A Unified Approach for Inverse and Direct Dynamics of Constrained Multibody Systems Based on Linear Projection Operator: Applications to Control and Simulation

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***Abstract*—This paper presents a unified approach for inverse and direct dynamics of constrained multibody systems that can serve as a basis for analysis, simulation, and control. The main advantage of the dynamics formulation is that it does not require the constraint equations to be linearly independent. Thus, a simulation may proceed even in the presence of redundant constraints or singular configurations, and a controller does not need to change its structure whenever the mechanical system changes its topology or number of degrees of freedom. A motion-control scheme is proposed based on a *projected inverse-dynamics* scheme which proves to be stable and minimizes the weighted Euclidean norm of the actuation force. The projection-based control scheme is further developed for constrained systems, e.g., parallel manipulators, which have some joints with no actuators (passive joints). This is complemented by the development of constraint force control. A condition on the inertia matrix resulting in a decoupled mechanical system is analytically derived that simplifies the implementation of the force control. Finally, numerical and experimental results obtained from dynamic simulation and control of constrained mechanical systems, based on the proposed inverse and direct dynamics formulations, are documented.**

***Index Terms*—Constrained multibody systems, constraint motion control, hybrid force/motion control.**

I. Introduction

MANY robotic systems are formulated as multibody systems with closed-loop topologies, such as manipulators with end-effector constraints [1]–[5], cooperative manipulators [6], [7], robotic hands for grasping objects [8], [9], parallel manipulators [10], [11], humanoid robot and walking robots [12], [13], and VR/Haptic applications [14]. Simulation and control of such systems call for corresponding direct dynamics and inverse-dynamics models, respectively. Mathematically, constrained mechanical systems are modeled by a set of n differential equations coupled with a set of m algebraic equations, i.e., differential algebraic equations (DAE). Although computing the dynamics model is of interest for both simulation and control, the research done in these two areas is rather divided. Surveys of the existing techniques for solving DAE may be found in [14]–[19], while model-based control of constrained manipulators can be found in [1]–[3] and [20]–[24].

The classical method to deal with DAE is to express the constraint condition at the acceleration level. This allows replacement of the original DAE system with an ordinary differential equation (ODE) system by augmenting the inertia matrix with the second derivative of the constraint equation. However, this method performs poorly in the vicinity of singularities [25]–[27], because the augmented inertia matrix is invertible only with a full-rank Jacobian matrix.

Other methods are based on coordinate partitioning [28]–[30] by using the fact that the n coordinates are not independent because of the m constraint equations. The motion of the system can be described by the independent coordinates which can be separated using an annihilator operator. Although this method may significantly reduce the number of equations, finding the annihilator operator is a complex task [14]. Moreover, the sets of independent and dependent coordinates should be determined first. But a fixed set of independent coordinates occasionally leads to ill-conditioned matrices [15], [31] when the system changes its topology or the number of degrees of freedom (DOFs). The concept of coordinate separation is used in [5] for controlling manipulator robots with constrained end-effectors. The augmented Lagrangian formulation proposed in [32]–[34] can handle redundant constraints and singular situations. However, this formulation solves the equations of motion through an iterative process. Nakamura *et al.* [35] developed a general algorithm that provides a way to partition the coordinates into independent and dependent ones even around the singular configuration, which is suitable for simulation of mechanical systems with structure-varying kinematic chains. This is a special case of the projection method proposed herein that allows generic constraints which cannot be handled by coordinate partitioning.

There are also efficient algorithms for solving direct dynamics of constrained systems that are suitable for parallel processing. Featherstone's work in [36] and [37] presents a recursive algorithm, which is called Divide-and-Conquer (DAC), for calculating the forward dynamics of a general *rigid-body* system on a parallel computer. The central formula for this DAC algorithm takes the equations of motions of two independent subassembly (rigid body) and also a description

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of how they are to be connected, and the output is the equation of motion of the assembly, i.e., those of two articulated body. Since the equation of acceleration of the assembly is written in terms of two independent equations of motions, the formulation is suitable for parallel processing, and one can apply the formula recursively to construct the articulated-body equations of motions of an entire rigid-body assembly from those of its constituent parts. The author claims that the DAC algorithm is computationally effective if a large number of processors, more than 100, is available.

Another group of researchers [27], [33], [38]–[40] focused on other techniques to deal with the problem of accurately maintaining the constraint condition. Blajer [39], [40] proposed an elegant geometric interpretation of constrained mechanical systems. Then the analysis was extended and modified in [41] for control application. The augmented Lagrangian formulation proposed in [32]–[34] can handle redundant constraints and singular situations. However, this formulation solves the equations of motion through an iterative process.

In the realm of control of constrained multibody systems, the vast majority of the literature is devoted to control of manipulators with constrained end-effectors. The hybrid position/force control concept was originally introduced in [2], and then the manipulator dynamic model was explicitly included in the control law in [1]. The constrained task formulation with inverse-dynamics controller is developed in [3] and [4] by assuming that the Cartesian constraints are linearly independent. Hybrid motion/force control proposed in [42]–[44] achieves a complete decoupling between channels of acceleration and force. In these approaches, all joints are assumed to have an actuator, and no redundancy was considered in the kinematic constraint.

In this paper, we propose a new formulation for the direct and the inverse dynamics of constrained mechanical systems based on the notion of a projection operator [45]. First, constraint reaction forces are eliminated by projecting the initial dynamic equations into the tangent space with respect to the constraint manifold. Subsequently, the direct dynamics, or the equations of motion, is derived in a compact form that relates explicitly the generalized force to the acceleration by introducing a *constraint inertia matrix*, which turns out to be always invertible. The constraint reactions can then be retrieved from the dynamics projection in the normal space. Unlike in the other formulations, the projection matrix is a square matrix of order equal to the number of dependent coordinates. Since the formulation of the projector operators is based on pseudoinverting the constraint Jacobian (the process not conditioned upon the maximal rank of the Jacobian), the present approach is valid also for mechanical systems with redundant constraints and/or singular configurations, which is unattainable with many other classical methods. A projected inverse-dynamics control (PIDC) scheme is developed based on the dynamics formulation. The motion control proves to be stable while minimizing the weighted Euclidean norm of actuation force. The notion of the projected inverse dynamics is further developed for control of constrained mechanical systems which have passive joints, i.e., joints with no actuator. This result is particularly important for control of parallel manipulators. Finally, a hybrid force/motion control scheme based on the proposed formulation is presented. Also, some useful insights are gained from the dynamics formulation. For instance, the condition on the inertia matrix for achieving a complete decoupling between force and motion equations is rigorously derived.

This paper is organized as follows. We begin with the notion of *linear operator equations* in Section II by reviewing some basic definitions and elementary concepts which will be used in the rest of the paper. Using the projection operator, we derive models of inverse and direct dynamics in Sections IV and V, which are used as a basis for developing strategies for simulation and control of constrained mechanical systems in Sections VI and VII. Section VII-B presents change of coordinate if there is inhomogeneity in the spaces of the force and velocity. In Section VII-D, the IDC scheme is extended for constrained systems which have some joints with no actuator (passive joints). Finally, Sections VIII and IX report some simulation and experimental results.

II. Linear Operator Equations

For any linear operator transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, range space and null space are defined as $\mathcal{R}(A) = \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n \ni y = Ax\}$ and $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$, respectively. The linear transformation maps vector space $\mathcal{X}$ into vector space $\mathcal{Y}$. Assume that the Euclidean inner product is defined in $\mathcal{X}$, that is, elements of vectors, such as x_1 and x_2, of $\mathcal{X}$ have homogeneous units. Then, by definition, the vectors are orthogonal iff their inner product is zero, i.e.,

$$\langle x_1, x_2 \rangle = x_1^T x_2 = x_2^T x_1 = 0 \tag{1}$$

where the superscript ^{T} denotes transpose. It follows that the orthogonal complement of any set $\mathcal{S}$, denoted by $\mathcal{S}^\perp$, is the set of vectors each of which is orthogonal to every vector in $\mathcal{S}$.

Theorem 1: [46], [47]: The fundamental relationships between the range space and the null space associated with a linear operator and its transpose are

$$\mathcal{R}(A^T)^\perp = \mathcal{N}(A) \tag{2}$$
$$\mathcal{R}(A) = \mathcal{N}(A^T)^\perp. \tag{3}$$

See Appendix I for a proof.

As will be seen in the following sections, it is desirable to be able to project any vector in $\mathbb{R}^n$ to the null space of A by a projector operator. Let $P \in \mathbb{R}^{n \times n}$ be the orthogonal projection onto the null space, i.e., $\mathcal{R}(P) = \mathcal{N}(A)$. Note that every orthogonal projection operator has these properties: $P^2 = P$ and $P^T = P$ [47].

The projection operator can be calculated by the *singular value decomposition* (SVD) method [47]–[49]. Assuming

$$r = \text{rank}(A)$$

then there exist unitary matrices $U = [U_1 \; U_2]$ and $V = [V_1 \; V_2]$ (i.e., $U^T U = I$ and $V^T V = I$) so that

$$A = [U_1 \quad U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$, and $\sigma_1 \geq \cdots \geq \sigma_r \geq 0$ are the singular values. The proof of this statement is straightforward and can be found, for example, in [47] and [48]. Since $\mathcal{N}(A) = \text{span}(V_2)$ [47], [48], the projection operator can be calculated by

$$P = V_2 V_2^T. \quad (4)$$

A. Orthogonal Decomposition and Norm

From the definition, one can show that projector operator $(I - P)$ projects onto the null space orthogonal $\mathcal{N}(A)^\perp$. Let us assume that the elements of a vector $x \in \mathbb{R}^n$ have homogeneous measure units, then the vector has a unique orthogonal decomposition

$$x = x_{\|} \oplus x_\perp$$

where $x_{\|} \in \mathcal{N}(A)$ and $x_\perp \in \mathcal{N}(A)^\perp$. The components of the decomposition can be obtained uniquely by using the projection operator as

$$x_{\|} := Px \quad \text{and} \quad x_\perp := (I - P)x. \quad (5)$$

The Euclidean norm is defined as

$$\|x\| := \langle x, x\rangle^{1/2} = (x^T x)^{1/2}. \quad (6)$$

Remark 1: From orthogonality of the subspaces, i.e., $\langle x_{\|}, x_\perp\rangle = 0$, we can say

$$\|x\|^2 = \|x_{\|}\|^2 + \|x_\perp\|^2. \quad (7)$$

Equation (7) forms the basis for finding an optimal solution.

1) Metric Tensor: The Euclidean inner product, and hence, the Euclidean norm defined in (6), are noninvariant quantities if there is inhomogeneity in the units of the elements of vector x. With the same token, the projection matrix (5) and the decomposition are not invariant, and hence, the minimum-norm solution may depend on the measure units chosen. This is because components with different units are added together in (5).

To circumvent the quandary of the measure units, we consider the following transformation:

$$x_W := W^{1/2} x \quad (8)$$

and assume that the vector x_W has components with the same physical units. Then, a physically consistent Euclidean inner product and Euclidean norm exists on the new space [43], i.e.,

$$\|x_W\| = \langle x_W, x_W\rangle^{1/2} = \langle x, Wx\rangle^{1/2}. \quad (9)$$

The symmetric, positive definite (p.d.) matrix W is called a *metric tensor* of the n-space. Note that the Euclidean norm of the new coordinate is tantamount to the weighted-norm, that is $\|x\|_W = \|x_W\|$, where

$$\|x\|_W = (x^T W x)^{1/2}. \quad (10)$$

Furthermore, denoting $A_W := AW^{1/2}$, one can say $A_W x_W = 0$. Let P_W be the projection operator onto the null space of A_W. Then, mapping P_W is dimensionless and invariant.

III. Decomposition of the Acceleration

The kinematics of a constrained mechanical system can be represented by a set of m nonlinear equations $\Phi(q) = [\phi_1(q), \ldots, \phi_m(q)]^T = 0$, where $q \in \mathbb{R}^n$ is the vector of the generalized coordinate, and $m \le n$. Without loss of generality, we consider time-invariant (scleronomic) constraint conditions, but the methodology can be readily extended to a time-varying case (rheonomic). By differentiating the constraint equation with respect to time, we have

$$A\dot{q} = 0 \quad (11)$$

where $A = \partial\Phi/\partial q$ is the Jacobian of the constraint equation with respect to the generalized coordinate. For brevity of notation, in the following, we assume that the elements of the force and velocity vectors have homogeneous units. This assumption will be relaxed in Section VII-B by changing of the coordinates similar to (8).

Equation (11) is expressed in form of the linear operation equation. This matrix equation specifies that any admissible velocity must belong to the null space of the Jacobian matrix, that is, $\dot{q} \in \mathcal{N}(A)$. Thus, the constraint equation (11) can be expressed by the notion of the projection operator, i.e.,

$$\dot{q}_\perp \equiv (I - P)\dot{q} = 0. \quad (12)$$

Time differentiation of the above equation yields

$$\ddot{q}_\perp \equiv (I - P)\ddot{q} = C\dot{q} \quad (13)$$

where $C := (d/(dt))P$ which, in turn, can be obtained from (4) by

$$C = S + S^T, \quad \text{where } S = \dot{V}_2 V_2^T. \quad (14)$$

It is apparent from (13) and (12) that, unlike the case of velocity, the null-space orthogonal component of the acceleration is not always zero—a physical interpretation of (13) is given in Section V-D. Equation (13) expresses the component of acceleration produced exclusively by the constraint and not by dynamics. As will be seen in Section V, this equation can complement the dynamics equation in order to provide sufficient independent equations for solving the acceleration.

A. Calculating P Based on Pseudoinversion

Many mature algorithms and numerical techniques are available for computing the *pseudoinverse* [47], [48], [50]. There are also computer programs that can solve SVD and pseudoinverse in real time and non-real time, for instance, the DSP Blockset of Matlab [51]. Therefore, it may be useful to calculate the matrices P and C based on pseudoinversion.

Let A^+ denote the pseudoinverse of A. Then, the projection operator can be calculated by

$$P = I - A^+ A. \quad (15)$$

Also, one can obtain matrix C through the pseudoinversion as follows. Differentiation of (11) with respect to time leads to

$$A\ddot{q} = -\dot{A}\dot{q}.$$

The theory of linear systems of equations [47], [50], [52] establishes that the particular solution, i.e., the $\mathcal{N}^{\perp}$ component of the acceleration, can be obtained from the above equation as $\ddot{q}_{\perp} = -A^{+}\dot{A}\dot{q}$. Hence

$$C\dot{q} = -A^{+}\dot{A}\dot{q}. \tag{16}$$

Assuming the elements of generalized coordinate have identical units, then matrices P and C have homogeneous units, i.e., P is dimensionless and the dimension of C is s^{-1}. Therefore, P and C are invariant under unit changes.

Note that the inconsistency problem which may arise in computing the pseudoinverse because of the existence of the components of different units can be solved by including the metric of the n-space in computing the pseudoinverse [39], [53].

IV. Projected Inverse Dynamics

Consider a constrained mechanical system with Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where $\mathcal{T} = (1/2)\dot{q}^T M\dot{q}$ and $\mathcal{V}(q)$ are the kinetic and the potential energy functions, and $M \in \mathbb{R}^{n\times n}$ is the inertia matrix. The fundamental equation of differential variational principles of a mechanical system containing a constraint can be written as [16]

$$\delta q^T f_{\mathrm{tot}} = 0 \tag{17}$$

where $f_{\mathrm{tot}} = \varphi - (f - \mathcal{F})$, $\varphi = (d/(dt))(\partial\mathcal{L})/(\partial\dot{q}) - (\partial\mathcal{L})/(\partial q)$, $f \in \mathbb{R}^n$ is the vector of generalized input force, and $\mathcal{F} \in \mathbb{R}^n$ is the *generalized constraint force*, which is related to the *Lagrange multipliers* $\lambda \in \mathbb{R}^m$ by

$$\mathcal{F} = A^T\lambda \in \mathcal{R}(A^T). \tag{18}$$

Then, the equations describing the system dynamics can be obtained as

$$M\ddot{q} + h(q,\dot{q}) = f - \mathcal{F} \tag{19}$$
$$\Phi(q) = 0 \tag{20}$$

where vector $h(q,\dot{q}) \in \mathbb{R}^n$ contains the Coriolis, centrifugal, and gravitational terms. In solving the DAE (19)–(20), it is typically assumed that: 1) the inertia matrix is p.d., and hence, invertible; and 2) the constraint equations are independent, i.e., the Jacobian matrix is not rank-deficient [14]–[19]. In this paper, we solve the equations without relying on the second assumption.

From (18) and by virtue of *Theorem 1*, one can immediately conclude that $\mathcal{F} \in \mathcal{N}(A)^{\perp}$. In other words, the projection operator P is an annihilator for the constraint force, i.e., $P\mathcal{F} = 0$. Therefore, the constraint force can be readily eliminated from (19) if the equation is projected on P, i.e.,

$$PM\ddot{q} = P(f - h). \tag{21}$$

Equation (21) is called the *projected inverse dynamics* of a constrained multibody system that is expressed in the so-called *descriptive form*. This is because matrix PM is singular, and hence, the acceleration cannot be computed from the equation through matrix inversion.

V. Direct Dynamics

As mentioned earlier, the acceleration cannot be determined uniquely from (21), because there are fewer independent equations than unknowns. Nevertheless, (13) and (21) are in orthogonal spaces and thus cannot cancel out each other. Therefore, a unique solution can be obtained by solving these two equations together. To this end, we simply multiply (13) by M and then add both sides of the equation to those of (21). After factorization, the resultant equation can be written concisely in the following form:

$$M_c\ddot{q} = P(f - h) + C_c\dot{q} \tag{22}$$

where $C_c := MC$, and $M_c \in \mathbb{R}^{n\times n}$ is called the *constraint inertia matrix*, which is related to the unconstrained inertia matrix M, assuming a symmetric inertia matrix, by

$$M_c := M + \tilde{M} \tag{23}$$
$$\tilde{M} := PM - (PM)^T. \tag{24}$$

Equation (22) constitutes the so-called *direct dynamics* of a constrained multibody system from which the acceleration can be solved. It is worth mentioning that if M commutes with P, then $\tilde{M} = 0$, and hence, $M_c = M$. To compute the acceleration from (22) requires that the constraint inertia matrix be invertible.

Theorem 2: If the unconstrained inertia matrix M is p.d., then the constraint inertia matrix M_c is p.d., too.

Proof: It is evident from (24) that $\tilde{M}$ is a skew-symmetric matrix, i.e., $\tilde{M}^T = -\tilde{M}$. Consequently, adding $\tilde{M}$ to the inertia matrix in (23) preserves the p.d. property of the inertia matrix. This is because, for any vector $z \in \mathbb{R}^n$, we can say

$$z^T\tilde{M}z = 0.$$

Therefore, one can conclude that $z^T Mz = z^T M_c z$, or

$$M \text{ is p.d.} \iff M_c \text{ is p.d.}$$

■

Theorem 2 is pivotal in showing the usefulness of the dynamics equation (22); it signifies that the constraint inertia matrix is always invertible, regardless of the constraint condition. Therefore, the acceleration can be always obtained from (22).

Remark 2: Equation (22) signifies that only the null-space component of the generalized input force contributes to the motion of a constrained mechanical system, as the projector in the right-hand-side (RHS) of the equation filters out all forces lying in the null-space orthogonal. This fact is exploited in Section VII-A for an optimal control scheme.

It can be envisaged from *Remark 2* that it is useful to decompose the generalized input force into two orthogonal components

$$f = f_{||} \oplus f_{\perp}$$

where $f_{||} \in \mathcal{N}$ and $f_{\perp} \in \mathcal{N}^{\perp}$ are called *acting input force* (potent) and *passive input force* (impotent), respectively. The decomposition of the generalized input force can be carried out by the projection operator according to (5). Now, the equation of motion can be written as

$$\ddot{q} = M_c^{-1}(f_{\|} - h_{\|} + C_c\dot{q}) \tag{25}$$

where the nonlinear vector is decomposed in the same way as the generalized force, i.e., $h = h_{\|} + h_{\perp}$. Equation (25) is the so-called *equation of motion* of a constrained mechanical system in a compact form. It is worth mentioning that only one matrix inversion operation is required in (25), which is one less than in the standard Lagrangian method (see Appendix II).

Remark 3: Since $\tilde{M}$ does not produce any kinetic energy, the total kinetic energy associated with the constrained system is $\mathcal{T} = (1/2)\dot{q}^T M_c \dot{q} = (1/2)\dot{q}^T M \dot{q}$.

A. Constraint Inertia Matrix

The constraint inertia matrix does not have a unique definition because there are many ways that (21) and (13) can be combined together. Although all dynamics formulations thus obtained are equivalent, each one may have a certain computational advantage over the others.

1) Symmetric Inertia Matrix: M_c is a p.d. matrix but not a symmetric one. In the following, we present an alternative dynamics formulation in which the inertia matrix appears both p.d. and symmetric. Equation (13), together with the decomposition of the acceleration, imply that $\ddot{q} = P\ddot{q} + C\dot{q}$, which can be substituted in (21) to give

$$PMP\ddot{q} = P(f - h) - PC_c\dot{q}. \tag{26}$$

Now premultiply (13) by $(I - P)M$, and then adding both sides of the equation thus obtained with those of (26) yields

$$M_c'\ddot{q} = P(f - h) + C_c'\dot{q} \tag{27}$$

where

$$M_c' := PMP + (I - P)M(I - P) \qquad C_c' := (I - 2P)C_c. \tag{28}$$

It is worth mentioning that $I - 2P$ represents a *reflection operator*.

Proposition 1: If matrix M is symmetric and p.d., then M_c' is symmetric and p.d., too.

Proof: It is apparent from (28) that M_c' is a symmetric matrix. Moreover, the positive-definiteness of matrix M_c' can be shown by an argument similar to the previous case. Again, for any vector $z \neq 0 \in \mathbb{R}^n$ and from definition (28), we can say $z^T M_c' z = z_{\|}^T M z_{\|} + z_{\perp}^T M z_{\perp}$, where $z_{\|} \in \mathcal{N}$ and $z_{\perp} \in \mathcal{N}^{\perp}$, and $z_{\|}^T M z_{\|} \geq 0$ and $z_{\perp}^T M z_{\perp} \geq 0$. Both decomposed components of the nonzero vector z cannot be zero, i.e., $z_{\|} = 0 \Rightarrow z_{\perp} \neq 0$ and vice versa. Therefore, only one of the quadratic functions can be zero, and that implies their summation is nonzero and positive. Thus M_c' is a p.d. matrix. ∎

2) Parameterized Inertia Matrix: Alternatively, the constraint inertia matrix can be parameterized in terms of an arbitrary scalar. To this end, let us first premultiply (13) by a scalar γ, and then add both sides of the resultant equation to those of (21). That gives the standard dynamics formulation similar to (27) with the following parameters:

$$M_c'' := PM + \gamma(I - P) \qquad C_c'' := \gamma C. \tag{29}$$

Proposition 2: If M is an invertible matrix, then M_c'' is always invertible, too.

Proof: In a proof by contradiction, we show that M_c'' should be a full-rank matrix. If matrix M_c'' is not of full rank, then there must exist at least one nonzero vector $\xi \neq 0$ lying in the matrix null space, that is $M_c''\xi = 0$, or

$$PM\xi + \gamma(I - P)\xi = 0. \tag{30}$$

The two terms of the above equation are in two orthogonal subspaces and cannot cancel out each other. Hence, in order to satisfy the equation, both terms must be identically zero, i.e., $PM\xi = 0$ and $\gamma(I - P)\xi = 0$. The former and the latter equations imply that $y = M\xi \in \mathcal{N}^{\perp}$ and $\xi \in \mathcal{N}$, respectively. Therefore, one can conclude that y is perpendicular to vector ξ, i.e., $\xi^T y = 0$, or that

$$\xi^T M \xi = 0 \tag{31}$$

which is a contradiction because M is a p.d. matrix. Therefore, the null set of M_c'' is empty and the matrix is always invertible, and this completes the proof. ∎

Since P is dimensionless, the scalar γ has the dimensions of mass. Therefore, the value of γ should be comparable to that of M to avoid any numerical pitfall in the matrix inversion—a logical choice is $\gamma = \|M\|$; yet, certain γ may lead to the minimum condition number of M_c, which is desired for matrix inversion.

3) A Comparison of Different Constraint Inertia Matrices: Theoretically, all the inverse-dynamics formulations presented here are equivalent, and they should yield the same result. However, from a numerical point of view, each has a certain advantage over the others that can lead to simplification of simulation or control. In summary:

- M_c is a p.d. matrix but not a symmetric one. If P commutes with M, then $M_c = M$;
- M_c' is a symmetric and p.d. matrix; hence, it physically exhibits the characteristic of an inertia matrix. However, computing M_c' involves three additional matrix multiplication operations, compared with M_c;
- M_c'' is an invertible matrix, but it is neither p.d. nor symmetric. Nevertheless, computing of M_c'' requires less computation effort, compared with the others.

B. Constraint Force and Lagrange Multipliers

Equation (25) expresses the generalized acceleration of a constrained multibody system in a compact form without any need for computing the Lagrange multipliers. Yet, in the following, we will retrieve the constraint force by projecting (19) onto $I - P$. That gives

$$\mathcal{F} = (f_{\perp} - h_{\perp}) - (I - P)M\ddot{q}.$$

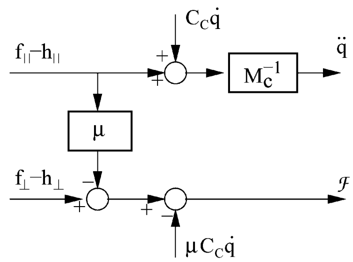

Fig. 1. Input/output realization of a constrained mechanical system based on decomposition of the generalized input force.

Now, substituting the acceleration from (25) into the above equation gives

$$\mathcal{F} = (f_\perp - h_\perp) - \mu(f_{||} - h_{||}) - \mu C_c \dot{q} \tag{32}$$

where $\mu = (I-P)\alpha$, and $\alpha = MM_c^{-1}$ is the ratio of the two inertia matrices.

Equation (32) implies that the constraint force can always be obtained uniquely, but this may not be true for the Lagrange multipliers. Having calculated the constraint force from (32), one may obtain the Lagrange multipliers from (18) through pseudoinversion, i.e., $\lambda = A^{+T}\mathcal{F} + \lambda_n$ where $\lambda_n \in \mathcal{N}(A^T)$ is the homogeneous solution. By virtue of *Theorem 1*, we can also say that $\lambda_n \in \mathcal{R}(A)^\perp$. This, in turn, implies that λ_n is a nonzero vector only if the Jacobian matrix is rank-deficient, i.e., $r < m$—recall that $r = \text{rank}(A)$.

Remark 4: The vector of Lagrange multipliers can be determined uniquely iff the Jacobian matrix is full rank. In that case, there is a one-to-one correspondence between $\mathcal{F}$ and λ. Otherwise, the $\mathcal{R}(A)^\perp$ component of the Lagrange multiplies is indeterminate.

C. Decoupling

Fig. 1 illustrates the input/output realization of a constrained mechanical system based on (25) and (32). The input channels $f_{||}$ and $f_\perp$ are the potent and the impotent components of the generalized input force, while the output channels are the acceleration and the constraint force, $\ddot{q}$ and $\mathcal{F}$, respectively. It is apparent from the figure that the acceleration is only affected by $f_{||}$, and not by $f_\perp$ whatsoever. However, the constraint force output, in general, can be affected by two inputs: by $f_\perp$ directly, and by $f_{||}$ through the cross-coupling channel μ. The cross-coupling channel is disabled if the inertia matrix satisfies a certain condition, which is stated in *Proposition 3*.

Proposition 3: The equations of the constraint force and the acceleration are completely decoupled, i.e., the cross-coupling μ vanishes if the null space of the constraint Jacobian is invariant under M. That is, the inertia matrix should have this property: $\{\forall x \in \mathcal{N}(A) : Mx \in \mathcal{N}(A)\}$.

The proof is given in Appendix IV.

Mechanical systems satisfying the condition in *Proposition 3* are called *decoupled constrained mechanical systems*. The equation of constraint force of such a system is reduced to

$$\mathcal{F} = (f_\perp - h_\perp) - \mu C_c \dot{q}.$$

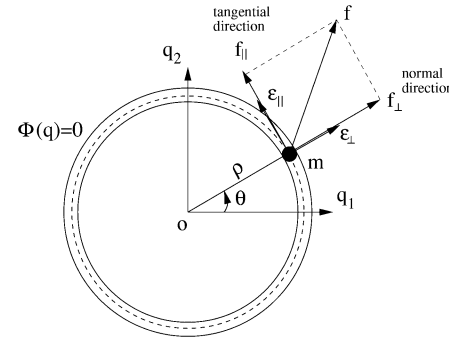

Fig. 2. Illustrative example.

In that case, the constraint force is determined exclusively by the passive input force that leads to a simple force-control scheme, as will be seen in Section VII-C.

D. Illustrative Example and Geometrical Interpretation

A particle of mass m moves on a circle of radius ρ; see Fig. 2. Assume $q = [q_1 \;\; q_2]^T$ to be the generalized coordinate. The constraint equation is

$$\Phi(q) = \left(q_1^2 + q_2^2\right)^{1/2} - \rho = 0$$

which yields the Jacobian and its time-derivative as $A = [q_1/\rho \;\; q_2/\rho]$ and $\dot{A} = [\dot{q}_1/\rho \;\; \dot{q}_2/\rho]$, and the pseudoinverse is $A^+ = [q_1/\rho \;\; q_2/\rho]^T$. Then, from (15), we have

$$\begin{aligned} P &= \frac{1}{\rho^2}\begin{bmatrix} q_2^2 & -q_1q_2 \\ -q_1q_2 & q_1^2 \end{bmatrix} \\ &= \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}. \end{aligned} \tag{33}$$

Let unit vectors $\epsilon_{||} = [-\sin\theta \;\; \cos\theta]^T$ and $\epsilon_\perp = [\cos\theta \;\; \sin\theta]^T$ represent the tangential and normal directions, as shown in Fig. 2. Then, from (33), we have

$$P = \epsilon_{||}\epsilon_{||}^T, \quad I - P = \epsilon_\perp\epsilon_\perp^T.$$

The above equations represent the geometrical interpretation of the projection operators. Note that the normal component of the acceleration, $\ddot{q}_\perp = C\dot{q}$, which can be interpreted as the centripetal acceleration, is given by

$$C\dot{q} = -\frac{1}{\rho^2}\begin{bmatrix} q_1(\dot{q}_1^2 + \dot{q}_2^2) \\ q_2(\dot{q}_1^2 + \dot{q}_2^2) \end{bmatrix} \tag{34}$$

$$= -\frac{\|\dot{q}\|^2}{\rho}\epsilon_\perp = -\rho\dot{\theta}^2\epsilon_\perp. \tag{35}$$

Let f denote the external force applied on the particle. Observe that the decomposition of the force is aligned with the tangential and the normal directions to the circular trajectory shown in Fig. 2. Finally, we arrive at the following set of dynamics equations:

$$\begin{aligned} m\ddot{q} &= f_{||} \\ \mathcal{F} &= f_\perp + m\frac{\|\dot{q}\|^2}{\rho}\epsilon_\perp. \end{aligned}$$

VI. Simulation of Constrained Multibody Systems

To simulate the dynamics of a constrained multibody system, one can make use of the acceleration model in (25). Having computed the generalized acceleration from the equation, one may proceed to a simulation by integrating the acceleration to obtain the generalized coordinates. However, the integration inevitably leads to drift that eventually results in a large constraint error. Baumgarte's stabilization term [19] is introduced to ensure exponential convergence of the constraint error to zero. However, this creates a very fast dynamics which tends to slow down the simulation. In this section, we use the pseudoinverse for correcting the generalized coordinate in order to maintain the constraint condition precisely. It should be noted that using the pseudoinverse here does not impose any extra computation burden, because the pseudoinverse has to be obtained to compute the acceleration anyway.

Having obtained the velocity through integrating the acceleration, one may obtain the generalized coordinate by integration

$$\{q\}_t = \{q\}_{t-\Delta T} + \int_{t-\Delta T}^{t} \dot{q}\, d\tau \tag{36}$$

where ΔT depicts the integration time step. However, the constraint condition may be violated slightly because of integration drift. Let q_0 denote the coordinate after a few integration steps and $\Phi(q_0) \neq 0$. Now, we seek a small compensation in the generalized coordinate $\delta q* = q^* - q_0$, such that the constraint condition is satisfied. That is, a set of nonlinear equations $\Phi(q^*) = 0$ must be solved in terms of q^*. The Newton–Raphson (NR) method solves a set of nonlinear equations iteratively based on linearized equations.

The constraint equation can be written by the first-order approximation as

$$\Phi(q_0 + \delta q) = \Phi(q_0) + A\delta q + \mathcal{O}(\delta q^2) = 0.$$

Neglecting the $\mathcal{O}(\delta q^2)$ term, one can obtain the solution of the linear system using any generalized inverse of the Jacobian. The pseudoinverse yields the minimum-norm solution, i.e., $\min_{A\delta q = -\Phi(q_0)} \|\delta q\|$. Therefore, the following loop:

$$q_{k+1} = q_k - A^+ \Phi(q_k) \tag{37}$$

may be worked out iteratively until the error in the constraint falls into an acceptable tolerance, e.g., $\|\Phi\| \leq \epsilon$.

The condition for local convergence of multidimensional NR iteration can found, e.g., in [15], [54], and [55]. Although it is known that the NR iteration will not always converge to a solution, the convergence is guaranteed if the initial approximation is close enough to a solution [54], [55].

Theorem 3: [54]: Assume that Φ is differentiable in an open set $\Omega \subset \mathbb{R}^n$, i.e., the Jacobian matrix A exists, and that A is Lipschitz continuous. Also assume that a solution $q^* \in \Omega$ exists, and that $A(q^*)$ is nonsingular. Then under these assumptions, if the initial start point is sufficiently close to the solution, the convergence is quadratic, that is $\exists K > 0$ such that

$$\|\Phi(q_{k+1})\| \leq K \|\Phi(q_k)\|^2.$$

One can expect that the drift, and hence, the inial constraint error, can be reduced by decreasing the integration time step. It should be pointed out that the iteration loop (37) corrects the error in the constraint coordinate caused by the integration process. Since the drifting error within a single integration time step ΔT is quite small, the initial estimate given by (36) cannot be far from the exact solution. Therefore, as shown by experiments, a fast convergence is achieved, even though the iteration loop (37) is called once every few time steps.

Finally, the simulation of a constrained mechanical system based on the projection method can be done by the following steps:

1) compute the acceleration from (25);
2) obtain the states $\{q, \dot{q}\}$ as a result of numerical integration of the acceleration;
3) in the case constraint error exceeds the tolerance, carry out iteration (37), upon convergence or counting the maximum number of iterations, go to step 1).

VII. Control of Constrained Multibody Systems

In this section, we discuss the position and/or force control of constrained multibody systems based on the proposed dynamics formulation. The input/output (I/O) realization of a constrained mechanical system is depicted in Fig. 1, which will be used subsequently as a basis for development of control algorithm. In fact, the topology of a control system can be inferred from the figure by considering the decomposed components of the generalized input force, $f_{\|}$ and $f_{\perp}$, as the corresponding control inputs for position and force feedback loops.

Due to the decoupled nature of the acceleration channel, an independent position feedback loop can be applied. The input channel $f_{\perp}$ is directly transmitted to the constraint force, $f_{\|}$, and the velocity enters as a disturbance, and hence, must be compensated for in a feedforward loop. Note that in the case of decoupled mechanical systems, where the cross-coupling channel vanishes, $f_{\perp}$ exclusively determines the constraint force.

A. Motion Control Using PIDC

Due to presence of only r independent constraints, the actual number of DOFs of the system is reduced to $k \leq n - r$. Thus, in principle, there must be k independent coordinates $\theta \in \mathbb{R}^k$ from which the generalized coordinates can be derived, i.e., $q = \psi(\theta)$. Now, differentiation of the given function with respect to time gives

$$\dot{q} = \Lambda \dot{\theta} \tag{38}$$

$$\ddot{q} = \Lambda \ddot{\theta} + \dot{\Lambda} \dot{\theta} \tag{39}$$

where $\Lambda = \partial \psi / \partial \theta \in \mathbb{R}^{n \times k}$. Since $\theta(q) = [\theta_1(q), \ldots, \theta_k(q)]^T$ constitutes a set of independent functions, the Jacobian matrix Λ must be of full rank. (The proof is in Appendix V). It is also important to note that any admissible function $\psi(\cdot)$ must satisfy the constraint condition, i.e.,

$$\Phi(\psi(\theta)) = 0 \quad \forall \theta \in \mathbb{R}^k.$$

Using the chain-rule, one can obtain the time-derivative of the above equation

$$\frac{\partial \Phi}{\partial q} \frac{\partial \psi}{\partial \theta} \dot{\theta} = A \Lambda \dot{\theta} = 0 \quad \forall \dot{\theta} \in \mathbb{R}^k. \tag{40}$$

Since Λ is a full-rank matrix, the only possibility for (40) to happen is that

$$\mathcal{R}(\Lambda) = \mathcal{N}(A). \tag{41}$$

Substituting the acceleration from (39) into the inverse-dynamics equation (21) gives the dynamics in terms of the reduced-dimensional coordinate

$$PM(\Lambda\ddot{\theta} + \dot{\Lambda}\dot{\theta}) = f_{\|} - h_{\|}. \tag{42}$$

Let $\{\theta_d(t), \dot{\theta}_d(t), \ddot{\theta}_d(t)\}$ denote the desired trajectory of the new coordinates. Now, we propose the *PIDC* law as follows:

$$f_{\|}^c = h_{\|} + PMu_p \tag{43}$$

where u_p is an auxiliary control input as

$$u_p = \dot{\Lambda}\dot{\theta} + \Lambda(\ddot{\theta}_d + G_D\dot{e}_p + G_P e_p) \tag{44}$$

$e_p = \theta_d - \theta$ is the position tracking error, and $G_P > 0$ and $G_D > 0$ are the proportional derivative (PD) feedback gains. In the following, superscript c is used to denote control input.

Theorem 4: While demanding minimum-norm control input, the PIDC law (43)–(44) stabilizes the position tracking error, i.e., $\theta(t) \to \theta_d(t)$ as $t \to \infty$.

Proof: First, we prove exponential stability of the position error. From (42)–(44), one can conclude that the proposed control law leads to the following equation for the tracking error:

$$PM\Lambda[\ddot{e}_p + G_D\dot{e}_p + G_P e_p] = 0. \tag{45}$$

To show that the expression within the bracket is zero, we need to show that the matrix $PM\Lambda$ is full rank. In the following, we will show that the matrix cannot have any null space, and hence, is full rank. If the matrix has a null space, then $\exists x \neq 0 \ni PM\Lambda x = 0$. Let us define $\xi = \Lambda x$. Recall that Λ is a full-rank matrix and that $\mathcal{R}(\Lambda) = \mathcal{N}(A)$—see (38). Hence, $\xi \neq 0$ and $\xi \in \mathcal{N}$. On the other hand, $PM\xi = 0$ implies that $M\xi \in \mathcal{N}(A)^{\perp}$, and hence, it is perpendicular to ξ, i.e., $\xi^T M\xi = 0$. But, this is a contradiction, because M is a p.d. matrix. Consequently, $\mathcal{N}(PM\Lambda) = \emptyset$, and it follows from (45) that

$$\ddot{e}_p + G_D\dot{e}_p + G_P e_p = 0.$$

Hence, the error dynamics can be stabilized by selecting adequate gains, that is, $\theta \to \theta_d$ as $t \to \infty$. Moreover, due to orthogonality of the decomposed generalized input force, we can say

$$\|f^c\|^2 = \|f_{\|}^c\|^2 + \|f_{\perp}^c\|^2.$$

From the above norm relation, it is clear that $f_{\|}^c$ is the minimum norm solution, since any other solution must have a component in $f_{\perp}^c$, and this would increase the overall norm. Therefore, setting $f_{\perp}^c = 0$ results in minimum norm of generalized input force subjected to producing the desired motion, i.e.,

$$f^c = f_{\|}^c \Longleftrightarrow \min_{\theta(t)\to\theta_d(t)} \|f^c\|. \tag{46}$$

■

B. Elements of Generalized Coordinate With Inhomogeneous Units

So far, we have assumed that the elements of the generalized velocity and the generalized input force have homogeneous units. Otherwise, the minimum-norm solution of the generalized force, (46), makes no physical sense if the manipulator has both revolute and prismatic joints. In this section, we assume the vector of the generalized force to have a combination of force and torque components, and the vector of the generalized velocity with of rotational and translational components. As mentioned in Section II-A.1, the minimization solution is not invariant with respect to changes in measure units if there is inhomogeneity of units in the spaces of the force and the velocity [56], [57]. To go around the quandary of inhomogeneous units, one can introduce a p.d. weight matrix by which the coordinates of the force vector is changed to

$$f_W = W^{-1/2} f.$$

Note that the corresponding change of coordinates for the velocity is $\dot{q}_W = W^{1/2}\dot{q}$ in order to preserve the force-velocity product [by virtue of (17)]. Therefore, the metric tensors for the force and velocity vectors are W^{-1} and W, respectively. The inertia matrix and the Jacobian with respect to the new coordinates are $M_W = W^{-1/2}MW^{-1/2}$ and $A_W = AW^{-1/2}$. Since $\dot{q}_W$ and f_W and the corresponding projection matrix $P_W = I - A_W^+ A_W$, where $A_W^+ = [AW^{-1/2}]^+$, is always dimensionless, and hence, invariant under the measure units chosen. The new force and velocity vectors have homogeneous units if the weight matrix is properly defined. Therefore, replacing the new parameters, which are now dimensionally consistent, in the optimal control (43) minimizes $\|f_W\|$, or equivalently, minimizes the weighted Euclidean norm of the generalized input force, i.e.,

$$\|f\|_W = (f^T W^{-1} f)^{1/2}. \tag{47}$$

A quiet direct structure for W is the diagonal one, i.e.,

$$W = \begin{bmatrix} \kappa^{-2} I & 0 \\ 0 & I \end{bmatrix} \tag{48}$$

where κ is a length, by which we divide the translational velocity (or multiply the force). Using this length is tantamount to the weighted norm as

$$\|f\|_W = (\kappa^2\|f_f\|^2 + \|f_t\|^2)^{1/2}$$

where the added terms are homogeneous, and f_f and f_t are the force and the torque components of f. It is worth pointing out that a *characteristic length* that arises naturally in the analysis and leads to invariant results was proposed in [58] and [59].

Alternatively, the weight matrix can be selected according to some engineering specifications. For instance, assume that the maximum force and torque generated by the actuators are limited to $f_{f\max}$ and $f_{t\max}$. Then, choosing $W^{1/2} = \mathrm{diag}\{f_{f\max}, f_{t\max}\}$ leads to the minimization of this cost function

$$\left(\left\| \frac{f_f}{f_{f\max}} \right\|^2 + \left\| \frac{f_t}{f_{t\max}} \right\|^2 \right)^{1/2}$$

which, in a sense, takes the saturation of the actuators into account.

C. Control of Constraint Force

The motion controller proposed in VII-A works well for mechanical systems with bilateral constraint. On the other hand, since the proposed controller does not guarantee that the sign of the constraint force will not change, a unilateral constraint condition may not be physically maintained under the control law. In this case, controlling the constraint force is a necessity.

Suppose that $\mathcal{F}_d$ represents the desired constraint force which can be derived from the desired Lagrange multipliers λ_d using

$$\mathcal{F}_d = A^T \lambda_d.$$

Then, considering $f_\perp^c$ as a control input, we propose the following control law:

$$f_\perp^c = h_\perp + \mu(f_\| - h_\| + C\dot{q}) + u_\mathcal{F} \tag{49}$$

where $u_\mathcal{F}$ is the auxiliary control input, which is traditionally chosen as

$$u_\mathcal{F} = \mathcal{F}_d + G_F e_f + G_I \int_0^t e_f \, \mathrm{d}\tau \tag{50}$$

in which $e_f = \mathcal{F}_d - \mathcal{F}$ is the force error, and $G_F > 0$ and $G_I > 0$ are the proportional integral (PI) feedback gains. It should be pointed out that the integral term is not necessary, but it improves the steady-state error. From (32), (49), and (50), one can obtain the error dynamics as

$$(G_F + 1)\dot{e}_f + G_I e_f = 0$$

which will be stable provided that the gains are p.d., i.e., $e_f \to 0$ as $t \to \infty$. Define $e_\lambda = \lambda_d - \lambda$, and $e_f = A^T e_\lambda$. Then

$$\underline{\sigma}(A)\|e_\lambda\| \le \|e_f\| \tag{51}$$

where $\underline{\sigma}(A)$ is the minimum singular value of the Jacobian, i.e.,

$$\underline{\sigma}(A) = (\min_{\text{eigenvalue}} A^T A)^{1/2}.$$

Therefore, one can conclude tracking of the Lagrange multipliers, i.e., $e_\lambda \to 0$ as $t \to \infty$, if the Jacobian is full rank or $\underline{\sigma}(A) \neq 0$.

Remark 5: The constraint force $\mathcal{F}$ is always controllable, while the Lagrange multipliers are controllable only if the Jacobian matrix is of full rank.

It is worth mentioning that unlike the traditional motion/force control schemes, which lead to coupled dynamics of force error and position error, our formulation yields two independent error equations. This is an advantage, because the motion control can be achieved regardless of the force control and vice versa. To this end, a hybrid motion/force control law can be readily obtained by combining (43) and (49)

$$\begin{aligned} f^c &= f_\|^c + f_\perp^c \\ &= h + \mu C\dot{q} + (I + \mu)PMu_p + u_\mathcal{F}. \end{aligned} \tag{52}$$

D. Control of Constrained Mechanical Systems With Passive Joints

Some constrained mechanical systems, e.g., parallel manipulators, have joints without any actuators. The joints with and without actuators are called *active joints* and *passive joints*, respectively. In this section, we use the notion of the linear projection operator to generalize the IDC scheme for constrained mechanical systems with passive joints. Assuming there are p active joints (and $n - p$ passive joints), the generalized input force has to have this form

$$f^c = \begin{bmatrix} f_1^c \\ \vdots \\ f_p^c \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{active joints} \\ \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{passive joints.} \end{matrix}$$

This implies that any admissible generalized force should satisfy

$$f^c \in \mathcal{R}(B) = \mathcal{B}, \quad \text{and} \quad B = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \tag{53}$$

where I_p is a $p \times p$ identity matrix. Note that B is a projection onto the *actuator space* $\mathcal{B}$, i.e., $B^2 = B$ and $Bf^c = f^c$.

Now, we need to modify the motion control law (43) so that the condition in (53) is fulfilled. If $\mathcal{N}(A) \subseteq \mathcal{R}(B)$, then (53) is automatically satisfied by choosing $f^c = f_\|^c$. Otherwise, we need to add a $\mathcal{N}^\perp$ component, say $f_\perp^c$, to $f_\|^c$ so that $f^c = f_\|^c + f_\perp^c \in \mathcal{R}(B)$. Since $f_\perp^c$ does not affect the system motion at all, the motion tracking performance is preserved by that enhancement, albeit control of constraint force may no longer be achievable. Let us assume

$$f_\perp^c = (I - P)\eta \tag{54}$$

where $\eta \in \mathbb{R}^n$. Then, we seek η such that

$$\begin{aligned} f^c \in \mathcal{R}(B) &\Leftrightarrow (I - B)f^c = 0 \\ &\Leftrightarrow (I - B)(f_\|^c + (I - P)\eta) = 0 \\ &\Leftrightarrow Q\eta = -(I - B)f_\|^c \end{aligned} \tag{55}$$

where $Q = I - B - P + BP$. Consider η as the unknown variable in (55). A solution exits if the RHS of (55) belongs to the range of Q, i.e.,

$$\mathcal{R}((I - B)P) \subseteq \mathcal{R}((I - B)(I - P)). \tag{56}$$

Then, the particular solution can be found via pseudoinversion, i.e.,

$$\eta = -Q^+(I - B)f_\|^c. \tag{57}$$

The above equation yields the minimum-norm solution, i.e., minimum $\|\eta\|$, which eventually minimizes the actuation force. Equations (54) and (57) give

$$f_\perp^c = Hf_\|^c \tag{58}$$

where

$$H = -(I - P)Q^+(I - B). \tag{59}$$

Finally, we arrive at the following control law for constrained mechanical systems with passive joints:

$$f^c = (I + H)f_\|^c \tag{60}$$

with $f_\|^c$ derived from (43).

1) Minimum-Norm Torque: A simple argument shows that the torque-control law (60), assuming the existence of a solution, yields a minimum-norm torque. Knowing that $||I - P|| = 1$, we have

$$\begin{aligned} ||f^c||^2 &= ||f^c_{||}||^2 + ||(I-P)\eta||^2 \\ &\leq ||f^c_{||}||^2 + ||\eta||^2 \end{aligned} \tag{61}$$

where both norms in the RHS of (61) are minimum.

2) Controllability: Because the existence of a solution is tantamount to the controllability condition of constrained mechanical systems under the proposed control law, it is important to find out when a solution to (55) exists, It can be inferred from (56) that

$$\text{controllability cond.} \Leftrightarrow \mathcal{N} \cap \mathcal{B}^\perp \subseteq \mathcal{N}^\perp \cap \mathcal{B}^\perp. \tag{62}$$

In general, the proposed control method for systems with passive joints works only if there exists a sufficient number of active joints. Since occurrence of the singularities gives rise to the number of DOFs, the system under the control law may no longer be controllable if there are not enough active joints. For instance, mechanical systems without any constraints at all cannot be controlled unless all joints are actuated. This is because no constraint means that $\mathcal{N}^\perp = \emptyset$ or $\mathcal{N} \in \mathbb{R}^n$; hence, according to (62), a controllable system requires that $\mathbb{R}^n \cap \mathcal{B}^\perp \subseteq \emptyset \Rightarrow \mathcal{B}^\perp = \emptyset$, which means there can be no passive joints.

Also, it is worth pointing out that choosing $\mathcal{N} \subseteq \mathcal{B}$ trivially results in a controllable system.

VIII. A Slider-Crank Case Study

In this section, we describe the results obtained from applying the proposed inverse and direct dynamics formulations for simulation and control of a slider-crank mechanism, Fig. 3(A). Assume that the dimension of the crank and the connecting rod are the same. Then, constraint singularity occurs at

$$q_1 = \frac{n\pi}{2} \quad n = \pm 1, \pm 3, \ldots \tag{63}$$

as seen in Fig. 3(B).

A. Equations of Direct Dynamics

As shown in Fig. 3(C), the closed loop is cut in the right-hand support, i.e., at joint C. Let vector $q = [q_1 \; q_2]^T$ denote the joint angles. Then, the vertical position of the point C is

$$y_C(q) = l(s_1 + s_{12}) \tag{64}$$

where l is the link length. Below, c_i, c_{12}, s_i, s_{12} are shorthand for $\cos q_i, \cos(q_1+q_2), \sin q_i$, and $\sin(q_1+q_2)$ using the notation in [60]. The vertical translational motion of point C is prohibited by imposing the following scleronomic constraint equation:

$$\Phi(q) = y_C(q) = 0. \tag{65}$$

Now, the derivation of the direct dynamics may proceed in the following steps.

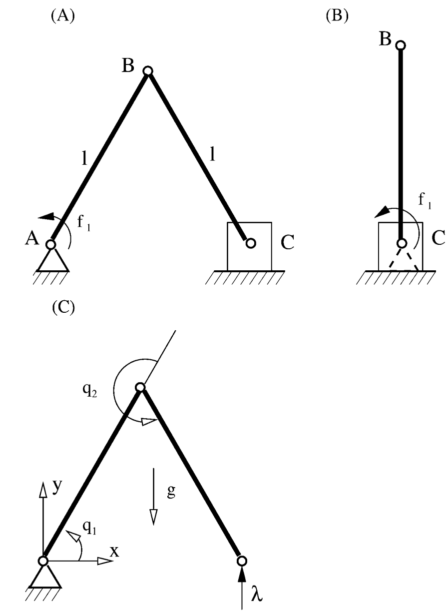

Fig. 3. Slider-crank mechanism.

Step 1) Obtain the dynamic parameters of the open chain system Fig. 3(C) (similar to the two-link manipulator case study in [60]) as

$$\begin{aligned} M &= ml^2 \begin{bmatrix} 3+2c_2 & 1+c_2 \\ 1+c_2 & 1 \end{bmatrix} \\ h &= \begin{bmatrix} -ml^2 s_2\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) + mlg(c_{12}+2c_1) \\ ml^2 s_2 \dot{q}_1^2 + mlgc_{12} \end{bmatrix} \end{aligned} \tag{66}$$

where m represents the mass of the link.

Step 2) Compute the projection matrix corresponding to the constraint (65). The Jacobian of the constraint is

$$A = l[c_1 + c_{12} \quad c_{12}]. \tag{67}$$

The projection matrix can be computed numerically (e.g., using the SVD block of the DSP Blockset in Matlab/Simulink [51]). Nevertheless, we obtain P in a closed form for this particular illustration to have some insight into how the SVD handles singularities. Since $q_2 = 2\pi - 2q_1$, the Jacobian matrix (67) can be simplified as $A = lc_1[2 \; 1]$ whose singular value is trivially $\sigma(A) = \sqrt{5}l|c_1|$. The SVD algorithm treats all singular values less than ϵ (a small value) as zeros, i.e., $A^+ = 0$ if $|c_1| < \epsilon/l\sqrt{5}$. Hence

$$P = \begin{cases} \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}, & \text{if } |c_1| < \frac{\epsilon}{l\sqrt{5}} \\ I, & \text{otherwise} \end{cases} \tag{68}$$

which indicates the constraint is virtually removed if the system is sufficiently close to the singular configurations (63).

Step 3) Now, assuming $\gamma = ml^2$, one can compute the constraint inertia matrix from (29) as

$$M_c'' = ml^2 \begin{bmatrix} 1 & (1+c_2)/5 \\ 0 & (3-2c_2)/5 \end{bmatrix}, \quad \text{if } |c_1| < \frac{\epsilon}{l\sqrt{5}} \tag{69}$$

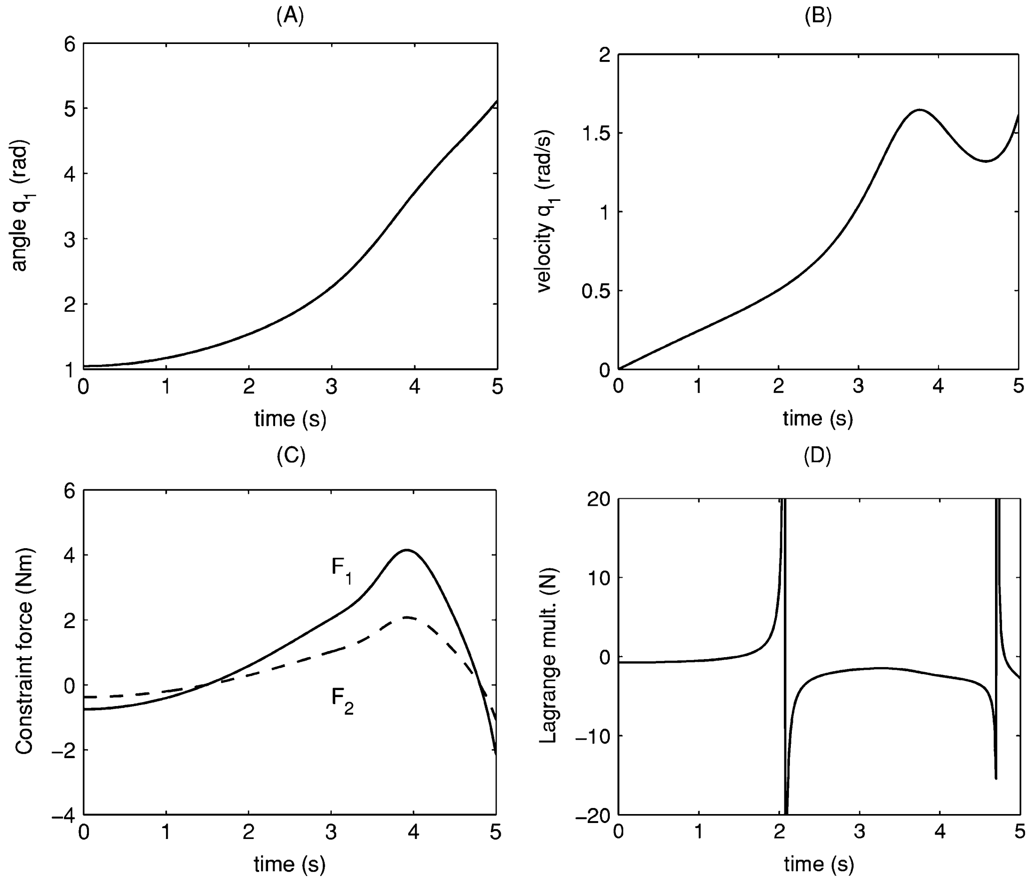

Fig. 4. Slider-crank mechanism: Trajectories of the state variables and constraint force reactions.

and $M_c' = M$ at the singular positions—note that $\det(M_c') = m^2l^4(3-2c_2)/5 \neq 0$. Finally, plugging P and M_c' from (68) and (69) into (22) yields the direct dynamics of the slider-crank mechanism.

1) Numerical Simulation: Simulation results are shown in Fig. 4 for an initial position $q_1 = ((2\pi)/3), f_1 = 1$ Nm, and system parameters $l = 1$ m and $m = 1$ kg. The standard ODE solver based on the *Runge–Kutta* algorithm in Matlab/Simulink with an integration step time $\Delta T = 0.01$ s was used to perform the simulation. The pseudoinverse is carried out using the DSP blockset of the Matlab/Simulink. It is evident from Fig. 4(A) and (B) that the motion goes smoothly through the singular configurations at $q_1 = \pi/2, 3\pi/2$, while abrupt changes in the value of λ_1 at the singular configuration can be observed in Fig. 4(D). Note that the calculation of acceleration does not require the value of Lagrange multipliers in our approach. Also, observe the smooth trajectories of the constraint forces in Fig. 4(C).

It is worth pointing out that, for some systems, a bifurcation of the system motion may happen if the system approaches a singular configuration with a vanishing velocity [61]. However, this does not happen in this particular example.

B. Control

1) Two Active Joints: Consider the slider-crank mechanism as a constrained two-link manipulator and assume both joints A and B are active. Let us take q_1 as the independent coordinate, i.e., $\theta = q_1$. From the geometry of the closed loop, we have $q_2 = 2\pi - 2q_1$. Hence

$$\Lambda = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \dot{\Lambda} = 0. \tag{70}$$

The motion control law minimizing actuation torque can be formed by plugging the parameters from (66), (68), and (70) into (43). After simplification, we have

$$\begin{aligned} f_{\|1}^c &= \frac{1}{5}(3-2c_2)u_1^* + \frac{1}{5}mglc_1 - \frac{2}{5}ml^2s_2\dot{q}_1 \\ f_{\|2}^c &= -2f_{\|1}^c \end{aligned} \tag{71}$$

where $u_1^* = \ddot{q}_{d1} + G_D\dot{e}_{q_1} + G_Pe_{q_1}$.

Similarly, the force control law can be derived by substituting the above parameters and (69) into (50), i.e.,

$$\begin{aligned} f_{\perp 1}^c &= \frac{2}{5}(1+c_2)u_1^* + u_2^* + \frac{14}{5}mglc_1 + \frac{2}{5}ml^2s_2\dot{q}_1 \\ f_{\perp 2}^c &= 0.5f_{\perp 1}^c \end{aligned} \tag{72}$$

where $u_2^* = 2l(c_1\lambda_d + G_Fc_1e_\lambda + G_i\int c_1e_\lambda\,dt)$. Combining (71) and (72) together yields a hybrid motion/force controller.

Fig. 5 shows the simulation results for $G_P = 25, G_D = 10, G_F = 1, G_I = 20$, and the mechanism moves from the initial position $\theta_0 = \pi/6$ to the desired position $\theta_d = \pi/3$. Fig. 5 shows the simulated position, the Lagrange multipliers (the contact force), and the joint torques for two cases: 1) Only

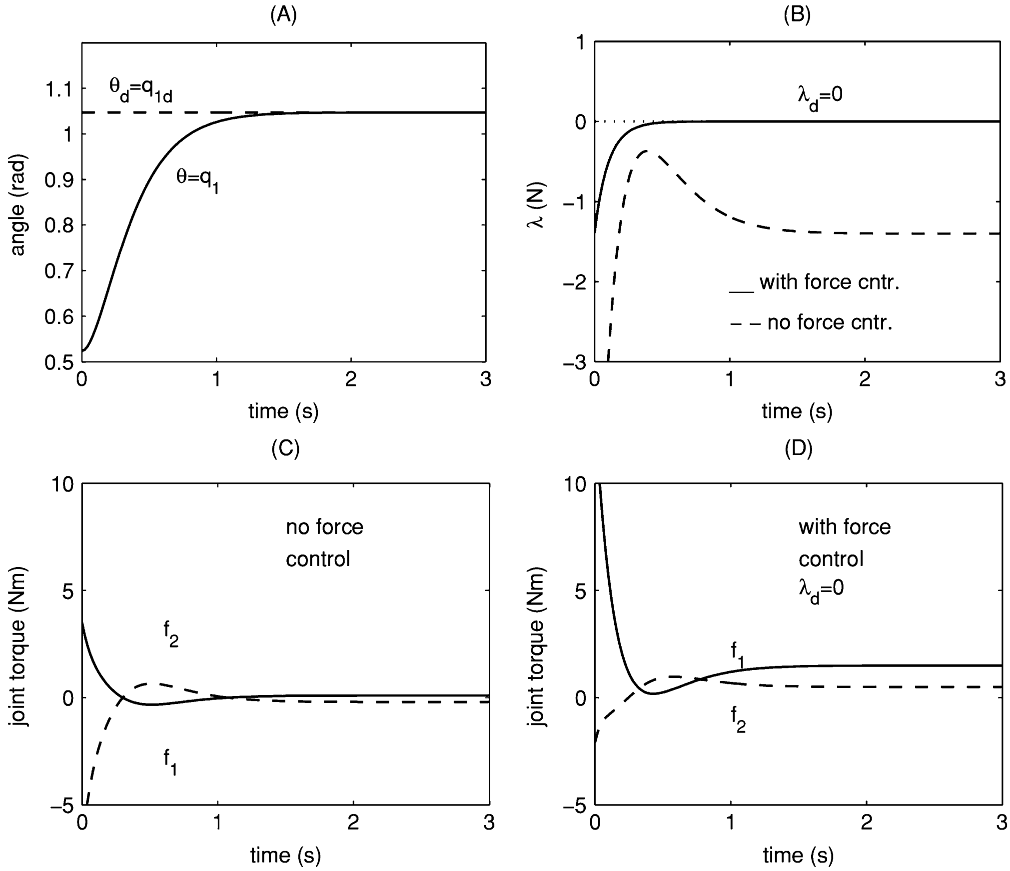

Fig. 5. Simulated position and force responses; two actuated joints.

the motion controller is applied; and 2) the hybrid motion/force controller where $\lambda_d = 0$ is applied.

2) One Active Joint: Now, we assume that joint A is active and joint B is passive (has no actuator), i.e.,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{73}$$

Then, the control syntheses can proceed by replacing P and B from (68) and (73) into (59); that yields

$$H = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}. \tag{74}$$

Note that at the instant of singularity, we have $P = I$ or $\mathcal{N}^\perp = \emptyset$. Therefore, according to (62), the system at the singular configuration with only one active joint is not controllable.

Now, assuming a controllable system, the motion control law for the system with a passive joint can be synthesized from (71) and (74) according to (60), i.e.,

$$f_1^c = -\frac{3}{5}(3 - 2c_2)u_1^* - \frac{3}{5}mglc_1 + \frac{6}{5}ml^2 s_2 \dot{q}_1$$
$$f_2^c = 0. \tag{75}$$

Fig. 6 shows the trajectories of the position and the joint torques. Observe that $f_2 = 0$, i.e., only one actuated joint exits. A comparison of the root mean square (RMS) norm[1] of the joint torque signals corresponding to different controllers is illustrated in Table I.

[1]The RMS norm of a vector signal is defined by

$$\|u\|_{\rm rms} = \left(\frac{1}{T} \int_0^T u(t)^T u(t)\, dt \right)^{1/2}.$$

TABLE I
RMS NORM OF THE JOINT TORQUE VECTOR

Controller	$\|f\|_{rms}$
Motion controller with 2 active joints	1.18
Motion controller with 1 active joint	2.44
Hybrid motion/force controller	2.65

IX. EXPERIMENT

In this section, we report comparative experimental results obtained from the constraint mechanical system shown in Fig. 7. The arm used for these experiments was a planar robot arm developed at the Canadian Space Agency with three revolute joints, which are driven by geared motors RH-8-6006, RH-11-3001, and RH-14-6002 from Hi-T Drive. The robot joints are equipped with optical encoders, and force sensor (gamma type from ATI) is installed in the robot wrist. The robot endpoint is connected to a slider by a hinge, which ensures that no constraint on the wrist rotation is imposed. The slider uses linear bearings to minimize friction along X axis motion, while the motion along Y axis is constrained—see Fig. 7.

Let $q^T = [q_1 \;\; q_2 \;\; q_3]$ represent the vector of joint angles, and $\{x, y, \theta\}$ represent the position and orientation of the robot endpoint. Then, from the kinematics, the constraint equation and the reduced-dimension coordinate can be specified by $\Phi = y(q) - y_0 = 0$, for $y_0 = -0.27$ m, and $\theta^T = [x(q), \theta(q)]$, respectively. The constraint force, or equivalently, the Lagrange multiplier, is measured by the ATI force sensor.

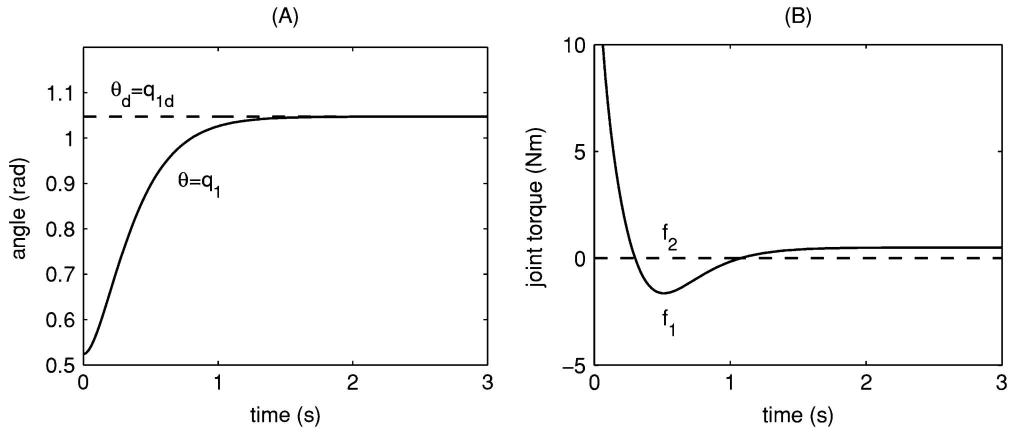

Fig. 6. Simulated position and force responses; one actuated joint.

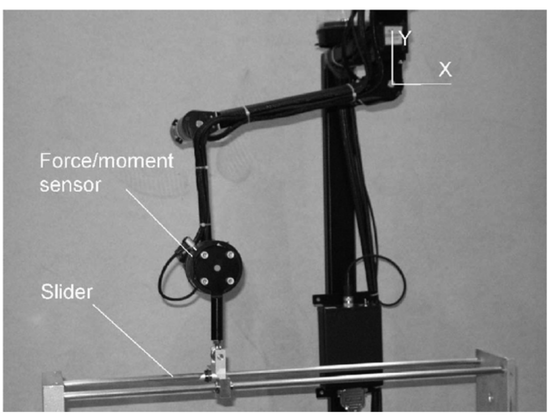

Fig. 7. Experimental setup.

A. Implementation

The controller has been developed using the Simulink diagram. From the Simulink model, Real-Time Workshop [62] generates portable C code that can be executed on the QNX real-time operating system. The kinematics and dynamics models of the robot are developed using Symofros [63]. They are then imported to the Simulink diagram as S-functions. Matrix manipulation including the pseudoinverting of the constraint Jacobian is carried out by using DSP Blockset of Matlab/Simulink [51]. This architecture allows us to achieve a 1000-Hz sampling rate on a 300-MHz Pentium computer.

B. Results

In this experiment, the position feedback gains are $G_P = 480$ and $G_D = 45$, which correspond to 3.5-Hz bandwidth of the closed-loop system. The force feedback gains are set to $G_F = 3$, and $G_I = 0.1$. The desired position trajectory is specified as $x_d(t) = -0.075 + 0.225\sin(\pi t + \pi/9)$ and $\theta_d = 0$, while $\lambda_d = 0$.

Three different control schemes are implemented: the PIDC, the hybrid force/motion control described in Section VII, and the standard IDC. All controllers demonstrated a good motion-tracking performance as illustrated in Fig. 8(A) and (B). Differences among the control schemes, however, is manifested in their force responses. Trajectories of the contact force and those of the Euclidean norm of the joint torque requested by the three controllers are plotted in Fig. 8(C) and (D), respectively. It is evident from Fig. 8(D) that the PIDC requires minimum joint torque at every instance, albeit it does not yield zero constraint force, as shown in Fig. 8(C). The constraint force is regulated to the desired value zero when the force-feedback law (49)–(50) is activated. However, this gives rise to the requested joint torque; see Fig. 8(D). The spikes in the contact force are attributed to joint friction, whose effects are entered as strong disturbances to the control system, especially when the joint velocities change direction. To complete the comparison, the traditional IDC is implemented that produces large forces. This is because, unlike PIDC which exhibits compliance in the constraint direction, the IDC tends to be stiff in all directions, and this causes a large force in case of position uncertainty in the constraint equation. Finally, Table II summarizes the RMS and the peak norms[2] of the joint torques requested by each controller. A comparison of the results shows that the use of the force feedback increases both the RMS norm and the peak norm of the requested joint torque by 75% and 21%, respectively.

X. Conclusion

A unified formulation applicable to both the direct dynamics (simulation) and inverse-dynamics (control) of constrained mechanical systems has been presented. The approach is based on projecting the Lagrangian dynamics equations into the tangent space with respect to the constraint manifold. This automatically eliminates the constraint forces from the equation, albeit the constraint forces can then be retrieved separately from dynamics projection into the normal space.

The novelty of the formulation lies in the definition of the projector operators which, unlike other formulations, are square matrices of orders equal to the number of the generalized coordinates. Therefore, the structure of the dynamics formulation does not change if the system changes its DOF or its topology. Moreover, since the process of computing projection operator is not conditioned upon the maximal rank of the constraint Jacobian, the direct and the inverse-dynamics formulation are valid also for mechanical systems with redundant constraints and/or

[2]The peak or $\mathcal{L}_\infty$ norm of a vector signal is defined by

$$\|u\|_\infty = \sup_{t\geq 0} \max_{1\leq i\leq n} |u_i(t)|.$$

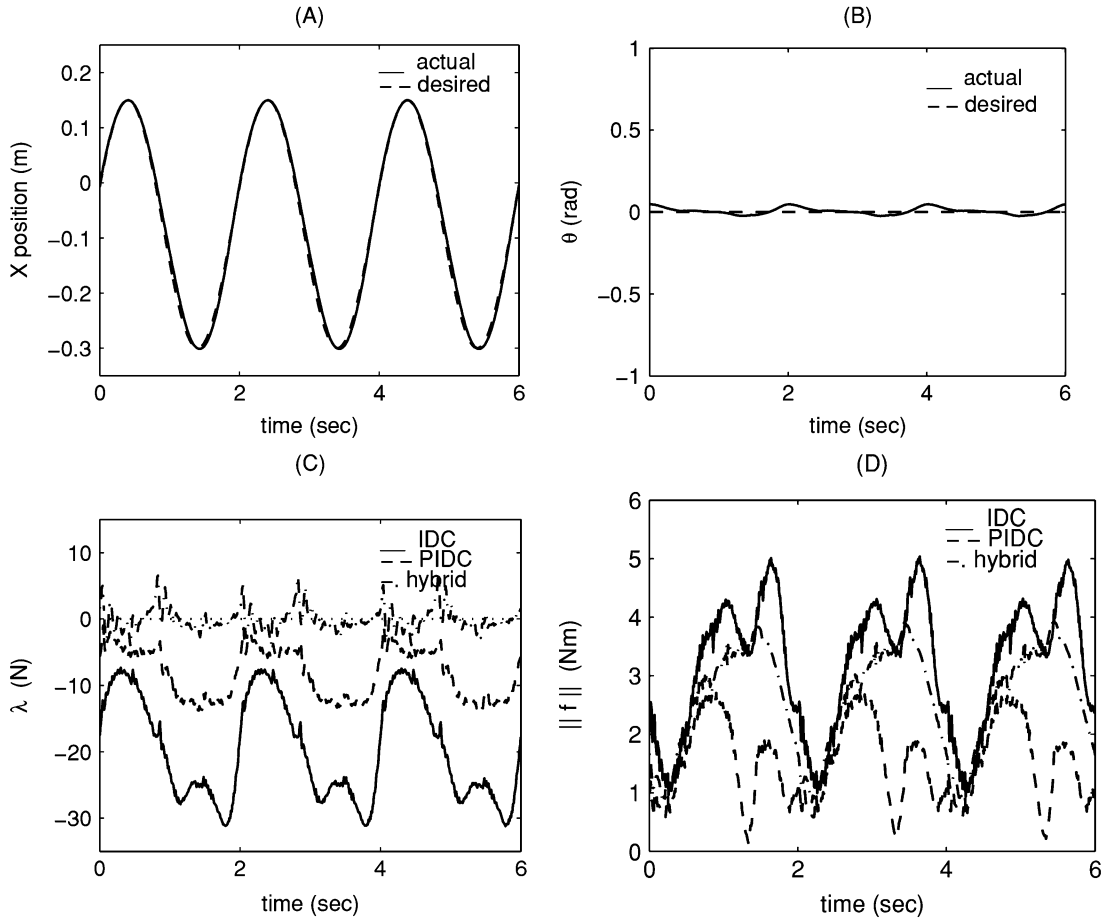

Fig. 8. Actual and desired trajectories of the position (A) and the orientation (B), trajectories of the contact force (Lagrange multipliers) (C), and those of the Euclidean norm of joint torque (D).

TABLE II
RMS NORM AND PEAK NORM OF THE JOINT TORQUE VECTOR SIGNAL OBTAINED FROM THE DIFFERENT CONTROLLERS

Controller	$\|f\|_{rms}$	$\|f\|_\infty$
IDC	3.18	4.90
PIDC	1.50	1.89
Hybrid	2.63	2.30

singular configurations, which is unattainable with many other classical approaches.

A motion control system has been developed based on the *PIDC*, which minimizes the actuation force, and also works for systems with unactuated joints (passive joints). To this end, a hybrid motion/force controller was developed.

In summary, particular features of the proposed formulation associated with simulation and control of constrained mechanical systems are listed below.

- A simulation may proceed even with presence of redundant constraint equations and/or singular configurations. With the same token, the projected inverse-dynamics motion controller can cope with changes in the system constraint, topology, or number of DOFs.
- The generalized formulation requires no knowledge of the constraint topology, i.e., description of how subassemblies are connected, and it works for rigid-body or flexible systems alike.
- The IDC scheme leads to a minimum weighted Euclidean norm of the control force input.
- The IDC scheme can be applied to constrained systems which have some unactuated joints.
- If the inertia matrix possesses a certain property, the system exhibits decoupling which leads to further simplification of the force control.
- Both redundant and flexible manipulators can be dealt with.

APPENDIX I

Note that $A^T : \mathbb{R}^m \rightarrow \mathbb{R}^n$. Then $\forall x \in \mathcal{R}(A^T)^\perp$ and $\forall y \in \mathbb{R}^m$, we have

$$\begin{aligned} &\langle x, A^T y\rangle = 0 \\ &\Leftrightarrow x^T(A^T y) = y^T A x = 0 \quad \forall y \in \mathbb{R}^m \qquad (76)\\ &\Leftrightarrow Ax = 0 \\ &\Leftrightarrow x \in \mathcal{N}(A) \qquad (77)\end{aligned}$$

where the inference (77) is concluded because (76) implies that vector Ax must be orthogonal to every vector in $\mathbb{R}^m$, which is possible only if vector Ax is identically zero. Thus (2) is proved. The proof of (3) can be shown by a similar argument.

Appendix II

Assuming that the Lagrange multipliers are known, the acceleration can be carried out from (19)

$$\ddot{q} = M^{-1}(f - h - A^T\lambda). \quad (78)$$

Substituting the acceleration from (78) into the second time derivative of the constraint equation

$$A\ddot{q} + \dot{A}\dot{q} = 0$$

gives the Lagrange multipliers as

$$\lambda = (AM^{-1}A^T)^{-1}[AM^{-1}(f-h) + \dot{A}\dot{q} + \dot{c}]. \quad (79)$$

This methods works only if the Cartesian inertia matrix $\mathcal{M} = (AM^{-1}A^T)$ is not singular. Finally, substituting (79) to (78) yields the acceleration.

Appendix III

Lemma 1: The subspace $\mathcal{W}$ is invariant under an invertible transformation $\mathcal{A}$ iff the subspace is invariant under $\mathcal{A}^{-1}$.

Proof: By definition, $\mathcal{W}$ is an invariant subspace under $\mathcal{A}$ iff $\mathcal{AW} \subseteq \mathcal{W}$. Moreover, the invertible mapping $\mathcal{A}$ cannot reduce the dimension of any subspace, that is, $\dim(\mathcal{AW}) = \dim(\mathcal{W})$; hence, $\mathcal{AW} = \mathcal{W}$. It follows that

$$\mathcal{AW} = \mathcal{W} \Leftrightarrow \mathcal{A}^{-1}\mathcal{W} = \mathcal{W}$$

which completes the proof. ■

Appendix IV

It is apparent from Fig. 1 that the decoupling is achieved iff

$$\mu P = (I - P)\alpha P = 0 \quad (80)$$

which implies that the null space must be invariant under α. Since α is an invertible mapping, it can be inferred from *Lemma 1* (see Appendix III) that the null space must be invariant under α^{-1} too. This means that (80) is equivalent to $(I-P)\alpha^{-1}P = 0$. Now, replacing M_c from (23) into the latter equation and after factorization, one can infer the following:

$$\begin{aligned} \text{decoupling} &\Leftrightarrow (I-P)[(M+\tilde{M})M^{-1}]P = 0 \\ &\Leftrightarrow [(I-P)MP][M^{-1}P] = 0 \\ &\Leftarrow (I-P)MP = 0 \end{aligned}$$

which completes the proof.

Appendix V

Since $\theta^T = [\theta_1(q), \ldots, \theta_k(q)]$ comprises a set of independent functions, the corresponding Jacobian matrix is full rank, i.e., rank $((\partial\theta)/(\partial q)) = k$. Moreover, by using the chain rule, we have $(\partial\theta)/(\partial q)\Lambda = I_k$, where I_k is the $k \times k$ identity matrix. Now, by virtue of the property of the rank operator $\text{rank}(AB) \leq \min[\text{rank}(A), \text{rank}(B)]$, one can say that

$$k \leq \min[\text{rank}(\Lambda), k]$$

or that

$$\text{rank}(\Lambda) = k.$$

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