small Doppler shifts the ambiguity function has large range sidelobes. From a radar imaging viewpoint, this means that a weak target can be masked by the sidelobes associated with a strong reflector. This motivates the following question.

Question 1: Is it possible to construct a *Doppler resilient* sequence or pulse train of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that

$$G(z,\theta) = \sum_{n=0}^{N-1} e^{jn\theta} |X_n(z)|^2 \approx \alpha(\theta) \tag{9}$$

where $\alpha(\theta)$ is some function of θ, independent of the delay operator z^{-1}, for a reasonable range of Doppler shifts θ?

Remark 1: The ambiguity function of the pulse train $x_0, x_1, \ldots, x_{N-1}$ has $2(N-1)$ range aliases (cross terms) which are offset from the zero-delay axis by nK, $n = \pm 1, \ldots, \pm(N-1)$, where K is the PRI. In this paper, we ignore the range aliasing effects and only focus on the *main lobe* of the ambiguity function, which corresponds to $G(z,\theta)$ given in (7). Range aliasing effects can be accounted for using standard techniques devised for this purpose (e.g., see [1]) and hence will not be further discussed.

III. Doppler Resilient Golay Pairs

In this section, we consider the design of Doppler resilient sequences of Golay pairs. More precisely, we describe how to select Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion of $G(z,\theta)$ around $\theta = 0$ the coefficients of all terms up to a certain order, say M, vanish at all nonzero delays. Consider the Taylor expansion of $G(z,\theta)$ around $\theta = 0$, i.e.,

$$G(z,\theta) = \sum_{m=0}^{\infty} C_m(z)(j\theta)^m \tag{10}$$

where

$$C_m(z) = \sum_{n=0}^{N-1} n^m |X_n(z)|^2, \quad \text{for } m = 0, 1, 2, \ldots. \tag{11}$$

The coefficient $C_0(z)$ is equal to NL and has no components at nonzero delays. The rest of the coefficients are two-sided polynomials in z^{-1} and can be expressed as

$$C_m(z) = \sum_{l=-(L-1)}^{L-1} c_{m,l} z^{-l}, \quad m = 1, 2, 3, \ldots. \tag{12}$$

For instance, the first coefficient $C_1(z)$ is

$$C_1(z) = 0|X_0(z)|^2 + 1|X_1(z)|^2 + 2|X_2(z)|^2 + \cdots + (N-1)|X_{N-1}(z)|^2. \tag{13}$$

Noting that $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are Golay pairs we can simplify $C_1(z)$ as

$$C_1(z) = N(N-2)L/2 + |X_1(z)|^2 + |X_3(z)|^2 + \cdots + |X_{N-1}(z)|^2. \tag{14}$$

Each term $|X_{2k+1}(z)|^2 = X_{2k+1}(z)X_{2k+1}^*(1/z^*)$ in (14) is a two-sided polynomial of degree $L-1$ in the delay operator z^{-1}, which cannot be matched with any of the other terms, as we have already taken into account all the Golay pairs. Consequently, $C_1(z)$ is a two-sided polynomial in z^{-1} of the form (12).

We wish to design the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that $c_{1,l}$ vanish for all nonzero l. More generally, we wish to design $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion in (10) the coefficients of all the terms up to a given order M vanish at all nonzero delays, i.e., $c_{m,l} = 0$, for all m $(1 \le m \le M)$ and for all nonzero l. Although not necessary, we continue to carry the term $0^m|X_0(z)|^2$ in writing $C_m(z)$ for reasons that will become clear. From here on, whenever we say a function $A(z)$ or $A(z,\theta)$, which is a polynomial in the delay operator z^{-1}, vanishes at all nonzero delays we simply mean that the coefficients of all z^{-l}, $l \neq 0$ in $A(z,\theta)$ are zero.

A. The Requirement That $C_1(z)$ Vanishes at All Nonzero Delays

To provide intuition, we first consider the case $N = 2^2 = 4$, where Golay pairs (x_0, x_1) and (x_2, x_3) are transmitted over four PRIs. Then, as the following calculation shows, $C_1(z)$ will vanish at all nonzero delays if the Golay pairs (x_0, x_1) and (x_2, x_3) are selected such that (x_1, x_3) is also a Golay pair:

$$C_1(z) = \underbrace{\underbrace{0|X_0(z)|^2 + |X_1(z)|^2}_{1|X_1(z)|^2} + \underbrace{2|X_2(z)|^2 + 3|X_3(z)|^2}_{2\times 2L+1|X_3(z)|^2}}_{3\times 2L}. \quad (15)$$

The trick is to break 3 into 2+1, and then pair the extra $|X_3(z)|^2$ with $|X_1(z)|^2$. Note that it is easy to choose the pairs (x_0, x_1) and (x_2, x_3) such that (x_1, x_3) is also a Golay pair. For example, let (x, y) be an arbitrary Golay pair, then $(x_0 = x, x_1 = y)$, $(x_2 = y, x_3 = x)$, and $(x_1 = y, x_3 = x)$ are Golay pairs. Other combinations of $\pm x$, $\pm\widetilde{x}$, $\pm y$, and $\pm\widetilde{y}$ are also possible. For instance, $(x_0 = x, x_1 = y)$ and $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$ also satisfy the extra Golay pair condition. The calculation in (15) shows that it is possible to make $C_1(z)$ vanish at all nonzero delays with $N = 2^{1+1}$ Golay sequences $x_0, \ldots, x_3$.

B. The Requirement That $C_1(z)$ and $C_2(z)$ Vanish at All Nonzero Delays

It is easy to see that when $N = 4$ it is not possible to force $C_2(z)$ $(M = 2)$ to zero at all nonzero delays. However, this is possible when $N = 2^{2+1} = 8$. As the calculations in (16) and (17) at the bottom of the page show, we can make both $C_1(z)$ and $C_2(z)$ vanish at all nonzero l if we select the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ such that (x_1, x_3), (x_5, x_7), and (x_3, x_7) are also Golay pairs.[2]

Note that it is easy to select the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ such that (x_1, x_3), (x_5, x_7), and (x_3, x_7) are also Golay pairs. For example, $(x_0 = x, x_1 = y)$, $(x_2 = y, x_3 = x)$, $(x_4 = y, x_5 = x)$, and $(x_6 = x, x_7 = y)$, where (x, y) is an arbitrary Golay pair, satisfy all the extra Golay pair conditions. Again, other combinations of $\pm x$, $\pm\widetilde{x}$, $\pm y$, and $\pm\widetilde{y}$ are also possible, e.g., $(x_0 = x, x_1 = y)$, $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$, $(x_4 = -\widetilde{y}, x_5 = \widetilde{x})$, and $(x_6 = x, x_7 = y)$. We notice that what allows us to make both $C_1(z)$ and $C_2(z)$ vanish at all nonzero l is the identity

$$3^m - 2^m - 1^m + 0^m = 7^m - 6^m - 5^m + 4^m \quad (18)$$

or alternatively

$$(0^m + 3^m + 5^m + 6^m) - (1^m + 2^m + 4^m + 7^m) = 0 \quad (19)$$

where $m = 1$ and $m = 2$ correspond to the calculations for $C_1(z)$ and $C_2(z)$, respectively. In other words, the reason $C_1(z)$ and $C_2(z)$ can be forced to zero at all nonzero delays is that the set $\mathbb{S} = \{0, 1, \ldots, 7\}$ can be partitioned into two disjoint subsets $\mathbb{S}_0 = \{0, 3, 5, 6\}$ and $\mathbb{S}_1 = \{1, 2, 4, 7\}$ whose elements satisfy (19) for $m = 1, 2$. This is a special case of the Prouhet (or Prouhet–Tarry–Escott) problem [46], [47] which we will discuss in more detail later in this section. But for now we just note that $\mathbb{S}_0$ is the set of all numbers in $\mathbb{S}$ that correspond to the zeros in the length-8 Prouhet–Thue–Morse (PTM) sequence [43]–[46]

$$(s_k)_{k=0}^{7} = 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad (20)$$

and $\mathbb{S}_1$ is the set of all numbers in $\mathbb{S}$ that correspond to the ones in $(s_k)_{k=0}^{7}$.

A key observation here is that the extra Golay pair conditions we had to introduce are all associated with pairs of the form (x_p, x_q) where p and q are odd, and $p \in \mathbb{S}_0$ and $q \in \mathbb{S}_1$. This suggests a close connection between the PTM sequence and the way Golay sequences $x_0, x_1, \ldots, x_{N-1}$ must be paired.

C. The Requirement That $C_1(z)$ Through $C_m(z)$ Vanish at All Nonzero Delays

We now address the general problem of selecting the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ to make $C_m(z)$, $m = 1, 2, \ldots, M$ vanish at all nonzero delays. We begin with some definitions and results related to the PTM sequence.

Definition 2: [43]–[46] The PTM sequence $\mathcal{S} = (s_k)_{k\geq 0}$ over $\{0, 1\}$ is defined by the following recursions:

[2]In writing (16) and(17) we have dropped the argument z on the right-hand side (RHS) of the equations for simplicity.

$$C_1(z) = \underbrace{\underbrace{0|X_0|^2 + 1|X_1|^2}_{[(1-0)=1]|X_1|^2} + \underbrace{2|X_2|^2 + 3|X_3|^2}_{2\times 2L+\,[(3-2)=1]|X_3|^2}}_{3\times 2L} + \underbrace{\underbrace{4|X_4|^2 + 5|X_5|^2}_{4\times 2L+\,[(5-4)=1]|X_5|^2} + \underbrace{6|X_6|^2 + 7|X_7|^2}_{6\times 2L+\,[(7-6)=1]|X_7|^2}}_{11\times 2L} \quad (16)$$

$$C_2(z) = \underbrace{\underbrace{\underbrace{0^2|X_0|^2 + 1^2|X_1|^2}_{[(1^2-0^2)=1]|X_1|^2} + \underbrace{2^2|X_2|^2 + 3^2|X_3|^2}_{4\times 2L+\,[(3^2-2^2)=5]|X_3|^2}}_{5\times 2L+\,[(3^2-2^2-1^2+0^2)=4]|X_3|^2} + \underbrace{\underbrace{4^2|X_4|^2 + 5^2|X_5|^2}_{16\times 2L+\,[(5^2-4^2)=9]|X_5|^2} + \underbrace{6^2|X_6|^2 + 7^2|X_7|^2}_{36\times 2L+\,[(7^2-6^2)=13]|X_7|^2}}_{61\times 2L+\,[(7^2-6^2-5^2+4^2)=4]|X_7|^2}}_{[(0^2+1^2+2^2+\cdots+7^2)=70]\times 2L} \quad (17)$$

1) $s_0 = 0$,
2) $s_{2k} = s_k$,
3) $s_{2k+1} = \overline{s}_k = 1 - s_k$,

for all $k > 0$, where $\overline{s} = 1 - s$ denotes the binary complement of $s \in \{0, 1\}$.

For example, the PTM sequence of length 16 is shown in (21) at the bottom of the page.

Prouhet Problem: [46], [47]. Let $\mathbb{S} = \{0, 1, \ldots, N-1\}$ be the set of all integers between 0 and $N-1$. The Prouhet problem (or Prouhet–Tarry–Escott problem) is the following. Given M, is it possible to partition $\mathbb{S}$ into two disjoint subsets $\mathbb{S}_0$ and $\mathbb{S}_1$ such that

$$\sum_{p \in \mathbb{S}_0} p^m = \sum_{q \in \mathbb{S}_1} q^m \tag{22}$$

for all $0 \leq m \leq M$? Prouhet proved that this is possible when $N = 2^{M+1}$ and that the partitions are identified by the PTM sequence.

Theorem 1 (Prouhet): [46], [47]. Let $\mathcal{S} = (s_k)_{k \geq 0}$ be the PTM sequence. Define

$$\mathbb{S}_0 = \{p \in \mathbb{S} = \{0, 1, 2, \ldots, 2^{M+1} - 1\} |\ s_p = 0\}$$
$$\mathbb{S}_1 = \{q \in \mathbb{S} = \{0, 1, 2, \ldots, 2^{M+1} - 1\} |\ s_q = 1\}.$$

Then, (22) holds for all m, $0 \leq m \leq M$.

Lemma 1: Let $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ be Golay pairs. Let $\mathbb{X}_0 = \{x_p | p \in \mathbb{S}_0\}$ and $\mathbb{X}_1 = \{x_q | q \in \mathbb{S}_q\}$. Then, neither $\mathbb{X}_0$ nor $\mathbb{X}_1$ contains any of the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$.

Proof: The Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are of the form (x_{2k}, x_{2k+1}), where $k = 0, 1, \ldots, N/2 - 1$. From the definition of the PTM sequence we have $s_{2k+1} = \overline{s}_k = \overline{s}_{2k}$. Therefore, x_{2k} and x_{2k+1} cannot be in the same set. □

Lemma 2: Assume that the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ are such that all pairs of the form $(x_{2k'+1} \in \mathbb{X}_0, x_{2k''+1} \in \mathbb{X}_1)$ are also Golay complementary. Then

$$|X_p(z)|^2 = |X_{p'}(z)|^2 \quad \text{and} \quad |X_q(z)|^2 = |X_{q'}(z)|^2 \tag{23}$$

for all p, $p' \in \mathbb{S}_0$ and for all q, $q' \in \mathbb{S}_1$, and all pairs of the form $(x_p \in \mathbb{X}_0, x_q \in \mathbb{X}_1)$ are Golay complementary.

Proof: Assume $p = 2k$ is even and $p \in \mathbb{S}_0$. Then $q = 2k + 1$ is odd and $q \in \mathbb{S}_1$. We know that the pair $(x_{p=2k} \in \mathbb{X}_0, x_{q=2k+1} \in \mathbb{X}_1)$ is Golay complementary, as all the original Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are of the form (x_{2k}, x_{2k+1}), hence

$$|X_p(z)|^2 + |X_q(z)|^2 = 2L. \tag{24}$$

Let $p' \in \mathbb{S}_0$ and assume p' is odd. Then, since $q = 2k + 1 \in \mathbb{S}_1$ and all pairs of the form $(x_{2k'+1} \in \mathbb{X}_0, x_{2k''+1} \in \mathbb{X}_1)$ are Golay complementary (from our assumption), we have

$$|X_{p'}(z)|^2 + |X_q(z)|^2 = 2L. \tag{25}$$

Subtracting (25) from (24) gives

$$|X_p(z)|^2 = |X_{p'}(z)|^2. \tag{26}$$

Since (26) is true for any even $p \in \mathbb{S}_0$ and any odd $p' \in \mathbb{S}_0$ it must be true for any $p, p' \in \mathbb{S}_0$, or equivalently, any $x_p, x_{p'} \in \mathbb{X}_0$. Similarly, we can prove that $|X_q(z)|^2 = |X_{q'}(z)|^2$ for all $x_q, x_{q'} \in \mathbb{X}_1$. Since at least one element from $\mathbb{X}_0$ forms a pair with one element in $\mathbb{X}_1$ (e.g., x_0 and x_1) then all pairs of the form $(x_p \in \mathbb{X}_0, x_q \in \mathbb{X}_1)$ must be Golay complementary. □

Remark 2: We note that to construct Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ that satisfy the conditions of Lemma 2 we can consider an arbitrary Golay pair (x, y) and then arbitrarily choose $x_p \in \mathbb{X}_0$ from the set $\{x, -x, \widetilde{x}, -\widetilde{x}\}$ and $x_q \in \mathbb{X}_1$ from the set $\{y, -y, \widetilde{y}, -\widetilde{y}\}$, for any $p \in \mathbb{S}_0$ and any $q \in \mathbb{S}_1$.

We now present the main result of this section by stating the following theorem.

Theorem 2: The coefficients $C_1(z), \ldots, C_M(z)$ in the Taylor expansion (10) will vanish at all nonzero delays if the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ are selected such that all pairs (x_p, x_q) where p and q are odd and $p \in \mathbb{S}_0$ and $q \in \mathbb{S}_1$ are also Golay complementary.

Proof: From Lemma 2, we have $|X_p(z)|^2 = |X_{p'}(z)|^2$ for all $p, p' \in \mathbb{S}_0$ and $|X_q(z)|^2 = |X_{q'}(z)|^2$ for all $q, q' \in \mathbb{S}_1$. Therefore, we can write $C_m(z)$ $(1 \leq m \leq M)$ as

$$\begin{aligned} C_m(z) &= \sum_{n=0}^{N-1} n^m |X_n(z)|^2 \\ &= (\sum_{p \in \mathbb{S}_0} p^m) |X_0(z)|^2 + (\sum_{q \in \mathbb{S}_1} q^m) |X_1(z)|^2. \end{aligned} \tag{27}$$

From the Prouhet theorem (Theorem 1), we have

$$C_m(z) = \beta(|X_0(z)|^2 + |X_1(z)|^2) = 2\beta L \tag{28}$$

where $\beta = \sum_{p \in \mathbb{S}_0} p^m$.

Definition 3: Let 0 represent x and 1 represent y. Then, a length-N $(N = 2^{M+1})$ *PTM pulse train* is a pulse train in which the transmission of x and y over N PRIs is coordinated according to the zeros and ones in the length-N PTM sequence. If $s_n = 0$ then the nth entry in the PTM pulse train is x, but if $s_n = 1$ then the nth entry is y.

It is easy to see that the length-2^{M+1} PTM pulse train constructed from an arbitrary Golay pair (x, y) satisfies all the con-

$$\mathcal{S} = (s_k)_{k=0}^{15} = 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0. \tag{21}$$

ditions of Theorem 2. For example, the length-8 PTM pulse train built from the Golay pair (x, y) is

$$x \quad y \quad y \quad x \quad y \quad x \quad x \quad y \tag{29}$$

which annihilates $M = 2$ Taylor coefficients $C_1(z)$ and $C_2(z)$ at all nonzero delays.

IV. Doppler Resilient Golay Pairs for Fully Polarimetric Radar Systems

Fully polarimetric radar systems are capable of simultaneously transmitting and receiving on two orthogonal polarizations. The use of two orthogonal polarizations increases the degrees of freedom and can result in significant improvement in detection performance. Recently, Howard *et al.* [14], [15] (also see [16]) proposed a novel approach to radar polarimetry that uses orthogonal polarization modes to provide essentially independent channels for viewing a target, and achieve diversity gain. Unlike conventional radar polarimetry, where polarized waveforms are transmitted sequentially and processed noncoherently, the approach in [14], [15] allows for instantaneous radar polarimetry, where polarization modes are combined coherently on a pulse-by-pulse basis. Instantaneous radar polarimetry enables detection based on full polarimetric properties of the target and hence can provide better discrimination against clutter. When compared to a radar system with a singly-polarized transmitter and a singly-polarized receiver, the instantaneous radar polarimetry can achieve the same detection performance (same false alarm and detection probabilities) with a substantially smaller transmit energy, or alternatively it can detect at substantially greater ranges for a given transmit energy [14], [15].

A key ingredient of the approach in [14], [15] is a unitary Alamouti matrix of Golay waveforms that has a perfect matrix-valued ambiguity function along the zero-Doppler axis. The unitary property of the waveform matrix allows for detection in range based on the full polarimetric properties of the target, without increasing the receiver signal processing complexity beyond that of single-channel-matched filtering. We show in this section that it is possible to design a sequence of Alamouti matrices of Golay waveforms, for which the range sidelobes associated with the matrix-valued ambiguity function vanish for modest Doppler shifts.

Fig. 1 shows the scattering model of the fully polarimetric radar system considered in [14], [15] where h_{VH} denotes the scattering coefficient into the vertical polarization channel from a horizontally polarized incident field. Howard *et al.* employ Alamouti signal processing [48] to coordinate the transmission of a Golay pair (x, y) over vertical and horizontal polarizations during two PRIs. The constructed waveform matrix is

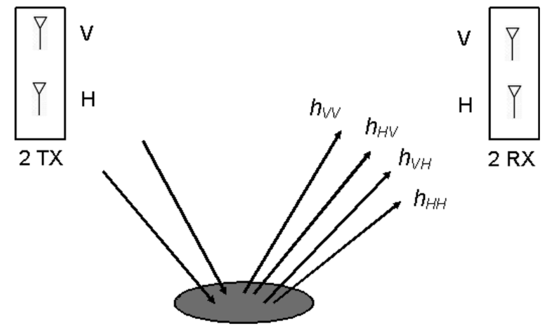


Fig. 1. Scattering model for a fully polarimetric radar system, with a dually-polarized transmit and a dually-polarized receive antenna.

$$\begin{pmatrix} X(z) & -\widetilde{Y}(z) \\ Y(z) & \widetilde{X}(z) \end{pmatrix}$$

where different rows correspond to vertical and horizontal polarizations, and different columns correspond to different time slots (PRIs).

Suppose now that $(N/2)$ Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are transmitted in the above fashion over N PRIs, where N is even. Then, the waveform matrix $\boldsymbol{X}(z)$ consists of a sequence of Alamouti matrices and is given by (30) shown at the bottom of the page .

The radar measurement matrix $\boldsymbol{R}(z)$ for this transmission scheme can be written as

$$\boldsymbol{R}(z) = z^{-d_0}\boldsymbol{H}\boldsymbol{X}(z)\boldsymbol{D}(\theta_0) + \boldsymbol{W}(z) \tag{31}$$

where $\boldsymbol{H}$ is the 2 by 2 target scattering matrix, with entries h_{VV}, h_{VH}, h_{HV}, and h_{HH}, $\boldsymbol{W}(z)$ is a 2 by N noise matrix, and $\boldsymbol{D}(\theta)$ is the diagonal Doppler modulation matrix introduced in (5).

If we process $\boldsymbol{R}(z)$ with a receiver matrix $\widetilde{\boldsymbol{X}}(z)$ given by (32), also shown at the bottom of the page, then the receiver output will be

$$\boldsymbol{R}(z)\widetilde{\boldsymbol{X}}(z) = z^{-d_0}\boldsymbol{H}\boldsymbol{G}(z, \theta_0) + \boldsymbol{W}(z)\widetilde{\boldsymbol{X}}(z). \tag{33}$$

The matrix $\boldsymbol{G}(z, \theta) = \boldsymbol{X}(z)\boldsymbol{D}(\theta)\widetilde{\boldsymbol{X}}(z)$ can be viewed as the z-transform of a matrix-valued ambiguity function for $\boldsymbol{X}(z)$. Along the zero-Doppler axis, where $\boldsymbol{D}(0) = \boldsymbol{I}$, due to the interplay between Alamouti signal processing and the Golay property, $\boldsymbol{G}(z, \theta)$ reduces to (34) shown at the bottom of the following page. This shows that $\boldsymbol{X}(z)$ has a perfect matrix-valued ambiguity function along the zero-Doppler axis; that is, along the zero-Doppler axis $\boldsymbol{G}(z, \theta)$ vanishes at all nonzero (integer)

$$\boldsymbol{X}(z) = \begin{pmatrix} X_0(z) & -\widetilde{X}_1(z) & \ldots & X_{2k}(z) & -\widetilde{X}_{2k+1}(z) & \ldots & X_{N-2}(z) & -\widetilde{X}_{N-1}(z) \\ X_1(z) & \widetilde{X}_0(z) & \ldots & X_{2k+1}(z) & \widetilde{X}_{2k}(z) & \ldots & X_{N-1}(z) & \widetilde{X}_{N-2}(z) \end{pmatrix}. \tag{30}$$

$$\widetilde{\boldsymbol{X}}(z) = \begin{pmatrix} \widetilde{X}_0(z) & -X_1(z) & \ldots & \widetilde{X}_{2k}(z) & -X_{2k+1}(z) & \ldots & \widetilde{X}_{N-2}(z) & -X_{N-1}(z) \\ \widetilde{X}_1(z) & X_0(z) & \ldots & \widetilde{X}_{2k+1}(z) & X_{2k}(z) & \ldots & \widetilde{X}_{N-1}(z) & X_{N-2}(z) \end{pmatrix}^T \tag{32}$$

delays, and is unitary at zero-delay. A consequence of (34) is that the full scattering matrix $\boldsymbol{H}$ can be made available on a pulse-by-pulse basis with a computational complexity comparable to that of single-channel matched filtering. Off the zero-Doppler axis however the property in (34) no longer holds, and the elements of the matrix-valued ambiguity function have large range sidelobes, even at small Doppler shifts.

We consider how the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ must be selected so that for small Doppler shifts we have

$$\boldsymbol{G}(z,\theta) = \boldsymbol{X}(z)\boldsymbol{D}(\theta)\widetilde{\boldsymbol{X}}(z) = \begin{pmatrix} G_1(z,\theta) & G_2(z,\theta) \\ \widetilde{G}_2(z,\theta) & G_1(z,\theta) \end{pmatrix} \approx \alpha(\theta) \begin{pmatrix} NL & 0 \\ 0 & NL \end{pmatrix} \quad (35)$$

where $\alpha(\theta)$ is some function of θ independent of the delay operator z^{-1}. $G_1(z,\theta)$ and $G_2(z,\theta)$ are given by

$$G_1(z,\theta) = \sum_{n=0}^{N-1} e^{jn\theta} |X_n(z)|^2, \quad (36)$$

and

$$G_2(z,\theta) = \sum_{k=0}^{N/2-1} (e^{j2k\theta} - e^{j(2k+1)\theta}) X_{2k}(z)\widetilde{X}_{2k+1}(z). \quad (37)$$

The diagonal term of $\boldsymbol{G}(z,\theta)$, i.e., $G_1(z,\theta)$, is equal to $G(z,\theta)$ in (7). Therefore, we can use Theorem 2 to design the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ such that in the Taylor expansion (10) the coefficients $C_m(z), m = 1, 2, \ldots, M$ vanish at all nonzero delays. Thus, from now on we only discuss how the off-diagonal term $G_2(z,\theta)$ can be forced to zero for small Doppler shifts.

Consider the Taylor expansion of $G_2(z,\theta)$ around $\theta = 0$, i.e.,

$$G_2(z,\theta) = \sum_{m=0}^{\infty} B_m(z)(j\theta)^m \quad (38)$$

where the coefficients $B_m(z)$, $m = 0, 1, 2, \ldots$, are given by

$$B_m(z) = \sum_{k=0}^{N/2-1} ((2k)^m - (2k+1)^m) X_{2k}(z)\widetilde{X}_{2k+1}(z). \quad (39)$$

In general, the coefficients $B_m(z)$, $m = 1, 2, 3, \ldots$ are two-sided polynomials in z^{-1} of the form

$$B_m(z) = \sum_{l=-(L-1)}^{L-1} b_{m,l} z^{-l}, \quad m = 1, 2, 3, \ldots. \quad (40)$$

For instance, the first coefficient $B_1(z)$ is

$$B_1(z) = (0-1)X_0(z)\widetilde{X}_1(z) + \cdots + ((N-2) - (N-1))X_{N-2}(z)\widetilde{X}_{N-1}(z). \quad (41)$$

Each term of the form $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ in (41) is a two-sided polynomial of degree $L-1$ in z^{-1}, and since, in general, the terms $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ for different values of k do not cancel each other, $B_1(z)$ is also a two-sided polynomial of degree $L-1$ in z^{-1}.

Suppose that the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ satisfy the conditions of Theorem 2 so that $C_1(z), \ldots, C_M(z)$ vanish at all nonzero delays. We wish to determine the extra conditions required for $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ to force $B_1(z), \ldots, B_M(z)$ to zero at all delays. As we show, again the PTM sequence is the key to finding the zero-forcing conditions. The zero-order term $B_0(z)$ is always zero and hence we do not consider it in our discussion.

A. The Requirement That $B_1(z)$ Vanishes

Again, to gain intuition, we first consider the case $N = 2^2 = 4$. Then, as the calculation in (42) at the bottom of the page shows, $B_1(z)$ will vanish if the Golay pairs (x_0, x_1) and (x_2, x_3) are selected so that $X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z)$. In summary, to make $C_1(z)$ vanish at all nonzero delays and to force $B_1(z)$ to zero at the same time, the Golay pairs (x_0, x_1) and (x_2, x_3) must be selected such that (x_1, x_3) is also a Golay pair and $X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z)$. If we let (x, y) be an arbitrary Golay pair then it is easy to see that $(x_0 = x, x_1 = y)$, $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$ satisfy these conditions. The Alamouti waveform matrix $\boldsymbol{X}(z)$ for this choice of Golay pairs is given by (43), shown at the bottom of the page. Other choices are also possible.

$$\boldsymbol{G}(z,0) = \boldsymbol{X}(z)\widetilde{\boldsymbol{X}}(z) = \begin{pmatrix} \sum_{n=0}^{N-1} |X_n(z)|^2 = NL & \sum_{k=0}^{N/2-1} (1-1)X_{2k}(z)\widetilde{X}_{2k+1}(z) = 0 \\ \sum_{k=0}^{N/2-1} (1-1)\widetilde{X}_{2k}(z)X_{2k+1}(z) = 0 & \sum_{n=0}^{N-1} |X_n(z)|^2 = NL \end{pmatrix}. \quad (34)$$

$$B_1(z) = \underbrace{(0-1)X_0(z)\widetilde{X}_1(z) + \underbrace{(2-3)X_2(z)\widetilde{X}_3(z)}_{-(2-3)X_0(z)\widetilde{X}_1(z)}}_{0} \quad (42)$$

$$\boldsymbol{X}(z) = \begin{pmatrix} X_0(z) = X(z) & -\widetilde{X}_1(z) = -\widetilde{Y}(z) & X_2(z) = -\widetilde{Y}(z) & -\widetilde{X}_3(z) = -X(z) \\ X_1(z) = Y(z) & \widetilde{X}_0(z) = \widetilde{X}(z) & X_3(z) = \widetilde{X}(z) & \widetilde{X}_2(z) = -Y(z) \end{pmatrix}. \quad (43)$$

$$B_1(z) = \underbrace{\underbrace{(0-1)X_0\widetilde{X}_1}_{[(0-1)=-1]X_0\widetilde{X}_1} + \underbrace{(2-3)X_2\widetilde{X}_3}_{[-(2-3)=1]X_0\widetilde{X}_1} + \underbrace{(4-5)X_4\widetilde{X}_5}_{[-(4-5)=1]X_0\widetilde{X}_1} + \underbrace{(6-7)X_6\widetilde{X}_7}_{[(6-7)=-1]X_0\widetilde{X}_1}}_{0}. \quad (44)$$

$$B_2(z) = \underbrace{\underbrace{\underbrace{(0^2-1^2)X_0\widetilde{X}_1}_{[(0^2-1^2)=-1]X_0\widetilde{X}_1} + \underbrace{(2^2-3^2)X_2\widetilde{X}_3}_{[-(2^2-3^2)=5]X_0\widetilde{X}_1}}_{[0^2-1^2-2^2+3^2=4]X_0\widetilde{X}_1} + \underbrace{\underbrace{(4^2-5^2)X_4\widetilde{X}_5}_{[-(4^2-5^2)=9]X_0\widetilde{X}_1} + \underbrace{(6^2-7^2)X_6\widetilde{X}_7}_{[(6^2-7^2)=-13]X_0\widetilde{X}_1}}_{[-4^2+5^2+6^2-7^2=-4]X_0\widetilde{X}_1}}_{0} \quad (45)$$

B. The Requirement That $B_1(z)$ and $B_2(z)$ Vanish

Let us now consider the case $N = 2^3 = 8$. Then, as the calculations in (44) and (45) at the top of the page show, both $B_1(z)$ and $B_2(z)$ will vanish if we select $(x_0, x_1), \ldots, (x_6, x_7)$ such that $X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z) = -X_4(z)\widetilde{X}_5(z) = X_6(z)\widetilde{X}_7(z)$.[3] Making $B_1(z)$ vanish we get equation (44) at the top of the page. Making $B_2(z)$ vanish we get (45) at the top of the page.

In summary, to make $C_1(z)$ and $C_2(z)$ vanish at all nonzero delays and to force $B_1(z)$ and $B_2(z)$ to zero at the same time, the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ must satisfy the conditions of Theorem 2, and the within-pair cross-spectral densities must satisfy

$$\begin{aligned} X_0(z)\widetilde{X}_1(z) &= -X_2(z)\widetilde{X}_3(z) \\ &= -X_4(z)\widetilde{X}_5(z) \\ &= X_6(z)\widetilde{X}_7(z). \end{aligned} \quad (46)$$

Let (x, y) be an arbitrary Golay pair. Then, it is easy to see that the Golay pairs in the waveform matrix $\boldsymbol{X}(z)$ given by

$$\boldsymbol{X}(z) = (\,\boldsymbol{X}_0(z) \quad \boldsymbol{X}_1(z) \quad \boldsymbol{X}_1(z) \quad \boldsymbol{X}_0(z)\,) \quad (47)$$

where

$$\boldsymbol{X}_0(z) = \begin{pmatrix} X(z) & -\widetilde{Y}(z) \\ Y(z) & \widetilde{X}(z) \end{pmatrix} \quad (48)$$

and

$$\boldsymbol{X}_1(z) = \begin{pmatrix} -\widetilde{Y}(z) & -X(z) \\ \widetilde{X}(z) & -Y(z) \end{pmatrix} \quad (49)$$

satisfy all the zero-forcing conditions.

The trick in forcing $B_1(z)$ and $B_2(z)$ to zero is to cleverly select the signs of the cross-correlation functions (cross-spectral densities) between the two sequences in every Golay pair relative to the cross-correlation function (cross-spectral density) for x_0 and x_1. If we let 0 and 1 correspond to the positive and negative signs, respectively; we observe that the sequence of signs in (46) corresponds to the length-4 PTM sequence. In Section V, we show that the PTM sequence is in fact the right sequence for specifying the relative signs of the cross-correlation functions between the Golay sequences in each Golay pair.

[3]We have dropped the argument z from the RHS of (44) and (45) for simplicity.

Remark 3: Representing $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ by 0 and 1, respectively, we notice that the placements of $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ in $\boldsymbol{X}(z)$ are also determined by the length-4 PTM sequence.

C. The Requirement That $B_1(z)$ Through $B_m(z)$ Vanish

We now consider the general case $N = 2^{M+1}$ where Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are used to construct a Doppler resilient waveform matrix $\boldsymbol{X}(z)$. We have the following theorem.

Theorem 3: Let $N = 2^{M+1}$ and let $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ be Golay pairs. Then, for any m between 1 and M, $B_m(z)$ will vanish if for all k, $0 \leq k \leq N/2 - 1$, we have

$$X_{2k}(z)\widetilde{X}_{2k+1}(z) = (-1)^{s_k} X_0(z)\widetilde{X}_1(z) \quad (50)$$

where s_k is the kth element in the PTM sequence.

Proof: For any m $(1 \leq m \leq M)$, $B_m(z)$ may be written as

$$\begin{aligned} B_m(z) &= \sum_{k=0}^{N/2-1} ((2k)^m - (2k+1)^m) X_{2k}(z)\widetilde{X}_{2k+1}(z) \\ &= \left[\sum_{k=0}^{N/2-1} (-1)^{s_k}((2k)^m - (2k+1)^m)\right] X_0(z)\widetilde{X}_1(z) \end{aligned} \quad (51)$$

where the second equality in (51) follows by replacing $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ with $(-1)^{s_k} X_0(z)\widetilde{X}_1(z)$. Since in the PTM sequence $s_k = s_{2k} = \overline{s}_{2k+1}$, we can rewrite (51) as

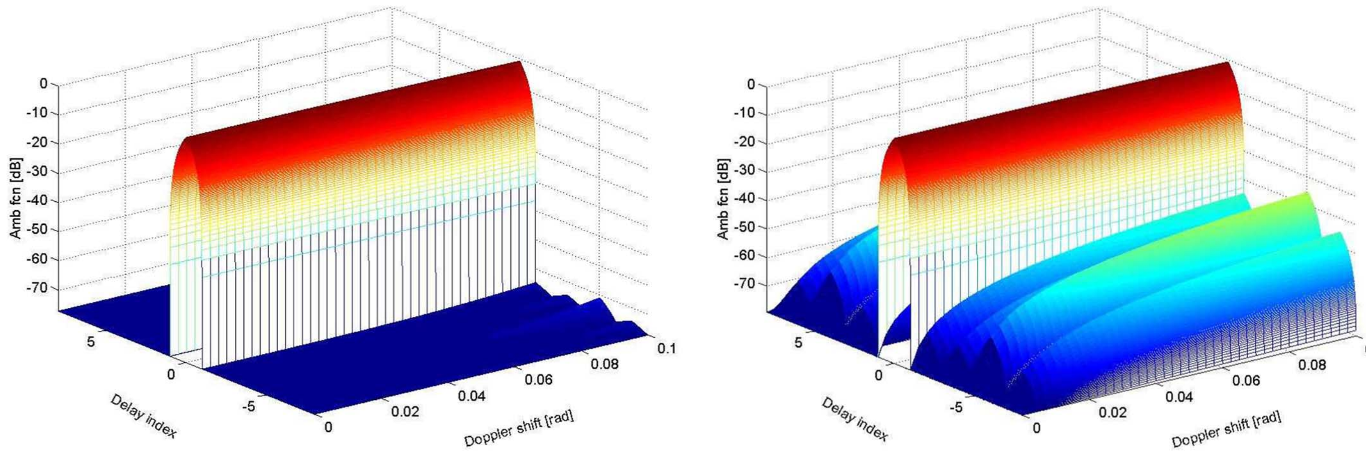
$$B_m(z) = \left[\sum_{k=0}^{N-1} (-1)^{s_k} k^m\right] X_0(z)\widetilde{X}_1(z). \quad (52)$$

However, from the Prouhet theorem (Theorem 1), it is easy to see that

$$\sum_{k=0}^{N-1} (-1)^{s_k} k^m = 0. \quad (53)$$

Therefore, $B_m(z) = 0$. □

Finally, we note that it is always possible to find Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ that satisfy the conditions of both Theorem 2 and Theorem 3. Suppose $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are built from an arbitrary Golay pair (x, y) (as



(a) $g(\ell,\theta)$ (b) $g_c(\ell,\theta)$

Fig. 2. (a) The plot of the ambiguity function $g(\ell,\theta)$ (corresponding to the Doppler resilient transmission scheme) versus delay index ℓ and Doppler shift θ. (b) The plot of the ambiguity function $g_c(\ell,\theta)$ (corresponding to the conventional transmission scheme) versus delay index ℓ and Doppler shift θ.

explained in Section III) to satisfy the conditions of Theorem 2. Then, we can apply the time reversal operator and change the sign of the elements within the pairs to satisfy the conditions of Theorem 3, as the Golay property is invariant to time reversal and changes in the signs of the Golay sequences within a pair. For example, a sequence of Alamouti matrices in which the placement of $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ is coordinated by the zeros and ones in the PTM sequence satisfies all the conditions of Theorems 2 and 3.

V. Numerical Examples

In this section, we present numerical examples to verify the results of Sections III and IV and compare our Doppler resilient design to a conventional scheme, where the same Golay pair is repeated.

A. Single-Channel Radar System

We first consider the case of a single-channel radar system. Following Theorem 2, we coordinate the transmission of eight Golay pairs $(x_0,x_1),\ldots,(x_{14},x_{15})$ over $N=16$ PRIs to make the Taylor expansion coefficients $C_1(z),\ldots,C_3(z)$ $(M=3)$ vanish at all nonzero delays. Starting from a Golay pair (x,y), it is easy to verify that the eight Golay pairs in the following waveform vector $\underline{\boldsymbol{x}}^T(z)=\underline{\boldsymbol{x}}^T$ satisfy the conditions of Theorem 2:

$$\underline{\boldsymbol{x}}^T = (\,\underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_1^T \quad \underline{\boldsymbol{x}}_1^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_1^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_1^T\,) \tag{54}$$

where $\underline{\boldsymbol{x}}_0^T=\underline{\boldsymbol{x}}_0^T(z)=[X(z)\quad Y(z)]$ and $\underline{\boldsymbol{x}}_1^T=\underline{\boldsymbol{x}}_1^T(z)=[-\widetilde{Y}(z)\quad \widetilde{X}(z)]$.

Remark 4: Representing $\underline{\boldsymbol{x}}_0^T(z)$ and $\underline{\boldsymbol{x}}_1^T(z)$ by 0 and 1, respectively, we notice that the placements of $\underline{\boldsymbol{x}}_0^T(z)$ and $\underline{\boldsymbol{x}}_1^T(z)$ in $\underline{\boldsymbol{x}}^T(z)$ are determined by the length-8 PTM sequence.

We compare the Doppler resilient transmission scheme in (54) with a conventional transmission scheme, where the same Golay pair $(x_0=x, x_1=y)$ is transmitted during all PRIs, resulting in a waveform vector $\underline{\boldsymbol{x}}_c^T(z)=\underline{\boldsymbol{x}}_c^T$ of the form

$$\underline{\boldsymbol{x}}_c^T = (\,\underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T \quad \underline{\boldsymbol{x}}_0^T\,) \tag{55}$$

with the ambiguity function (in z-domain)

$$\begin{aligned} G_c(z,\theta) &= \underline{\boldsymbol{x}}_c^T(z)\boldsymbol{D}(\theta)\widetilde{\underline{\boldsymbol{x}}}_c(z) \\ &= \left(\sum_{k=0}^{N/2-1} e^{j(2k)\theta}\right)|X_0(z)|^2 \\ &\quad + \left(\sum_{k=0}^{N/2-1} e^{j(2k+1)\theta}\right)|X_1(z)|^2. \end{aligned} \tag{56}$$

The pair (x,y) used in constructing $\underline{\boldsymbol{x}}^T(z)$ and $\underline{\boldsymbol{x}}_c^T(z)$ can be any Golay pair. Here, we choose (x,y) to be the following length-8 $(L=8)$ Golay pair

$$\begin{aligned} x[l] &= \{1,1,-1,1,1,1,1,-1\} \\ y[l] &= \{-1,-1,1,-1,1,1,1,-1\} \end{aligned} \tag{57}$$

with z-transforms

$$\begin{aligned} X(z) &= 1+z^{-1}-z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}-z^{-7} \\ Y(z) &= -1-z^{-1}+z^{-2}-z^{-3}+z^{-4}+z^{-5}+z^{-6}-z^{-7}. \end{aligned}$$

Referring to the Taylor expansion of $G(z,\theta)$ in (10), it is easy to verify that $C_1(z)$, $C_2(z)$, and $C_3(z)$ vanish at all nonzero delays for the Doppler resilient design in (54).

Fig. 2 shows the plots of the ambiguity functions $g(\ell,\theta)=\mathcal{Z}^{-1}\{G(z,\theta)\}$ and $g_c(\ell,\theta)=\mathcal{Z}^{-1}\{G_c(z,\theta)\}$ versus delay

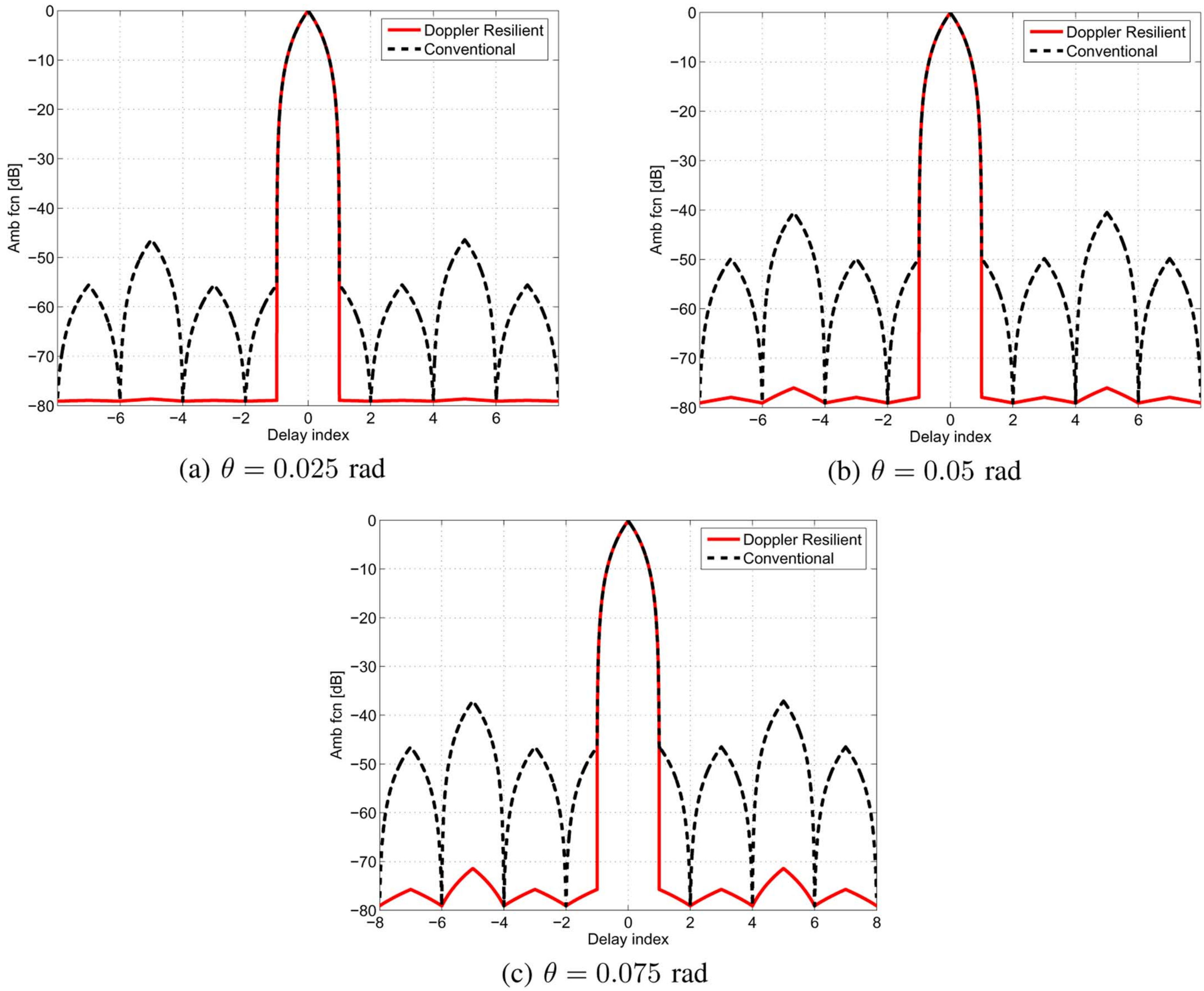


Fig. 3. Comparison of the ambiguity functions $g(\ell, \theta)$ and $g_c(\ell, \theta)$ at Doppler shifts (a) $\theta = 0.025$ rad, (b) $\theta = 0.05$ rad, and (c) $\theta = 0.075$ rad.

index ℓ and Doppler shift θ.[4] Comparison of $g(\ell, \theta)$ and $g_c(\ell, \theta)$ at Doppler shifts $\theta = 0.025$ rad, $\theta = 0.05$ rad, and $\theta = 0.075$ rad is provided in Fig. 3(a)–(c), where the solid lines correspond to $g(\ell, \theta)$ (Doppler resilient scheme) and the dashed lines correspond to $g_c(\ell, \theta)$ (conventional scheme). We notice that the peaks of the range sidelobes of $g(\ell, \theta)$ are at least 24 dB (for $\theta = 0.025$ rad), 28 dB (for $\theta = 0.05$ rad), and 29 dB (for $\theta = 0.075$ rad) smaller than those of $g_c(\ell, \theta)$. These plots clearly show the Doppler resilience of the waveform vector in (54).

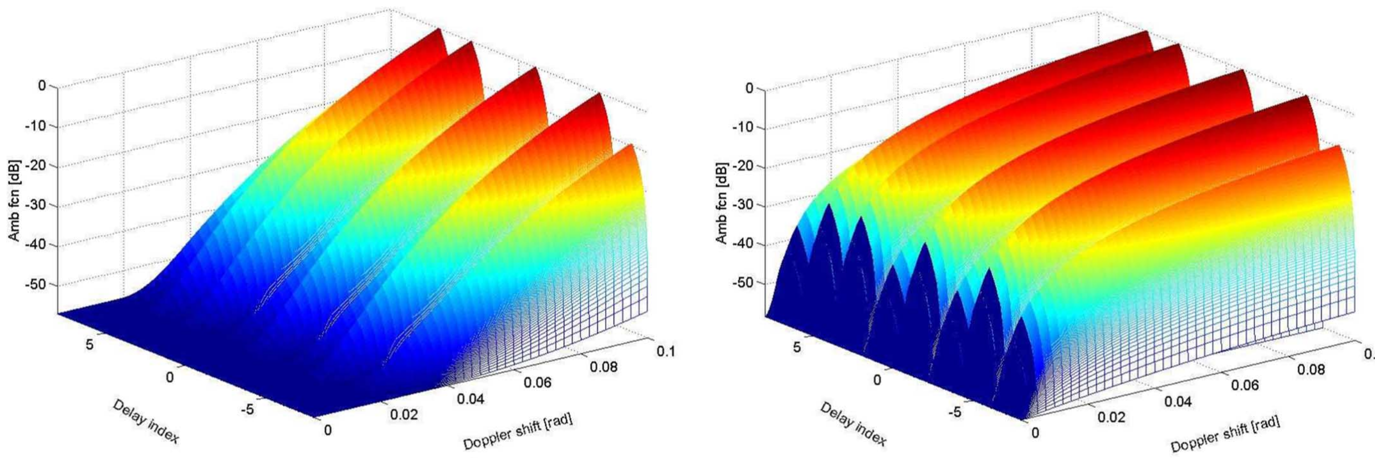
Remark 5: By increasing the number of PRIs N (in powers of two) more of the Taylor expansion coefficients can be zero-forced (at all nonzero delays) and the width of the Doppler resilient interval can be increased. In practice, however, N cannot be made arbitrarily large, as we have a limited amount of time to interrogate a range cell. We note that finding an exact relationship between the width of the Doppler resilient interval and the number of PRIs (or, equivalently, the length of the PTM sequence) requires an in-depth analysis of the Taylor expansion in (10) and is beyond the scope of this paper.

B. Fully Polarimetric Radar System

We now consider the matrix-valued ambiguity function corresponding to the fully polarimetric radar system described in Section IV. Following Theorems 2 and 3, we coordinate the transmission of eight Golay pairs $(x_0, x_1), \ldots, (x_{14}, x_{15})$ across vertical and horizontal polarizations and over $N = 16$ PRIs, so that in the Taylor expansions of $G_1(z, \theta)$ (the diagonal element of $\boldsymbol{G}(z, \theta)$) and $G_2(z, \theta)$ (the off-diagonal element of $\boldsymbol{G}(z, \theta)$) the coefficients $C_1(z)$, $C_2(z)$, and $C_3(z)$ vanish at all nonzero delays and $B_1(z)$, $B_2(z)$, and $B_3(z)$ vanish at all delays. Letting $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ be the Alamouti matrices in (48) and (49), then it is easy to check that the Golay pairs in the following waveform matrix $\boldsymbol{X}(z) = \boldsymbol{X}$ satisfy all the conditions of Theorems 2 and 3:

$$\boldsymbol{X} = (\,\boldsymbol{X}_0 \quad \boldsymbol{X}_1 \quad \boldsymbol{X}_1 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_1 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_1\,). \tag{58}$$

[4]The ambiguity plots are interpolated in delay index for ease in visual inspection.



(a) $g_2(\ell, \theta)$ (b) $g_{c2}(\ell, \theta)$

Fig. 4. (a) The plot of $g_2(\ell, \theta)$ (corresponding to the Doppler resilient transmission scheme) versus delay index ℓ and Doppler shift θ. (b) The plot of $g_{c2}(\ell, \theta)$ (corresponding to the conventional transmission scheme) versus delay index ℓ and Doppler shift θ.

Remark 6: Representing $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ by 0 and 1, respectively, we notice that the placements of $\boldsymbol{X}_0(z)$ and $\boldsymbol{X}_1(z)$ in $\boldsymbol{X}(z)$ are determined by the length-8 PTM sequence.

We compare the Doppler resilient transmission scheme in (58) with a conventional transmission scheme, where the Alamouti waveform matrix built from a single Golay pair ($x_0 = x$, $x_1 = y$) is repeated and the waveform matrix $X_c(z) = X_c$ is given by

$$\boldsymbol{X}_c = (\boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0 \quad \boldsymbol{X}_0). \tag{59}$$

The matrix-valued ambiguity function (in z domain) for $\boldsymbol{X}_c(z)$ is given by

$$\begin{aligned}\boldsymbol{G}_c(z,\theta) &= \boldsymbol{X}_c(z)\boldsymbol{D}(\theta)\widetilde{\boldsymbol{X}}_c(z)\\ &= \begin{pmatrix} G_{c1}(z,\theta) & G_{c2}(z,\theta)\\ \widetilde{G}_{c2}(z,\theta) & G_{c1}(z,\theta)\end{pmatrix}\end{aligned} \tag{60}$$

where

$$\begin{aligned}G_{c1}(z,\theta) = &\left(\sum_{k=0}^{N/2-1} e^{j(2k)\theta}\right)|X_0(z)|^2\\ &+\left(\sum_{k=0}^{N/2-1} e^{j(2k+1)\theta}\right)|X_1(z)|^2\end{aligned} \tag{61}$$

and

$$G_{c2}(z,\theta) = \left(\sum_{n=0}^{N-1}(-1)^n e^{jn\theta}\right) X_0(z)\widetilde{X}_1(z). \tag{62}$$

The Golay pair (x, y) used in building both $\boldsymbol{X}(z)$ and $\boldsymbol{X}_c(z)$ is the length-8 Golay pair in (57).

The diagonal elements of $\boldsymbol{G}(z,\theta)$ and $\boldsymbol{G}_c(z,\theta)$, i.e., $G_1(z,\theta)$ and $G_{c1}(z,\theta)$, are equal to $G(z,\theta)$ and $G_c(z,\theta)$, respectively. Therefore, the plots in Figs. 2 and 3 apply for comparing the diagonal elements. Thus, in this example, we only need to consider the off-diagonal terms $G_2(z,\theta)$ and $G_{c2}(z,\theta)$.

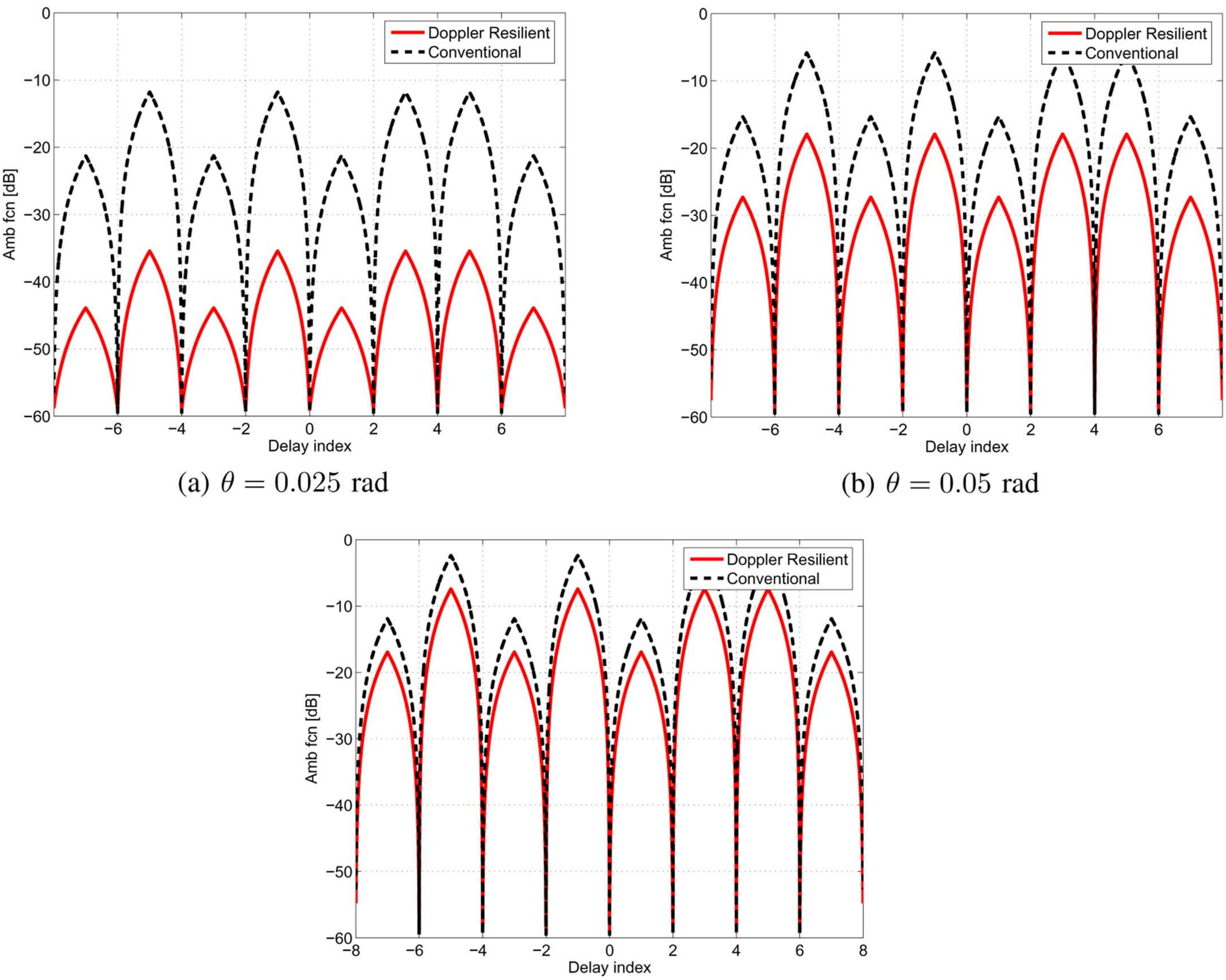
Referring to the Taylor expansion of $G_2(z,\theta)$ in (38), it is easy to verify that $B_1(z)$, $B_2(z)$, and $B_3(z)$ vanish at all delays for the Doppler resilient design in (58).

Fig. 4 shows the plots of $g_2(\ell,\theta) = \mathcal{Z}^{-1}\{G_2(z,\theta)\}$ and $g_{c2}(\ell,\theta) = \mathcal{Z}^{-1}\{G_{c2}(z,\theta)\}$ versus delay index ℓ and Doppler shift θ. Comparison of $g_2(\ell,\theta)$ and $g_{c2}(\ell,\theta)$ at Doppler shifts $\theta = 0.025$ rad, $\theta = 0.05$ rad, and $\theta = 0.075$ rad is provided in Fig. 5(a)–(c), where the solid lines correspond to $g_2(\ell,\theta)$ (Doppler resilient scheme) and the dashed lines correspond to $g_{c2}(\ell,\theta)$ (conventional scheme). We notice that the peaks of the range sidelobes of $g_2(\ell,\theta)$ are at least 24 dB (for $\theta = 0.025$ rad), 12 dB (for $\theta = 0.05$ rad), and 5 dB (for $\theta = 0.075$ rad) smaller than those of $g_{c2}(\ell,\theta)$. These plots together with the plots in Fig. 3(a)–(c) show the Doppler resilience of the waveform matrix in (58).

Remark 7: The range sidelobes due to a point scatterer correspond to the sum of the range sidelobes of $g_1(\ell,\theta)$ and $g_2(\ell,\theta)$, weighted by the target scattering coefficients $(h_{\mathrm{VV}}, h_{\mathrm{VH}})$ or $(h_{\mathrm{HV}}, h_{\mathrm{HH}})$. In the example considered here, the range sidelobes corresponding to the off-diagonal term, shown in Figs. 4(a) and 5, are considerably larger than the range sidelobes for the diagonal term, shown in Figs. 2(a) and 3. Therefore, here the overall range sidelobe improvement of the Doppler resilient design is determined by the improvement for the off-diagonal term.

VI. Conclusion

We have constructed a Doppler resilient sequence of Golay complementary waveforms, for which the pulse train ambiguity function is free of range sidelobes at modest Doppler shifts. We have extended our results to the design of Doppler resilient Alamouti matrices of Golay complementtary waveforms for instantaneous radar polarimetry. The main contribution is a method for selecting Golay complementary sequences to force the low-order terms of the Taylor expansion of an ambiguity function to zero. The PTM sequence was found to be the key to constructing the Doppler resilient sequences of Golay



(a) $\theta = 0.025$ rad

(b) $\theta = 0.05$ rad

(c) $\theta = 0.075$ rad

Fig. 5. Comparison of $g_2(\ell, \theta)$ and $g_{c2}(\ell, \theta)$ corresponding to the conventional and Doppler resilient schemes at Doppler shifts (a) $\theta = 0.025$ rad, (b) $\theta = 0.05$ rad, and (c) $\theta = 0.075$ rad.

pairs. Numerical examples were presented, demonstrating the resilience of PTM sequences of Golay pairs to modest Doppler shifts.

ACKNOWLEDGMENT

The authors would like to thank Louis Scharf for his comments.

REFERENCES

[1] N. Levanon and E. Mozeson, *Radar Signals*. New York: Wiley, 2004.
[2] R. C. Heimiller, "Phase shift pulse codes with good periodic correlation properties," *IRE Trans. Inf. Theory*, vol. IT-7, no. 4, pp. 254–257, Oct. 1961.
[3] R. L. Frank, "Polyphase codes with good nonperiodic correlation properties," *IEEE Trans. Inf. Theory*, vol. IT-9, no. 1, pp. 43–45, Jan. 1963.
[4] D. C. Chu, "Polyphase codes with good periodic correlation properties," *IEEE Trans. Inf. Theory*, vol. IT-18, no. 4, pp. 531–532, Jul. 1972.
[5] R. H. Barker, "Group synchronizing of binary digital sequences," in *Communication Theory*, W. Jackson, Ed. London, U.K.: Butteworth, 1953, pp. 273–287.
[6] S. W. Golomb and R. A. Scholtz, "Generalized Barker sequences," *IEEE Trans. Inf. Theory*, vol. IT-11, no. 4, pp. 533–537, Oct. 1965.
[7] M. J. E. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. IT-7, no. 2, pp. 82–87, Apr. 1961.
[8] R. J. Turyn, "Ambiguity functions of complementary sequences," *IEEE Trans. Inf. Theory*, vol. IT-9, no. 1, pp. 46–47, Jan. 1963.
[9] G. R. Welti, "Quaternary codes for pulsed radar," *IRE Trans. Inf. Theory*, vol. IT-6, no. 3, pp. 400–408, Jun. 1960.
[10] Y. Taki, M. Miyakawa, M. Hatori, and S. Namba, "Even-shift orthogonal sequences," *IEEE Trans. Inf. Theory*, vol. IT-15, no. 2, pp. 295–300, Mar. 1969.
[11] C. C. Tseng and C. L. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. IT-18, no. 5, pp. 644–652, Sep. 1972.
[12] R. Sivaswami, "Multiphase complementary codes," *IEEE Trans. Inf. Theory*, vol. IT-24, no. 5, pp. 546–552, Sep. 1978.
[13] R. L. Frank, "Polyphase complementary codes," *IEEE Trans. Inf. Theory*, vol. IT-26, no. 6, pp. 641–647, Nov. 1980.
[14] S. D. Howard, A. R. Calderbank, and W. Moran, "A simple polarization diversity technique for radar detection," in *Proc. 2nd Int. Conf. Waveform Diversity and Design*, Lihue, HI, Jan. 2006.
[15] S. D. Howard, A. R. Calderbank, and W. Moran, "A simple signal processing architecture for instantaneous radar polarimetry," *IEEE Trans. Inf. Theory*, vol. 53, no. 4, pp. 1282–1289, Apr. 2007.

[16] A. R. Calderbank, S. D. Howard, W. Moran, A. Pezeshki, and M. Zoltowski, "Instantaneous radar polarimetry with multiple dually-polarized antennas," in *Conf. Rec. 40th Asilomar Conf. Signals, Systemas and Computers*, Pacific Grove, CA, Oct. 2006, pp. 757–761.

[17] S. J. Searle and S. D. Howard, "A novel polyphase code for sidelobe suppression," in *Proc. Int. Waveform Diversity and Design Conf.*, Pisa, Italy, Jun. 2007, pp. 377–381.

[18] S. J. Searle and S. D. Howard, "A novel nonlinear technique for sidelobe suppression in radar," in *Proc. Int. Conf. Radar Systems*, Edinburgh, U.K., Oct. 2007.

[19] K. Harman and B. Hodgins, "Next generation of GUIDAR technology," *IEEE Aerosp. Electron. Syst. Mag.*, vol. 20, no. 5, pp. 16–26, Mar. 2005.

[20] N. Suehiro and M. Hatori, "N-shift cross-orthogonal sequences," *IEEE Trans. Inf. Theory*, vol. 34, no. 1, pp. 143–146, Jan. 1988.

[21] T. A. Wilkinson and A. E. Jones, "Minimization of the peak-to-mean envelope power ratio of multicarrier transmission scheme by block coding," in *Proc. IEEE Vehicular Technology Conf. (VTC)*, Chicago, IL, Jul. 1995, pp. 825–829.

[22] R. J. van Nee, "OFDM codes for peak-to-average power reduction and error correction," in *Proc. IEEE Global Telecom. Conf. (GLOBECOM)*, London, U.K., Nov. 1996, pp. 740–744.

[23] H. Ochiai and H. Imai, "Block coding scheme based on complementary sequences for multicarrier signals," *IEICE Trans. Fundamentals*, vol. E80-A, pp. 2136–2146, 1997.

[24] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed–Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 6, pp. 2397–2417, Nov. 1999.

[25] K. G. Paterson, "Generalized Reed–Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, no. 1, pp. 104–120, Jan. 2000.

[26] K. Schmidt, "On cosets of the generalized first-order Reed–Muller code with low PMEPR," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 3220–3232, Jun. 2006.

[27] K. Schmidt and A. Finger, "Constructions of complementary sequences for power-controlled OFDM transmission," in *Proc. Workshop on Coding and Cryptography (WCC) 2005 (Lecture Notes in Computer Science)*. Berlin, Germany: Springer-Verlag, 2006.

[28] M. Ku and C. Huang, "A complementary code pilot-based transmitter diversity technique for OFDM systems," *IEEE Trans. Wireless Commun.*, vol. 5, no. 3, pp. 504–508, Mar. 2006.

[29] H. H. Chen, J. F. Yeh, and N. Seuhiro, "A multi-carrier CDMA architecture based on orthogonal complementary codes for new generation of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, no. 10, pp. 126–135, Oct. 2001.

[30] H. H. Chen, H. W. Chiu, and M. Guizani, "Orthogonal complementary codes for interference-free CDMA technologies," *IEEE Wireless Commun. Mag.*, vol. 13, no. 1, pp. 68–79, Feb. 2006.

[31] S. M. Tseng and M. R. Bell, "Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences," *IEEE Trans. Commun.*, vol. 48, no. 1, pp. 53–59, Jan. 2000.

[32] L. Lu and V. K. Dubey, "Extended orthogonal polyphase codes for multicarrier CDMA system," *IEEE Commun. Lett.*, vol. 8, no. 12, pp. 700–702, Dec. 2004.

[33] H. P. Lim, "Cyclic shifted orthogonal complementary codes for multicarrier CDMA systems," *IEEE Commun. Lett.*, vol. 10, no. 6, pp. 1–3, Jun. 2006.

[34] X. Li, "Contention resolution in random-access wireless networks based on orthogonal complementary codes," *IEEE Trans. Commun.*, vol. 52, no. 1, pp. 82–89, Jan. 2004.

[35] R. Sivaswami, "Self-clutter cancellation and ambiguity properties of subcomplementary sequences," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 2, pp. 163–181, Mar. 1982.

[36] J. Guey and M. R. Bell, "Diversity waveform sets for delay-Doppler imaging," *IEEE Trans. Inf. Theory*, vol. 44, no. 4, pp. 1504–1522, Jul. 1998.

[37] F. F. Kretschmer and B. L. Lewis, "Doppler properties of polyphase coded pulse-compression waveforms," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-19, no. 4, pp. 521–531, Apr. 1983.

[38] P. B. Rapajik and R. A. Kennedy, "Merit factor based comparison of new polyphase sequences," *IEEE Commun. Lett.*, vol. 2, no. 10, pp. 269–270, Oct. 1998.

[39] T. Felhauer, "New class of polyphase pulse compression code with unique characteristics," *Electron. Lett.*, vol. 28, no. 8, pp. 769–771, Apr. 1992.

[40] T. Felhauer, "Design and analysis of new $P(n,k)$ polyphase pulse compression codes," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-30, no. 3, pp. 865–874, Jul. 1994.

[41] H. A. Khan, Y. Zhang, C. Ji, C. J. Stevens, D. J. Edwards, and D. O'Brien, "Optimizing polyphase sequences for orthogonal netted radar," *IEEE Signal Process. Lett.*, vol. 13, no. 10, pp. 589–592, Oct. 2006.

[42] H. Deng, "Polyphase code design for orthogonal netted radar systems," *IEEE Trans. Signal Process.*, vol. 52, no. 11, pp. 3126–3135, Nov. 2004.

[43] E. Prouhet, "Mèmoire sur quelques relations entre les puissances des nombres," *C. R. Acad. Sci. Paris Sèr.*, vol. I 33, p. 225.

[44] M. Morse, "Recurrent geodesics on a surface of negative curvature," *Trans. Amer. Math. Soc.*, vol. 22, pp. 84–100, 1921.

[45] J. P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[46] J. P. Allouche and J. Shallit, "The ubiquitous Prouhet-Thue-Morse sequence," in *Sequences and Their Applications, Proc. SETA'98*, T. H. C. Ding and H. Niederreiter, Eds. Berlin, Germany: Springer-Verlag, 1999, pp. 1–16.

[47] D. H. Lahmer, "The Tarry-Escott problem," *Scripta Math.*, vol. 13, pp. 37–41, 1947.

[48] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, no. 8, pp. 1451–1458, Oct. 1998.