Thrust acceleration estimation using on-line non-linear recursive least squares algorithm

N Ghahramani[1], **A Naghash**[2], and **F Towhidkhah**[2*]
[1]Department of Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran
[2]Department of Biomedical Engineering, Amirkabir University of Technology, Tehran, Iran

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Abstract: In this article, a robust non-linear recursive algorithm, featuring a highly reduced computational load, is proposed to estimate the thrust acceleration of a typical flight vehicle. The robustness of this new algorithm allows a significant increase of deviations in initial values of the estimated parameters. In the proposed algorithm, at first, a transformation of the non-linear measurement equation to a linear one, without any approximation, is obtained. Then the recursive least squares algorithm is applied to the transformed equation. The maximum achievable accuracy of the estimation for the non-linear problem is obtained analytically by the Cramer–Rao lower bound and is compared with simulation results. Extensive simulations showed that the new method provides an unbiased as well as a more robust thrust acceleration estimate in comparison with the extended Kalman filter. Moreover, the proposed method is beneficial in that it has a lower number of parameters and results in a simple design with less computational effort.

Keywords: thrust acceleration, non-linear recursive least squares, extended Kalman filter, robustness

1 INTRODUCTION

Power plant is one of the major systems of any flight vehicle, and thrust deviations cause the perturbations about the nominal trajectory. Thrust deviations usually result from uncertainties associated with propellant compositions as well as nozzle geometry alterations [**1**] that are very difficult to predict and quantify [**2**]. Therefore, *estimation* of the actual thrust acceleration parameters is the major of the guidance and control system of a flight vehicle. This estimation is also used to reconstruct the boost trajectory [**3**, **4**] and to monitor the condition of reusable launch vehicles [**5**].

In the powered flight stage of every flight vehicle, thrust acceleration is a non-linear function of mass parameters. In general, the non-linear estimation problem requires a complex iterative scheme to obtain the optimal solution [**6**]. The most straightforward way is to linearize the non-linear prediction function and to apply the linear estimation approaches. In some cases, a non-linear problem can be segmented into two or more separate linear problems, and then the recursive least squares (RLS) method is used to solve each problem [**7**]. A drawback of the estimators based on an approximated linear model is their sensitivity to modelling error.

Some other non-linear estimation approaches such as the extended Kalman filter (EKF) and the Gaussian second-order filter [**8**, **9**] address the non-linearities of the measurement and process dynamics models by utilizing Taylor series expansions of the non-linear equations. A good survey of non-linear estimation methods is given in [**10**]. The EKF has been applied extensively to real-time aerospace problems [**11**–**14**]. Thrust estimation and failure detection can be achieved using the EKF. However, it often requires an accurate process model, good initializing, and process noise to compensate for modelling errors and approximations. Consequently, in this method the tracking performance of the filter is highly dependent

Corresponding author: Department of Biomedical Engineering, Amirkabir University of Technology, Hafez Avenue, Tehran, Iran. email: Towhidkhah@aut.ac.ir; Towhidkhah@gmail.com

on *a priori* information. Poor performance or even divergence arising from the linearization in the EKF has led to the development of other non-linear filters such as the two-step optimal estimator [**15**]. One significant limitation of most of these non-linear estimation problems for the on-line applications is that the computational complexity has increased.

In this article, an on-line non-linear recursive least squares (NRLS) algorithm is proposed to estimate the thrust acceleration of a typical flight vehicle with emphasis on the robustness of the approach in the presence of uncertainties. In this approach, first, a linearizing transformation [**9**, **16**] for the non-linear measurement function of acceleration is obtained, and then the parameters using the RLS method are estimated without any approximation. The advantages of this approach are: (a) it may be obtained by direct calculations with less computational effort, whereas non-linear estimation procedures require complex iterative schemes and (b) there is no approximation in the linearizing transformation. Therefore, this approach is more accurate and robust to estimate the parameters of a class of non-linear processes.

The maximum achievable accuracy of the proposed estimator for non-linear problems is obtained by the
Q1 Cramer–Rao lower bound (CRLB) [**17**, **18**] analytically. CRLB is used in simulation to compare the EKF and NRLS accuracies with a lower bound on the error covariance that can be achieved by any unbiased estimate for a non-linear stochastic system in terms of the Fisher information matrix.

The article is organized as follows: in section 2, the problem is formulated, in section 3 the EKF is reviewed, the proposed on-line NRLS estimator is presented in section 4, covariance analysis of the non-linear estimation problem using the Cramer–Rao bound (CRB) is given in section 5, and in section 6 the results obtained from the simulation of the proposed approach are presented. Finally, conclusions are presented in section 7.

2 PROBLEM FORMULATION

The objective is to estimate the thrust acceleration of a typical flight vehicle using the non-linear measurement model in the presence of the measurement noise.

2.1 Accelerometer output model

The applied non-gravitational acceleration vector $\boldsymbol{a}$ of a vehicle has the following components

$$\boldsymbol{a} = \boldsymbol{a}_{\mathrm{T}} + \boldsymbol{a}_{\mathrm{A}} \tag{1}$$

in which $\boldsymbol{a}_{\mathrm{T}}$ is the thrust acceleration and $\boldsymbol{a}_{\mathrm{A}}$ is the aerodynamic acceleration of the vehicle. An accelerometer is insensitive to the gravitational acceleration (g) and thus provides an output proportional to the non-gravitational force per unit mass to which the sensor is subjected along its sensitive axis [**19**]. The thrust and drag can be combined into the specific thrust [**4**]. From equation (1) the axial acceleration is

$$\boldsymbol{a}(t) = \frac{T}{m(t)} = \frac{T}{m_0 - \dot{m}t} \tag{2}$$

in which m_0 is the initial mass of the vehicle, $\dot{m}$ the mass flowrate of the propellant, T the thrust, and t the time from motor ignition. The resultant momentum and pressure thrust is given by [**20**]

$$T = \left(\frac{V_{\mathrm{e}} + (P_{\mathrm{e}} - P_{\mathrm{a}})A_{\mathrm{e}}}{\dot{m}}\right)\dot{m} \tag{3}$$

where V_{e} is the exhaust velocity of the propellant, P_{e} the nozzle exit pressure, P_{a} the ambient pressure, and A_{e} the nozzle exit area. The effective exhaust velocity, $c = V_{\mathrm{e}} + (P_{\mathrm{e}} - P_{\mathrm{a}})A_{\mathrm{e}}/\dot{m}$ frequently defines motor performance. In a solid motor, the thrust-level value may change as much as ± 10 per cent of its nominal value due to temperature and pressure variations [**21**]. For a solid motor, equation (2) can be written as

$$\boldsymbol{a}(t) = \begin{cases} f(t) = \dfrac{1}{x_1 + x_2\, t} & t \leqslant t_{\mathrm{b}} \\ \mathrm{Tail}(t) & t > t_{\mathrm{b}} \end{cases} \tag{4}$$

where t_{b} is the burn time and

$$x_1 = \frac{m_0}{T}, \quad x_2 = -\frac{\dot{m}}{T} = -\frac{1}{c} \tag{5}$$

In equation (4), $\mathrm{Tail}(t)$ is a function that describes the tail behaviour of the acceleration, and it can be approximated by a linear function. Generally, the time deviation of the tail is small and its initial value can be determined, $\mathrm{Tail}(t_{\mathrm{b}}) = f(t_{\mathrm{b}})$. Therefore, in this article, only the parameters of $f(t)$ are estimated.

2.2 Performance index

Suppose that there are N measurements from a fixed-regressor non-linear model with a known relationship $f(\cdot)$. Thus

$$y_i = f_i(\boldsymbol{x}) + v_i, \quad i = 1, 2, \ldots, N \tag{6}$$

where $y_i \in R$ is the ith measurement, $\boldsymbol{x} \in R^p$ the vector of p unknown parameters or states and $\{v_i\}$ a random sequence of zero-mean Gaussian white noise process with variance given by σ^2. Assume that function $f(\cdot)$ is differentiable with respect to x. The least squares estimate of $\boldsymbol{x}$, denoted by $\hat{\boldsymbol{x}}$, minimizes the error sum of squares

$$C(\boldsymbol{x}) = \sum_{i=1}^{N} [y_i - f_i(\boldsymbol{x})]^2 \tag{7}$$

This is a non-linear least squares problem. To obtain the optimal solution, the non-linear estimation problem generally requires the complex iterative scheme with high computational effort. If the estimation can be done off-line, various non-linear search and iterative batch algorithms can be used to minimize the cost function, resulting in an optimal estimate [**16**]. However, many applications require real-time parameter estimates obtained by a recursive algorithm. A number of approximate recursive algorithms have been developed [**8**]. Standard non-linear recursive estimators, such as the EKF, linearize the cost function to use the well-known linear Kalman filter equations. This approximation, however, results in an estimate that is suboptimal and biased, i.e. the expected value of the estimator is not the true value of the parameters.

In some cases, a transformation of variables $z = \varphi(y)$ can be introduced in such a way that the resulting function $\varphi[f(\boldsymbol{x})]$ is linear in $\boldsymbol{x}$. Then the linear least squares method can be applied to estimate $\boldsymbol{x}$. In this article, a linearizing transformation is proposed in order to convert the non-linear acceleration function into a linear one without approximation. The advantage of this approach is that the estimation may be accurately obtained by less computational effort, whereas non-linear estimation procedures require complex iterative schemes.

3 REVIEW OF THE EKF

Consider the measurement y_i and the non-linear observation vector $\boldsymbol{h}_i(\boldsymbol{x})$

$$y_i = f_i(x) + v_i, \quad i = 1, 2, \ldots, N$$
$$\boldsymbol{h}_i(\boldsymbol{x}) = \frac{\partial y_i}{\partial \boldsymbol{x}} \tag{8}$$

Measurement noise $\{v_i\}$ is a random sequence of zero-mean Gaussian white noise with variance given by r. After equation linearization, the EKF state estimate $\hat{\boldsymbol{x}}_i$ and covariance matrix $\mathbf{P}_i$ are updated sequentially when a measurement y_i becomes available at discrete time i [**8**, **22**]

$$K_i = \frac{\mathbf{P}_i(-)\boldsymbol{h}_i(-)}{\boldsymbol{h}_i^{\mathrm{T}}(-)\mathbf{P}_i(-)\boldsymbol{h}_i(-) + r} \tag{9a}$$

$$\mathbf{P}_i(+) = [\mathbf{I} - K_i\boldsymbol{h}_i^{\mathrm{T}}(-)]\mathbf{P}_i(-) \tag{9b}$$

$$\hat{\boldsymbol{x}}_i(+) = \hat{\boldsymbol{x}}_i(-) + K_i e_i \tag{9c}$$

$$e_i = y_i - f(\hat{\boldsymbol{x}}_i) \tag{9d}$$

where K_i is the Kalman filter gain. In the above equations, $(-)$ indicates the value immediately before the update and $(+)$ indicates the value immediately after the update. According to equations (9), the estimates are updated with the non-linear measurement function $f(\hat{\boldsymbol{x}}_i)$, whereas the EKF gains K_i depend on the measurement observation vector. Between successive updates, the non-linear dynamic system may be integrated forward from i to the next measurement update $i + 1$

$$\hat{\boldsymbol{x}}_{i+1}(-) = g(\hat{\boldsymbol{x}}_i(-)) + w_i \tag{10a}$$

$$\mathbf{P}_{i+1}(-) = \mathbf{F}_i\mathbf{P}_i(-)\mathbf{F}_i^{\mathrm{T}} + \mathbf{Q} \tag{10b}$$

in which w is the Gaussian white noise process with zero-mean and covariance matrix $E\{ww^T\} = \mathbf{Q}, \hat{\boldsymbol{x}}_i(+)$ and $\mathbf{P}_i(+)$ are initial conditions, and the linearized dynamics matrix $\mathbf{F}_i$ may be determined by $\mathbf{F}_i = \partial g/\partial \boldsymbol{x}$. $\mathbf{P}_{i+1}(-)$ and $\hat{\boldsymbol{x}}_{i+1}(-)$ are updated using equations (9b) and (9c) when the next measurement becomes available. An important limitation of the EKF is that the prior covariance $\mathbf{P}_i(-)$ is not updated using the actual statistics of e_i. Instead, these statistics are inferred from the relation

$$E[e_i^2] = s_i, \quad s_i = \boldsymbol{h}_i^{\mathrm{T}}(-)\mathbf{P}_i(-)\boldsymbol{h}_i(-) + r \tag{11}$$

Although this relation is valid for a linear process, non-linear effects can cause differences between the actual and modelled statistics of the filter such that $E[e_i^2] \neq s_i$. Therefore, $\mathbf{P}_i(+)$ may not accurately indicate the statistics of actual estimates' errors.

4 ON-LINE NRLS ESTIMATOR

Here, an algorithm using a transformation is presented to solve the non-linear least squares problem shown in equation (7). In the first step, the non-linear measurement model $y = f(\boldsymbol{x})$ is transformed by a function $z = \varphi(y)$ in such a way that the function $\varphi[f(\boldsymbol{x})]$ is linear in parameter vector $\boldsymbol{x}$. Then, the least squares method [**23**, **24**] is applied to estimate $\boldsymbol{x}$.

4.1 RLS approach

The RLS method calculates a new update for the **Q2**
parameter vector $\hat{\boldsymbol{x}}$ each time new data are received, and it requires time for the computation of each parameter. For time-varying systems, good tracking capability can be ensured by introducing a forgetting

factor $\lambda \leqslant 1$ [**24**]. It can easily be shown that the RLS update is

$$K_i = \frac{\mathbf{P}_{i-1}\boldsymbol{h}_i}{\boldsymbol{h}_i^{\mathrm{T}}\mathbf{P}_{i-1}\boldsymbol{h}_i + \lambda} \tag{12a}$$

$$\mathbf{P}_i = \frac{(I - K_i\boldsymbol{h}_i^{\mathrm{T}})\mathbf{P}_{i-1}}{\lambda} \tag{12b}$$

$$\hat{\boldsymbol{x}}_i = \hat{\boldsymbol{x}}_{i-1} + K_i e_i \tag{12c}$$

$$e_i = y_i - \boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{x}}_{i-1} \tag{12d}$$

where e_i is the one-step prediction error and $\boldsymbol{h}_i$ the new measurement for all regressors. The RLS algorithm has two sets of variables that must be initialized and then updated at each step: the process parameters $\hat{\boldsymbol{x}}_i$ and the covariance matrix $\mathbf{P}_i$. It can be shown that the Kalman filter reduces to the weighted least squares fit for the static problem [**25**] and provides an optimal recursive alternative.

4.2 NRLS approach using linearizing transformation

In this section, the proposed NRLS algorithm is formulated. The structure of this new formulation is similar to that of the RLS, but the error equation of the proposed algorithm $e_i = y_i - \boldsymbol{h}_i^{\mathrm{T}}\hat{\boldsymbol{x}}_{i-1}$ is corrected, and the error of the transformed measurement $\phi(y_i)$ and the transformed model $\varphi[f(\hat{\boldsymbol{x}}_{i-1})]$ is used as follows

$$K_i = \frac{\mathbf{P}_{i-1}\boldsymbol{h}_i}{\boldsymbol{h}_i^{\mathrm{T}}\mathbf{P}_{i-1}\boldsymbol{h}_i + \lambda} \tag{13a}$$

$$\mathbf{P}_i = \frac{(I - K_i\boldsymbol{h}_i^{\mathrm{T}})\mathbf{P}_{i-1}}{\lambda} \tag{13b}$$

$$\hat{\boldsymbol{x}}_i = \hat{\boldsymbol{x}}_{i-1} + K_i e_i \tag{13c}$$

$$e_i = \varphi(y_i) - \varphi[f(\hat{\boldsymbol{x}}_{i-1})] \tag{13d}$$

Notice that in this algorithm, although the states and covariance matrix are updated for the transformed linear equations, there is no approximation as in the case of EKF. This important feature improves the robustness of the proposed approach in the presence of noise and uncertainties as well as large deviations in initial conditions of the estimated unknown parameters.

The motivation for the proposed NRLS is to devise a robust recursive estimator for non-linear measurement of the accelerometer that provides a better performance than the EKF. It can be shown that the EKF is biased and suboptimal [**15**] because it approximates the original cost function by linearizing about the previous estimate, whereas the NRLS algorithm, proposed in this article, minimizes the desired cost function at each measurement update by transforming the non-linear measurement function (moving any
Q3 approximation), resulting in a more robust algorithm.

4.3 Thrust acceleration parameter estimation using the NRLS algorithm

The measurement model of the proposed estimator can be written as

$$y = f(\boldsymbol{x}) = \frac{1}{x_0^* + x_1^* t} + n(t) \tag{14}$$

in which $\boldsymbol{x} = [x_0^*, x_1^*]^{\mathrm{T}}$ is the vector of the nominal model parameters obtained from the nominal values of the thrust and mass flowrate and $n(t)$ is the measurement noise. This model generates data for the various test conditions described in section 6. To implement equations (13), we have

$$\varphi[f(\hat{\boldsymbol{x}})] = \boldsymbol{h}^{\mathrm{T}}\hat{\boldsymbol{x}} \tag{15}$$

in which $\hat{\boldsymbol{x}} = [\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1]^{\mathrm{T}}$, $\boldsymbol{h} = [1, t]^{\mathrm{T}}$ and also

$$\varphi[y] = \frac{1}{1 + 1/(x_0^* + x_1^* t) + n(t)} \tag{16}$$

The proposed equations (13) to (16) describe the NRLS algorithm to estimate the thrust acceleration at current discrete time

$$\hat{\boldsymbol{a}}(t_i) = \frac{1}{\hat{x}_0(t_i) + \hat{x}_1(t_i)t_i}$$

Figure 1 shows the block diagram of the on-line NRLS algorithm for estimating the thrust acceleration model parameters.

5 COVARIANCE ANALYSIS

One of the simple properties of the linear-Gaussian estimation problem is that there is an explicit formula for the estimation error covariance. This is unfortunately not the case for general non-linear problems. There are, however, a number of approximations and bounds that have been developed for error covariance for such problems, and one of the most basic and useful approaches is the CRB. It can be used to give us a lower bound on the expected errors between the estimated quantities and the true values from the known statistical properties of the measurements errors. This lower bound indicates the theoretically maximum achievable accuracy of the estimate.

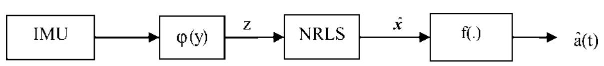

Fig. 1 On-line NRLS algorithm for estimating the thrust acceleration model parameters

5.1 Cramer–Rao inequality

The Cramer–Rao inequality for an unbiased estimator $\hat{\boldsymbol{x}}$ is given by [**17**]

$$\mathbf{P} = E[(\hat{\boldsymbol{x}} - \boldsymbol{x})(\hat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}] \geqslant \mathbf{J}^{-1} \tag{17}$$

where the Fisher information matrix $\mathbf{J}$ is given by

$$\mathbf{J} = E\left\{\left[\frac{\partial}{\partial \boldsymbol{x}} C(\boldsymbol{x})\right]\left[\frac{\partial}{\partial \boldsymbol{x}} C(\boldsymbol{x})^{\mathrm{T}}\right]\right\} = -E\left[\frac{\partial^2}{\partial \boldsymbol{x} \partial \boldsymbol{x}^{\mathrm{T}}} C(\boldsymbol{x})\right] \tag{18}$$

When the inequality in (17) is satisfied, the estimator $\hat{\boldsymbol{x}}$ is said to be efficient. This is useful for the investigation of the quality of a particular estimator.

5.2 Covariance analysis of the transformed non-linear model

To analyse the estimation error covariance of the approach, the following model is considered

$$y_i = \frac{1}{x_0 + x_1 t_i} + v_i, \quad i = 1, 2, \ldots, N \tag{19}$$

where N is the number of measurements and v_i is a zero-mean Gaussian white noise process with variance given by σ^2. In this non-linear estimation problem, the parameter vector $\boldsymbol{x} = [x_0, x_1]^{\mathrm{T}}$ has to be estimated from the measurement vector$y = [y_1, y_2, \ldots, y_N]^{\mathrm{T}}$. Therefore, the covariance of the estimated error is given by [**26**]

$$\mathbf{P} = \sigma^2(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1} \tag{20}$$

where

$$\mathbf{H} \triangleq \frac{\partial y}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{-1}{(x_0 + x_1 t_1)^2} & \dfrac{-t_1}{(x_0 + x_1 t_1)^2} \\ \dfrac{-1}{(x_0 + x_1 t_2)^2} & \dfrac{-t_2}{(x_0 + x_1 t_2)^2} \\ \vdots & \vdots \\ \dfrac{-1}{(x_0 + x_1 t_N)^2} & \dfrac{-t_N}{(x_0 + x_1 t_N)^2} \end{bmatrix} \tag{21}$$

Matrix $\mathbf{P}$ is also equivalent to the CRLB. Now, the transformation $z = \varphi(y)$ is considered to imply linear least squares to determine the parameters x_0 and x_1. However, by this transformation the measurement noise will certainly not be Gaussian anymore. Therefore, the main question is: how optimal is this solution? This question is answered by studying the effects of applying this approach. Expanding z in a first-order series gives (see Appendix 2 for details)

$$z_i \cong \boldsymbol{h}_i^{\mathrm{T}} \boldsymbol{x}_i + \varepsilon_i \tag{22}$$

where $\boldsymbol{h}_i = [1, t_i]^{\mathrm{T}}, \quad \boldsymbol{x}_i = [x_0, x_1]^{\mathrm{T}}$ and

$$\varepsilon_i = -(x_0 + x_1 t_i)^2 v_i \tag{23}$$

ε_i is the first-order expansion of the new measurement noise. The linear least squares "$\mathbf{H}$ matrix", denoted by $\bar{\mathbf{H}}$ is given by

$$\bar{\mathbf{H}} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_N \end{bmatrix} \tag{24}$$

and the estimated error covariance of the linear approach is given by

$$\bar{\mathbf{P}} = \left(\bar{\mathbf{H}}^{\mathrm{T}} \operatorname{diag}\left[\xi_1^{-1}, \xi_2^{-1}, \ldots, \xi_N^{-1}\right] \bar{\mathbf{H}}\right)^{-1} \tag{25}$$

where

$$\xi_i = (x_0 + x_1 t_i)^4 \sigma^2 \tag{26}$$

Comparing equations (20) and (25) shows that $\mathbf{P}$ is equivalent to $\bar{\mathbf{P}}$ and, therefore, the CRLB is achieved and the linear approach leads to an efficient estimator. This shows how the Cramer–Rao inequality can be useful to help quantify the errors introduced by using an approximate solution instead of the optimal approach.

6 SIMULATION STUDY AND DISSCUSION

In this section, the simulation results of the EKF and the proposed NRLS algorithms are compared and discussed. This study focuses on the robustness of both algorithms to prove the effectiveness of the proposed NRLS algorithm. Since there is no approximation in the NRLS method, it will be shown that the method has better robustness than the EKF.

6.1 Parameters of the model and the filters

A typical solid neutral burning motor was considered to study the performances of the estimation algorithms. Many motors in aerospace industry are neutral burning, and their thrust, pressure, and burning surface remain approximately constant during burn time [**1**]. The thrust can be modelled as simplified rectangular. In this section, both rectangular and realistic time-varying models of thrust are considered, to compare the estimators.

The constant parameters of the acceleration model were considered: $\bar{T} = 10\,000 \times 9.8\,\mathrm{N}$, $\bar{\dot{m}} = -80\,\mathrm{kg/s}$,

$t_b = 20\,\text{s}$, and $m_0 = 3000\,\text{kg}$. $\bar{T}$ and $\bar{\dot{m}}$ are the average values of thrust and mass flowrate, respectively. Time step Δt and the initial values of the covariance matrix of both filters were set as $\Delta t = 0.02\,\text{s}$, $\mathbf{P}_0 = 2.0\,\text{I}$ m/s^2. For the proposed NRLS, the forgetting factor was set as $\lambda = 0.98$. For the EKF the linearized dynamics matrix $\mathbf{F}$ and the linearized measurement vector $\boldsymbol{h}_i$ were obtained

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{h}_i = \begin{bmatrix} \dfrac{-1}{(x_0 + x_1 t_i)^2} & \dfrac{-t_i}{(x_0 + x_1 t_i)^2} \end{bmatrix}^{\mathrm{T}} \tag{27}$$

The measurement noise r and the process noise matrix $\mathbf{Q}$ were set as

$$r = 0.25, \quad \mathbf{Q} = \beta \begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{-11} \end{bmatrix} \tag{28}$$

β is a scalar, and for the nominal condition $\beta = 1$. We will use it for the sensitivity analysis. These conditions were used as the baseline for the entire study.

6.2 CRLB of the simulation results

The CRLB plays an important role in the evaluation of estimation. The effects of the parameter-estimate error and other error sources such as non-linearities and sensor accuracies increase the covariance error. Therefore, the CRLB is a conservative bound. Here, for the acceleration model given in section 6.1, the performance of the EKF and NRLS methods achieved by the simulation is compared with the theoretical CRLB. It is defined as the trace of the covariance matrix shown in equation (20).

Figure 2 compares the root of CRB, $\sqrt{\text{CRB}}$(solid line), the root mean square (r.m.s.) of the estimated acceleration model parameters using the EKF (dashed line), and the NRLS (dotted line) algorithms. A very close agreement is observed between the theoretical and simulation results for the proposed NRLS algorithm, and the r.m.s. of estimation error (or the root of covariance matrix trace) of the NRLS is greater than $\sqrt{\text{CRB}}$. This is due to the linearization of the measurement model and the applied noise level. This figure also shows that the r.m.s. of estimation error of NRLS is less than that of the EKF method.

6.3 Comparison of robustness of the EKF and NRLS methods

The methods were compared with respect to uncertain initial values of the estimated parameters, the process noise, and thrust deviations.

6.3.1 Robustness with respect to the deviations of the initial values

In this problem, the sensitivity of both filters with respect to the initial values of x_1 is less than that of x_0. Thus, only the sensitivity of the estimators for a set of the initial values of parameter x_0 was given. Figure 3 illustrates the sensitivity of the EKF for the initial condition set $\boldsymbol{x}_0\,(0) = [0.01, 0.03, 0.05, 0.06]^{\mathrm{T}}$. The initial value $x_0 = 0.03$ (solid line) is nearest to the nominal value. In this case, parameters x_0, x_1, and thrust acceleration tend to their nominal values, and therefore, the tracking performance of the EKF is achieved. Increasing the distance of x_0 from the nominal value ($x_0 = 0.01$ (dashed line) and $x_0 = 0.05$ (dotted-dashed line)) results in increase of the estimation tracking error for x_0. Finally, in the case of $x_0 = 0.06$ (dotted line) the EKF diverges. As a result, it is a sensitive estimator

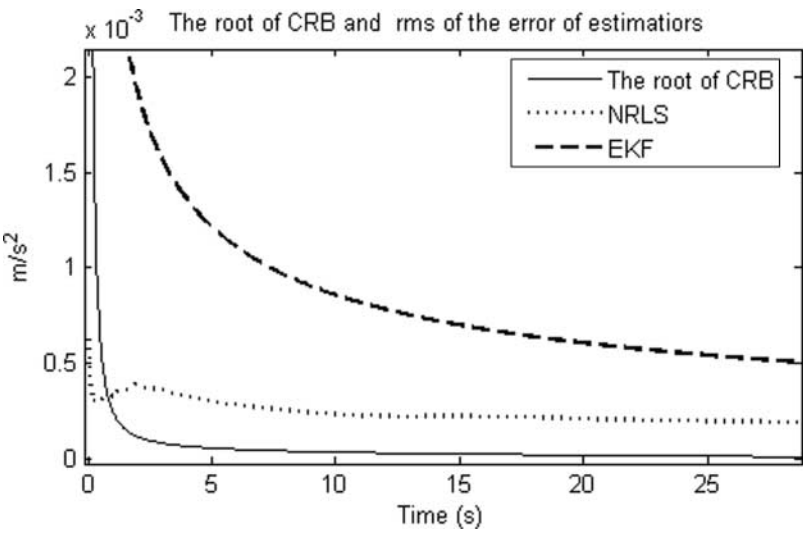

Fig. 2 The root of CRB (solid line), the r.m.s. of the estimated acceleration model parameters using the EKF (dashed line), and the NRLS (dotted line) algorithms

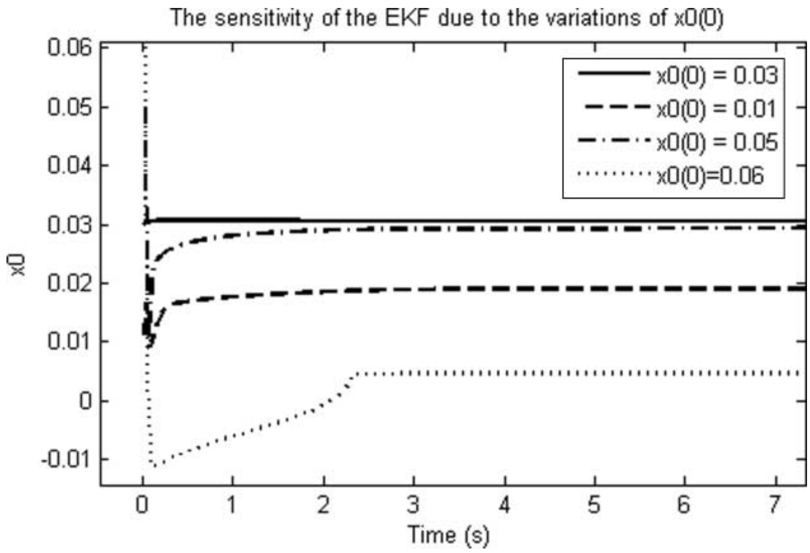

Fig. 3 The sensitivity of the EKF for the initial condition set $x_0(0) = [0.01, 0.03, 0.05, 0.06]^T$

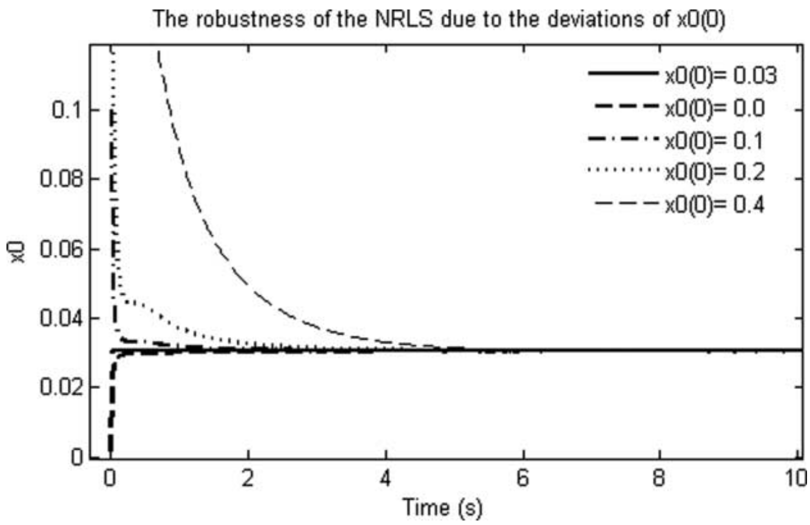

Fig. 4 Time history of x_0 for a large value of the initial condition set $x_0(0) = [0.0, 0.03, 0.10, 0.20, 0.40]^T$ to demonstrate the high robustness of the proposed NRLS

with respect to the parameters' initial conditions, and they must be tuned accurately.

Figure 4 demonstrates the high robustness of the proposed NRLS for the initial condition set $\boldsymbol{x}_0(0) = [0.0, 0.03, 0.10, 0.20, 0.40]^T$. The set has larger values than the previous one. For the values less than 0.10, the proposed filter has the desired tracking performance. The rise time of the filter increases to between 0.10 and 0.20, but the tracking error tends to zero. In this case, the filter has acceptable performance.

To illustrate higher robustness of the NRLS, an interval of 0.20 until 0.40 for x_0 was considered. Although the rise time of the NRLS has increased sharply in this interval, the filter parameters still tend to their nominal values. Figures 5 and 6 show further information such as the thrust, mass flowrate, and the acceleration for two initial conditions 0.2 and 0.4 of x_0, and these figures confirm that the NRLS has a higher robustness compared with the EKF. This is theoretically referred to the transformation of the non-linear measurement function instead of its linearization.

6.3.2 Sensitivity of the EKF to the variations of the process noise level

In the EKF algorithm, a process noise matrix is generally included to compensate for non-linear effects, however, because of the initializing of the states and covariance matrix, the tuning procedure of the process noise matrix can be cumbersome. In contrast, the **Q4**
NRLS algorithm does not need to use the process noise matrix in the algorithm. To study the sensitivity of the

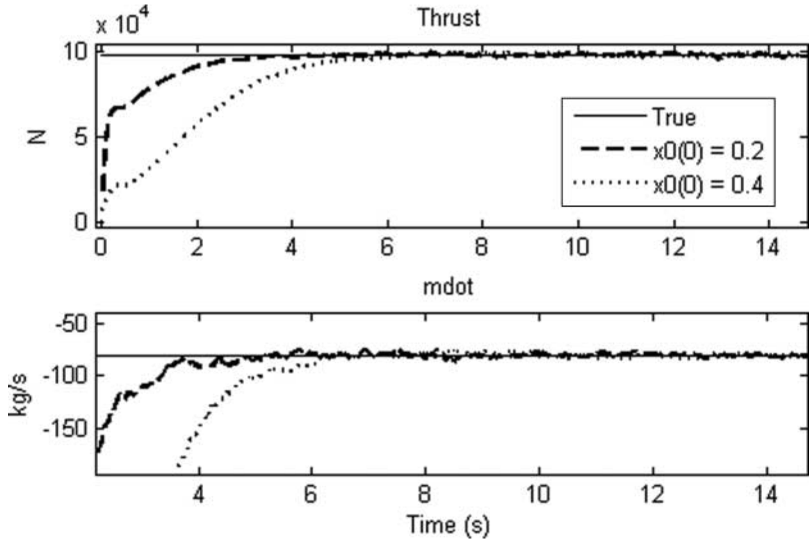

Fig. 5 Time history of the thrust and mass flowrate obtained from estimated parameters for a large value of the initial condition set $x_0(0) = [0.20, 0.40]^T$ – NRLS

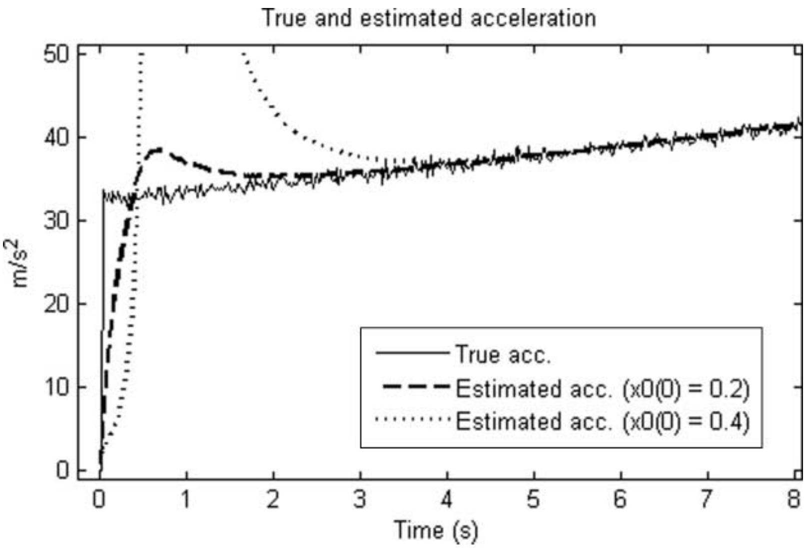

Fig. 6 Comparison of the true acceleration and the acceleration obtained from estimated parameters for a large value of the initial condition set $x_0(0) = [0.20, 0.40]^T$ – NRLS

EKF algorithm due to the deviation of the process noise level, the values of parameter β of the process noise matrix $\mathbf{Q}$ in equation (28) were considered to be 100 and 1.

The mean and the r.m.s. of the estimated acceleration model parameters (x_0, x_1) along the nominal deterministic parameters were computed. The EKF algorithm was used and the deviation of the process noise level was considered. Figures 7(a) and (b) show the mean and r.m.s. of the estimated parameters x_0 and x_1, respectively. Solid line indicates the true deterministic parameters. Dashed and dotted lines indicate the parameters for the nominal case ($\beta = 1$) and the deviated case ($\beta = 100$), respectively. Although the mean of parameter x_1 in both cases converges to its nominal value and its r.m.s. tends to zero, in the deviated case the mean and r.m.s. of parameter x_0 diverge. In other words, the EKF for non-linear problem is a biased filter again, due to the deviations of the process noise matrix.

6.3.3 *Performance of the estimators for a time-varying model of thrust*

Based on the parameters of the acceleration model described in section 6.1, a realistic time-varying thrust

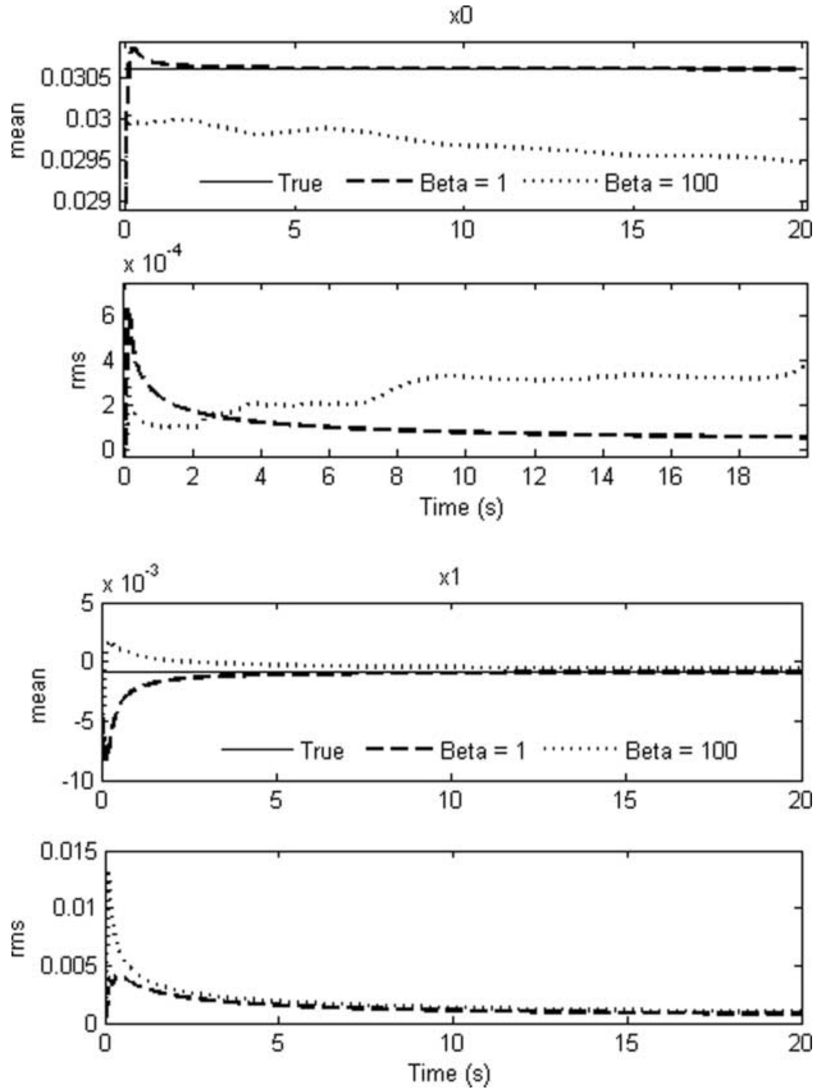

Fig. 7 (a) The sensitivity of the mean and r.m.s. of the estimated acceleration parameter x_0 using the EKF algorithm due to the deviation of the process noise matrix $\mathbf{Q}$ (b) The sensitivity of the mean and r.m.s. of the estimated acceleration parameter x_1 using the EKF algorithm due to the deviation of the process noise matrix $\mathbf{Q}$

was modelled as follows

$$T(t) = \begin{cases} (1 - 0.01t)\bar{T} & 0 \leqslant t < 5 \\ (0.95 + 0.01(t-5))\bar{T} & 5 \leqslant t < 20 \end{cases} \quad (29)$$

A similar function was also written for $\dot{m}$. The bias of the accelerometer was considered to be 10 mg (0.1 m/s^2) and the standard deviation of the measurement Gaussian noise σ was assumed to be 15 mg. Figure 8 illustrates the measured acceleration (dotted line) and the estimated acceleration (solid line) using the proposed NRLS method. This experiment demonstrates that the method can also accurately estimate the thrust acceleration when the motor has time-varying thrust.

Figure 9 compares the estimated acceleration error for the NRLS (solid line) and EKF (dotted line) methods. As it can be seen, before 5 s (when the sign of thrust rate is changed) the estimation errors of both methods are close to each other, but after this time the estimation error of the NRLS converges to the accelerometer bias faster than that of the EKF. Moreover, after 5 s the maximum deviation value of the NRLS (i.e. 0.2 m/s^2) is much less than that of the EKF (i.e. 0.5 m/s^2). In other words, the NRLS method produces a faster as well as more accurate output result than the EKF. The figure also shows the effect of the accelerometer bias on the estimation accuracy. Recall that the accelerometer bias can be removed by a posterior calibration procedure.

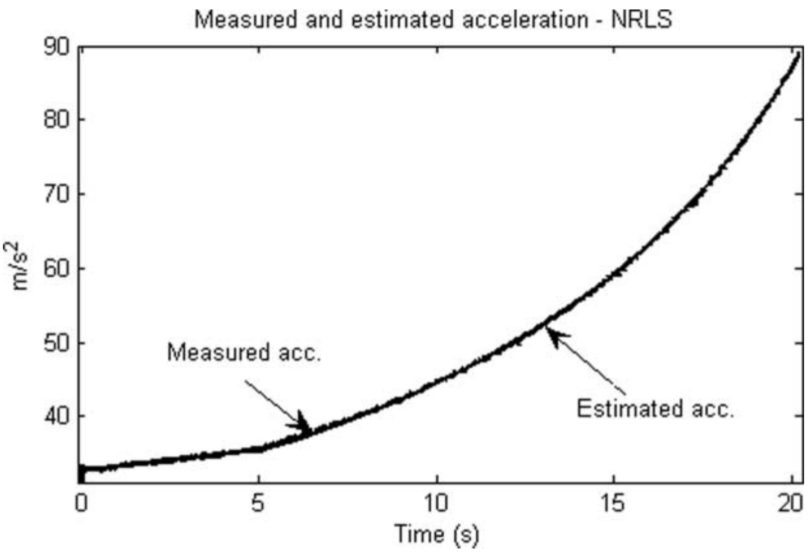

Fig. 8 The measured acceleration (dotted line) and the estimated acceleration (solid line) using the proposed NRLS method for time-varying thrust

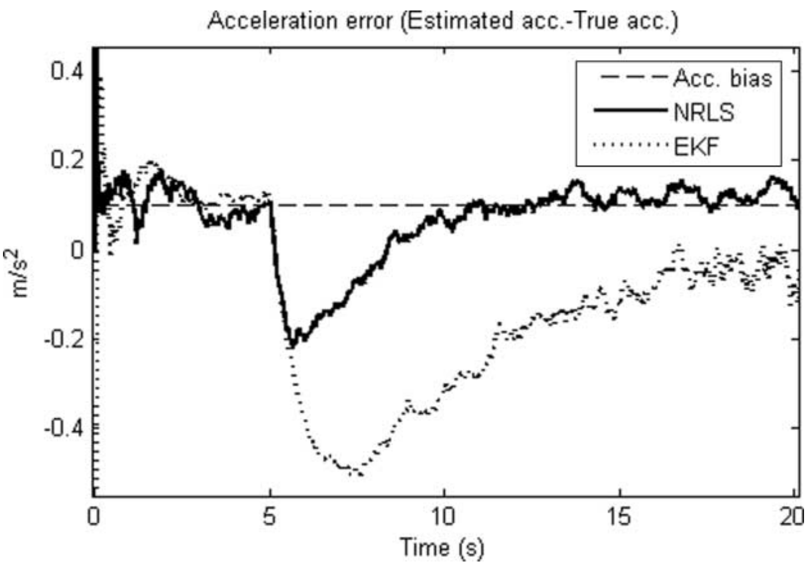

Fig. 9 The error of estimated acceleration and true acceleration for the NRLS (solid line) and EKF (dotted line) methods for time-varying thrust

According to the results presented in this section, the EKF differs significantly from the optimal estimate and is highly dependent on the initial guess and the process noise matrix. In contrast, the proposed NRLS algorithm is a more robust and accurate estimator due to the initial conditions and thrust deviations.

7 CONCLUSIONS

In this article, an on-line NRLS algorithm is proposed to estimate the thrust acceleration of a typical flight vehicle. A linearizing transformation is suggested to transform the non-linear measurement model to a linear one and then, the RLS algorithm is applied to the problem. In the error equation of the proposed algorithm, there is no approximation as in the case of EKF. For this reason, the proposed NRLS algorithm improves the robustness and accuracy of the estimation. The CRB is obtained for the non-linear problem to quantify the errors introduced using an approximate solution instead of the optimal approach. Covariance analysis and simulation results show that the proposed non-linear estimator is an efficient estimator.

To demonstrate the effectiveness of the NRLS method, it is examined in the presence of uncertainties in the initial conditions of the parameters and variations of thrust profile. Comparison with the EKF shows that the EKF is a biased estimator and differs significantly from the optimal estimate. It is also highly dependent on the initial guess and the process noise matrix. In contrast, the proposed NRLS algorithm is nearly independent of the initial guess, and it is a more accurate estimator for the thrust
Q4 acceleration because of moving any approximation. Moreover, this algorithm is easy to implement with less computational effort and design parameters.

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APPENDIX 1

Notation

$\boldsymbol{a}(t)$	axial acceleration
A_e	nozzle exit area
$C(x)$	cost function
e	prediction error
$\mathbf{F}$	linearized dynamics matrix
$\boldsymbol{h}$	observation vector
$\mathbf{J}$	Fisher information matrix
K	Kalman filter gain
m_0	initial mass of the vehicle
$\dot{m}$	mass flowrate of the propellant
$\mathbf{P}$	covariance matrix
P_e	nozzle exit pressure
P_a	ambient pressure
$\mathbf{Q}$	process noise matrix
r	measurement noise variance
T	thrust
V_e	exhaust velocity of the propellant
$\hat{x}$	estimated parameters
x	parameters of estimation
y	measurement
$\varphi(\cdot)$	linearizing transformation
λ	forgetting factor

APPENDIX 2

Expanding z_i in a first-order series

In equation (19) z_i is a non-linear function of the measurement noise v_i as follows

$$z_i = \frac{1}{y_i} = \frac{1}{1/(x_0 + x_1 t_i) + v_i} \tag{30}$$

A first-order expansion of this equation with respect to v_i is given by

$$\begin{aligned} z_i &= (x_0 + x_1\, t_i)\frac{1}{1 + (x_0 + x_1\, t_i)v_i} \\ &\cong (x_0 + x_1\, t_i)(1 - (x_0 + x_1\, t_i)v_i) \\ &= (x_0 + x_1\, t_i) - (x_0 + x_1\, t_i)^2 v_i \end{aligned} \tag{31}$$

The variance of $\varepsilon_i \triangleq -(x_0 + x_1\, t_i)^2 v_i$, denoted by ξ_i^2, is derived from

$$E[\varepsilon_i] = -(x_0 + x_1 t_i)^2 E[v_i] = 0 \tag{32}$$

$$\xi_i^2 \triangleq \mathrm{Var}(\varepsilon_i) = E[\varepsilon_i^2] - E^2[\varepsilon_i] = (x_0 + x_1\, t_i)^4 \sigma^2 \tag{33}$$

It is seen that $\varepsilon(k)$ is a zero-mean Gaussian white noise process with variance ξ_i^2; in other words, the noise characteristic is not changed.

JAERO371

Queries

N Ghahramani, A Naghash, and F Towhidkhah

Q1 The abbreviation of Cramer-Rao lower bound CRB is changed to CRLB to distinguish it from Cramer-Rao bound (CRB). Please check and confirm.

Q2 Please check the edit of the sentence "The RLS method calculates. . ."

Q3 Please clarify what the phrase "moving any approximation" means.

Q4 Please check the edit ". . .because of. . ."

Q5 Please provide the place of proceedings and publisher for refs. [3,4].