

Modeling airline frequency competition for airport congestion mitigation

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Abstract: Demand often exceeds capacity at the congested airports. Airline frequency competition is partially responsible for the growing demand for airport resources. We propose a game-theoretic model for airline frequency competition under slot constraints. The model is solved to obtain a Nash equilibrium using a successive optimizations approach, wherein individual optimizations are performed using a dynamic programming-based technique. The model predictions are validated against actual frequency data, with the results indicating a close fit to reality. We use the model to evaluate different strategic slot allocation schemes from the perspectives of the airlines and the passengers. The most significant result of this research shows that a small reduction in the total number of allocated slots translates into a substantial reduction in flight and passenger delays, and also a considerable improvement in airlines' profits.

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1 Introduction

Airport congestion is imposing a tremendous cost on the world economy. In the recently concluded *Total Delay Impact Study* (Ball et al., 2010) commissioned by the Federal Aviation Administration (FAA), researchers estimated the total cost of domestic air traffic delays to be around \$31.2 billion for calendar year 2007, including \$8.3 billion in additional aircraft operating costs, \$16.7 billion in passenger delay costs, and an estimated \$6.2 billion in other indirect costs of delays. The magnitude of these delay costs can be properly grasped by noting that during the same period, the aggregate profits of US domestic airlines were \$5.0 billion (ATA, 2008). For the year 2007, Bureau of Transportation Statistics (BTS, 2010c) categorized delays to around 50% of the delayed flights as delays caused by the National Aviation System (NAS). Weather and volume were the top two causes of NAS delays, together responsible for 84.5% of the NAS delays. Delays due to volume are those caused by scheduling more airport operations than the available capacity, while the delays due to weather are those caused by airport capacity reductions under adverse weather conditions. Both these types of delays are due to scheduling more operations than the realized capacity. Such mismatches between demand and capacity are a primary cause of flight delays in the United States.

These delays are disproportionately distributed across airports and metropolitan areas in the country. Congestion at a few major airports is responsible for a large proportion of overall delays. An analysis of air traffic patterns and delays by the Brookings Institution (Tomer and Puentes, 2009) suggests that almost 65% of the delayed flight arrivals are concentrated in the 25 largest metropolitan areas. Moreover, operations across an airline's network are interrelated due to linkages in aircraft, crew and passenger movements. Therefore, delays originating at these major airports propagate across the airline networks causing system-wide disruptive impacts. In the summer of 2007, according to the New York Aviation Rulemaking Committee (2007) report, three-quarters of the nationwide flight delays were generated from the air congestion surrounding New York. This suggests that mitigation of demand-capacity imbalance at a handful of congested airports should yield system-wide benefits in terms of delay alleviation.

1.1 Demand Management

Increasing capacity and decreasing demand are the two natural ways of bringing the demand-capacity mismatch into balance. Capacity enhancement measures such as building new airports, construction of new runways, etc. are investment intensive, require long-time horizons, and might not be feasible in many cases due to geographic, environmental, socio-economic and political issues associated with such large projects. On the other hand, demand management strategies such as administrative slot controls, market-based mechanisms, or any combinations thereof, have the potential to restore the demand-capacity balance over a medium- to short-time horizon with comparatively little investment. Demand management strategies refer to any administrative or economic policies and regulations that restrict airport access to users. All the demand management strategies proposed in the literature and practiced in reality can be broadly categorized as administrative controls and market-based mechanisms, although various hybrid schemes have also been proposed. The demand management problem involves two types of decisions, namely, (1) slot determination, which involves deciding the total number of slots to be allocated, and (2) slot allocation, which involves the decision on distribution of these slots among the different users. These decisions can be taken either sequentially, such as in an auction or administrative mechanism, or simultaneously, such as in a congestion pricing mechanism.

Administrative Controls: Currently, four major airports in the United States, namely, LaGuardia (LGA), John F. Kennedy (JFK), and Newark (EWR) airports in the New York region, and Reagan (DCA) airport at Washington D.C., have administrative controls limiting the number of flight operations. Outside of the US, administrative controls are commonplace at busy airports. Several major airports in Europe and Asia are 'schedule-coordinated', where a central coordinator allocates the airport slots to airlines based on a set of pre-determined rules. Under the current practices, both in and outside of the US, the criteria governing

the slot allocation process are typically based on historical precedents and use-it-or-lose-it rules. Under these rules, an airline is entitled to retain a slot that was allocated to it in the previous year, contingent on the fact that the slot was utilized for at least a certain minimum fraction of time over the previous year. An airline failing to utilize a slot frequently enough, however, is in danger of losing it.

One fundamental problem with the current administrative slot allocation procedures is that they are economically inefficient because they create barriers to entry by new carriers (Dot Econ Ltd., 2001) and encourage airlines to over-schedule in order to avoid losing the slots (Harsha, 2008). Another problem, as pointed out by Ball et al. (2006), is the implicit need to make a tradeoff between delays and resource utilization. Specifically, current approaches require ascertaining the 'declared' capacity of an airport beforehand even though the actual capacity on the day of operations is a function of prevalent weather conditions. Declaring too high a value for capacity poses the danger of large delays under bad weather conditions (Instrument Meteorological Conditions (IMC)) and declaring too low a value leads to wastage of resources under good weather conditions (Visual Meteorological Conditions (VMC)). Declared capacity, that is, the total number of allocated slots per time period, ultimately determines the congestion and delays at an airport.

Congestion Pricing: Researchers have shown that market-based mechanisms, if implemented properly, result in efficient allocation of airport resources. Congestion pricing and slot auctions are two of the most popular market mechanisms proposed in the literature. Classical studies such as Vickrey (1969) and Carlin and Park (1970) proposed congestion pricing based on the marginal cost of delays. Such pricing schemes, in theory, maximize the social welfare through optimal allocation of public resources. Under congestion pricing, the total cost to the user includes the delay cost as well as the congestion price. The notion of equilibrium congestion prices relies on the existence of a demand function, that is, an expression that gives the aggregate demand for airport resources as a function of total cost to the user. Some researchers, such as Morrison (1987) and Daniel (1995), performed numerical experiments under some specific assumptions about the underlying demand function, while others, like Carlin and Park (1970), have acknowledged the problems in estimating demand as a function of congestion prices with any level of reliability because of lack of sufficient data.

Beyond the unavailability of data, however, there is an even more basic issue associated with accurate demand estimation. Under congestion pricing, the aggregate demand for slots at an airport is the sum of the number of slots demanded by each airline. Assuming profit maximizing airlines, the number of slots demanded by an airline can be obtained by equating the incremental profitability of the last slot to the congestion price per slot. In reality, among other factors, the profitability of an airline depends on its own

schedule as well as on competitor schedules. It is easy to see that the incremental profitability of having an extra flight in a particular market largely depends on the number of additional passengers that the airline will be able to carry because of the additional flight, which in turn depends on the schedule of flights offered by the competitor airlines in the same market. So given a set of congestion prices, the total demand for slots should reflect these competitive interactions. Some recent congestion pricing studies by transportation economists such as Pels and Verhoef (2004) and Brueckner (2002), have modeled competitive effects through Cournot (1897) type models of firm competition. However, these models do not incorporate the inverse dependence of one airline's market share on competitor airlines' frequencies, which is a critical component of such competitive interactions.

Slot Auctions: The idea of airport slot auctions was first proposed by Grether et al. (1979). Rassenti, Smith, and Bulfin (1982) showed how combinatorial auction design is suitable for airport slot auctions and highlighted the associated efficiency gains through experiments. Since then, several researchers (Cramton et al., 2007; Ball, Donohue, and Hoffman, 2006; Dot Econ Ltd., 2001; Harsha, 2008, to name a few) have shown the advantages of slot auctions. The reader is referred to Ball, Donohue, and Hoffman (2006) and Harsha (2008) for detailed accounts of various commonly raised concerns regarding slot auctions and ways of addressing them. In spite of the many attractive properties of the auctioning mechanisms, an auction by itself does not alleviate airport congestion, but rather allocates a fixed set of resources in a more efficient way. So, to that extent, auctions are similar to administrative controls, as they too pose an implicit need to make a tradeoff between delays and resource utilization.

Once the number of slots to be allocated is determined through some procedure, slot auctions, in theory, should maximize the social welfare by allocating the slots to those who value them the most. But the determination of the actual value of a package of slots to an airline is a complicated problem. Harsha (2008) proposed a valuation model for estimating the value of a package of slots. However, the formulation does not capture any effects of airline competition.

In summary, in an auction or administrative mechanism, slot allocation must be explicitly preceded by some process for slot determination. It is this previous step that primarily determines the congestion level. Existing literature has typically focused on the second step and the first step has not received much attention. Furthermore, much of the discussion of the second step excludes any effects of frequency competition. Although congestion pricing tackles both these decisions simultaneously and hence implicitly handles the slot determination step, existing literature on congestion pricing does not capture important elements of frequency competition. In this research, we propose a framework for assessing different strategic demand management schemes while explicitly modeling the effects of frequency

competition. In our first experiment, we evaluate the impacts of slot determination step in terms of airline profits and passengers carried by varying the total number of allocated slots. In our second experiment, for a fixed number of total slots, we focus on the problem of slot allocation and evaluate the impacts of two different simple strategies for slot allocation on the various stakeholders.

Le (2006) showed that the delays at congested airports such as LaGuardia airport at New York are caused in large part due to the inefficient slot controls. Instead of modeling airline competition, this study assumed a hypothetical “single benevolent airline” and proved the existence of profitable flight schedules at LGA that can accommodate the passenger demand while reducing flight delays substantially. Vaze and Barnhart (2011) solved a large-scale mixed integer optimization problem to obtain delay-minimizing schedules for the air transportation network of the entire United States. This study concluded that effective administrative and/or market-based mechanisms for slot control have the potential to reduce delays while satisfying all passenger demand given the available airport capacity. Our conclusions confirm the findings of these previous studies. In this paper, we explicitly model airline frequency competition and propose tangible mechanisms for achieving profitable schedules that accommodate passenger demand and significantly reduce delays.

1.2 Airline Frequency Competition

Since the deregulation of US domestic airline business in 1978, apart from fare, service frequency has become the most important competitive weapon at an airline's disposal. Frequency planning is the part of the airline schedule development process that involves decisions about the number of flights to be operated on each route. By providing more frequency on a route, an airline attracts more passengers. Given an estimate of total demand on a route, the market share of each airline depends on its own frequency as well as on the competitor frequency. Market share can be modeled according to the so-called S-curve or sigmoidal relationship between the market share and frequency share, which is a popular notion in the airline industry (O'Connor, 2001; Belobaba, 2009a). Empirical evidence of the relationship was documented in some early post-deregulation studies and regression analysis was used to estimate the model parameters (Taneja, 1968; 1976; Simpson, 1970). Over the years, there have been several references to the S-curve including Kahn (1993) and Baseler (2002). The most commonly used mathematical expression for the S-curve relationship (Simpson, 1970; Belobaba, 2009a) is given by,

$$MS_i = \frac{FS_i^\alpha}{\sum_{j=1}^n FS_j^\alpha}. \quad (1)$$

In this equation, MS_i is the market share of airline i , FS_i is the frequency share of airline i , n is the number of competing airlines, and $\alpha \geq 1$ is a model parameter.

Some of the more recent empirical and econometric literature has focused on investigating the validity of the S-curve as the structure of airline business has evolved over the last few decades. The conclusions are mostly mixed. Wei and Hansen (2005) have provided statistical support for the S-curve, based on a nested Logit model for non-stop duopoly markets. They conclude that by increasing the service frequency, an airline can get a disproportionately high share of the market and hence there is an incentive for operating more frequent flights with smaller aircraft. In another recent study, Button and Drexler (2005) observed limited evidence of the S-curve phenomenon in the 1990s. But in the early 2000s, they found the relationship between market share and frequency share is not S-shaped but rather along an upward-sloping straight line with slope 1.0. This can be characterized by setting $\alpha = 1$ in equation (1). They, however, caution that the absence of empirical evidence for the S-curve does not necessarily mean that it does not affect airline behavior in a significant way. In an industry study, Binggeli and Pompeo (2006) concluded that the S-curve still very much exists in markets dominated by legacy carriers. However, there is very little measurable evidence of the S-curve in markets where low cost carriers (LCCs) compete with each other and a straight line relationship is a better approximation for such markets. They call for a rethinking of the S-Curve principle that has been “hard-wired” in the heads of many network planners over the years.

In summary, recent evidence confirms that the market share is an increasing function of the frequency share and hence competition considerations affect the frequency decisions in an important way. However, the evidence is mixed about the exact shape of the relationship, in particular the exact value of the parameter α for different types of markets.

Despite the continuing interest in frequency competition based on the S-curve phenomenon, literature on the game-theoretic aspects of such competition is limited. In most of the previous studies involving game-theoretic analysis of frequency competition, market share is modeled using Logit or nested Logit type models, with utility typically being an affine function of the inverse of frequency. Depending on the exact values of the utility parameters, such relationships can be considerably different from the S-shaped relationship between market share and frequency share. In this research, we use the most popular characterization of the S-curve model, as given by equation (1). By varying the value of α according to the characteristics of the individual markets, airline scheduling decisions can be well captured using the model given by equation (1).

1.3 Literature Review

The existing body of literature on airline frequency competition can be categorized into three broad groups- econometric, theoretical and computational studies. Studies by transportation economists such as Brander and Zhang (1993), Aguirregabiria and Ho (2008), Norman and Strandenes (1994) etc. employ econometric methods to estimate the parameters in the airline competition models using large datasets and use the calibrated models for gaining critical insights into the competitive behavior of the airlines and for answering policy-related broad questions. These studies do not deal with the issues of existence, computation and empirical validation of the equilibrium predictions. Theoretical studies including Brueckner (2010), Brueckner and Flores-Fillol (2007), Hendricks, Piccione and Tan (1999), Pels, Nijkamp, and Rietveld (2000), Hong and Harker (1992) etc. investigate analytically solvable game-theoretic models of airline competition and derive theoretical results that provide insights into important characteristics of equilibria and the comparative statics. These studies do not deal with real datasets. Computational studies such as Hansen (1990), Wei and Hansen (2007), Dobson and Lederer (1993), Adler (2001, 2005) etc. employ mathematical models and solution algorithms for obtaining Nash equilibria of airline competition games. Our research falls within this third category.

Dobson and Lederer (1993) model schedule and fare competition as a strategic form game for a sample problem comprising six airports and two airlines. Adler (2001) models airline competition on fares, frequencies and aircraft sizes as an extensive form game and presents equilibrium results for a network comprising four airports and two airlines. Subsequently, Adler (2005) considers the decisions on hub locations and decisions about fares, frequencies and aircraft sizes in a two-stage extensive form game framework for a reasonably sized-problem consisting of three airlines with two hubs for each airline. None of these studies provides any empirical justification of suitability of Nash equilibrium outcome. Hansen (1990) analyzes frequency competition in a hub-dominated environment using a strategic form game model and presents results for a large network of realistic size involving multiple airlines. This study reports significant disparities between model predictions and the state of the actual system. Each of these four studies adopts a successive optimizations approach to solve for a Nash equilibrium. In this paper, we also use a successive optimizations approach for the computation of a Nash equilibrium. We assess the impact of starting point on the equilibrium being reached. We also provide empirical validation of our equilibrium predictions.

Furthermore, in most of the previous research, scheduling decisions on one segment are not constrained by the schedule on other segments. (We define a segment as an origin and destination pair for non-stop flights.) This is a good approximation for a situation where an airport is not congested, and takeoff and

landing slots are freely available. But some congested US airports and several major airports in Europe and Asia are slot constrained. With projected passenger demand in the US expected to outpace the development of new airport capacity, there is a possibility of many more airports in the US employing some form of demand management in the future. At a slot constrained airport, increasing the frequency of flights on one segment usually requires the airline to decrease the frequency on some other segment from that airport. To the best of the authors' knowledge, no previous study has incorporated slot constraints into airline competition models.

1.4 Contributions

The main contributions of this paper fall into five categories. First, we propose a game-theoretic model of frequency competition as an evaluation methodology for slot determination and allocation schemes. Second, we provide a solution algorithm with good computational performance for solving the problem to equilibrium. Third, we provide justification of the credibility of the Nash equilibrium solution concept in two different ways, through empirical validation of the model outcome and through convergence properties of the learning dynamics for non-equilibrium situations. **Fourth, we address the slot determination problem indirectly through detailed computational experiments and sensitivity analyses.** Finally, under simple slot allocation schemes, we evaluate the system performance from the perspectives of the passengers and the competing airlines, and provide insights to guide the demand management policy decisions.

Market-based mechanisms lead to socially efficient resource allocation. But the problems such as calculating the equilibrium congestion prices or designing an efficient auction are computationally challenging, even without considering any competitive interactions among the carriers. Therefore, we approach the problem in a different way. We do not try to integrate schedule competition into the slot allocations problem. Instead, given a slot allocation, we provide a framework for predicting the airline schedules and estimating the impact on passengers and competing airlines.

The airline planning process involves a large number of decision variables. Considerations such as network effects and demand uncertainty introduce further complications in the process. More tactical decisions such as pricing and revenue management often interact with these planning decisions and hence should be considered in evaluating an airline's response to any slot allocation scheme. Therefore, any tractable mathematical model of airline decisions involves substantial simplifications and approximations of reality. In this paper, we present the models of airline competition along with a brief discussion of the underlying assumptions and the extent of their validity. After presenting the numerical results, we analyze and estimate the direction and magnitude of the impacts of the main assumptions on the results. In section

2, we provide details of our game-theoretic model of frequency competition under slot constraints. In section 3, we describe an efficient algorithm for equilibrium computation. In section 4, we provide empirical and learning-based justifications of the Nash equilibrium outcome. Finally, in section 5, we consider two different slot allocation schemes and evaluate their performance based on multiple criteria. In section 6, we conclude with a summary and discussion of the main results.

2 Model

In this section, we describe the relevant notation and formulate the model. In sub-sections 2.1 and 2.2, we present two important extensions to this model.

We will first formulate the frequency planning problem as an optimization problem from a single airline's point of view. Let us consider an airline a . Consider an airport which is slot constrained, that is, the number of flights arriving at and departing from that airport is restricted by slot availability. A slot available to an airline can be used for a flight to or from any other airport, but the total number of slots available to each airline is limited. In this model, we will consider only the flight arrivals at a slot constrained airport and assume that the departure airports are not slot constrained. This assumption is quite reasonable in the US context, where only a handful of airports are slot constrained. The timing of a slot is also an important aspect of its attractiveness from an airline's point of view. In our model, we focus only on the daily allocation of slots while ignoring the time-of-the-day aspects.

We will calculate airlines' operating profits under the full fare assumption, in which it is assumed that the entire fare of a connecting passenger contributes to the operating profits of each of the segments in the passenger's itinerary. In sub-section 5.5, we will analyze the impact of alternate profit calculation methods on our results. To begin with, we will consider frequency planning decisions while assuming that the aircraft sizes remain constant for each segment. We will analyze the impact of this assumption in sub-section 5.3. We propose a multi-player model of frequency competition where each airline's decision problem is represented as an optimization problem. From here onwards, this model will be referred to as the *basic model*. In this *basic model*, the only decision variables are the numbers of non-stop flights of airline a on each segment with destination at the slot constrained airport. This *basic model* is applicable for situations where the fares and other factors are similar among the competing airlines and the main differentiating factor between different airlines is the service frequency. We will relax this assumption in model extension 1 proposed in sub-section 2.1.

Let \mathbb{S}_a be the set of potential segments with destination at the slot constrained airport. Let p_{as} be the average fare charged by airline a on segment s . Let Q_{as} be the number of passengers carried by airline a on segment s . In general, a passenger might travel on more than one segment to go from his origin to destination, which in some cases involves connecting between flights at an intermediate airport. However, we will assume segment-based demand, that is, a passenger traveling on two different segments will be considered as a part of the demand on each segment. This assumption is quite reasonable for the airports in New York City area where nearly 75% of the passengers are non-stop (BTS, 2010d), but not very accurate for major transfer hubs such as the Chicago O'Hare airport. We will analyze the extent of impact of this assumption in sub-section 5.5. Let the total passenger demand on segment s be M_s . C_{as} is the operating cost per flight for airline a on segment s . f_{as} is the daily number of flights operated by airline a on segment s . S_{as} is the seating capacity of each flight of airline a on segment s .

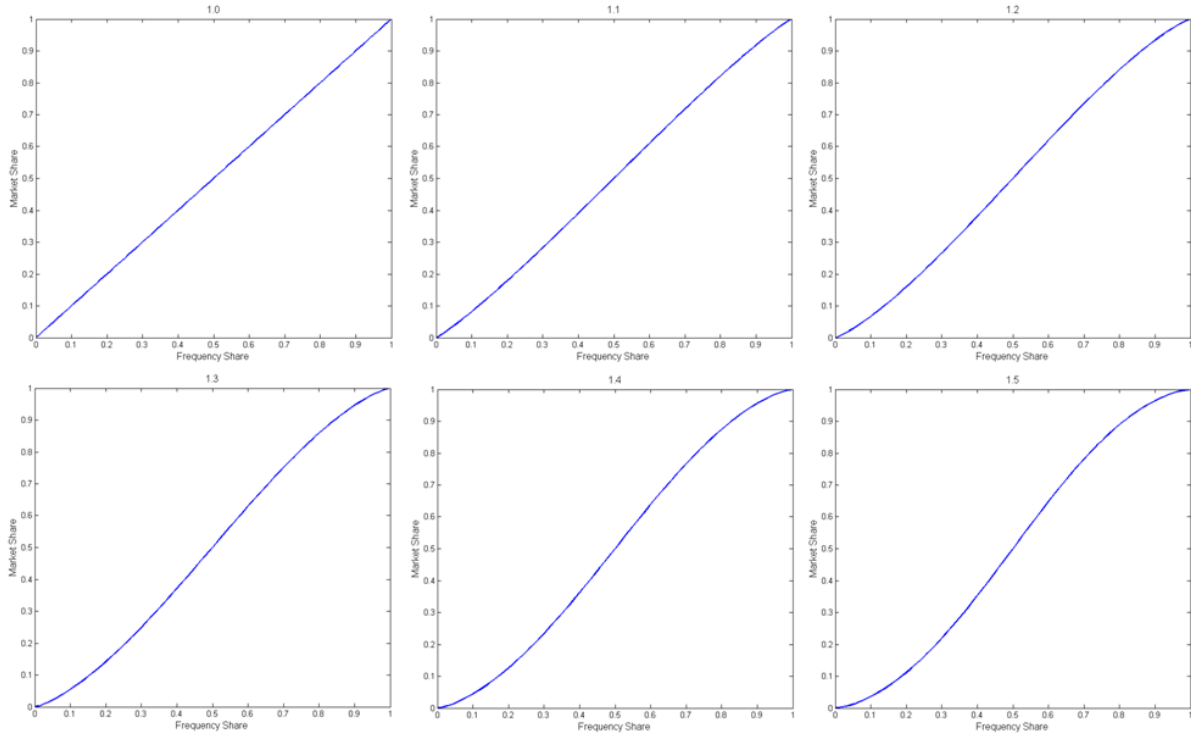


Figure 1: Shapes of S-curve for different values of α_s

Let α_s be the exponent in the S-curve relationship between the market share and the frequency share on the non-stop segment s . The value of α_s depends on the market's characteristics such as long-haul/short-haul, proportion of business/leisure passengers, etc. In short-haul markets and in markets dominated by business passengers, the value of α_s is expected to be higher and in long-haul markets and in markets dominated by leisure passengers, the value of α_s is expected to be lower. Figure 1 shows the shape of the

S-curve for different values of α_s ranging from 1.0 to 1.5. A higher α_s value leads to an S-curve which is farther away from being a straight line. Consequently, for a high α_s value, an airline player with less than 50% share of the frequency would enjoy a substantial gain in market share due to a smaller gain in frequency share. Thus, higher α_s markets provide greater incentive for airlines to add frequency rather than to upgauge their flights.

The vector of decision variables for airline a is $[f_{as}]_{s \in \mathbb{S}_a}$. Because the destination airport is slot constrained, the maximum number of flights that can be scheduled by airline a is restricted to U_a . Often, under the current set of administrative policies based on use-it-or-lose-it type rules, there are restrictions on the minimum number of slots that must be utilized by an airline in order to avoid losing slots for the next year. So there may be a lower limit on the number of slots that must be used. Let L_a be the minimum number of slots that must be utilized by airline a . Let \mathbb{A} be the set of all airlines and let \mathbb{A}_s be the set of airlines operating flights on segment s .

As defined by the S-curve relationship, the market share of airline a on non-stop segment s equals $\frac{f_{as}^{\alpha_s}}{\sum_{a \in \mathbb{A}_s} f_{as}^{\alpha_s}}$, which provides an upper bound on the number of passengers for a specific carrier on a specific segment. This restriction is imposed by constraint (3) in the model that follows. Obviously, the number of passengers on a segment cannot exceed the number of seats. Moreover, due to demand uncertainty and due to the effects of revenue management, the airlines are rarely able to sell all the seats on an aircraft. Assuming a maximum average segment load factor of LF_{max} , the seating capacity restriction is modeled by constraint (4). We present results assuming 85% as the maximum average segment load factor value. In sub-section 5.1, we test the sensitivity of the impacts of different slot allocation schemes to variations in this value. The objective function (2) to be maximized is the total operating profit, which is total fare revenue minus total flight operating cost. We have assumed average fares and deterministic demand. We will analyze the impacts of these two assumptions in sub-section 5.4. Our objective function does not include delay costs because average flight delay depends on the total number of operations at the airport, which is assumed to be a constant. Furthermore, any second-order variations in flight delay costs due to differences in sizes of aircraft used by an airline at the slot constrained airport can also be considered to be negligible. This is explained in sub-section 4.2 in more detail. The overall optimization model is as follows,

$$\text{maximize } \sum_{s \in \mathbb{S}_a} p_{as} Q_{as} - C_{as} f_{as} \quad (2)$$

$$\text{subject to: } Q_{as} \leq \frac{f_{as}^{\alpha_s}}{\sum_{a' \in \mathbb{A}_s} f_{a's}^{\alpha_s}} M_s \quad \forall s \in \mathbb{S}_a \quad (3)$$

$$Q_{as} \leq LF_{max} S_{as} f_{as} \quad \forall s \in \mathbb{S}_a \quad (4)$$

$$\sum_{s \in \mathbb{S}_a} f_{as} \leq U_a \quad (5)$$

$$\sum_{s \in \mathbb{S}_a} f_{as} \geq L_a \quad (6)$$

$$f_{as} \in \mathbb{Z}^+ \quad \forall s \in \mathbb{S}_a \quad (7)$$

The market share available to each airline depends on the frequency of other competing airlines in the same market, which in turn are decision variables of those other airlines. Therefore, this is multi-agent model. The optimization problem given by (2) through (7) can only be solved for a given set of values of competitors' frequencies.

We now propose two extensions to the *basic model*. The first extension is applicable to segments where the competing carriers differ in terms of fare charged or in some other important way. The second extension is applicable to segments on which only one carrier operates non-stop flights.

2.1 Model Extension 1: Fare Differentiation

The *basic model* assumes that the market share on each segment depends solely on the frequency share on that segment. This assumption is reasonable in many markets where the competitor fares are very close to each other and the competing airlines are similar from the perspectives of the passengers in most other ways. However, for markets where the fares are different, the basic S-curve relationship can be a poor approximation of actual market shares. Consider a market where the competing airlines are differentiated in both fare and frequency. Different types of the passengers would react differently to these attributes. While some passengers value lower fares more, others give more importance to higher frequency and the associated greater flexibility in scheduling their travel. In addition, there could be other airline-specific factors that impact the passenger share. For example, some passengers might have a preference for the big legacy carriers operating wide-body or narrow-body fleets over the regional carriers operating turbo-prop aircraft or small regional jets. To incorporate these effects, we propose an extension of inequality (3). Let there be T types of passengers. Let γ_{st} be the fraction of segment s passengers belonging to type t such that $\sum_{t=1}^T \gamma_{st} = 1$. Let α_{st} be the frequency exponent corresponding to the type t passengers for segment s , which serves the same purpose as the exponent α_s of the S-curve in the *basic model*. Let β_{st} be the fare

exponent corresponding to the type t passengers for segment s . Obviously, we expect the passengers to prefer higher frequency at least as much as lower frequency implying that α_{st} is expected to be non-negative. Similarly, we expect passengers to prefer a lower fare at least as much as a higher fare implying that β_{st} is expected to be non-positive. Let θ_a be the airline-specific factor for airline a . Inequality (3) can then be extended as,

$$Q_{as} \leq \sum_{t=1}^T \frac{\theta_a f_{as}^{\alpha_{st}} p_{as}^{\beta_{st}}}{\sum_{a' \in A_s} \theta_{a'} f_{a's}^{\alpha_{st}} p_{a's}^{\beta_{st}}} \gamma_{st} M_s \quad (8)$$

In contrast to the *basic model* of the S-curve in which airlines are differentiated solely based on their flight frequencies, extension 1 introduces additional differentiating features that collectively determine airline market shares. The market share of each airline is now a function of the fares, frequencies, and airline specific factors of all competing airlines. This model incorporates the effects of different fares and frequencies on the passenger shares. Also, it can model multiple passenger types such as leisure vs. business, by specifying different exponents for fare and frequency for different types of passengers. Finally, the remaining airline specific factors are captured through the θ_a parameter.

2.2 Model Extension 2: Market Entry Deterrence

This second extension is similar to the *basic model* except that the player decisions are now sequential rather than simultaneous. The idea of modeling the frequency competition as an extensive form game was proposed by Wei and Hansen (2007) where, for contractual or historical reasons, one airline has the privilege of moving first, i.e., deciding the frequency on a segment. The other airline responds upon observing the action by the first player. The *basic model* and the first extension implicitly assumed the existence of at least two competing airlines on a segment. However, frequency decisions in markets with only one existing airline are not completely immune to competition and the incumbent airline must consider the possibility of entry by another competitor while deciding the optimal frequency. Such situations can be modeled using the idea of Stackelberg equilibrium (von Stackelberg, 1952) or a subgame perfect Nash equilibrium of an extensive form game. In this situation, the incumbent carrier is the Stackelberg leader and the potential entrant is the follower. A potential entrant (a') is assumed to be a rational player. Inequality (3) can be extended as,

$$Q_{as} \leq \frac{f_{as}^{\alpha_s}}{f_{as}^{\alpha_s} + f_{a's}^{\alpha_s}} M_s \quad (9)$$

$$f_{as} = \arg \max_{f \in \mathbb{Z}^+} \left(\min \left(\frac{f^{\alpha_s}}{f_{as}^{\alpha_s} + f^{\alpha_s}} M_s, LF_{max} S_{as} f \right) p_{as} - C_{as} f \right) \quad (10)$$

3 Solution Algorithm

We use the Nash equilibrium solution concept to predict the outcome of this airline frequency competition game. In this section, we describe the solution algorithm used for solving this problem. In section 4, we will provide justification for using the Nash equilibrium outcome.

The objective function for each airline is the sum of profits on each segment and the frequencies of an airline on different segments are interrelated through the constraints on the minimum and maximum number of slots. The effect of competitors' frequencies on the profitability of an airline, as described by the *basic model*, can be fully captured through the notion of effective competitor frequency. Let us define the effective competitor frequency for airline a on segment s as $f_{as}^{eff} = \left(\sum_{a' \in \mathbb{A}_s, a' \neq a} f_{as'}^{\alpha_s} \right)^{\frac{1}{\alpha_s}}$. So constraint (3) in the *basic model* can be more succinctly expressed as $Q_{as} \leq \frac{f_{as}^{\alpha_s}}{f_{as}^{\alpha_s} + f_{as}^{eff \alpha_s}} M_s \quad \forall s \in \mathbb{S}_a$.

In a two-airline market, f_{as}^{eff} for either airline is nothing but the frequency of the other airline in that market. In case of markets with three or more airlines, if there is a dominant competitor, then f_{as}^{eff} tends to be slightly higher than the frequency of the dominant competitor. In such cases, f_{as}^{eff} is highly dependent on the frequency of the dominant competitor. For example, in a market with three competitors with frequencies of 10, 2 and 2 respectively, f_{as}^{eff} equals 11.2 (assuming $\alpha_s = 1.5$). If the frequency of one of the marginal competitors increases from 2 to 3, f_{as}^{eff} changes from 11.2 to 11.6. But if the frequency of the dominant competitor changes from 10 to 11, then the f_{as}^{eff} value changes from 11.2 to 12.1. Furthermore, the dependence of f_{as}^{eff} on the dominant competitor's frequency increases with increasing α_s value. On the other hand, in balanced competitive markets, the dependence of f_{as}^{eff} on the frequencies of all the competitors is comparable.

Figure 2 shows the typical form of the segment profit function under the *basic model* for a fixed value of effective competitor frequency, ignoring slot constraints and integrality constraints. Under the same assumptions, Figure 3 shows the typical shape of the optimal segment frequency (best response) as a function of effective competitor frequency. **Under these assumptions, segment revenue is proportional to market share, which is an S-shaped function of frequency, for a fixed value of effective competitor frequency. Also, segment operating cost is linear in frequency. Therefore, the segment profit function,**

which is the difference between segment revenue and segment operating cost, also has an S-shape. The best response function has three distinct parts. For low values of effective competitor frequency, best response corresponds to operating flights at the maximum possible load factor. For intermediate values of effective competitor frequency, best response is driven more directly by the S-curve and corresponds to operating flights at a load factor less than the maximum. For high values of effective competitor frequency, the optimal strategy is to operate no flights at all, which results in the discontinuity observable in Figure 3. For more details on, and intuition behind, these shapes, the reader is referred to Vaze and Barnhart (2010). The profit function and the best response function get further complicated by slot constraints, integrality constraints, and extensions 1 and 2 to the *basic model*. The optimization problem has discrete variables, and as visible from Figure 2, its continuous relaxation is non-convex. In addition, optimal decisions for each airline depend on the frequency decisions by other airlines. Therefore, the problem of computing an outcome of this multi-agent model can be very challenging. The strategy space for a typical problem size for a major airport is very large, with the number of potential candidates for equilibrium solutions being of the order of 10^{50} . To solve this problem, we propose a heuristic based on the idea of myopic best response, which employs successive optimizations, and individual optimization problems are solved to full optimality using a dynamic programming-based technique. In sub-section 3.1, we describe the myopic best response algorithm and in sub-section 3.2 we describe the dynamic programming formulation for individual optimizations.

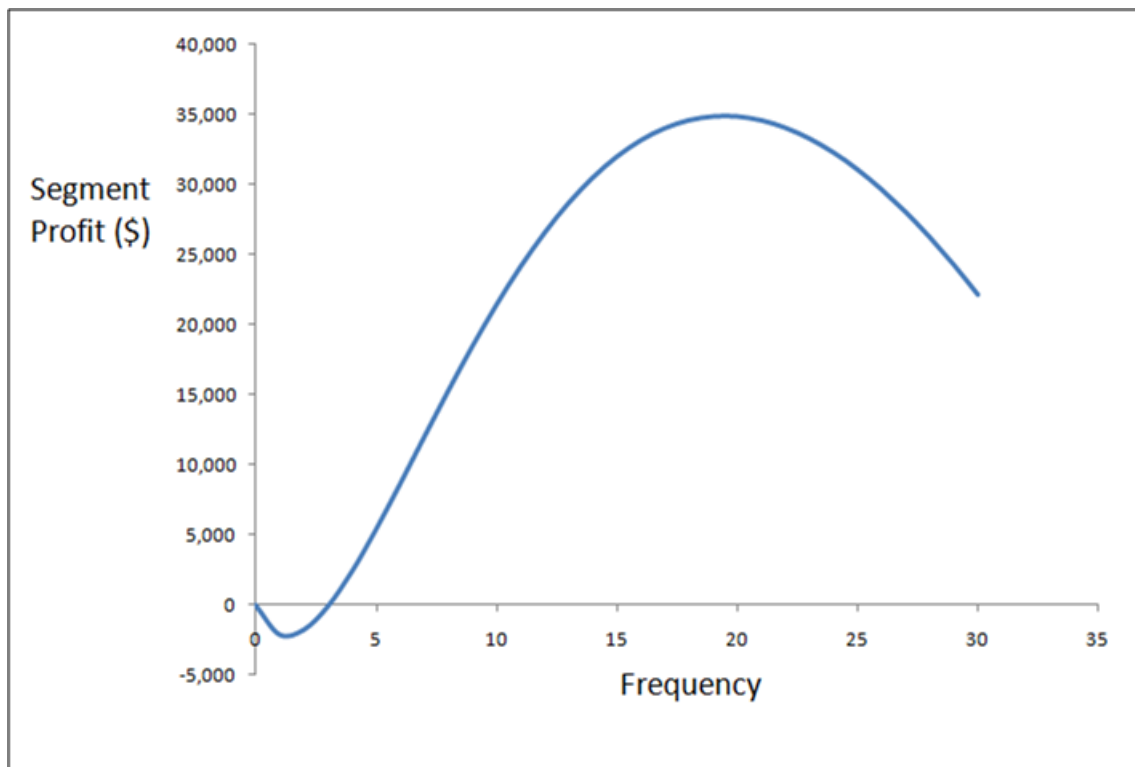


Figure 2: Typical shape of the segment profit function

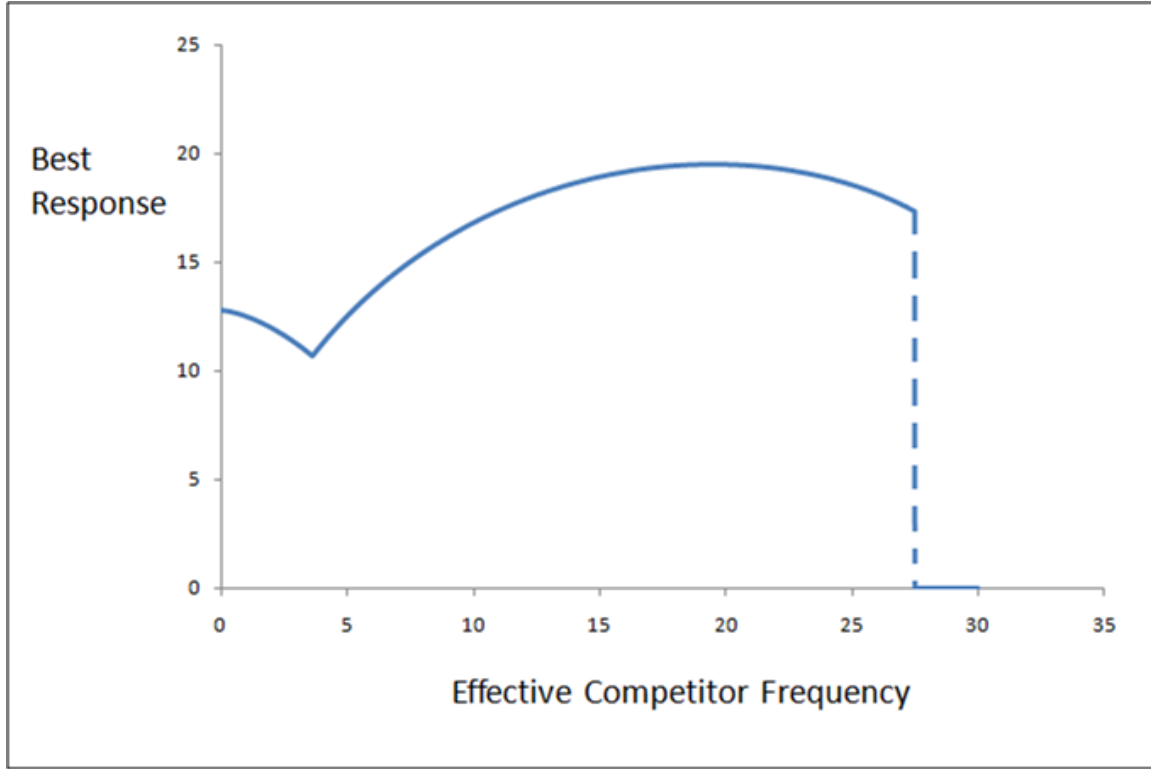


Figure 3: Typical shape of the best response function

3.1 Myopic Best Response Algorithm

Let $f_a = [f_{as}]_{s \in \mathbb{S}_a}$ be the vector of frequencies for carrier a . Let $f_{-a} = [f_{a'}]_{a' \in \mathbb{A}, a' \neq a}$ be the vector formed by concatenating the frequency vectors of all competitors of airline a . So any outcome of this problem can be compactly denoted as $f = (f_a, f_{-a})$. Then the myopic best response algorithm (a heuristic) is described as follows,

```

while there exists a carrier  $a$  for whom  $f_a$  is not a best response to  $f_{-a}$  do
     $f'_a \leftarrow$  some best response by  $a$  to  $f_{-a}$ 
     $f \leftarrow (f'_a, f_{-a})$ 
return

```

This heuristic is based on the idea of myopic best response. Some classes of games have certain desirable properties which make them solvable to equilibrium using an algorithm where each player successively optimizes his own decisions while assuming that the decisions of other players remain constant. Obviously, if such a heuristic converges to some outcome, then it must be a Nash equilibrium. In general, there is no guarantee that it will converge. Further, even if such an algorithm converges to some Nash equilibrium, there is no guarantee that the equilibrium will be unique. We discuss issues regarding its

convergence, and the existence and uniqueness of equilibrium for the game model under consideration, in sub-section 4.3.

3.2 Dynamic Programming Formulation

The main building block of the myopic best response algorithm is the calculation of an optimal response of airline a to the competitors' frequencies. Given the frequencies of all the competing carriers on all the segments, the problem of calculating a best response is an optimization problem. This problem can have a large solution space. For typical problem sizes, the number of discrete solutions in the solution space can be of the order of 10^{10} . As mentioned earlier, this problem is non-convex and discrete. However, this problem has a nice structure. Slot restrictions are the only coupling constraints across different segments and the objective function is additive across segments. Therefore, the problem structure is amenable to solution using dynamic programming.

Let $\Pi_s(f)$ denote the profit from operating f flights on segment s . We order the segments arbitrarily and number them from 1 to $|\mathbb{S}_a|$. Segments are considered in order and each segment corresponds to a stage in dynamic programming. Each state, (s, f) , corresponds to the combination of the last segment being considered, s , and the cumulative number of flights, f , operated on all the segments considered before and including the last segment being considered. Let $R(s, f)$ be the maximum profit that can be obtained from operating a total of f flights on the first s segments. We initialize $R(0,0) = 0$ and $R(0, f) = -\infty$ for $f \geq 1$. For any $s \geq 1$, the Bellman equation is given by,

$$R(s, f) = \max_{0 \leq f' \leq f} (R(s-1, f') + \Pi_s(f - f')).$$

The optimal value of total profit for airline a is given by,

$$\max_{L_a \leq f \leq U_a} R(|\mathbb{S}_a|, f).$$

4 Validity of Nash Equilibrium Outcome

Similar to our work, almost all the previous studies on airline competition have used the concept of Nash equilibrium (or one of its refinements) for predicting the outcome of a competitive situation. The traditional explanation for Nash equilibrium is that it results from introspection and detailed analysis by the players assuming that the rules of the game, the rationality of the players, and the profit functions of players are all common knowledge. A Nash equilibrium outcome is attractive mainly because of the fact that unilateral deviation by any of the players does not yield any additional benefit to that player. So given

an equilibrium outcome, the players do not have any incentive to deviate from the equilibrium decisions. However, in the absence of any apriori knowledge of an equilibrium outcome, given complicated profit functions such as the ones in this case, it isn't immediately clear that airlines would take the equilibrium decisions. In this section, we substantiate the predictive power of the equilibrium outcome using two different approaches, in sub-sections 4.2 and 4.3 respectively, and then verify the robustness of our model's fit to reality in sub-section 4.4. Before presenting any results, we describe the data sources that we used and the process of model calibration in sub-section 4.1.

4.1 Data Sources and Model Calibration

All the numerical results presented in sections 4 and 5 correspond to LaGuardia (LGA) airport as the slot controlled airport. For reasons of computational tractability we decided to restrict our analysis to all the segments of all airlines with destination at the LGA airport. LGA is one of the most congested airports in the US. Furthermore, a very high proportion of non-stop passengers on segments to LGA airport makes it comparatively easier to separate the airlines' decisions at LGA from the rest of the network. We discuss the impacts of passenger connections and network effects on our results in more detail in sub-section 5.5.

Flight schedules for the US domestic segments are available on the Bureau of Transportation Statistics (BTS) website (BTS, 2010c) for each certified US carrier with at least 1% of total domestic passenger revenue. The data on flight frequencies, aircraft sizes and segment passengers are obtained from the T100 Segment Database (BTS, 2010a). Average fare values for each market are obtained from the Airline Origin Destination Survey database (BTS, 2010d). **Operating costs and total airborne hours for each aircraft type for each carrier obtained from the Schedule P-5.2 information (BTS, 2010b) are used as estimates of hourly operating cost for that aircraft type, which in turn are used to calculate the segment-level operating costs based on the average airborne hours for a non-stop flight on each segment.**

Public data on segment passengers and operating expenses is available on a monthly aggregate level, while data on average fares is only available on a quarterly aggregate level. Unfortunately, more disaggregate values of these entities, such as on a daily level, are not available publicly. Also the daily values often tend to fluctuate due to various types of cyclical variations. In order to avoid biases in our model estimates because of choice of certain days over others, and also to circumvent the data unavailability issue, we ran our experiments on quarterly average values. All results in sections 4 (except sub-section 4.4) and 5 are for the first quarter of 2008. In sub-section 4.4, we verify the robustness of our model's fit to reality by running our model for the 2nd, 3rd and 4th quarters of 2008.

In order to estimate the flight delay reduction for experiment 2 presented in section 5, we use realized values of airport capacity for an entire year, which were made available by Metron Aviation®, and actual flight delay data obtained from the Airline On-time Performance Database available on the BTS website (BTS 2010c). Details of the delay reduction estimation procedure are described in sub-section 5.1.

Our dataset consists of all segment-carrier combinations with destination at LGA operating at least one flight per day on average. Thus our dataset encompasses 96.11% of all the flights destined for LGA. For all segments where only one carrier provides non-stop service, we use the market share function given by model extension 2. We use the market share function given by model extension 1 for segments on which: 1) the competitors' average fares differ by more than 5%; and/or 2) one or more major carriers operating a narrow- or a wide-body fleet compete against one or more regional carriers operating small jets. For all the other segments, we use the market share function given by inequality (3) in the *basic model*.

We conducted several test runs to choose the parameters such that the frequency estimates given by the model match the actual frequency values closely. The final set of parameters was chosen based on the results from these test runs as well as practical insights based on our experience, intuition and prior literature. The exact value of α_s for a particular market can be expected to depend on the importance of frequency in that market (Belobaba, 2009a). For the *basic model*, we used different α_s values for different markets. In general, flights into LGA tend to be short-haul flights. In our dataset, there was not a single flight from the west-coast airports. The range of α_s values that we used varied from 1.5 for very short-haul markets to 1.2 for the comparatively long-haul markets. We used higher α_s values for short-haul, business-intensive markets such as Washington DC, Boston MA, etc., and comparatively lower values for long-haul, leisure-intensive markets such as Orlando FL and Miami FL.

For model extension 1, we considered 2 types of passengers: 1) business passengers, and 2) non-business passengers, i.e. $T = 2$. The magnitudes of α_{st} values are expected to be high for business passengers because business passengers tend to give particularly high importance to more frequent flights. The magnitudes of β_{st} values are expected to be high for non-business passengers because non-business passengers are typically more sensitive to fares. We used the following values for the exponents in model extension 1: $\alpha_{s1} = 1.3$, $\beta_{s1} = -0.5$, $\alpha_{s2} = 0.3$, and $\beta_{s2} = -1.2$. The value of airline specific factor (θ_a) can be expected to be higher for airlines which have a bigger brand name and a better track record. The value of θ_a was taken to be 0.3 for all regional carriers operating turbo-props or small regional jets, and 1.0 for all other carriers. In general, the γ_{st} values depend on the business/leisure composition of specific markets. However, for simplicity, the fraction of passengers belonging to type 1 (business passengers) was taken to be $\gamma_{s1} = 0.3$, and hence, $\gamma_{s2} = 0.7$, for all markets for which model extension

1 was used. Given that in most cases, the average fares of competing airlines on each segment into LGA were very close to each other, we assumed the average fare (f_{as}) of the potential entrant in model extension 2 to be the same as the fare charged by the existing operator on that segment. The seating capacity (S_{as}) and the operating cost (C_{as}) of the potential entrant were taken to be those corresponding to the most profitable combination (decided by the minimum ratio of operating cost to seating capacity) available across all the fleet types operated by all airlines into LGA.

Because we ascertained the values of the model parameters using a heuristic process, it is very important to investigate the sensitivity of our results to changes in these parameter values. The results of the sensitivity analyses to various model parameter values are presented in sub-section 5.2. Additionally, in order to avoid over-fitting the model parameters to a certain dataset, we used the same model parameters to compare the error between the model's frequency predictions and the actual frequency values for the 2nd, 3rd, and 4th quarters of 2008. These results are presented in sub-section 4.4.

4.2 Empirical Validation

To validate our model against actual frequency data, we compared the equilibrium frequencies predicted by the model against the actual values. At LGA, the maximum number of slots for each airline is restricted and each airline usually wants to make use of all the slots available to it in order to avoid losing any slots in subsequent seasons. The minimum and maximum numbers of slots available to an airline, that is, L_a and U_a , are assumed to be equal. So the total number of slots allocated to each airline is fixed. The airline needs only to decide the number of slots to allocate to flights from each of its origin airports. As a result, average flight delay is independent of an individual airline's frequency decisions. Also, the fleet of aircraft operated by each airline at LGA is fairly homogeneous; the coefficient of variation (the ratio of standard deviation to mean) of aircraft seating capacities is between 0.00 and 0.13 across all the airlines in our data, with an average of 0.06. As a result, an airline's total delay cost is almost entirely unaffected by the variables controllable by that airline. This is the reason why we did not explicitly include delay costs in our model's objective function. Let f_{as} be the actual frequency of airline a on segment s and \widehat{f}_{as} be the equilibrium frequency as predicted by the model. The model ensures that the total frequency for each airline remains constant. Therefore, when the model overestimates the frequency on one segment it necessarily underestimates the frequency on some other segment corresponding to the same carrier. In order to measure the model's fit to reality, we will use Mean Absolute Percentage Error (MAPE) defined as,

$$\text{MAPE} = \frac{\sum_{a \in \mathbb{A}} \sum_{s \in \mathbb{S}_a} |\widehat{f}_{as} - f_{as}|}{\sum_{a \in \mathbb{A}} \sum_{s \in \mathbb{S}_a} f_{as}}$$

Figure 4 compares the actual frequency (on x-axis) and the frequency predicted by the model (on y-axis) for each carrier from each origin. The weight of each point in this figure indicates the number of observations corresponding to that point. As shown in the Figure 4, most of the observations are on or very close to the 45° line. The overall MAPE was found to be 14.72%. The model predictions thus match actual frequencies reasonably well.

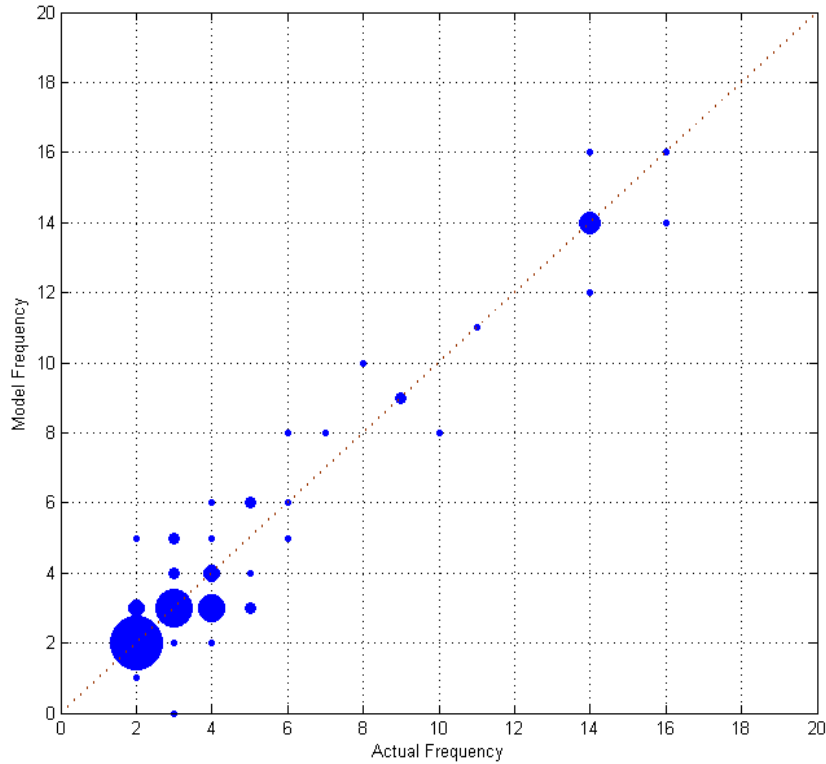


Figure 4: Empirical validation of frequency predictions

4.3 Game Dynamics

Airlines typically operate flights on similar sets of segments year after year. The group of competitors on each segment and the general properties of markets stay constant over long periods of time. Therefore, the airlines have opportunities to adapt their decisions primarily by fine-tuning the frequency values to optimize their profits. Such adjustments can be captured by modeling the dynamics of the game. In a previous paper, Vaze and Barnhart (2010) used a simplified version of the frequency competition model used in this paper and proved the convergence of best response dynamics in the two-player case without slot constraints. The key factor responsible for convergence of the myopic best response algorithm was the flat shape of the best response function near equilibrium. In other words, the magnitude of the derivative of the optimal frequency with respect to the effective competitor frequency is very small. Therefore, the best response for a large range of competitor frequency values is very close to the

equilibrium frequency, resulting in strong convergence properties of the best response dynamics. The *basic model* of frequency competition used in this research is the same as the model used by Vaze and Barnhart (2010), except for the addition of slot constraints and integrality constraints. Though their convergence results are not directly applicable to this complicated model, they provide some intuition.

In this paper, we have used the best response algorithm for computation of an equilibrium. For the results of empirical validation presented in the previous sub-section, we used the vector of actual frequency values as the starting point of the best response algorithm and the algorithm converged to an equilibrium in just 2 iterations per player (per airline). Let us term this equilibrium solution as the *base equilibrium*. In this sub-section, we present the impact of variation in the starting point on the computed equilibrium prediction.

For each starting point, the algorithm was run for at most 10 iterations per player. In most of the following cases, the algorithm converged to an equilibrium and terminated in fewer than 10 iterations. However, in the few cases that the algorithm did not converge within 10 iterations, it was terminated after 10 iterations. Starting from the actual frequency values, we perturbed each dimension of the frequency vector uniformly between $-x\%$ to $+x\%$ of the original value. For each x value, we drew 1000 samples of starting points randomly from this uniform distribution. Values presented in Table 1 are the average MAPE values across the 1000 runs obtained by comparing the solution computed by the best response algorithm to the actual frequencies as well as to the *base equilibrium*. These results indicate that the model predictions are quite insensitive even to large perturbations in the starting point. The algorithm converges to, or comes very close to, the equilibrium solution within very few iterations, irrespective of the starting point. This also suggests that the best response dynamics displays good convergence properties. Therefore, even assuming less than perfectly rational players, an equilibrium outcome can be reached through a simple myopic learning procedure. Empirical validation results, coupled with these desirable convergence properties, make a strong case for using the Nash equilibrium solution concept for predicting the outcome of this airline frequency competition game.

Maximum Perturbation	MAPE Compared to Actual Frequencies	MAPE Compared to <i>Base Equilibrium</i>
0%	14.72%	0.00%
10%	14.72%	0.00%
20%	14.78%	0.06%
30%	15.04%	0.34%
40%	15.25%	0.56%

50%	15.65%	0.96%
60%	15.81%	1.72%
70%	16.14%	2.50%
80%	16.22%	2.95%
90%	16.43%	3.59%
100%	16.76%	4.27%

Table 1: Stability of algorithm results to starting point perturbations

4.4 Robustness Verification

In sub-sections 4.2 and 4.3, we presented empirical results using data from the 1st quarter of 2008. In this sub-section, we will verify the robustness of our models by validating them against empirical data from the 2nd, 3rd, and 4th quarters of 2008.

Table 2 presents the discrepancy between the frequency values predicted by the model and the actual frequency decisions taken by the airlines using the MAPE measure of error for each quarter of 2008. Table 2 shows that the error in frequency predictions is very stable across different time periods. Table 3 presents the stability properties of the algorithm across the four quarters of 2008. We find that the performance is stable across different quarters, which means that the model predictions are robust to large perturbations to the starting point for different time periods.

Quarter	Mean Absolute Percentage Error
1	14.72%
2	16.12%
3	15.67%
4	14.38%

Table 2: Model prediction error across different quarters

Maximum Perturbation	MAPE Compared to Actual Frequencies				MAPE Compared to <i>Base Equilibrium</i>			
	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4
0%	14.72%	16.12%	15.67%	14.38%	0.00%	0.00%	0.00%	0.00%
10%	14.72%	16.12%	15.67%	14.38%	0.00%	0.00%	0.00%	0.00%
20%	14.78%	16.12%	16.05%	14.45%	0.06%	0.20%	0.73%	0.13%
30%	15.04%	16.12%	16.63%	14.57%	0.34%	0.22%	1.90%	0.40%

40%	15.25%	16.12%	17.07%	14.99%	0.56%	0.23%	2.71%	1.28%
50%	15.65%	16.12%	17.64%	15.35%	0.96%	0.34%	3.91%	2.06%
60%	15.81%	16.19%	18.05%	16.32%	1.72%	0.80%	4.91%	4.99%
70%	16.14%	16.28%	18.53%	16.85%	2.50%	1.53%	6.04%	6.55%
80%	16.22%	16.36%	18.84%	17.23%	2.95%	2.29%	6.71%	7.51%
90%	16.43%	16.72%	19.04%	17.88%	3.59%	3.40%	7.44%	8.77%
100%	16.76%	16.90%	19.47%	18.32%	4.27%	4.05%	8.33%	9.58%

Table 3: Stability of algorithm results to starting point perturbations across different quarters

Table 4 presents the MAPE with respect to actual frequencies as the values of key model parameters vary between -25% to +25% of the values listed in sub-section 4.1. The first row lists the changes in MAPE with different percentage changes in α_s values for *all* segments. The next six rows list the variation in MAPE with changes in parameters of model extension 1. The remaining three rows list the variation in MAPE with operating cost, seating capacity and average fare of the potential entrant in model extension 2. As shown in Table 4, the MAPE values vary between 13.5% and 19.6% and are reasonably stable to significant variations in model parameters. Additionally, in sub-section 5.2 we present results on sensitivity of the impacts of slot reduction to variations in these parameters.

Parameter	-25%	-20%	-15%	-10%	-5%	0%	5%	10%	15%	20%	25%
α_s	17.2%	16.0%	16.0%	15.3%	15.3%	14.7%	17.2%	17.8%	19.0%	19.0%	19.6%
θ_a	17.8%	17.8%	16.6%	15.3%	15.3%	14.7%	17.8%	17.8%	17.8%	17.8%	18.4%
α_{s1}	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	15.3%	15.3%	17.8%	17.8%	16.6%
β_{s1}	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	15.3%	15.3%	15.3%	15.3%	15.3%
γ_{s1}	15.3%	15.3%	15.3%	15.3%	14.7%	14.7%	15.3%	16.0%	17.8%	17.8%	17.8%
α_{s2}	15.3%	15.3%	15.3%	15.3%	15.3%	14.7%	15.3%	15.3%	18.4%	18.4%	16.0%
β_{s2}	17.8%	17.8%	17.8%	17.8%	14.7%	14.7%	15.3%	15.3%	16.0%	16.0%	16.0%
$C_{a/s}$	19.6%	16.6%	17.8%	14.7%	13.5%	14.7%	14.1%	13.5%	14.1%	14.1%	14.1%
$S_{a/s}$	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%	14.7%
$p_{a/s}$	14.1%	14.1%	14.1%	14.1%	14.1%	14.7%	13.5%	14.7%	17.8%	17.2%	16.6%

Table 4: Sensitivity of prediction accuracy (in MAPE) to model parameters

5 Evaluation of Simple Slot Reduction Strategies

In this section, we propose two different strategies for allocating the available slots among different airlines and evaluate the performance of each strategy under the Nash equilibrium modeling framework.

Proportionate Allocation Scheme: Under the existing administrative controls, airlines often receive a similar number of slots from year to year. Historical precedent is usually used as the main criterion for slot allocation. There is opposition from the established carriers to any significant redistribution of slots. In the spirit of maintaining much of the status quo, our first slot distribution strategy involves proportionate allocation of slots. We vary the total number of slots at an airport while always distributing them among different carriers in the same ratio as that of actual flight schedules. For example, if the total number of slots at an airport is reduced from 100 to 80 and if the 100 slots were distributed as 40 and 60 between two carriers, then under our proportionate allocation scheme, the 80 slots will be distributed as 32 and 48 between the same two carriers.

Reward-based Allocation Scheme: While the proportionate allocation scheme is likely to be considered more acceptable by major carriers, it ignores the level of efficiency with which an airline utilizes its slots. Airlines differ, often substantially, in the number of passengers carried per flight or per slot. The idea behind the reward-based allocation is to reward those airlines which carry more passengers per slot, due to larger planes and/or higher load factors, and penalize those who carry fewer passengers per slot. Under this scheme, the number of slots allocated to each airline is proportional to the total number of passengers carried by that airline. In the previous example, if the first airline currently carries 140 passengers per slot and the second airline currently carries 120 passengers per slot, then under our reward-based allocation scheme, when the total number of slots is reduced to 80, the first airline will receive 35 slots and the second airline will receive 45 slots.

Next we present results of the impacts of slot reduction assuming that these strategies are implemented as administrative slot controls. Alternatively, these slot allocations can also be results of some more complicated demand management strategy, such as a market mechanism. If the market mechanism involves monetary payments, the resulting airline profits will have to be adjusted to account for the payments. In sub-section 5.1, we present the impacts of slot reduction on various important metrics using the models presented in section 2. Subsequently, in sub-sections 5.2 through 5.5, we test the sensitivity of our results to the various parameters and assumptions underlying our models.

5.1 Numerical Results

We conducted the following two experiments. In the first experiment, we varied the total number of allocated slots at LGA and studied the impact on two important metrics, namely, the total operating profits of all the airlines and the total number of passengers carried. Figures 5 and 6 show the changes in total operating profits of all the airlines with slot reductions under the proportionate and reward-based allocation schemes, respectively. Figures 7 and 8 show the change in the total number of passengers carried, assuming that the aircraft type (and seating capacity) for each airline on each segment remains unchanged upon slot reduction. The total number of passengers carried decreases as the number of slots decreases, but at a much lower rate. For the proportionate allocation scheme, up to a 30% slot reduction, each 1% reduction in slots leads to, on average, just a 0.27% reduction in the total passengers. A 30% reduction in slots leads to approximately 8% reduction in total passengers. Beyond 30%, the rate of decrease in passengers nearly quadruples, with each 1% reduction in slots leading to about 1.12% reduction in total passengers. Also, the total operating profit for the proportionate allocation scheme increases with increasing slot reduction percentage, up to 30% slot reduction. Beyond that point, the operating profit starts to decrease. Very similar patterns are observed for the reward-based allocation scheme. Up to a 40% reduction in slots, each 1% reduction in slots leads to, on average, just a 0.25% reduction in the total passengers. A 40% slot reduction results in about 10.2% reduction in total passengers. However, beyond that point, each 1% reduction in slots results in about 0.86% reduction in total passengers. Similarly, total operating profit increases up to a 40% reduction and decreases thereafter.

These effects are easy to understand intuitively. Given that aircraft sizes remain constant, the initial reduction in the number of slots results primarily in increases in load factors and hence, under our constant fares assumption, operating costs decrease at a faster rate than the rate of decrease in total revenue. So the operating profit increases. This effect continues until a point where the aircraft size constraint becomes binding and reduces the number of passengers almost proportionally to the number of slots. Therefore the operating revenue decreases at almost the same rate as the operating cost decrease, causing the operating profit to decrease.

Also, the rate of change in total profits and in passengers carried is comparable for both the proportionate and the reward-based strategies at small levels of slot reduction. However, at higher slot reduction percentages, the decrease in total passengers carried is smaller for the reward-based strategy, which makes sense given that the reward-based strategy allocates a greater proportion of slots to carriers who carry more passengers per slot. As a result, the increase in total profits is also greater for the reward-based strategy at higher slot reduction percentages.

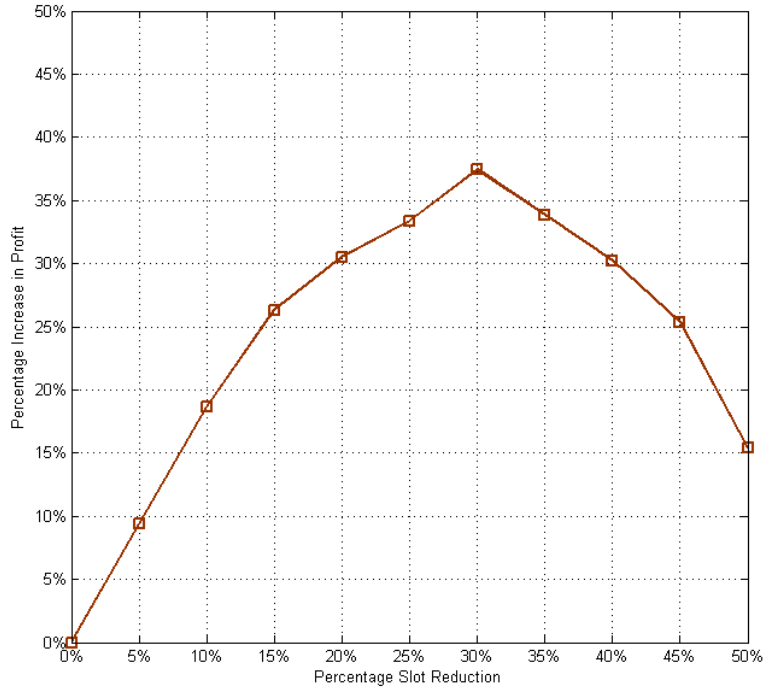


Figure 5: Total operating profit as a function of slot reductions under a proportionate allocation scheme assuming constant aircraft sizes

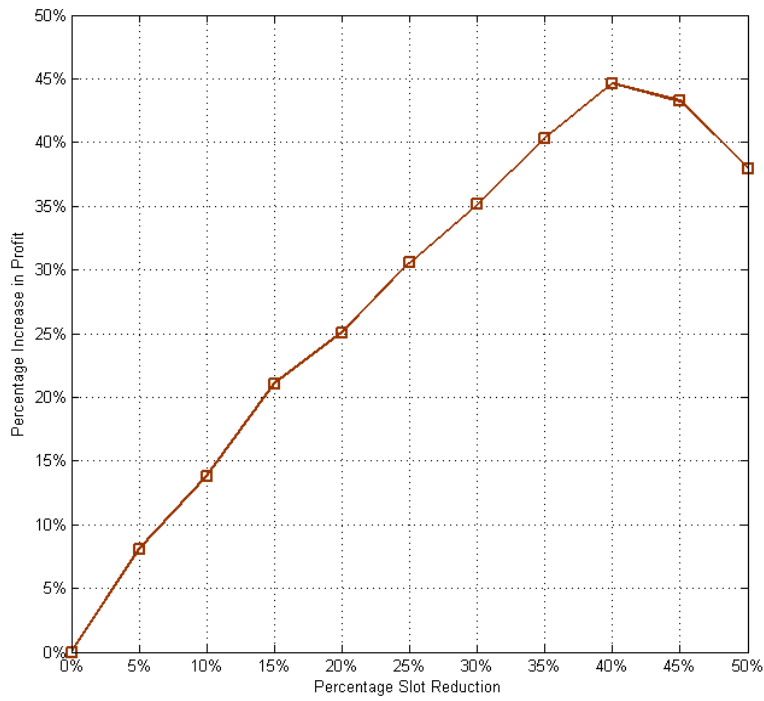


Figure 6: Total operating profit as a function of slot reductions under a reward-based allocation scheme assuming constant aircraft sizes

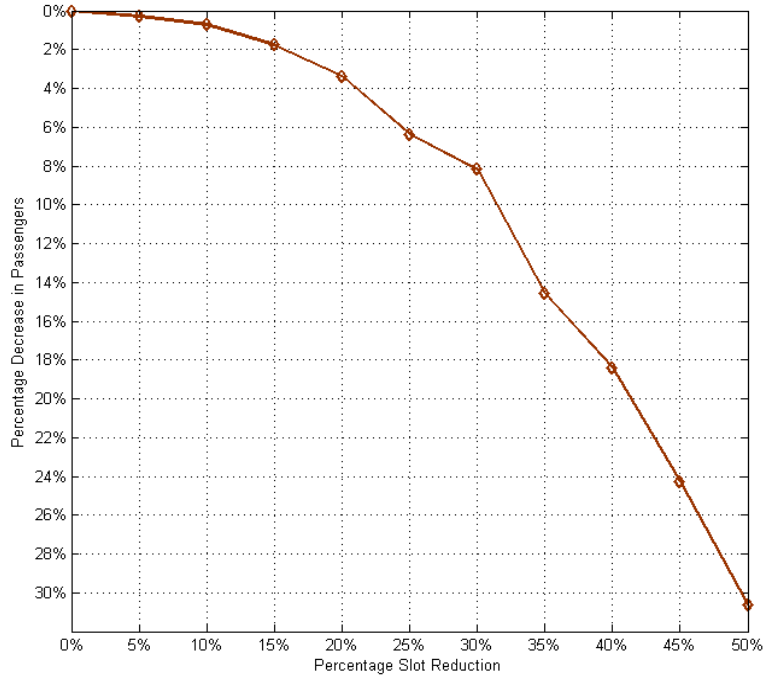


Figure 7: Total number of passengers carried as a function of slot reductions under a proportionate allocation scheme assuming constant aircraft sizes

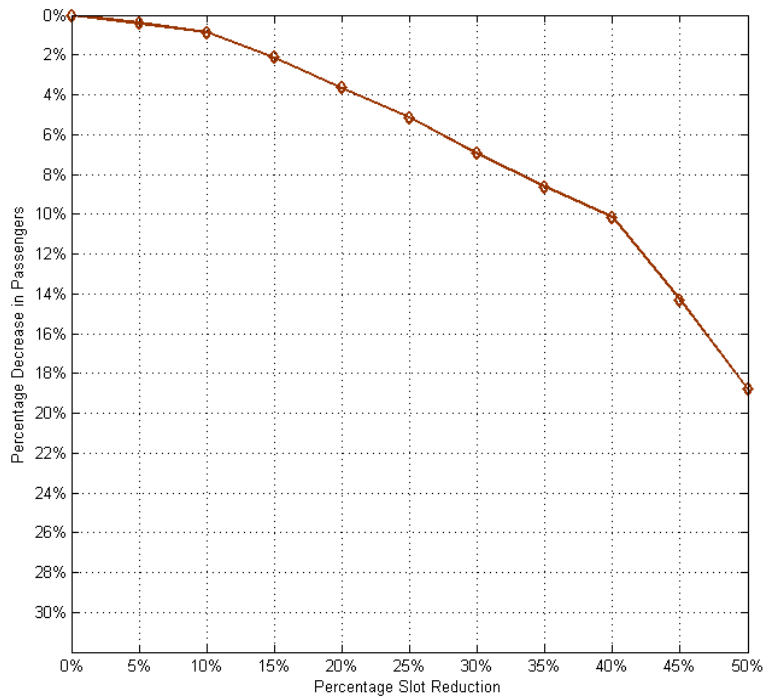


Figure 8: Total number of passengers carried as a function of slot reductions under a reward-based allocation scheme assuming constant aircraft sizes

In our second experiment, we fixed a particular level of slot reduction and evaluated its system-wide impacts on the airlines (both individually and as a group), and on the passengers, based on multiple metrics. We considered the impact on the following metrics: airline operating profits, average flight delays, average passenger delays, total number of passengers carried, and average schedule displacement for passengers. The airport capacity benchmark report published by the Federal Aviation Administration (FAA, 2004), sets the IMC capacity of LaGuardia airport at approximately 87.7% of its VMC capacity. Currently, the number of operations scheduled at LaGuardia is close to the VMC capacity. We chose to evaluate the case of a 12.3% reduction in slots, which approximately corresponds to scheduling at IMC capacity instead of at VMC capacity. This policy is very similar to that currently followed at many major European airports.

Next, we describe the procedures used to estimate the average flight delays, the average passenger delays and the average schedule displacement. In order to estimate the impact on the average flight delays, we used the information on ground delay programs (GDPs) for an entire year (made available from Metron Aviation®) and actual flight delay data (obtained from the airline on-time performance database available on the BTS website (BTS, 2010c)). All the delay computations were performed only for the NAS delay component of flight delays. The number of operations scheduled at LaGuardia, was close to its VMC capacity for the 1 year time period for which we have the data. We assumed that the realized capacity equaled the IMC capacity during the period when a GDP was implemented at LGA, and it equaled the VMC capacity otherwise. We calculated the average NAS delays to flights landing at LaGuardia for both GDP and non-GDP periods for the entire year. Thus, the calculated average NAS delay during non-GDP periods is the average NAS delay when capacity is at the VMC level and scheduled demand is also at the VMC level. Let us denote it by $d_{non-GDP}$. Similarly, the calculated average NAS delay during GDP periods is the average NAS delay when capacity is at the IMC level and scheduled demand is at the VMC level. Let us denote it by d_{GDP} . We calculated the overall average NAS delay as the expected value of delays under VMC and IMC capacities for the case where the scheduled demand is at the VMC level, i.e., the case without slot reduction. If we use t_{GDP} to denote the fraction of time for which a GDP was in place at LGA, then the overall average NAS delay for the case without slot reduction equals $d_{GDP} * t_{GDP} + d_{non-GDP} * (1 - t_{GDP})$.

We assumed that the average NAS delays under IMC capacity with 12.3% slot reduction equal the average NAS delays under the VMC capacity without slot reduction. With the 12.3% slot reduction, the average NAS delays under VMC capacity will be lower than those under VMC capacity without slot reduction. However, in order to be conservative in our delay reduction estimates, we assumed that the average NAS delays under VMC capacity remain unchanged with a 12.3% slot reduction. In other words,

the calculated average NAS delay during non-GDP periods was used as an estimate for the average NAS delay when scheduled demand is at the IMC level regardless of whether the capacity is at the VMC level or at the IMC level. Thus, the overall average NAS delay for the case with a 12.3% slot reduction simply equals $d_{non-GDP}$ as per our conservative estimation methodology.

In addition to flight delays, passenger itinerary disruptions due to flight cancellations and missed connections are responsible for a significant component of passenger delays. Barnhart, Fearing and Vaze (2010) estimated the ratio of average passenger delay to average flight delay in the domestic US to be 1.97. We used this representative value for computing the average passenger delays from the average flight delays.

The total trip time for the passengers is also affected by what is known as schedule displacement (Belobaba, 2009a) or schedule delay. Schedule displacement is a measure of the difference between the time when a passenger wishes to travel and the actual time when he/she can travel given a flight schedule. The higher the daily frequency of flights, the lower is the schedule displacement. Due to slot reduction, the flight frequency on some segments is expected to reduce, which affects schedule displacement adversely. Schedule displacement is expressed as $\frac{K}{F}$, where F is the flight frequency and K is a constant which depends on the distribution of flight departure times and the distribution of desired times when passengers wish to travel. In this research, we assume both these distributions to be uniform. Let T be the duration of time over which the frequency F is distributed. Under the uniform distribution assumption for flights, if we divide the time T into F intervals of equal size, then there will be one flight scheduled at the midpoint of each interval. So the schedule displacement for all the passengers with desired departure times in that interval will vary uniformly between 0 and $\frac{1}{2} \frac{T}{F}$, with an average value of $\frac{T}{4F}$. Therefore, K equals $\frac{T}{4}$. We assume T to be 16 hours because 97% of all the arrivals at LGA are concentrated in the 16 hour time duration from 8:00 am to midnight (BTS, 2010c).

Table 5 summarizes the impacts of slot reduction to airlines and passengers based on various metrics. Values in the 3rd and 4th columns correspond to those under the actual frequencies and the *base equilibrium* before slot reduction, respectively. **The base equilibrium before slot reduction will be referred to as the no reduction equilibrium.** The results in the 5th and 6th columns correspond to a 12.3% reduction in slots for proportionate and reward-based allocation schemes respectively. The values in parentheses indicate the percentage change in each metric with respect to the *no reduction equilibrium*. The level of congestion depends on the total number of slots and not on the distribution of these slots among different airlines. Therefore, the delay reduction is the same under both proportionate and reward-based slot

allocation schemes. Under either strategy, slot reductions lead to substantial reductions in flight delays as well as passenger delays. The total operating profits across all carriers increase substantially. There is a small reduction in the total passengers carried. However, this is partly because we have assumed that aircraft sizes on each segment for each airline remain unchanged upon slot reduction. We will investigate the impact of relaxing this restriction in sub-section 5.3. The average schedule displacement increases by just over 2 minutes. The total travel time for passengers arriving at LGA includes not only the schedule displacement and the duration of the flight into LGA, but also the airport access and egress times, and in cases of connecting passengers, the layover times and duration of the first flight in their itineraries. For flights into LGA airport, the average flight duration itself is 185.38 minutes. Therefore, in comparison, the increase in schedule displacement is negligibly small.

A potential negative impact of slot reduction is the loss of non-stop service on segments corresponding to some of the smaller markets. The results from our second experiment showed that a 12.3% proportionate reduction in slots did not lead to loss of non-stop service on any of the 38 segments included in our data, while a 12.3% reward-based reduction in slots resulted in loss of non-stop service on 1 out of the 38 segments, a segment which corresponded to less than 0.3% of all the passengers traveling to LGA. Of course, a loss of non-stop service on a segment does not rule out the possibility that some of the passengers traveling on that segment simply shift to a one-stop route to LGA. Thus, we found that the impact of slot reduction in terms of loss of service to small markets was small.

Stakeholder	Metrics	Actual Frequencies	<i>No Reduction Equilibrium</i>	12.3% Reduction	
				Proportionate	Reward-based
Airline	Total Operating Profits (Excluding Flight Delay Costs)	\$1,228,749	\$1,281,663	\$1,550,565 (20.98%)	\$1,501,100 (17.12%)
	NAS Delay per Flight	12.74 min	12.74 min	7.52 min (- 40.97%)	7.52 min (- 40.97%)
Passengers	Total Passengers Carried	22,896	22,965	22,678 (-1.25%)	22,661 (-1.32%)
	Average Passenger Delay (due to NAS Delays)	25.10 min	25.10 min	14.81 min (- 40.97%)	14.81 min (- 40.97%)
	Average Schedule Displacement	24.23 min	25.05 min	27.68 min (10.50%)	27.32 min (9.06%)

Table 5: Effect of a 12.3% slot reduction on system-wide performance metrics

Table 6 presents the distribution of operating profits across different carriers. Profits of all carriers that account for at least 1% of the operations at LGA are included. The impacts of a 12.3% slot reduction on the remaining carriers are negligible because their slots do not get reduced under a 12.3% reduction due to integral rounding. The 2nd, 3rd, 4th and 5th columns correspond to actual frequencies, *no reduction equilibrium*, 12.3% reduction under proportionate allocation scheme, and 12.3% reduction under reward-based allocation scheme respectively. Again, the values in parentheses represent the percentage increases in profits compared with the *no reduction equilibrium*. When the total number of slots is reduced under either allocation scheme, the operating profit of each carrier is strictly greater compared to that under the *no reduction equilibrium*. The relative increase in operating profits is largest for the regional carriers operating small regional jets into LGA. This is primarily because they had very low operating profit margins at LGA under the *no reduction equilibrium*. In fact, for one of regional carriers, the slot reduction under reward-based allocation helps achieve an operating profit instead of an operating loss, which is the case under the *no reduction equilibrium*. On the other hand, the network legacy carriers achieve the maximum absolute increase in operating profit per carrier. This is primarily because the average number of slots per day for network legacy carriers (34.50) itself is nearly 27% higher than that for the remaining carriers (27.25), and the average operating profits for the network legacy carriers per day (\$192,276) are much higher than that for the remaining carriers (\$23,037) under the *no reduction equilibrium*. If we compare against the actual frequencies case, then the profit for all but one carrier increases after slot reduction under both allocation schemes.

Carrier	Actual Frequencies	<i>No Reduction Equilibrium</i>	12.3% Reduction	
			Proportionate	Reward-based
Network Legacy Carrier 1	\$349,363	\$390,735	\$452,701 (15.86%)	\$423,466 (8.38%)
Low Cost Carrier 1	\$47,090	\$26,684	\$34,487 (29.24%)	\$31,887 (19.50%)
Network Legacy Carrier 2	\$49,693	\$71,922	\$79,550 (10.61%)	\$75,823 (5.42%)
Network Legacy Carrier 3	\$202,489	\$206,315	\$299,385 (45.11%)	\$282,772 (37.06%)
Low Cost Carrier 2	\$54,000	\$59,927	\$79,766 (33.11%)	\$73,541 (22.72%)
Regional Carrier 1	\$29,836	\$34,461	\$39,480 (14.56%)	\$39,473 (14.54%)
Network Legacy Carrier 4	\$91,772	\$85,708	\$113,736 (32.70%)	\$106,451 (24.20%)
Regional Carrier 2	-\$28,493	-\$28,923	-\$2,227 (n.a.)	\$8,513 (n.a.)
Network Legacy Carrier 5	\$200,796	\$200,796	\$201,786 (0.49%)	\$208,280 (3.73%)
Network Legacy Carrier 6	\$196,346	\$198,180	\$216,043 (9.01%)	\$214,913 (8.44%)
Total for Network Legacy	\$1,090,459	\$1,153,656	\$1,363,201	\$1,311,705

Carriers			(18.16%)	(13.70%)
Total for Low Cost Carriers	\$101,090	\$86,611	\$114,253 (31.92%)	\$105,428 (21.73%)
Total for Regional Carriers	\$1,343	\$5,539	\$37,254 (572.61%)	\$47,986 (766.39%)

Table 6: Increase in operating profits due to a 12.3% slot reduction

These results in Tables 5 and 6 are obtained assuming a maximum average segment load factor (LF_{max}) of 85%. Now, we will present results on the sensitivity of slot reduction impacts to this assumption. We will focus on the sensitivity of the results of the second experiment. Table 7 describes the sensitivity of total profits and total number of passengers carried to variations in the maximum average segment load factor value. Obviously, the average flight delays, average passenger delays and average schedule displacements do not change, because they depend only on the scheduled number of flight operations. The increase in total operating profit varies between 16.56% and 21.99% and the decrease in number of passengers varies between 0.49% and 3.71%. With increases in the maximum average segment load factor, we observe a general trend towards smaller reductions in total passengers due to slot reduction, which is intuitively reasonable. Consequently, we also observe a general trend towards greater increases in total profits with increases in the maximum average segment load factor. Due to the integrality constraints on the number of slots, the results don't vary smoothly in some cases.

Maximum Average Segment Load Factor	Increase in Total Profits		Decrease in Total Passengers Carried	
	Proportionate	Reward-based	Proportionate	Reward-based
75%	20.25%	16.65%	3.71%	3.36%
80%	19.18%	16.56%	2.59%	2.14%
85%	20.98%	17.12%	1.25%	1.32%
90%	21.14%	17.93%	0.94%	0.94%
95%	21.99%	17.86%	0.49%	0.95%

Table 7: Sensitivity of slot reduction impacts to the maximum average segment load factor value under a 12.3% slot reduction

5.2 Sensitivity to Model Parameters

Table 4 in sub-section 4.4 showed that the model’s prediction accuracy is not highly sensitive to variations in model parameters. In this sub-section, we present the sensitivity of slot reduction impacts to variations in the values of various parameters in the *basic model* and in the model extensions 1 and 2. The parameters are varied within -25% to +25% of their values listed in sub-section 4.1. Tables 8 and 9 present the percentage increase in total profits and the percentage decrease in passengers carried, respectively, under a 12.3% reduction with the proportionate allocation scheme. Even with significant variations in the parameter values, slot reduction results in at least a 17.3% increase in total operating profits with at most a 1.9% reduction in passengers carried (assuming constant aircraft sizes). We also performed similar sensitivity analyses of our slot reduction results under the reward-based allocation scheme and found the results to be similarly stable.

Parameter	-25%	-20%	-15%	-10%	-5%	0%	5%	10%	15%	20%	25%
α_s	20.3%	20.9%	20.4%	20.7%	21.5%	21.0%	21.3%	18.9%	18.3%	19.1%	18.7%
θ_a	21.5%	21.2%	21.0%	20.8%	20.5%	21.0%	21.6%	21.0%	22.3%	22.4%	22.4%
α_{s1}	20.8%	20.7%	20.7%	20.5%	20.8%	21.0%	20.9%	20.5%	20.3%	21.6%	20.3%
β_{s1}	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	20.8%	20.9%	20.9%	20.8%	20.8%
γ_{s1}	20.7%	20.7%	20.7%	20.7%	20.8%	21.0%	20.8%	20.6%	20.4%	20.4%	20.4%
α_{s2}	20.7%	20.7%	20.7%	20.8%	20.9%	21.0%	20.9%	20.5%	20.3%	20.3%	20.2%
β_{s2}	20.3%	21.3%	21.0%	21.0%	21.0%	21.0%	20.8%	20.5%	20.1%	21.1%	20.5%
$C_{a/s}$	22.8%	27.7%	19.2%	17.3%	19.3%	21.0%	23.7%	24.2%	24.1%	25.5%	25.8%
$S_{a/s}$	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%	21.0%
$p_{a/s}$	25.8%	25.8%	25.5%	24.1%	23.7%	21.0%	19.3%	17.3%	20.4%	24.0%	27.7%

Table 8: Sensitivity of increase in total profit due to slot reduction to model parameters (under 12.3% reduction with proportionate allocation)

Parameter	-25%	-20%	-15%	-10%	-5%	0%	5%	10%	15%	20%	25%
α_s	1.17%	1.23%	1.41%	1.63%	1.30%	1.25%	1.29%	1.77%	1.71%	1.41%	1.52%
θ_a	1.03%	1.30%	1.13%	1.45%	1.40%	1.25%	0.91%	1.17%	1.30%	1.25%	1.30%
α_{s1}	1.29%	1.30%	1.28%	1.33%	1.26%	1.25%	1.22%	1.46%	1.43%	1.43%	1.45%

β_{s1}	1.25%	1.25%	1.26%	1.25%	1.25%	1.25%	1.27%	1.23%	1.22%	1.23%	1.23%
γ_{s1}	1.30%	1.31%	1.30%	1.30%	1.26%	1.25%	1.23%	1.39%	1.39%	1.39%	1.39%
α_{s2}	1.30%	1.30%	1.30%	1.27%	1.26%	1.25%	1.23%	1.46%	1.42%	1.33%	1.46%
β_{s2}	1.30%	1.01%	1.14%	1.14%	1.24%	1.25%	1.22%	1.40%	1.40%	1.36%	1.50%
$C_{a/s}$	1.37%	0.85%	1.59%	1.90%	1.35%	1.25%	1.03%	0.99%	0.82%	0.78%	0.60%
$S_{a/s}$	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%	1.25%
$p_{a/s}$	0.60%	0.60%	0.78%	0.82%	1.03%	1.25%	1.35%	1.90%	1.37%	0.88%	0.85%

Table 9: Sensitivity of decrease in passengers carried due to slot reduction to model parameters (under 12.3% reduction with proportionate allocation)

5.3 Effect of Aircraft Upgauges

Results in sub-section 5.1 were obtained under the assumption that, even when the total number of slots available to an airline is reduced, the airline will continue to operate the same-size aircraft as it did in the absence of slot reduction. This assumption might be realistic for very small reductions in the number of slots, but for significant reductions, it is reasonable to expect that the airlines will operate larger aircraft on some of the segments in order to accommodate more passengers and therefore increase profit. The main problem with modeling aircraft size decisions is that such decisions depend on the fleet availability. We estimate the impact of aircraft size upgauges by allowing for a limited number of upgauges for each airline. We sort all the available types of aircraft operated into LGA by any of the airlines in increasing order of seating capacity. We allow a certain maximum percentage of an airline's fleet (flying into LGA) to be upgauged to the next bigger-sized aircraft. This constraint indirectly models the fact that an airline cannot arbitrarily increase aircraft sizes due to fleet availability constraints. We calculate the equilibrium frequency solution under the slot reduction scenario as described in sub-section 5.1 and subsequently perform, for each airline, the most profitable flight upgauges subject to the limits on maximum allowable upgauge percentage. As before, we assume a maximum average segment load factor of 85%.

Figure 9 describes the impact of a limited number of aircraft upgauges on the reductions in passengers carried when the total number of slots is reduced by 12.3%, and the proportionate allocation scheme is used. The maximum allowable upgauge percentage is on the x-axis, which represents the maximum percentage of an airline's flights that can be upgauged to the next bigger aircraft size. The percentage reduction in the total number of passengers varies from 1.25%, assuming 0% upgauges, to 0.39% assuming at most 8% upgauges for each airline. This shows that even with a small fraction of flights

upgauging to a larger-sized aircraft, most of the reduction in the number of passengers disappears. The remaining reduction in the number of passengers is primarily attributable to the fact that there is only a limited number of aircraft sizes available; and on some segments, the number of passengers who are denied a seat due to a smaller aircraft size is not large enough to justify a profitable upgauge to the next bigger aircraft size.

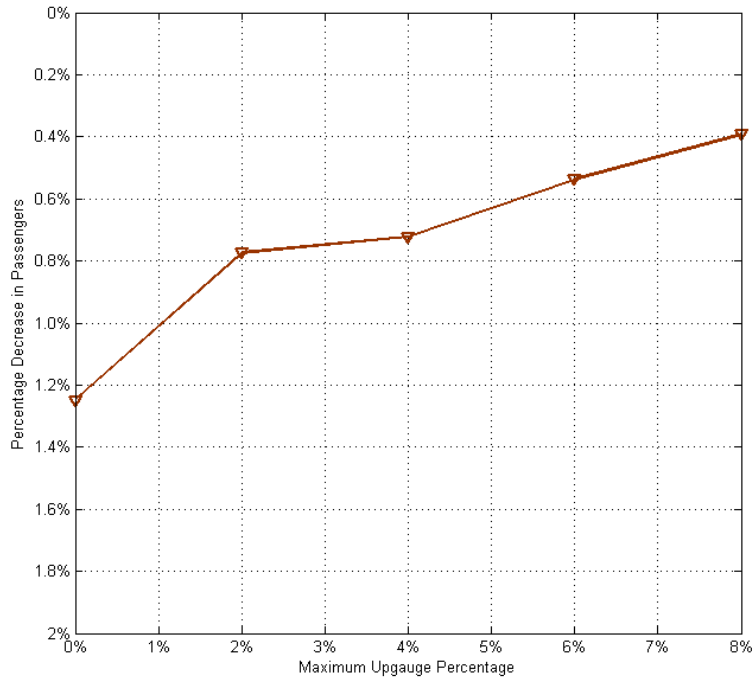


Figure 9: Effect of limited upgauging on total number of passengers under a 12.3% slot reduction

5.4 Effects of Demand Uncertainty and Revenue Management Practices

The results presented in sub-section 5.1 assume deterministic passenger demand and the existence of a fixed fare value for each flight. In reality, passenger demand varies from day-to-day and these variations affect the number of passengers transported. Spill (or spilled passengers) is defined as the portion of passenger demand that cannot be accommodated because of the limited capacity of the aircraft. Models of spill estimation typically assume some distribution of demand and calculate the expected value of this distribution truncated by the seating capacity to obtain the expected number of passengers carried (Belobaba and Farkas, 1999). In our models presented thus far, to account for demand stochasticity, we put a hard constraint on the maximum number of seats sold on each segment at 85% of the total seating capacity of all flights on the segment. This is a fairly conservative value, considering the fact that in the year 2007, across all 7,452 combinations of segments and carriers in the domestic U.S. with at least 1 flight per day, 923 combinations had an average load factor of greater than 85%. Moreover, the higher the

demand factor, the higher is the average load factor, where demand factor is defined as the ratio of average passenger demand to seating capacity. If the total number of seats offered on a segment is reduced, which is likely to be the case under a slot reduction scenario, the demand factor increases further. Therefore, we expect our method to introduce a downward bias in the expected number of passengers carried.

The revenue management methods practiced by the airlines affect the average fares of the spilled passengers by ensuring that the spilled passengers are predominantly the low-fare passengers. Until now, we have ignored this effect. Therefore, our method can also be expected to introduce a downward bias in the average fare values, and therefore, in the operating profit estimates. Moreover, as pointed out by Belobaba and Farkas (1999), in addition to impacting the average fares of the spilled passengers, revenue management practices also affect the number of spilled passengers. In this sub-section, we estimate the extent and direction of this bias, in the number of passengers carried and in the airline profits, introduced by our simplified assumptions about demand uncertainty and fares.

Under a slot reduction scenario, the total segment seating capacity is reduced, unless there is a substantial increase in aircraft sizes. Revenue management systems used by the airlines are expected to readjust their seat allocation decisions across different fare classes by spilling the low-fare passengers, resulting in some increase in average fares. In order to estimate the effect of demand uncertainty and revenue management on expected spills and average fares, we use passenger spills and average fares of spilled passengers (called spill fares) estimated by Belobaba and Farkas (1999) using a multiple nested fare class, single booking period model. Belobaba and Farkas (1999) have estimated the expected spills and spill fares for a 5-fare class example using different values of demand factors and fare discount ratios, where a fare discount ratio is defined as the ratio of the fare of a class to the fare of the immediately higher fare class.

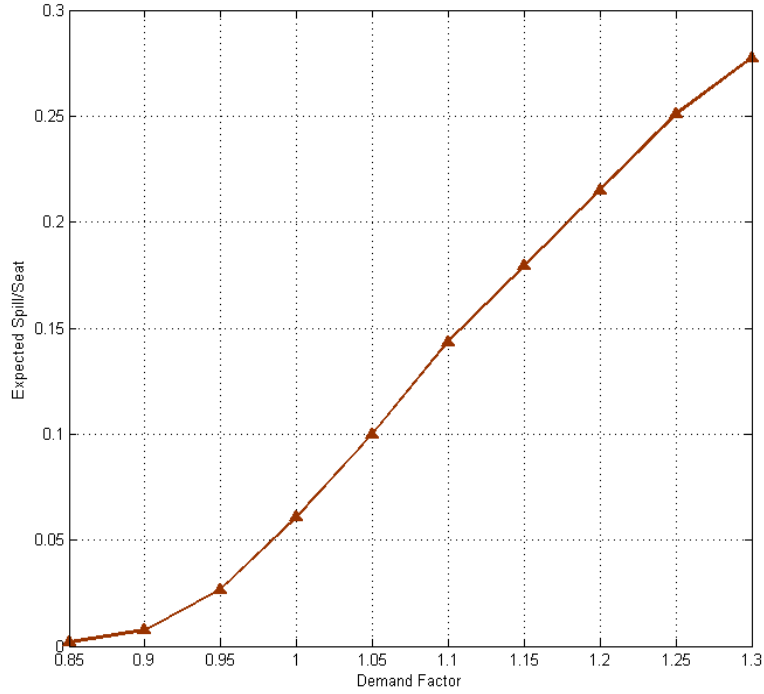


Figure 10: Expected spill per seat obtained using the spill modeling approach for a fare discount ratio of 0.7 (Source: Belobaba and Farkas, 1999)

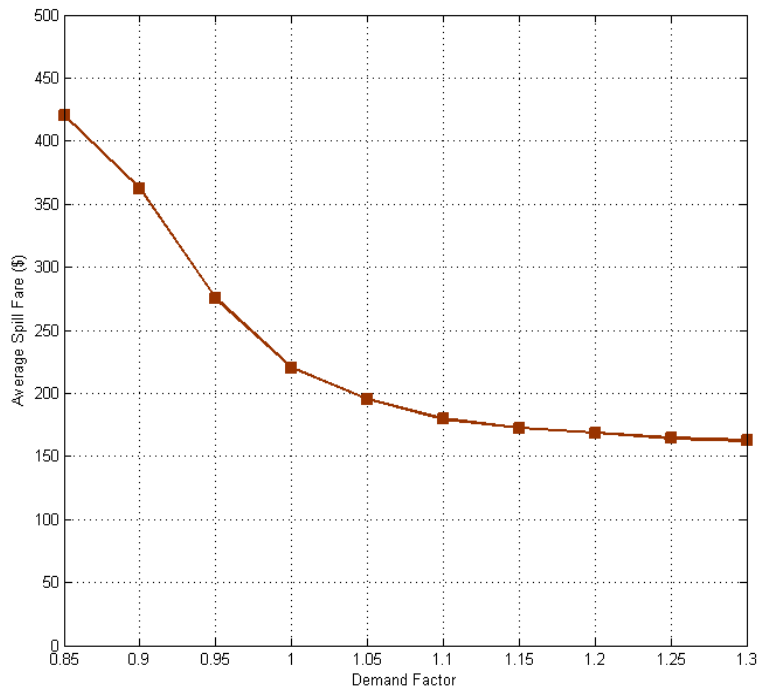


Figure 11: Average spill fare obtained using the spill modeling approach for a fare discount ratio of 0.7 (Source: Belobaba and Farkas, 1999)

Figures 10 and 11 show the expected spill and spill fare for different values of demand factors and fare discount ratios estimated by Belobaba and Farkas (1999). In both figures, different curves correspond to different fare discount ratios. The demand factor is on the x-axis in both figures. Figure 10 has the ratio of the expected number of spilled passengers to the aircraft seating capacity on the y-axis while Figure 11 has the average fare of spilled passengers on the y-axis. Note that Figure 11 corresponds to a maximum fare of \$600. For other fare values, the average spill fare can be calculated by simply rescaling the y-axis accordingly.

We will call our default approach as the *deterministic approach* and this new approach as the *spill modeling approach*. Under the *spill modeling approach*, for a given set of flight frequencies, the demand for each airline on each segment is computed using the appropriate market share model (given by constraint (3), constraint (8), or by constraints (9) and (10)). The expected number of spilled passengers on each segment is calculated using Figure 10 and the average fare of the spilled passengers is calculated using Figure 11. The airline’s revenue is obtained by subtracting the product of the expected spill and spill fare from the product of the demand and the overall average fare across the unconstrained demand for each airline on each segment.

For the equilibrium frequency solution with and without slot reductions, we recalculated the airline profits using the *spill modeling approach*. Table 10 shows the impact of a 12.3% slot reduction, using both proportionate and reward-based allocation, in terms of total number of passengers carried and total operating profits using the *spill modeling approach*. The numbers in parentheses represent the percentage change as compared to the *no reduction equilibrium*. We used a fare discount ratio of 0.7, which was determined to be a reasonably representative value for the US airline fare structures (Belobaba and Farkas, 1999).

Metrics	Actual Frequencies	No Reduction Equilibrium	12.3% Reduction	
			Proportionate	Reward-based
Total Operating Profit (Excluding Flight Delay Costs)	\$1,263,650	\$1,326,149	\$1,617,158 (21.90%)	\$1,566,471 (18.10%)
Total Passengers Carried	23,085	23,129	22,927 (-0.87%)	22,897 (-1.00%)

Table 10: Effect of a 12.3% slot reduction under *spill modeling approach*

The results in Table 10 confirm that our *deterministic approach* introduces a slight downward bias in number of passengers carried as well as in the total operating profits. This can be observed by comparing

these results with those in Table 5. Furthermore, as the demand factor increases, the magnitude of this bias increases, which means that the bias is greater for the slot reduction scenario than for the no reduction scenario. So, our *deterministic approach*, in fact, slightly underestimates the benefits of slot reduction. This is apparent from the greater increase in total operating profits and smaller reduction in passengers carried due to slot reduction as reported in Table 10 compared to that in Table 5.

The preceding discussion about increases in average fares assumes that the total seating capacity on a segment is reduced due to slot reduction. However, in light of the possibility of aircraft upgauges as discussed in sub-section 5.3, the decrease in seats due to frequency reduction might be partially or completely compensated by increases in seats per flight, depending on the extent of upgauging. Therefore, even the results presented in Table 10 are likely to be overestimates of the reduction in passengers carried and underestimates of the increase in total operating profits.

In addition to the effects of demand uncertainty and revenue management, one can consider the possibility of changes in the actual fare values due to a reduction in the total number of seats offered on a non-stop segment. Slot reduction can have an impact on fares in different ways. Borenstein (1989) noted that an airline's share of passengers on a route and at an airport affects fares. Slot reduction might affect the passenger shares of different airlines differently, thus leading to an increase in fares for some combinations of routes and carriers, and a decrease for some other combinations. However, there is no reason to expect any significant change in the overall fare levels because of this effect. Additionally, the delay reduction resulting from slot reduction strategies can, in turn, be expected to have an impact on fares in multiple ways. Britto, Dresner and Voltes (2010) found that an increase in flight delays increases airline costs and hence has an increasing effect on fares. According to their results, for every minute of reduction in average flight delays, the fares decrease by about \$0.04. On the other hand, Forbes (2008) analyzed a rather extreme delay situation for LGA airport and concluded that an increase in flight delays deteriorates the quality of air service and therefore decreases the fares. According to the findings of this study, the fares increase by about \$1.42 for every minute of reduction in average flight delays. Detailed analysis of these various, often opposite, effects is beyond the scope of this paper. However, the possibility of net increases in fares upon slot reduction suggests that our results are likely to be conservative, that is, the actual increase in total profits of the airlines due to slot reduction strategies could be even higher than what we report in Table 5.

In addition to the inherent demand stochasticity, the total demand for travel on each segment into LGA (M_S), is likely to be affected by slot controls in a complicated way. A significant proportion of passengers travelling to New York City have other travel alternatives to flying into LGA, e.g. travel by car, travel by

rail, and travel by air into other New York area airports such as Newark (EWR) and Kennedy (JFK). Upon slot reduction at LGA, the impact on total passenger demand can be several-fold. Substantial delay reduction is expected to make LGA more attractive while a corresponding reduction in frequency on some routes and a possible increase in fares are expected to make LGA less attractive. While the net effect is difficult to predict, it is reasonable to expect some increase in net attractiveness of LGA unless EWR and/or JFK also implement similar slot reduction strategies. This is yet another reason the passenger demand and actual number of passengers carried into LGA after slot reduction can be expected to be higher than those presented in Table 5.

5.5 Effects of Passenger Connections

Our models of frequency competition assume segment-based demand. In reality, passengers demand seats on itineraries, which might be combinations of two or more flights. Accordingly, the airlines also charge fares on an itinerary basis rather than on a flight basis. So calculation of the fare revenue generated by a segment is not a straightforward process. Belobaba (2009b) acknowledged that different assumptions made by airline planners can lead to very different estimates of profitability. The results presented in subsection 5.1 were calculated under the full fare assumption. One potential issue with full fare assumption is that the fare of a one-stop passenger gets counted twice, once for each flight in the passenger's itinerary. In this research, we have considered only those segments which have one end-point at LGA airport. Because a very small fraction of passengers (~5%) connect at LaGuardia (BTS, 2010d), there will be very little double-counting of the revenues. Moreover, whatever double counting occurs has a comparable effect on the results under the slot reduction scenario and the scenario with no slot reduction. So the issue of double-counting does not affect the percentage increase in total profits significantly.

Another issue with the full fare assumption is that when a passenger is spilled, the entire fare revenue corresponding to that passenger is assumed to be lost. This fails to capture the possibility of having an additional non-stop passenger on the other segment, which has seats available. For example, when a reduction in the frequency of American Airlines' flights from Dallas-Fort Worth (DFW) airport to LGA results in spilling a passenger traveling from Los Angeles (LAX) to LGA via DFW due to lack of seats on the DFW-LGA segment, there is still a possibility of recovering a part of that revenue by carrying an additional non-stop passenger on the LAX-DFW segment. An alternative method for fare revenue calculation is the complete proration approach, in which the fare is completely prorated based on distance. Under this assumption, if the airline is not able to carry a connecting passenger, it will only lose the revenue equal to a fraction of the passenger's full fare. This is not necessarily the most valid representation of segment profits either, because of the possibility that the seat vacated by that passenger

on the other segment may not be filled by another non-stop or connecting passenger. In the aforementioned example, the seat on the LAX-DFW segment vacated by the spilled passenger going from LAX to LGA via DFW may not get filled by another passenger. Moreover, another issue with the distance-based proration is that the fares are not necessarily well correlated with distances.

While neither the full fare assumption nor the complete proration assumption is very accurate, these represent the two extremes of possible methods for calculating segment profitability. Any reasonable method of fare revenue estimation is expected to lie somewhere in between these two extremes. To measure the difference between these extreme scenarios, we evaluated the impacts of slot reduction strategies under the distance-based complete proration assumption. The results are presented in Table 11. As expected, the profitability values are considerably lower than those corresponding to the full fare assumption, i.e. those in Table 5. However, the absolute increase in total profits due to slot reduction changes only slightly.

Metrics	Actual Frequencies	<i>No Reduction Equilibrium</i>	12.3% Reduction	
			Proportionate	Reward-based
Total Operating Profit (Excluding Flight Delay Costs)	\$901,346	\$959,944	\$1,247,750 (29.98%)	\$1,174,736 (22.38%)
Total Passengers Carried	22,895	22,971	22,676 (-1.29%)	22,532 (-1.91%)

Table 11: Effect of a 12.3% slot reduction under completely prorated fares approach

In addition to the fare proration issue, another limitation of the S-curve model of market share is that it does not estimate the connecting passenger shares accurately. The model predicts that when the frequency of one carrier decreases compared to that of another competing carrier, a fraction of its passengers will shift to the other carrier. But this does not make sense for connecting passengers if the other carrier does not offer a connecting itinerary to that passenger's final destination. For example, if American Airlines reduces its flight frequency from DFW to LGA, the segment-based S-curve model predicts that some passengers on the DFW-LGA segment will shift to other airlines on DFW-LGA segment. But in reality, a passenger traveling from LAX to LGA via DFW may not shift to the DFW-LGA flights of another airline if that other airline does not offer service from LAX to DFW. In fact, with changes in frequency values, the passengers might shift to traveling through a completely different hub of the same carrier or some other carrier to their final destinations. For example, with a reduction in the frequency of American Airlines' flights from DFW to LGA, a passenger traveling from LAX to LGA on American Airlines

through its DFW hub may decide to instead travel on Continental Airlines by connecting at Houston (IAH) or to continue travelling on American Airlines but through the Chicago O'Hare (ORD) hub rather than DFW. Therefore, such interactions can get extremely complicated.

Of all passengers on flights arriving at LaGuardia, approximately 75% are non-stop passengers, only 20% are connecting passengers with a final destination at LGA and the remaining 5% connect at LaGuardia itself (BTS, 2010d). Of all the connecting passengers with a final destination at LGA, nearly 97% are carried by seven major airlines and nearly 89% of them connect through the two biggest hubs of each of these seven airlines. We expect the effects of connecting passengers to be fairly well distributed across these major airlines, without any obvious advantages or disadvantages to any particular carriers. Furthermore, these connecting passenger effects should have a similar impact on results with and without slot reduction. So we expect that the passenger connections would not affect the main results of this study significantly.

6 Summary and Discussion

Any demand management strategy implicitly or explicitly involves deciding the total capacity to be allocated and the distribution of this capacity among different airlines. In this research, we explicitly consider these two stages separately. Although there is extensive literature on airport demand management strategies, none of the previous studies have captured critical elements of frequency competition among carriers. To the best of the authors' knowledge, this is the first study that tries to model airline competition under demand management strategies.

We developed a game-theoretic model of airline frequency competition based on the S-curve relationship, which is a popular model of market share in the airline literature. Due to the discreteness of the problem and the non-convexity of its continuous relaxation, the optimization problem for each airline is complicated. Furthermore, due to competitive interactions among different players, the problem becomes one of computing a Nash equilibrium. The large size of the solution space makes it very challenging to solve. We propose an efficient solution algorithm for obtaining a Nash equilibrium. We justify the predictive power of the Nash equilibrium solution concept using empirical validation of the model estimates under existing slot allocation. Irrespective of the starting point, the best response algorithm was found to approach the equilibrium outcome within a very few iterations. This shows that even less than perfectly rational carriers can reach the equilibrium outcome through simple myopic learning dynamics, and thus provides further justification of the predictive power of Nash equilibrium outcome.

We evaluated two simple slot reduction strategies. The results showed that in addition to a substantial reduction in flight and passenger delays, small reductions in total allocated capacity can improve the operating profits of carriers considerably. While the two strategies led to some differences in the actual profitability increases across individual carriers, the aggregate impacts were similar. Under each strategy, slot reduction led to a substantial increase in the profits of all carriers across the board, and substantial reductions in flight delays and passenger delays. It also led to a small reduction in the number of passengers carried. However, most of the reduction in total passengers carried was eliminated when the possibility of a limited number of aircraft upgauges was introduced. The increase in schedule displacement due to the slot reduction was negligibly small compared to the overall travel times of the passengers.

We tested the sensitivity of the results to various assumptions, parameter values and changes in time periods and datasets. In most of the cases, the model results varied only slightly. Also in most of the cases, our assumptions were found to be conservative, that is, relaxing these assumptions is expected to make the slot reduction an even more attractive strategy.

In summary, we showed that a small reduction in the allocated slots at a congested airport resulted in a large (over 40%) reduction in average passenger delays. While it is a challenging task to predict the associated fare increase (if any), most prior estimates indicate a small value on the order of at most 3% to 4%. Other potential detrimental passenger effects, such as increases in schedule displacement, loss of service to small markets and reductions in the number of passengers carried are shown to be negligibly small. Thus almost all passengers get transported to their respective destinations, with negligible increases in schedule displacement and significantly lower average passenger delays, with the possibility of a small increase in fares. Therefore, the net effect on passengers can be concluded to be positive. A small slot reduction also benefits the carriers, all of whom experience reductions in delay costs as well as increases in planned operating profits. It is also beneficial to the airport operators because congestion and airport delays are reduced substantially. From the perspective of the entire system, slot reduction strategies lead to almost all passengers being transported with many fewer flights and lower total cost. Hence, slot reduction strategies are also attractive from the perspective of overall social welfare.

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