Fixed-Final-Time-Constrained Optimal Control of Nonlinear Systems Using Neural Network HJB Approach

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***Abstract*—In this paper, fixed-final time-constrained optimal control laws using neural networks (NNS) to solve Hamilton–Jacobi–Bellman (HJB) equations for general affine in the constrained nonlinear systems are proposed. An NN is used to approximate the time-varying cost function using the method of least squares on a predefined region. The result is an NN nearly -constrained feedback controller that has time-varying coefficients found by *a priori* offline tuning. Convergence results are shown. The results of this paper are demonstrated in two examples, including a nonholonomic system.**

***Index Terms*—Constrained input systems, finite-horizon optimal control, Hamilton–Jacobi–Bellman (HJB), neural network (NN) control.**

I. Introduction

THE constrained input optimization of dynamical systems has been the focus of many papers during the last few years. Several methods for deriving constrained control laws are found in [50], [56], and [10]. However, most of these methods do not consider optimal control laws for general constrained nonlinear systems. Constrained-input optimization possesses challenging problems; a great variety of versatile methods have been successfully applied in [4], [11], [17], and [51]. Many problems can be formulated within the Hamilton–Jacobi–Bellman (HJB) and Lyapunov's frameworks, but the resulting equations are difficult or impossible to solve, such as [40]–[42].

Successful neural networks (NNs) controllers not based on optimal techniques have been reported in [15], [32], [53], [22], [47], and [49]. It has been shown that NN can effectively extend adaptive control techniques to nonlinearly parameterized systems. NN applications to optimal control via the HJB equation were first proposed by Werbos [43].

We were motivated by the important results in [1], [8], and [36]–[40]. However, [1] focuses on constrained policy iteration control with infinite horizon and [8] focuses on unconstrained policy iteration with finite-time horizon. The authors of [36]–[42] showed how to formulate constrained input in terms of a nonquadratic performance index, but did not provide formal solution algorithms. In contrast to these works, we study finite-time horizon system with constrained control without policy iteration, establishing an innovative methodology that incorporates control constraints into the framework of the HJB philosophy. We use NN to approximately solve the time-varying HJB equation for constrained control nonlinear systems. In [16], we considered this problem. In this paper, we extend the results to the case of constrained controls. It is shown that using an NN approach, one can simply transform the problem into solving a nonlinear ordinary differential equation (ODE) backwards in time. The coefficients of this ODE are obtained by the weighted residuals method. We provide uniform convergence results over a Sobolev space.

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II. Background on Fixed-Finite-Time HJB Optimal Control

Consider an affine in the control nonlinear dynamical system of the form

$$\dot{x} = f(x) + g(x)u(x) \tag{1}$$

where $x \in \Re^n$, $f(x) \in \Re^n$, $g(x) \in \Re^{n\times m}$, and the input $u(t) \in R^m$. The dynamics $f(x)$ and $g(x)$ are assumed to be known and $f(0) = 0$. Assume that $f(x) + g(x)u(x)$ is Lipschitz continuous on a set $\Omega \subseteq \Re^n$ containing the origin, and that system (1) is stabilizable in the sense that there exists a continuous control on Ω that asymptotically stabilizes the system. It is desired to find the constrained input control $u(x)$ that minimizes a generalized functional

$$V(x(t_0), t_0) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [Q(x) + W(u)]\, dt \tag{2}$$

with $Q(x)$ and $W(u)$ positive definite on Ω, i.e., $\forall\, x \neq 0$, $x \in \Omega$, $Q(x) > 0$, and $x = 0 \Rightarrow Q(x) = 0$.

Definition 1 (Admissible Controls): A control $u(x)$ is defined to be admissible with respect to (2) on Ω, denoted by $u \in \Psi(\Omega)$, if $u(x)$ is continuous on Ω, $u(0) = 0$, $u(x)$ stabilizes (1) on Ω, and $\forall\, x_0 \in \Omega$, $V(x(t_0), t_0)$ is finite.

Under regularity assumptions, i.e., $V(x,t) \in C^1(\Omega)$, an infinitesimal equivalent to (2) is [33]

$$-\frac{\partial V(x,t)}{\partial t} = L + \left(\frac{\partial V(x,t)}{\partial x}\right)^T (f(x) + g(x)u(x)) \tag{3}$$

where $L = Q(x) + W(u)$. This is a time-varying partial differential equation with $V(x,t)$ being the cost function for any given $u(x)$ and it is solved backward in time from $t = t_f$. By setting $t_0 = t_f$ in (2), its boundary condition is seen to be

$$V(x(t_f), t_f) = \phi(x(t_f), t_f). \quad (4)$$

According to Bellman's optimality principle [33], the optimal cost is given by

$$-\frac{\partial V(x,t)^*}{\partial t} = \min_{u(t)}\left(L + \left(\frac{\partial V(x,t)^*}{\partial x}\right)^T \cdot (f(x) + g(x)u(x))\right). \quad (5)$$

Assuming $W(u) = u^T R u$, this yields the optimal control

$$u^*(x,t) = -\frac{1}{2}R^{-1}g(x)^T \frac{\partial V(x,t)^*}{\partial x} \quad (6)$$

where $V^*(x,t)$ is the optimal value function and R is positive definite and assumed to be symmetric for simplicity of analysis. Substituting (6) into (5) yields the well-known time-varying HJB equation [33]

$$\frac{\partial V(x,t)^*}{\partial t} + \frac{\partial V(x,t)^*}{\partial x} f(x) + Q(x) - \frac{1}{4}\frac{\partial V(x,t)^{*T}}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V(x,t)^*}{\partial x} = 0. \quad (7)$$

Equations (6) and (7) provide the solution to fixed-final-time optimal control for affine nonlinear systems. However, closed-form solution for (7) is, in general, impossible to find. In [16], we showed how to approximately solve this equation using NN.

III. HJB Equation With Constraints on the Control System

Consider now the case when the control input is constrained by a saturated function $\boldsymbol{\varphi}(\cdot)$, e.g., $\tanh$, etc. To guarantee bounded controls, [1], Lyshevski [36] introduced a generalized nonquadratic functional

$$W(u) = 2\int_0^u \boldsymbol{\varphi}^{-T}(v) R dv \quad (8)$$

where $W(u)$ is a scalar

$$\boldsymbol{\varphi}(v) = [\phi(v_1)\cdots\phi(v_m)]^T$$
$$\boldsymbol{\varphi}^{-T}(u) = \left[\phi^{-1}(u_1)\ldots\phi^{-1}(u_m)\right]$$

$v \in \Re^m$, and $\boldsymbol{\varphi} \in \Re^m$ is a bounded one-to-one function that belongs to $C^p(p \geq 1)$ and $L_2(\Omega)$. Define notation $w(v) = \phi^{-1}(v)R$, where $\int_0^u p^T(v)dv \equiv \int_0^{u_1} p_1(v_1)dv_1 + \int_0^{u_2} p_2(v_2)dv_2 + \cdots + \int_0^{u_m} p_m(v_m)dv_m$ is a scalar, for $u \in R^m$, $v \in R^m$, and $p(v) = [p_1 \ \ldots \ p_m] \in R^m$. Moreover, $\boldsymbol{\varphi}(\cdot)$ is a monotonic odd function with its first derivative bounded by a constant M. Note that $W(u)$ is positive definite because $\boldsymbol{\varphi}^{-1}(u)$ is monotonic odd and R is positive definite.

When (8) is used, (2) becomes

$$V(x(t_0), t_0) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f}\left[Q(x) + 2\int_0^u \boldsymbol{\varphi}^{-T}(v)Rdv\right]dt \quad (9)$$

and (5) becomes

$$-\frac{\partial V(x,t)^*}{\partial t} = \min_{u(t)}\left(Q(x) + 2\int_0^u \boldsymbol{\varphi}^{-T}(v)Rdv + \frac{\partial V(x,t)^{*T}}{\partial x}(f(x,t) + g(x)u(x))\right).$$

Minimizing the Hamiltonian of the optimal control problem with regard to u gives

$$g^T(x)\frac{\partial V(x,t)^*}{\partial x} + 2\boldsymbol{\varphi}^{-1}(u^*)R = 0$$

so

$$u(x,t)^* = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g(x)^T\frac{\partial V(x,t)^*}{\partial x}\right), \quad u \in U \subset \Re^m. \quad (10)$$

This is constrained as required.

Lemma 1: The smooth bounded control law (10) guarantees at least a strong relative minimum for the performance cost (9) for all $x \in X \subset \Re$ on $\lfloor t_0, t_f)$. Moreover, if an optimal control exists, it is unique and represented by (10).

Proof: See [40].

When (10) is used, (5) becomes

$$\text{HJB}\left(V(x,t)^*\right) = \frac{\partial V(x,t)^*}{\partial t} + \frac{\partial V(x,t)^{*T}}{\partial x} f(x) + 2\int_0^u \boldsymbol{\varphi}^{-T}(v)Rdv - \frac{\partial V(x,t)^{*T}}{\partial x} \cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g(x)^T\frac{\partial V(x,t)^*}{\partial x}\right) + Q(x) = 0. \quad (11)$$

If this HJB equation can be solved for the value function $V(x,t)$, then (10) gives the optimal-constrained control. This HJB equation cannot generally be solved. There is currently no method for rigorously solving for the value function of this constrained optimal control problem.

Remark 1: The HJB equation requires that $V(x,t)$ is continuously differentiable function. Usually, this requirement is not satisfied in constrained optimization because the control function is piecewise continuous. But control problems do not necessarily have smooth or even continuous value functions [24], [6]. Lio [34] used the theory of viscosity solutions to show that for infinite-horizon optimal control problems with unbounded cost functional, under certain continuity assumptions of the dynamics, the value function is continuous on some set Ω, $V^*(x,t) \in C(\Omega)$. Bardi [6] showed that if the Hamiltonian is strictly convex and if the continuous viscosity solution is semiconcave, then $V^*(x,t) \in C^1(\Omega)$ satisfying the HJB equation everywhere. In this paper, all derivations are performed under the assumption of smooth solutions to (7). A similar assumption was made by Van der schaft [57] and Isidori [26].

IV. Nonlinear Fixed-Final-Time HJB Solution by NN Least Squares Approximation

The HJB (11) is difficult to solve for the cost function $V(x,t)$. In this section, NNs are used to solve approximately for the value function in (11) over Ω by approximating the cost function $V(x,t)$ uniformly in t. The result is an efficient, practical, and computationally tractable solution algorithm to find nearly optimal state feedback controllers for nonlinear systems.

A. NN Approximation of the Cost Function $V(x,t)$

It is well known that an NN can be used to approximate smooth time-invariant functions on prescribed compact sets [23]. Since the analysis required here is restricted to the region of asymptotical stability (RAS) of some initial stabilizing controller, NNs are natural for this application. In [52], it is shown that NNs with time-varying weights can be used to approximate uniformly continuous time-varying functions. We assume that $V(x,t)$ is smooth, and so uniformly continuous on a compact set. Therefore, one can use the following equation to approximate V for $t \in \lfloor t_0, t_f \rfloor$ on a compact set $\Omega \subset \Re^n$:

$$V_L(x,t) = \sum_{j=1}^{L} w_j(t)\,\sigma_j(x) = w_L^T(t)\,\sigma_L(x). \qquad (12)$$

This is an NN with activation functions $\sigma_j(x) \in C^1(\Omega)$, $\sigma_j(0) = 0$. The NN weights are $w_j(t)$ and L is the number of hidden-layer neurons. $\boldsymbol{\sigma}_L(x) \equiv [\sigma_1(x)\,\sigma_2(x)\ldots\sigma_L(x)]^T$ is the vector of activation function and $\mathbf{w}_L(t) \equiv [w_1(t)\,w_2(t)\ldots w_L(t)]^T$ is the vector of NN weights.

It is assumed that L is large enough so that $V_L(x(t_f), t_f) = \mathbf{w}_L^T(t_f)\,\delta_L(x(t_f)) = \boldsymbol{\sigma}(x(t_f), t_f)$, i.e., there exist weights $w_j(t_f)$ that exactly satisfy the approximation at $t = t_f$.

The next result shows that initial conditions $x(t_0)$ can be selected to guarantee that $x(t) \in \Omega$ for $t \in [t_0, t_f]$.

Lemma 2: Let $\Omega \subset \Re^n$ be a compact set. Then $\exists \Omega_0 \subset \Omega$, s.t., for system (1), $x(t) \in \Omega$, $t \in [t_0, t_f]$, $\forall\, x(t_0) \in \Omega_0$.

The set $\sigma_j(x)$ is selected to be independent. Then, without loss of generality, they can be assumed to be orthonormal, i.e., select equivalent basis functions to $\sigma_j(x)$ that are also orthonormal [8]. The orthonormality of the set $\{\sigma_j(x)\}_1^\infty$ on Ω implies that, if a real-valued function $\psi(x,t) \in R$, then

$$\psi(x,t) = \sum_{j=1}^{\infty} \langle \psi(x,t), \sigma_j(x)\rangle_\Omega\, \sigma_j(x)$$

where $\langle f, g\rangle_\Omega = \int_\Omega g \cdot f^T dx$ is an outer product, f and g are continuous functions, and the series converges pointwise, i.e., for any $\varepsilon > 0$ and $x \in \Omega$, one can choose N sufficiently large to guarantee that $\left|\sum_{j=N+1}^{\infty} \langle \psi(x,t), \sigma_j(x)\rangle_\Omega\, \sigma_j(x)\right| < \varepsilon$ for all $t \in [t_0, t_f]$; see [9].

Note that, since one requires $\partial V(x,t)/\partial t$ in (11), the NN weights are selected to be time varying. This is similar to methods such as assumed mode shapes in the study of flexible mechanical systems [5]. However, here, $\boldsymbol{\sigma}_L(x)$ is an NN activation vector, not a set of eigenfunctions. That is, the NN approximation property significantly simplifies the specification of $\boldsymbol{\sigma}_L(x)$. For the infinite final-time case, the NN weights are constant [1]. The NN weights will be selected to minimize a residual error in a least squares sense over a set of points sampled from a compact set Ω_0 inside the RAS of the initial stabilizing control [21].

Note that

$$\frac{\partial V_L(x,t)}{\partial x} = \frac{\partial \boldsymbol{\sigma}_L^T(x)}{\partial x}\mathbf{w}_L(t) \equiv \nabla \boldsymbol{\sigma}_L^T(x)\,\mathbf{w}_L(t) \qquad (13)$$

where $\nabla \boldsymbol{\sigma}_L(x)$ is the Jacobian $\partial \boldsymbol{\sigma}_L(x)/\partial x$ and that

$$\frac{\partial V_L(x,t)}{\partial t} = \dot{\mathbf{w}}_L^T(t)\,\boldsymbol{\sigma}_L(x). \qquad (14)$$

Therefore, approximating $V(x,t)$ by $V_L(x,t)$ uniformly in t in the HJB (11) results in

$$\begin{aligned} &-\dot{\mathbf{w}}_L^T(t)\,\boldsymbol{\sigma}_L(x) - \mathbf{w}_L^T(t)\,\nabla\boldsymbol{\sigma}_L(x)\,f(x) - 2\int_0^u \boldsymbol{\varphi}^{-T}(v)\,R\,dv \\ &\quad + \mathbf{w}_L^T(t)\,\nabla\boldsymbol{\sigma}_L(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right) \\ &\quad - Q(x) = e_L(x,t) \end{aligned} \qquad (15)$$

or

$$\mathrm{HJB}\left(V_L(x,t) = \sum_{j=1}^{L} w_j(t)\,\sigma_j(x)\right) = e_L(x,t) \qquad (16)$$

where $e_L(x,t)$ is a residual equation error. From (10), the corresponding constrained optimal control input is

$$u_L(x,t) = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\,\nabla\boldsymbol{\sigma}_L^T(x)\,\mathbf{w}_L(t)\right). \qquad (17)$$

To find the least squares solution for $\mathbf{w}_L(t)$, the method of weighted residuals is used [21]. The weight derivatives $\dot{\mathbf{w}}_L(t)$ are determined by projecting the residual error onto $\partial e_L(x,t)/\partial \dot{\mathbf{w}}_L(t)$ and setting the result to zero $\forall\, x \in \Omega_0$ and $\forall\, t \in [t_0, t_f)$ using the inner product, i.e.,

$$\left\langle \frac{\partial e_L(x,t)}{\partial \dot{\mathbf{w}}_L(t)}, e_L(x,t)\right\rangle_\Omega = 0. \qquad (18)$$

From (15), we can get

$$\frac{\partial e_L(x,t)}{\partial \dot{\mathbf{w}}_L} = \boldsymbol{\sigma}_L(x). \qquad (19)$$

Therefore, we obtain

$$\begin{aligned} &\left\langle -\dot{\mathbf{w}}_L^T(t)\,\boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega \\ &\quad + \left\langle -\mathbf{w}_L^T(t)\,\nabla\boldsymbol{\sigma}_L(x)\,f(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega \\ &\quad + \left\langle -2\int_0^u \boldsymbol{\varphi}^{-T}(v)\,R\,dv, \boldsymbol{\sigma}_L(x)\right\rangle_\Omega \\ &\quad + \Bigg\langle \mathbf{w}_L^T(t)\,\nabla\boldsymbol{\sigma}_L(x)\cdot g(x) \\ &\qquad\quad \cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\,\nabla\boldsymbol{\sigma}_L^T(x)\,\mathbf{w}_L(t)\right), \boldsymbol{\sigma}_L(x)\Bigg\rangle_\Omega \\ &\quad + \left\langle -Q(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = 0 \end{aligned} \qquad (20)$$

so that

$$\begin{aligned}\dot{\mathbf{w}}_L(t) = &-\langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^{-1} \\ &\cdot \langle \nabla\boldsymbol{\sigma}_L(x) f(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega \cdot \mathbf{w}_L(t) \\ &- \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^{-1} \left\langle 2\int_0^u \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle_\Omega \\ &+ \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^{-1} \\ &\cdot \left\langle \mathbf{w}_L^T(t) \nabla\boldsymbol{\sigma}_L(x) \right. \\ &\left. \cdot g(x) \cdot \boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega \\ &- \langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^{-1} \cdot \langle Q(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega \end{aligned} \tag{21}$$

with boundary condition $V(x(t_f), t_f) = \phi(x(t_f), t_f) = \mathbf{w}_L^T(t_f)\boldsymbol{\sigma}_L(x(t_f))$. Note that, given a mesh of $x(t_f)$ (see Section IV-C), the boundary condition allows one to determine $\mathbf{w}_L(t_f)$.

Therefore, the NN weights are simply found by integrating this nonlinear ODE backwards in time.

We now show that this procedure provides a nearly optimal solution for the time-varying optimal control problem if L is selected large enough.

B. Uniform Convergence in t for Time-Varying Function of the Method of Least Squares

In what follows, one shows convergence results as L increases for the method of least squares when NNs are used to uniformly approximate the cost function in t. The following definitions and facts are required.

Let $F(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition

$$\|F(x,t) - F(y,.t)\| \le L_1 \|x - y\|$$

$\forall\, x, y \in B = \{x \in R^n \, \|x - x_0\| \le r\}$, $\forall\, t \in [t_0, t_1]$. Then, there exists some $\delta > 0$ such that the state equation $\dot{x} = F(x,t)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$. Provided the Lipschitz condition holds uniformly in t for all t in a given interval of time, function $F(x,t)$ is called globally Lipschitz if it is Lipschitz on R^n [27].

Definition 2 (Convergence in the Mean for Time-Varying Functions): A sequence of functions $\{f_n(x,t)\}$ that is Lebesgue integrable on a set Ω, $L_2(\Omega)$, is said to converge (uniformly in t) in the mean to $f(x,t)$ on Ω if $\forall\, \varepsilon > 0$, $\forall\, t$, $\exists N(\varepsilon)$: $n > N \Rightarrow |f_n(x,t) - f(x,t)|_{L_2(\Omega)} < \varepsilon$.

Definition 3 (Uniform Convergence for Time-Varying Functions): A sequence of functions $\{f_n(x,t)\}$ converges to $f(x,t)$ (uniformly in t) on a set Ω if $\forall\, \varepsilon > 0$, $\forall\, t$, $\exists N(\varepsilon)$: $n > N \Rightarrow |f_n(x,t) - f(x,t)| < \varepsilon\ \forall\, x \in \Omega$, or equivalently $\sup_{x\in\Omega}|f_n(x,t) - f(x,t)| < \varepsilon$.

Definition 4 (Sobolev Space): $H^{m,p}(\Omega)$: Let Ω be an open set in $\Re^n$ and let $u \in C^m(\Omega)$. Define a norm on u by

$$\|u\|_{m,p} = \sum_{0\le|\alpha|\le m}\left(\int_\Omega |D^\alpha u(x)|^p dx\right)^{1/p}, \qquad 1 \le p < \infty.$$

This is the Sobolev norm in which the integration is Lebesgue. The completion of $u \in C^m(\Omega)$: $\|u\|_{m,p} < \infty$ with respect to $\|\cdot\|_{m,p}$ is the Sobolev space $H^{m,p}(\Omega)$. For $p = 2$, the Sobolev space is a Hilbert space.

The convergence proofs of the least squares method are done in the Sobolev function space $H^{1,2}(\Omega)$ setting [2], since one requires to prove the convergence of both $V_L(x,t)$ and its gradient. The following technical lemmas are required.

Technical Lemma 1: Given a linearly independent set of L functions $\{f_L\}$, then for the series $a_L^T f_L$, it follows that

$$\left\|a_L^T f_L\right\|_{L_2(\Omega)}^2 \to 0 \Leftrightarrow \|a_L\|_{L_2(\Omega)}^2 \to 0.$$

Proof: See [1]. ■

Technical Lemma 2: Suppose that $\{\nabla\sigma_j(x)\}_1^L \ne 0$, then $\{\sigma_j(x)\}_1^L$-linearly independent $\Rightarrow$ $\{\nabla\sigma_j(x)\}_1^L$-linearly independent.

Proof: See [8]. ■

Technical Lemma 3: If $W(x,t) = \sum_{j=1}^\infty w_j(t)\phi_j(x)$ and $\phi_j(x)$ are continuous on Ω, then $\sum_{j=L+1}^\infty w_j(t)\phi_j(x)$ converges to zero uniformly in t on Ω iff the following are true:

1) $W(x,t)$ is continuous on Ω;
2) $\sum_{j=1}^\infty w_j(t)\phi_j(x) \in PD(\Omega)$;

where $PD(\Omega)$ means pointwise decreasing on Ω.

Proof: See [8]. ■

The following assumptions are required.

Assumption 1: The system's dynamics and the performance integrands $Q(x) + W(u)$ are such that the solution of the cost function is continuous and differentiable and belongs to the Sobolev space $V \in H^{1,2}(\Omega)$. Here, $Q(x)$ and $W(u)$ satisfy the requirement of existence of smooth solutions.

Assumption 2: We can choose a complete coordinate elements $\{\sigma_j(x)\}_1^\infty \in H^{1,2}(\Omega)$ such that the solutions $V(x,t) \in H^{1,2}(\Omega)$ and $\{\partial V(x,t)/\partial x_1, \ldots, \partial V(x,t)/\partial x_n\}$ can be uniformly approximated in t by the infinite series built from $\{\sigma_j(x)\}_1^\infty$.

Assumption 3: The coefficients $|w_j(t)|$ are uniformly bounded in t for all L.

The first two assumptions are standard in optimal control and NNs control literature. Completeness follows from [23].

We now show the following convergence results.

Lemma 3 (Convergence of Approximate HJB Equation): Given $u \in \psi(\Omega)$, let $V_L(x,t) = \sum_{j=1}^L w_j^T(t)\sigma_j(x)$ satisfy $\langle HJB(V_L(x,t)), \boldsymbol{\sigma}_L(x)\rangle_\Omega = 0$ and $\langle V_L(t_f), \boldsymbol{\sigma}_L(x)\rangle_\Omega = 0$, and let $V(x,t) = \sum_{j=1}^\infty c_j^T(t)\sigma_j(x)$ and $\mathbf{c}_L(t) \equiv [c_1(t)\, c_2(t) \ldots c_L(t)]^T$ satisfy $HJB(V(x,t)) = 0$ and $V(x(t_f), t_f) = \phi(x(t_f), t_f)$. Then

$$|\mathrm{HJB}(V_L(x,t))| \to 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

Proof: The hypotheses imply that $HJB\left(V_L\left(x,t\right)\right)$ are in $L_2\left(\Omega\right)$. Note that

$$\begin{aligned}\left\langle HJB\left(V_L\left(x,t\right)\right),\sigma_j\left(x\right)\right\rangle_\Omega &= \sum_{k=1}^{L}\dot{w}_k\left(t\right)\left\langle\sigma_k\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega \\ &+ \sum_{k=1}^{L}w_k\left(t\right)\left\langle\nabla\sigma_k\left(x\right)f\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega \\ &+ \sum_{k=1}^{L}\left\langle 2\int_0^{u_k}\boldsymbol{\varphi}^{-T}\left(v\right)Rdv,\sigma_j\right\rangle_\Omega \\ &- \sum_{k=1}^{L}\left\langle w_k\left(t\right)\nabla\sigma_k\left(x\right)\cdot g\left(x\right)\right. \\ &\qquad \left.\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T\left(x\right)\nabla\boldsymbol{\sigma}_L^T\left(x\right)\mathbf{w}_L\left(t\right)\right),\sigma_j\left(x\right)\right\rangle_\Omega \\ &+ \left\langle Q\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega.\end{aligned} \tag{22}$$

Since the set $\left\{\sigma_j\left(x\right)\right\}_1^\infty$ is orthogonal, $\left\langle\sigma_k\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega = 0$.

Therefore, (23), shown at the bottom of the page, holds, where the first equation shown at the bottom of the next page, also holds. Assumptions 2–4 imply that Ω_0 is compact and the functions $\nabla\boldsymbol{\sigma}_L\left(x\right)f\left(x\right)$, $2\int_0^u\boldsymbol{\varphi}^{-T}\left(v\right)Rdv$, $\nabla\boldsymbol{\sigma}_L\left(x\right)g\left(x\right)\boldsymbol{\varphi}\left((1/2)R^{-1}g^T\left(x\right)\nabla\boldsymbol{\sigma}_L^T\left(x\right)\mathbf{w}_L\left(t\right)\right)$, and $Q\left(x\right)$ are continuous on Ω and are in $L_2\left(\Omega\right)$, and the coefficients $\left|w_j\left(t\right)\right|$ are uniformly bounded for all L, so the orthonormality of the set $\left\{\sigma_j\left(x\right)\right\}_1^\infty$ implies that $B\left(x\right)$, $D\left(x\right)$, $E\left(x\right)$, and the fourth term on the right-hand side can be made arbitrarily small by an appropriate choice of L. Therefore, $A\cdot B\left(x\right) + C\cdot D\left(x\right) + AE\left(x\right) \to 0$ and $\left|\sum_{j=L+1}^{\infty}\left\langle Q\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\sigma_j\left(x\right) \to 0\right|$.

This means that $\left|HJB\left(V_L\left(x,t\right)\right)\right| \to 0$ uniformly in t on Ω_0 as L increases. ■

Lemma 4 (Convergence of NN Weights): Given $u \in \Psi\left(\Omega\right)_0$, suppose the hypotheses of Lemma 3 hold. Then, $\left\|\mathbf{w}_L\left(t\right) - \mathbf{c}_L\left(t\right)\right\|_2 \to 0$ uniformly in t as L increases.

Proof: Define

$$\begin{aligned}u_L\left(x,t\right) &= -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T\left(x\right)\frac{\partial V_L\left(x,t\right)}{\partial x}\right) \\ u\left(x,t\right) &= -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T\left(x\right)\frac{\partial V\left(x,t\right)}{\partial x}\right) \\ e_L\left(x,t\right) &= \mathrm{HJB}\left(V_L\left(x,t\right)\right)\end{aligned}$$

$$\begin{aligned}\left|HJB\left(V_L\left(x,t\right)\right)\right| &= \left|\sum_{j=1}^{\infty}\left\langle HJB\left(V_L\left(x,t\right)\right),\sigma_j\left(x\right)\right\rangle_\Omega\sigma_j\left(x\right)\right| \\ &\le \left|\sum_{j=L+1}^{\infty}\left(\sum_{k=1}^{L}w_k\left(t\right)\left\langle\nabla\sigma_k\left(x\right)f\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\right)\cdot\sigma_j\left(x\right)\right| \\ &\quad + \left|\left(\sum_{j=L+1}^{\infty}\left(\sum_{k=1}^{L}\left\langle 2\int_0^{u_k}\boldsymbol{\varphi}^{-T}\left(v\right)Rdv,\sigma_j\left(x\right)\right\rangle_\Omega\right)\cdot\sigma_j\left(x\right)\right)\right| \\ &\quad + \left|\left(\sum_{j=L+1}^{\infty}\left(\sum_{k=1}^{L}\left\langle -w_k\left(t\right)\nabla\sigma_k\left(x\right)g\left(x\right)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T\left[x\right]\nabla\boldsymbol{\sigma}_L^T\left(x\right)\mathbf{w}_L\left(x\right)\right),\sigma_j\left(x\right)\right\rangle_\Omega\right)\cdot\sigma_j\left(x\right)\right)\right| \\ &\quad + \left|\sum_{j=L+1}^{\infty}\left\langle Q\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\sigma_j\left(x\right)\right| \\ &= \left|\left(\sum_{k=1}^{L}w_k\left(t\right)\sum_{j=L+1}^{\infty}\left\langle\nabla\sigma_k\left(x\right)f\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\cdot\sigma_j\left(x\right)\right)\right| \\ &\quad + \left|\left(\sum_{k=1}^{L}\sum_{j=L+1}^{\infty}\left\langle 2\int_0^{u_k}\boldsymbol{\varphi}^{-T}\left(v\right)Rdv,\sigma_j\left(x\right)\right\rangle_\Omega\cdot\sigma_j\left(x\right)\right)\right| \\ &\quad + \left|\left(\sum_{k=1}^{L}w_k\left(t\right)\sum_{j=L+1}^{\infty}\left\langle\nabla\sigma_k\left(x\right)g\left(x\right)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T\left(x\right)\nabla\boldsymbol{\sigma}_L^T\left(x\right)\mathbf{w}_L\left(t\right)\right),\sigma_j\left(x\right)\right\rangle_\Omega\cdot\sigma_j\left(x\right)\right)\right| \\ &\quad + \left|\sum_{j=L+1}^{\infty}\left\langle Q\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\sigma_j\left(x\right)\right| \\ &\le AB\left(x\right) + CD\left(x\right) + AE\left(x\right) + \left|\sum_{j=L+1}^{\infty}\left\langle Q\left(x\right),\sigma_j\left(x\right)\right\rangle_\Omega\sigma_j\left(x\right)\right|\end{aligned} \tag{23}$$

and

$$\hat{e}_L(x(t_f), t_f) = V_L(x(t_f), t_f) - \phi(x(t_f), t_f). \quad (24)$$

Then, $\langle e_L(x,t), \boldsymbol{\sigma}_L(x)\rangle_\Omega = \langle \hat{e}_L(x,t), \boldsymbol{\sigma}_L(x)\rangle_\Omega = 0$. From the hypotheses, one has that

$$\begin{aligned} \text{HJB}(V_L(x,t)) - \text{HJB}(V(x,t)) &= e_L(x,t) \\ V_L(x(t_f), t_f) - V(x(t_f), t_f) &= \hat{e}_L(x(t_f), t_f) \end{aligned} \quad (25)$$

substituting the series expansion for $V_L(x,t)$ and $V(x,t)$, and moving the terms in the series that are greater than L to the right-hand side one obtains (26), shown at the bottom of the page.

The final condition is

$$\left(\mathbf{w}_L^T(t_f) - \mathbf{c}_L^T(t_f)\right)\boldsymbol{\sigma}_L(x) = \hat{e}_L(x(t_f), t_f) + \sum_{j=L+1}^{\infty} c_j(t_f)\sigma_j(x). \quad (27)$$

Taking the inner product of both sides over Ω_0 and taking into account the orthonormality of the set $\{\sigma_j(x)\}_1^\infty$, one obtains the third equation shown at the bottom of the page, with the final condition

$$\mathbf{w}_L(t_f) - \mathbf{c}_L(t_f) = 0. \quad (28)$$

Let $A = \langle \nabla\boldsymbol{\sigma}_L(x) f(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^T$, where A is scalar.

Define $\xi = \mathbf{w}_L(t) - \mathbf{c}_L(t)$ and consider

$$\begin{aligned} \dot{\xi} + A\cdot\xi + f(\xi,t) &= 0 \\ \xi(t_f) &= 0 \end{aligned} \quad (29)$$

where the fourth equation, shown at the bottom of the page, is continuously differentiable in a neighborhood of a point (ξ_0, t_0). Since this is an ordinary differential equation, satisfying a local Lipschitz condition [27], it has a unique solution,

$$\begin{aligned} A &= \max_{1\le k\le L,} |w_k(t)| \\ B(x) &= \sup_{(t,x)\in[t_0,T]\times\Omega} \left| \sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \langle \nabla\sigma_k(x) f(x), \sigma_j(x)\rangle_\Omega \cdot \sigma_j(x) \right| \\ C &= 1 \\ D(x) &= \sup_{(t,x)\in[t_0,T]\times\Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \left\langle 2\int_0^{u_k} \boldsymbol{\phi}^{-T}(v) R dv, \sigma_j(x) \right\rangle_\Omega \cdot \sigma_j(x) \right) \right| \\ E(x) &= \sup_{(t,x)\in[t_0,T]\times\Omega} \left| \left(\sum_{k=1}^{L} \sum_{j=L+1}^{\infty} \left\langle \nabla\sigma_k(x) g(x) \boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right), \sigma_j(x) \right\rangle_\Omega \sigma_j(x) \right) \right| \end{aligned}$$

$$\begin{aligned} &\left(\dot{\mathbf{w}}_L^T(t) - \dot{\mathbf{c}}_L^T(t)\right)\boldsymbol{\sigma}_L(x) + \left(\mathbf{w}_L^T(t) - \mathbf{c}_L^T(t)\right)\nabla\boldsymbol{\sigma}_L(x) f(x) \\ &+ \left(2\int_0^{u_L} \boldsymbol{\varphi}^{-T}(v) R dv - 2\int_0^{u} \boldsymbol{\varphi}^{-T}(v) R dv\right) - \left(\mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x) g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right)\right. \\ &\left. - \mathbf{c}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right)\right) = e_L(x,t) + \sum_{j=L+1}^{\infty}\dot{c}_j(t)\sigma_j(x) + \sum_{j=L+1}^{\infty} c_j(t)\nabla\sigma_j(x) f(x) \\ &+ \sum_{j=L+1}^{\infty} 2\int_0^{u_j} \boldsymbol{\varphi}^{-T}(v) R dv - \sum_{j=L+1}^{\infty} c_j(t)\nabla\sigma_j(x) g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right). \end{aligned} \quad (26)$$

$$\begin{aligned} &\left(\dot{\mathbf{w}}_L(t) - \dot{\mathbf{c}}_L(t)\right) + \langle \nabla\boldsymbol{\sigma}_L(x) f(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^T \left(\mathbf{w}_L(t) - \mathbf{c}_L(t)\right) \\ &+ \left\langle 2\int_0^{u_L} \boldsymbol{\varphi}^{-T}(v) R dv - 2\int_0^{u} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L \right\rangle_\Omega^T - \left\langle \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x) g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right)\right. \\ &\left. - \mathbf{c}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega^T = \sum_{j=L+1}^{\infty} c_j(t)\langle \nabla\sigma_j(x) f(x), \boldsymbol{\sigma}_L(x)\rangle_\Omega^T \\ &+ \sum_{j=L+1}^{\infty} \left\langle 2\int_0^{u_j} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle_\Omega^T - \sum_{j=L+1}^{\infty} c_j(t)\cdot\left\langle \nabla\sigma_j(x) g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega^T \end{aligned}$$

$$\begin{aligned} f(\xi,t) = &\left\langle 2\int_0^{u_L} \boldsymbol{\varphi}^{-T}(v) R dv - 2\int_0^{u} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle_\Omega^T - \left\langle \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T\mathbf{w}_L(t)\right)\right. \\ &- \mathbf{c}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)\cdot g(x) \\ &\left.\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega^T \end{aligned}$$

namely, $\xi(t) = 0, \forall\, t \in [t_0, t_f]$. Noting that the first equation, shown at the bottom of the page, is continuous in t, one invokes the standard result from the theory of ordinary differential equations [3] that a continuous perturbation in the system equations and the initial state imply a continuous perturbation of the solution [2]. There exists a $\rho_L(t) > 0$ such that, $\forall\, t \in [t_0, t_f]$, the second equation shown at the bottom of the page holds. From Technical Lemma 3, $\rho_L(t) \to 0$ as L increases. Therefore, for all $\varepsilon > 0$, $\exists L$ such

$$\Rightarrow \|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 < \varepsilon. \tag{30}$$

This means that $\|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 \to 0$ uniformly in t on Ω_0 as L increases. ■

Now, we are in a position to prove our main results.

Theorem 1 (Convergence of Approximate Value Function): Under the hypotheses of Lemma 3, one has

$$\|V_L(x,t) - V(x,t)\|_{L_2(\Omega)} \to 0 \text{ uniformly in } t \text{ on } \Omega \text{ as } L \text{ increases.}$$

Proof: From Lemma 4, we have $\|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 \to 0$

$$\begin{aligned}\|V_L(x,t) - V(x,t)\|^2_{L_2(\Omega)} &= \int_\Omega |V_L(x,t) - V(x,t)|^2 dx \\ &\le \int_\Omega \left|(\mathbf{w}_L(t) - \mathbf{c}_L(t))^T \boldsymbol{\sigma}_L(x)\right|^2 dx \\ &\quad + \int_\Omega \left|\sum_{j=L+1}^{\infty} c_j(t)\sigma_j(x)\right|^2 dx \\ &= \left(\mathbf{w}_L^T(t) - \mathbf{c}_L^T(t)\right)\left\langle \boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L^T(x)\right\rangle_\Omega \\ &\quad \cdot (\mathbf{w}_L(t) - \mathbf{c}_L(t)) \\ &\quad + \int_\Omega \left|\sum_{j=L+1}^{\infty} c_j(t)\sigma_j(x)\right|^2 dx. \end{aligned} \tag{31}$$

By the mean value theorem, Technical Lemmas 3, $\exists \xi \in \Omega$ such that the third equation shown at the bottom of the page holds. ■

Theorem 2 (Convergence of Value Function Gradient): Under the hypotheses of Lemma 3

$$\left\|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right\|_{L_2(\Omega)} \to 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

Proof: From Lemma 4, we have $\|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 \to 0$ and the equation shown at the bottom of the next page. By the mean value theorem, Technical Lemmas 1–3, $\exists \xi \in \Omega$ such that

$$\begin{aligned}\left\|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right\|^2_{L_2(\Omega)} &= \left\|\nabla \boldsymbol{\sigma}_L^T(x)(\mathbf{w}_L(t) - \mathbf{c}_L(t))\right\|^2_{L_2(\Omega)} \\ &\quad + \lambda(\Omega)\left|\sum_{j=L+1}^{\infty} \nabla\sigma_j^T(x) c_j(t)\right|^2.\end{aligned}$$

Since $\nabla\boldsymbol{\sigma}_L^T(x)$ is linearly independent and $\|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2 \to 0$, then

$$\left\|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right\|_{L_2(\Omega)} \to 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

$$\sum_{j=L+1}^{\infty} c_j(t)\langle \nabla\sigma_j(x) f(x), \boldsymbol{\sigma}_L(x)\rangle^T_\Omega + \sum_{j=L+1}^{\infty}\left\langle 2\int_0^{u_j} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega - \sum_{j=L+1}^{\infty} c_j(t)\left\langle \nabla\sigma_j(x) g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega$$

$$\begin{aligned}&\Bigg\|\sum_{j=L+1}^{\infty} c_j(t)\cdot\langle \nabla\sigma_j(x) f(x), \boldsymbol{\sigma}_L(x)\rangle^T_\Omega + \sum_{j=L+1}^{\infty}\left\langle 2\int_0^{u_j} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega \\ &\quad - \sum_{j=L+1}^{\infty} c_j(t)\cdot\left\langle \nabla\sigma_j(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega\Bigg\|_{L_2(\Omega)} \\ &\le \left\|\sum_{j=L+1}^{\infty} c_j(t)\cdot\langle \nabla\sigma_j(x) f(x), \boldsymbol{\sigma}_L(x)\rangle^T_\Omega\right\|_{L_2(\Omega)} + \left\|\sum_{j=L+1}^{\infty}\left\langle 2\int_0^{u_j} \boldsymbol{\varphi}^{-T}(v) R dv, \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega\right\|_{L_2(\Omega)} \\ &\quad + \left\|\sum_{j=L+1}^{\infty} c_j(t)\cdot\left\langle \nabla\sigma_j(x)\cdot g(x)\cdot\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle^T_\Omega\right\|_{L_2(\Omega)} \\ &\le \rho_L(t)\end{aligned}$$

$$\|V_L(x,t) - V(x,t)\|^2_{L_2(\Omega)} = \|\mathbf{w}_L(t) - \mathbf{c}_L(t)\|_2^2 + \lambda(\Omega)\cdot\left|\sum_{j=L+1}^{\infty} c_j(t)\sigma_j(\xi)\right|^2 \to 0 \text{ uniformly in } t \text{ on } \Omega_0 \text{ as } L \text{ increases.}$$

Through Theorem 1 and 2, we have shown that the HJB approximating solution (12) guarantees convergence in Sobolev space $H^{1,2}$. ∎

Theorem 3 (Convergence of Control Inputs): If the conditions of Lemma 3 are satisfied and

$$u_L(x,t) = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V_L(x,t)}{\partial x}\right)$$
$$u(x,t) = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V(x,t)}{\partial x}\right)$$

then $\|u_L(x,t) - u(x,t)\|_{L_2(\Omega)} \to 0$ in t on Ω_0 as L increases.

Proof: Denote $\alpha_L(x) = -(1/2)R^{-1}g^T(x)(\partial V_L(x,t)/\partial x)$ and $\alpha(x) = -(1/2)R^{-1}g^T(x)(\partial V(x,t)/\partial x)$.

By Theorem 2 and the fact that $g(x)$ is continuous and, therefore, bounded on Ω, hence

$$\left\|-\frac{1}{2}R^{-1}g^T(x)\frac{\partial V_L(x,t)}{\partial x} + \frac{1}{2}R^{-1}g^T\frac{\partial V(x,t)}{\partial x}\right\|^2_{L_2(\Omega)} \le \left\|-\frac{1}{2}R^{-1}g^T(x)\right\|^2_{L_2(\Omega)}\left\|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right\|^2_{L_2(\Omega)} \to 0 \Rightarrow \alpha_L(x) - a(x) \to 0.$$

Because $\boldsymbol{\varphi}(\cdot)$ is smooth and under the assumption that its first derivative is bounded, we have $\|\boldsymbol{\varphi}(\alpha_L(x)) - \boldsymbol{\varphi}(\alpha(x))\| \le M\|(\alpha_L(x) - \alpha(x))\|$. Therefore

$$\|\alpha_L(x) - \alpha(x)\|_{L_2(\Omega)} \to 0 \Rightarrow \|\boldsymbol{\varphi}(\alpha_L(x)) - \boldsymbol{\varphi}(\alpha(x))\|_{L_2(\Omega)} \to 0$$

hence $\|u_L(x,t) - u(x,t)\|_{L_2(\Omega)} \to 0$ in t on Ω_0 as L increases. ∎

At this point, we have proven uniform convergence in t in the mean of the approximate HJB equation, the NN weights, the approximate value function, and the value function gradient. This demonstrates uniform convergence in t in the mean in Sobolev space $H^{1,2}(\Omega)$. In fact, the next result shows even stronger convergence properties, namely, uniform convergence in both x and t.

Lemma 5 (Uniform Convergence): Since a local Lipschitz condition holds on (29), then

$$\sup_{x\in\Omega}|V_L(x,t) - V(x,t)| \to 0$$
$$\sup_{x\in\Omega}\left|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right| \to 0 \text{ and } \sup_{x\in\Omega}|u_L(x,t) - u(x,t)| \to 0.$$

Proof: This follows by noticing that $\|\mathbf{w}_L(t) - c_L(t)\|_2^2 \to 0$ uniformly in t and the series with c_j is uniformly convergent in t and Technical Lemma 1. ∎

The final result shows that if the number L of hidden layer units is large enough, the proposed solution method yields an admissible control.

Theorem 4 [Admissibility of $u_L(x,t)$]: If the conditions of Lemma 3 are satisfied, then $\exists L_0 : L \ge L_0, u_L \in \Psi(\Omega_0)$.

Proof: Define $V(x,u) = \phi(t_0, t_f, x(t_f), u) + \int_{t_0}^{t_f}[Q(x) + W(u)]\,dt$. We must show that for L sufficiently large, $V(x,u_L) < \infty$ when $V(x,u) < \infty$. However, the solution of (1) depends continuously on u, i.e., small variations in u result in small variations in solution of (1). Also, since $\|u_L(\cdot)\|^2_{L_2(\Omega_0)}$ can be made arbitrarily close to $\|u(\cdot)\|^2_{L_2(\Omega_0)}$, $V(x,u_L)$ can be made arbitrarily close to $V(x,u)$. Therefore, for L sufficiently large, $V(x,u_L) < \infty$ and, hence, $u_L(x,t)$ is admissible. ∎

C. Optimal Algorithm Based on NN Approximation

Solving the integration in (20) is expensive computationally, since evaluation of the L_2 inner product over Ω_0 is required. This can be addressed using the collocation method [21]. The integrals can be well approximated by discretization. A mesh of points over the integration region can be introduced on Ω_0 of size Δx. The terms of (21) can be rewritten as follows:

$$A = \left[\boldsymbol{\sigma}_L(x)|_{x_1} \ldots \boldsymbol{\sigma}_L(x)|_{x_p}\right]^T$$
$$B = \left[\boldsymbol{\sigma}_L(x)f(x)|_{x_1} \ldots \boldsymbol{\sigma}_L(x)f(x)|_{x_p}\right]^T$$
$$C = \left[2\int_0^{u_L}\boldsymbol{\varphi}^{-T}(v)Rdv|_{x_1} \ldots 2\int_0^{u_L}\boldsymbol{\varphi}^{-T}(v)Rdv|_{x_p}\right]^T$$
$$D = \left[\nabla\boldsymbol{\sigma}_L(x)g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right)|_{x_1} \ldots \nabla\boldsymbol{\sigma}_L(x)g(x)\boldsymbol{\varphi} \cdot \left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{c}_L(t)\right)|_{x_p}\right]^T$$
$$E = \left[Q(x)|_{x_1} \ldots Q(x)|_{x_p}\right]^T$$

where p in x_p represents the number of points of the mesh. Reducing the mesh size, we have (32)–(34), shown at the bottom of the next page. This implies that (20) can be converted to

$$-A^TA\dot{\mathbf{w}}_L(t) - A^TB\mathbf{w}_L(t) - A^TC + A^TD\mathbf{w}_L(t) - A^TE = 0 \quad (35)$$

$$\left\|\frac{\partial V_L(x,t)}{\partial x} - \frac{\partial V(x,t)}{\partial x}\right\|^2_{L_2(\Omega)} = \left\|\nabla\boldsymbol{\sigma}_L^T(x)(\mathbf{w}_L(t) - \mathbf{c}_L(t)) - \sum_{j=L+1}^{\infty}\nabla\sigma_j^T(x)c_j(t)\right\|^2_{L_2(\Omega)}$$
$$\le \left\|\nabla\boldsymbol{\sigma}_L^T(x)(\mathbf{w}_L(t) - \mathbf{c}_L(t))\right\|^2_{L_2(\Omega)} + \left\|\sum_{j=L+1}^{\infty}\nabla\sigma_j^T(x)c_j(t)\right\|^2_{L_2(\Omega)}$$
$$= \left\|\nabla\boldsymbol{\sigma}_L^T(x)(\mathbf{w}_L(t) - \mathbf{c}_L(t))\right\|^2_{L_2(\Omega)} + \int_\Omega\left|\sum_{j=L+1}^{\infty}\nabla\sigma_j^T(x)c_j(t)\right|^2 dx$$

then

$$\dot{\mathbf{w}}_L(t) = -\left(A^T A\right)^{-1} A^T B \mathbf{w}_L(t) - \left(A^T A\right)^{-1} A^T + \left(A^T A\right)^{-1} A^T D \mathbf{w}_L(t) - \left(A^T A\right)^{-1} A^T E. \quad (36)$$

This is a nonlinear ODE that can easily be integrated backwards using final condition $\mathbf{w}_L(t_f)$ to find the least squares optimal NN weights. Then, the nearly optimal value function is given by

$$V_L(x,t) = \mathbf{w}_L^T(t)\boldsymbol{\sigma}_L(x)$$

and the nearly optimal control by

$$u_L(x,t) = -\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T(x)\mathbf{w}_L(t)\right). \quad (37)$$

Note that in practice, we use a numerically efficient least squares relatively to solve (35) without matrix inversion.

V. Simulation

We now show the power of our NN control technique for finding nearly optimal fixed-final-time-constrained controllers. Two examples are presented.

A. Linear System

1) We start by applying the algorithm obtained previously for the linear system

$$\begin{aligned}\dot{x}_1 &= 2x_1 + x_2 + x_3\\ \dot{x}_2 &= x_1 - x_2 + u_2\\ \dot{x}_3 &= x_3 + u_1.\end{aligned} \quad (38)$$

Define performance index

$$V(x(t_0),t_0) = \phi(x(t_f),t_f) + \int_{t_0}^{T}\left(Q(x) + 2\int_0^u \boldsymbol{\varphi}^{-T}(v)Rdv\right)dt. \quad (39)$$

Here, $Q = 10^* I_{3\times3}$ and $R = I_{2\times2}$, where I is an identity matrix. It is desired to control the system with input constraints $|u_1| \le 5$ and $|u_2| \le 20$. In order to ensure the constrained control, a nonquadratic cost performance term (9) is used. To show how to do this for the general case of $|u| \le 5$, we use $A\tanh(1/A\cdots)$ for $\boldsymbol{\varphi}(\cdots)$. Hence, the nonquadratic cost is

$$W(u) = 2\int_0^u A\tanh^{-T}\left(\frac{v}{A}\right)Rdv.$$

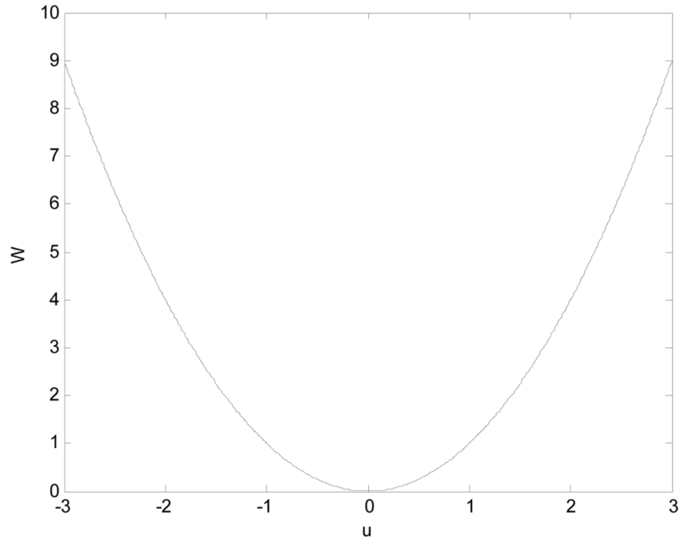

Fig. 1. Nonquadratic cost.

The plot is shown in Fig. 1. This nonquadratic cost performance is used in the algorithm to calculate the optimal-constrained controller. The algorithm is run over the region Ω_0 defined by $-2 \le x_1 \le 2$, $-2 \le x_2 \le 2$, and $-2 \le x_3 \le 2$. To find a nearly optimal time-varying controller, the following smooth function is used to approximate the value function of the system:

$$V(x_1,x_2) = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 + w_4x_1x_2 + w_5x_1x_3 + w_6x_2x_3.$$

This is an NN with polynomial activation functions, and hence, $V(0) = 0$.

In this example, six neurons are chosen and $\mathbf{w}_L(t_f) = [10, 10, 10, 0, 0, 0]$. Our algorithm was used to determine the nearly optimal time-varying-constrained control law by backwards integrating to solve (35). The required quantities A, B, C, D, and E in (35) were evaluated for 5000

$$\left\langle -\dot{\mathbf{w}}_L^T(t)\boldsymbol{\sigma}_L(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -\left(A^T A\right)\cdot\dot{\mathbf{w}}_L(t)\cdot\Delta x \quad (32)$$

$$\left\langle -\mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)f(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -\left(A^T B\right)\cdot\mathbf{w}_L(t)\cdot\Delta x \quad (33)$$

$$\left\langle -2\int_0^{u_L}\boldsymbol{\varphi}^{-T}(v)Rdv, \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -A^T C\cdot\Delta x$$

$$\left\langle \mathbf{w}_L^T(t)\nabla\boldsymbol{\sigma}_L(x)g(x)\boldsymbol{\varphi}\left(\frac{1}{2}R^{-1}g^T(x)\nabla\boldsymbol{\sigma}_L^T\mathbf{w}_L(t)\right), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = \lim_{\|\Delta x\|\to 0} A^T D\mathbf{w}_L(t)\cdot\Delta x$$

$$\left\langle -Q(x), \boldsymbol{\sigma}_L(x)\right\rangle_\Omega = \lim_{\|\Delta x\|\to 0} -\left(A^T E\right)\cdot\Delta x. \quad (34)$$

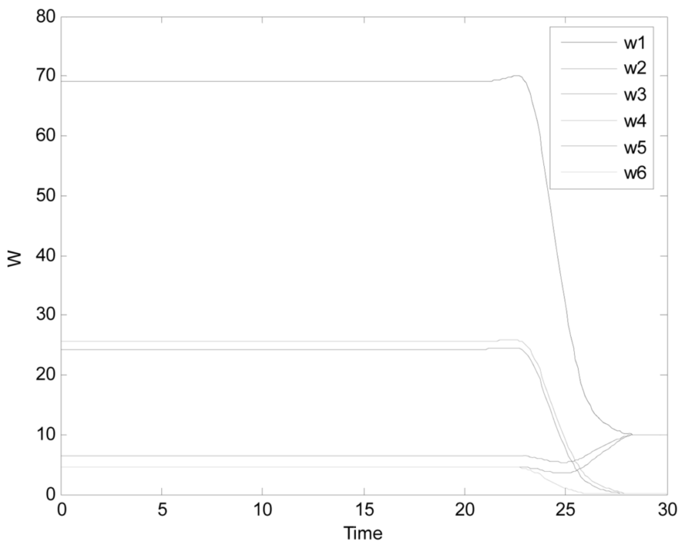

Fig. 2. Constrained linear system weights.

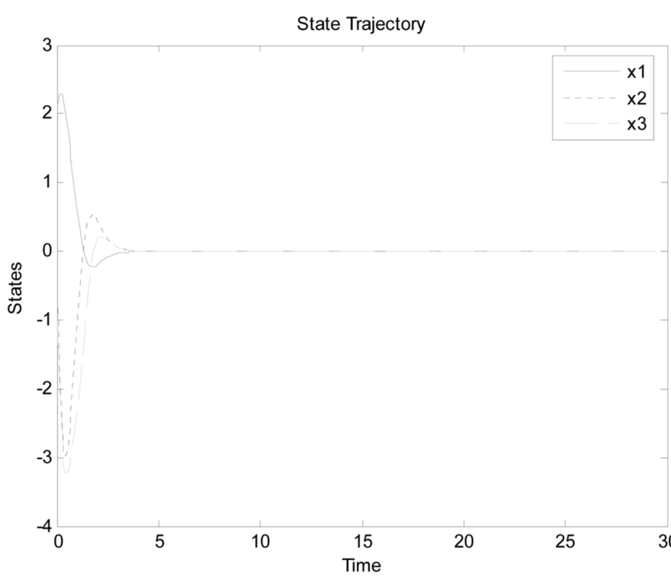

Fig. 3. State trajectory of linear system with bounds.

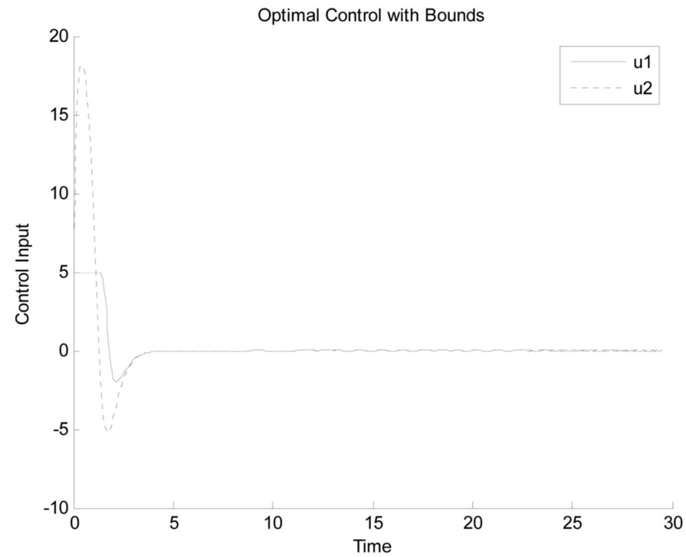

Fig. 4. Optimal NN control law with bounds.

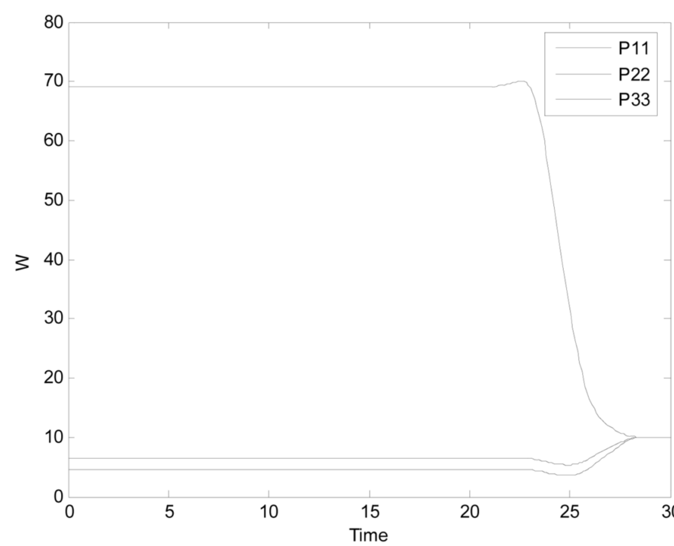

Fig. 5. Unconstrained control system weights.

points in Ω_0. A least square algorithm from MATLAB was used to compute $\dot{\mathbf{w}}_L(t)$ at each integration time. The solution was obtained in 30 s. From Fig. 2, it is obvious that about 25 s from t_f, the weights converge to constant. The states and control signal obtained by a forward integration of (38) using these weights in (37) are shown in Figs. 3 and 4. The control is bounded as required.

2) Now, let $A = 100$ so that the control constraints are effectively removed. The algorithm is run and the plots of P_{11}, P_{22}, and P_{33} and function of time are shown in Fig. 5. These plots converge to steady-state values of $P_{11} = 69.0573$, $P_{22} = 4.6208$, and $P_{22} = 6.5008$. These correspond exactly to the algebraic Riccati equation solution obtained by standard optimal control methods [33], which is

$$P = \begin{bmatrix} 69.0573 & 12.8164 & 12.1491 \\ 12.8164 & 4.6208 & 2.2448 \\ 12.1491 & 2.2448 & 6.5008 \end{bmatrix}.$$

B. Nonlinear Chained Form System

One can apply the results of this paper to a mobile robot, which is a nonholonomic system [29]. It is known [14] that there does not exist a continuous time-invariant feedback control law that minimizes the cost. Some methods for deriving stable controls of nonholonomic systems are found in [12], [13], [18]–[20], [45], [46], [48], and [55]. Our method will yield a time-varying gain. From [32], under some sufficient conditions, a nonholonomic system can be converted to chained form as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2. \end{aligned} \tag{40}$$

Define performance index (39). Here, Q and R are chosen as identity matrices. It is desired to control the system with control limits of $|u_1| \leq 1$ and $|u_2| \leq 2$. A similar nonquadratic cost performance term is used as in the last example. Here, the region

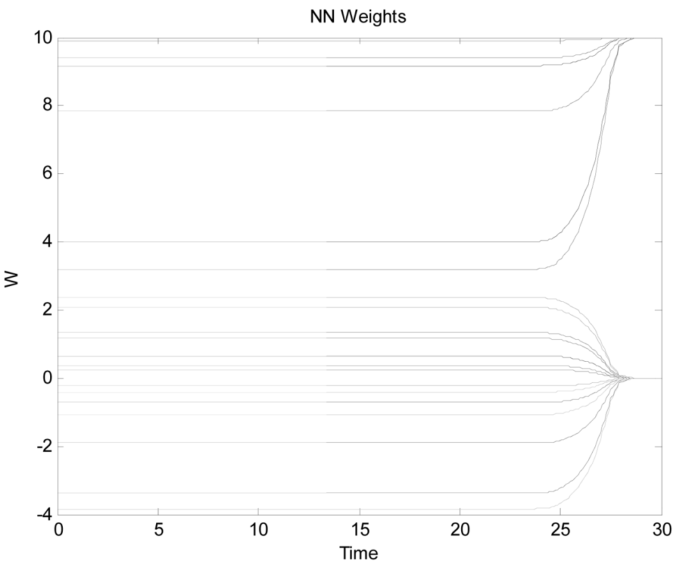

Fig. 6. Nonlinear system weights.

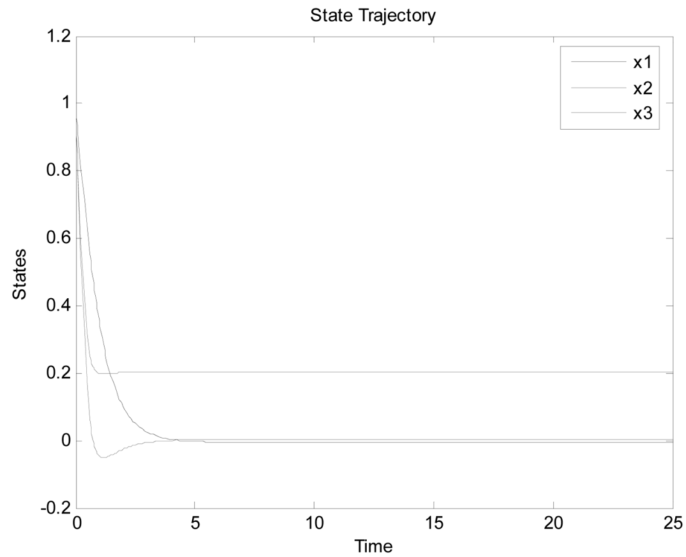

Fig. 7. State trajectory of nonlinear system.

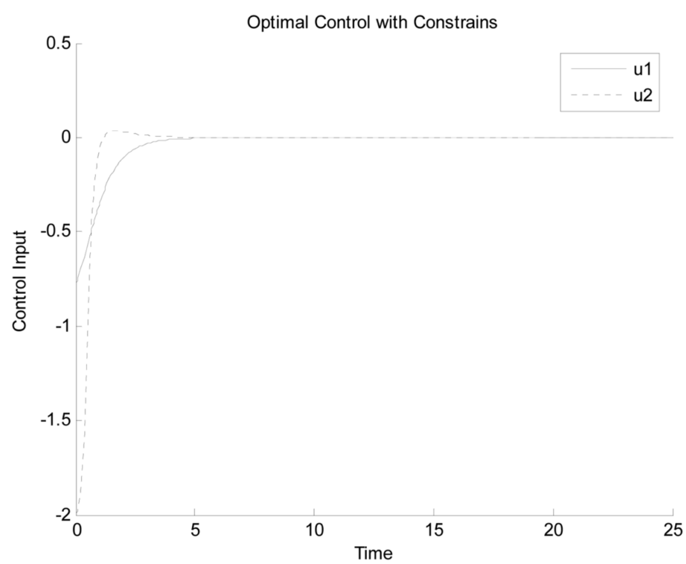

Fig. 8. Optimal NN-constrained control law.

Ω_0 is defined by $-2 \leq x_1 \leq 2$, $-2 \leq x_2 \leq 2$, and $-2 \leq x_3 \leq 2$. To solve for the value function of the related optimal control problem, we selected the smooth approximating function

$$\begin{aligned} V(x_1, x_2, x_3) = \; & w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + w_4 x_1 x_2 + w_5 x_1 x_3 \\ & + w_6 x_2 x_3 + w_7 x_1^4 + w_8 x_2^4 + w_9 x_3^4 \\ & + w_{10} x_1^2 x_2^2 + w_{11} x_1^2 x_3^2 + w_{12} x_2^2 x_3^2 \\ & + w_{13} x_1^2 x_2 x_3 + w_{14} x_1 x_2^2 x_3 + w_{15} x_1 x_2 x_3^2 \\ & + w_{16} x_1^3 x_2 + w_{17} x_1^3 x_3 + w_{18} x_1 x_2^3 \\ & + w_{19} x_1 x_3^3 + w_{20} x_2 x_3^3 + w_{21} x_2^3 x_3. \end{aligned} \quad (41)$$

The selection of the NN is usually a natural choice guided by engineering experience and intuition. This is an NN with polynomial activation functions, and hence, $V(0) = 0$. This is a power series NN with 21 activation functions containing powers of the state variable of the system up to the fourth order. Convergence was not observed using an NN with only second-order powers of the states. The number of neurons required is chosen to guarantee the uniform convergence of the algorithm. In this example, $\mathbf{w}_L(t_f) = [10; 10; 10; 0; 0; 0; 10; 10; 10; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0]$ and $t_f = 30$ s. The required quantities A, B, C, D, and E in (35) were evaluated for 5000 points in Ω_0. Fig. 6 indicates that the weights converge to constants when they are integrated backwards. The time-varying controller (37) is then applied to (40). Fig. 7 shows that the systems' states responses, including x_1, x_2, and x_3, are all bounded. It can be seen that the states do converge to a value close to the origin. Fig. 8 shows the optimal control is constrained as required and converges to zero.

VI. Conclusion

We use NN to approximately solve the time-varying HJB equation for constrained input nonlinear systems. The technique can be applied to both linear and nonlinear systems. Full conditions for convergence have been derived. Simulation examples have been carried out to show the effectiveness of the proposed method.

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