CONSENSUS-BASED DISTRIBUTED PARTICLE FILTERING ALGORITHMS FOR COOPERATIVE BLIND EQUALIZATION IN RECEIVER NETWORKS

Claudio J. Bordin Jr.

UFABC, Brazil
claudio.bordin@ufabc.edu.br

Marcelo G. S. Bruno

Instituto Tecnológico de Aeronáutica, Brazil
bruno@ita.br

ABSTRACT

We describe in this paper novel consensus-based distributed particle filtering algorithms which are applied to cooperative blind equalization of frequency-selective channels in a network with one transmitter and multiple receivers. The proposed algorithms employ parallel consensus averaging iterations to evaluate the product of some node-dependent quantities across the receiver network, thus eliminating the need for message broadcasts beyond each receiver's local neighborhood. Additionally, parallel minimum consensus iterations are used to assess the convergence of the quantized consensus averages and ensure accordingly the coherence of particle sets across the different network nodes. We verify via computer simulations that the consensus-based schemes exhibit a small performance gap compared to both centralized and communication-intensive broadcast solutions.

***Index Terms*—** Distributed Algorithms, Particle Filters, Blind Equalization, Consensus Averaging.

1. INTRODUCTION

The ubiquity of sensor networks where each node is equipped with computing and communication capabilities has stimulated the development of distributed algorithms for solving a variety of problems. We consider here a setup where a single transmitter broadcasts a sequence of discrete-valued symbols to a network of multiple remotely located receivers. Rather than forwarding the local observations to other nodes or to a data fusion center, we aim instead at deriving algorithms in which the different nodes process their own local observations independently, but also cooperate with each other to approximate the optimal estimate of the transmitted sequence given all measurements in the network.

Previous distributed filtering algorithms are mostly limited to linear estimation frameworks [1], [2], being not ideally suited for distributed equalization of digital broadcast channels, as the optimal MAP estimate of the transmitted data sequence may significantly differ from the LMMSE estimates obtained by conventional adaptive or Kalman filters. That limitation may be circumvented by using nonlinear particle filters that converge asymptotically in the number of particles to the desired MAP estimate. The development of distributed particle filters has been hindered, however, by the fact that, unless further approximations are made, all nodes must obtain the same set of particles and weights. To comply with this restriction, most methods developed so far [3],[4], [5] rely on broadcast of messages across the network, an undesirable feature in many scenarios with communication constraints.

In this paper, we eliminate the broadcast requirement by introducing new consensus-based, fully distributed particle filtering algorithms. The algorithms employ consensus averaging [6] to evaluate the product of some node-observation-dependent probability densities across the receiver network, assessing the convergence of quantized approximations to those quantities at all nodes via minimum consensus [7]. An alternative consensus-based approach to distributed particle filtering was introduced in [8] in the context of a target tracking application. Unlike our work, reference [8] assumes that the parameters of the observation models are perfectly known at every node and uses both a different distributed importance function and a different consensus strategy.

The remaining text is organized as follows: in Sec. 2 we describe the problem setup, briefly introducing in Sec. 3 a centralized particle filter approach to its solution. In Section 4, we present the new consensus-based distributed particle filter schemes (CB-I and II), whose performance is assessed in Sec. 5. Our conclusions are summarized in Sec. 6.

2. PROBLEM SETUP

Denote by $\{b_n\}$ an independent, identically distributed (i.i.d.) binary bit sequence and by $\{x_n\}$, $x_n \in \{\pm 1\}$, the corresponding differentially encoded symbols. We assume that the observations $y_{r,0:n} \triangleq \{y_{r,0}, \ldots, y_{r,n}\}$ at the $r-$th node of a network of R receivers are obtained as the output of the additive noise frequency-selective FIR channel

$$y_{r,n} = \mathbf{h}_r^H \mathbf{x}_n + v_{r,n} \ , \quad (1)$$

where $\mathbf{h}_r \in \mathbb{C}^{L\times 1}$ is a vector with the (time-invariant) channel impulse response terms, $\mathbf{x}_n \triangleq [x_n \ldots x_{n-L+1}]^T$, and $v_{r,n}$ represents an i.i.d zero-mean complex Gaussian random process of variance σ_r^2.

The *unknown, random* parameters $\mathbf{h}_r$ and σ_r^2, $1 \le r \le R$, are assumed to be independent for $r \neq s$, and distributed a priori as $\sigma_r^2 \sim \mathcal{IG}(\sigma_r^2|\alpha;\beta)$ and $\mathbf{h}_r \mid \sigma_r^2 \sim \mathcal{N}_L(\mathbf{h}_r|0; I\sigma_r^2/\epsilon^2)$, where $\mathcal{N}_L$ and $\mathcal{IG}$ denote respectively an $L-$variate Gaussian and an inverse Gamma p.d.f., and $\{\alpha, \beta, \epsilon\}$ are the model's hyperparameters.

Under these hypotheses, we aim at developing a recursive method for obtaining smoothed MAP estimates $\hat{b}_{n-d} = \arg\max_{b_{n-d}} p(b_{n-d}|y_{1:R,0:n})$, where $d \ge 0$ and $y_{1:R,0:n} \triangleq \{y_{1,0:n} \cdots y_{R,0:n}\}$.

3. CENTRALIZED SOLUTION VIA PARTICLE FILTERS

Particle filters allow one to approximate the posterior probability mass function (p.m.f) of the transmitted bits as

$$p(b_{n-d}|y_{1:R,0:n}) \approx \sum_{q=1}^{Q} w_n^{(q)} \mathcal{I}\left\{b_{n-d} = b_{n-d}^{(q)}\right\}, \quad (2)$$

where $\mathcal{I}\{\cdot\}$ denotes for the indicator function, Q the number of particles $b_n^{(q)}$, sampled from an importance function $\pi(\cdot)$, and $w_n^{(q)}$ are the importance weights. Exploiting that each distinct bit sequence $b_{-L:n-1}^{(q)}$ uniquely defines a corresponding state sequence $\mathbf{x}_{0:n}^{(q)}$, the so-called optimal importance function [9] can be written as $\pi(b_n|b_{-L:n-1}^{(q)}, y_{1:R,0:n}) = p(\mathbf{x}_n \mid \mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n})$, which can be determined as

$$p(\mathbf{x}_n \mid \mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n}) = \frac{p(\mathbf{x}_n, \mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n})}{\sum_{\mathbf{x}_n} p(\mathbf{x}_n, \mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n})}. \quad (3)$$

The *importance weights* can in turn be recursively propagated [9] as

$$w_n^{(q)} \propto w_{n-1}^{(q)} \sum_{\mathbf{x}_n} \frac{p(\mathbf{x}_n, \mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n})}{p(\mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n-1})}. \quad (4)$$

From the a priori independence of the unknown parameters for each receiver's channel, one deduces [5] that

$$p(\mathbf{x}_{0:n}^{(q)}, y_{1:R,0:n}) \propto \prod_{r=1}^{R} p(\mathbf{x}_{0:n}^{(q)}, y_{r,0:n}). \quad (5)$$

Finally, under the assumptions of Sec. 2, one can show after some algebraic manipulations [9] that

$$\begin{aligned} p(\mathbf{x}_{0:n}^{(q)}, y_{r,0:n}) &= \int_{\mathbb{R}^+}\int_{\mathbb{C}^L} p(\mathbf{x}_{0:n}^{(q)}, y_{r,0:n}, \mathbf{h}_r, \sigma_r^2)\, d\mathbf{h}_r\, d\sigma_r^2 \\ &\propto |\boldsymbol{\Sigma}_n^{(q)}| \left[\beta_{r,n}^{(q)}\right]^{-\alpha_n}, \end{aligned} \quad (6)$$

where $\Sigma_n^{(q)}$, $\beta_{r,n}^{(q)}$, and α_n can be recursively computed via

$$\alpha_n = \alpha_{n-1} + 1, \quad (7)$$

$$\beta_{r,n}^{(q)} = \beta_{r,n-1}^{(q)} + \|e_{r,n}^{(q)}\|^2/\gamma_n^{(q)}, \quad (8)$$

$$\bar{\mathbf{h}}_{r,n}^{(q)} = \bar{\mathbf{h}}_{r,n-1}^{(q)} + \boldsymbol{\Sigma}_{n-1}^{(q)} \mathbf{x}_n^{(q)} (e_{r,n}^{(q)})^*/\gamma_n^{(q)}, \quad (9)$$

$$\boldsymbol{\Sigma}_n^{(q)} = \boldsymbol{\Sigma}_{n-1}^{(q)} - \boldsymbol{\Sigma}_{n-1}^{(q)} \mathbf{x}_n^{(q)} (\mathbf{x}_n^{(q)})^H \boldsymbol{\Sigma}_{n-1}^{(q)}/\gamma_n^{(q)}, \quad (10)$$

with $\alpha_{-1} = \alpha$, $\beta_{r,-1}^{(q)} = \beta$, $\bar{\mathbf{h}}_{r,-1}^{(q)} = 0$, $\boldsymbol{\Sigma}_{-1}^{(q)} = \mathbf{I}\epsilon^{-2}$, $e_{r,n}^{(q)} \triangleq y_{r,n} - (\bar{\mathbf{h}}_{r,n-1}^{(q)})^H \mathbf{x}_n^{(q)}$, and $\gamma_n^{(q)} \triangleq 1 + (\mathbf{x}_n^{(q)})^H \boldsymbol{\Sigma}_{n-1}^{(q)} \mathbf{x}_n^{(q)}$.

3.1. Cooperative Approach

Substituting (5) into (3), the expression of the optimal importance function can be rewritten as

$$p(\mathbf{x}_n|\mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n}) = \frac{\prod_{r=1}^{R} \lambda_{r,n}^{(q)}(\mathbf{x}_n)}{\sum_{\mathbf{x}_n} \prod_{r'=1}^{R} \lambda_{r',n}^{(q)}(\mathbf{x}_n)} \quad (11)$$

where $\lambda_{r,n}^{(q)}(\mathbf{x}_n) \triangleq p(\mathbf{x}_n, \mathbf{x}_{0:n-1}^{(q)}, y_{r,0:n})$. Likewise, plugging in (5) into (4) leads to the weight update rule

$$w_n^{(q)} \propto w_{n-1}^{(q)} \sum_{\mathbf{x}_n} \prod_{r=1}^{R} \frac{\lambda_{r,n}^{(q)}(\mathbf{x}_n)}{\lambda_{r,n-1}^{(q)}(\mathbf{x}_{n-1}^{(q)})}. \quad (12)$$

The DcPF-II algorithm in [5] is an exact decentralized implementation of (11)-(12). Despite its asymptotic optimality in the number of particles, that algorithm has the undesirable feature of relying on broadcasting particle-dependent quantities across the receiver network.

4. CONSENSUS-BASED ALGORITHMS

Equations (11) and (12) can be rewritten respectively as

$$p(\mathbf{x}_n|\mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n}) = \frac{\exp\left(\Lambda_n^{(q)}(\mathbf{x}_n)\right)}{\sum_{\mathbf{x}_n} \exp\left(\Lambda_n^{(q)}(\mathbf{x}_n)\right)}, \quad (13)$$

$$w_n^{(q)} \propto w_{n-1}^{(q)} \sum_{\mathbf{x}_n} \exp\left(\Lambda_n^{(q)}(\mathbf{x}_n) - \Lambda_{n-1}^{(q)}(\mathbf{x}_{n-1}^{(q)})\right), \quad (14)$$

where $\Lambda_{r,n}^{(q)}(\mathbf{x}_n) \triangleq \log_e\left(\lambda_{r,n}^{(q)}(\mathbf{x}_n)\right)$ and $\Lambda_n^{(q)}(\mathbf{x}_n) \triangleq \sum_{r=1}^{R} \Lambda_{r,n}^{(q)}(\mathbf{x}_n)$. The latter sum can be evaluated via $2Q$ parallel[1] *consensus averaging* iterations [6] as

$$\Lambda_n^{(q)}(\mathbf{x}_n) = \lim_{k\to\infty} \tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n), \quad \forall r \quad (15)$$

where k is the consensus algorithm iteration index (independent of n) and

$$\begin{aligned} \tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n) = \tilde{\Lambda}_{r,n}^{(k-1,q)}(\mathbf{x}_n) + \\ \sum_{s\in\mathbf{N}(r)} a_{rs}\left(\tilde{\Lambda}_{r,n}^{(k-1,q)}(\mathbf{x}_n) - \tilde{\Lambda}_{s,n}^{(k-1,q)}(\mathbf{x}_n)\right). \end{aligned} \quad (16)$$

[1] Observe that one consensus algorithm must be employed for each of the Q particles and each possible value of $\mathbf{x}_n^{(q)}$ given $\mathbf{x}_{n-1}^{(q)}$ (2, for binary signal).

In (16), $\mathbf{N}(r)$ denotes the neighborhood of node r and a_{rs} are real-valued weights such that $a_{rs} \geq 0$, $\forall (r, s)$, and $a_{rs} = a_{sr}$. By stacking the terms $\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)$, $r = 1, \ldots, R$, into a long $R \times 1$ vector $\underline{\tilde{\Lambda}}_n^{(k,q)}(\mathbf{x}_n)$, equation (16) may be rewritten in compact matrix notation as

$$\underline{\tilde{\Lambda}}_n^{(k,q)}(\mathbf{x}_n) = \mathbf{A}\ \underline{\tilde{\Lambda}}_n^{(k-1,q)}(\mathbf{x}_n) \tag{17}$$

where $\mathbf{A}$ is, by construction, an $R \times R$ doubly stochastic matrix whose rows and columns both add up to one. If the weights a_{rs} are additionally chosen such that matrix $\mathbf{A}$ is primitive then, as $k \to \infty$, $\mathbf{A}^k$ converges [10] to a matrix with identical entries equal to $1/R$. It suffices then to initialize (16) with the initial conditions $\tilde{\Lambda}_{r,n}^{(0,q)}(\mathbf{x}_n) \triangleq R\Lambda_{r,n}^{(q)}(\mathbf{x}_n)$ to achieve the desired limit in (15).

In general, for any finite k, $\varepsilon_{r,n}^{(k,q)} \triangleq \tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n) - \Lambda_n^{(q)}(\mathbf{x}_n) \neq 0$. Therefore, a direct application of consensus averaging would result in distinct values for the importance function and weights at each node for the same particle q. This would, in general, result in distinct particle sets at each node, even if synchronous sampling/resampling [3] is employed, violating previous assumptions. To guarantee coherence in the particle sets across the network nodes, some form of quantization step has to be used, as we describe in the sequel.

4.1. CB-I Algorithm

In this Section, we propose a new technique for detecting consensus. Let $\mathcal{Q}(\cdot)$ denote a deterministic quantizer. At the node r, one can evaluate the function

$$\iota_{r,n}^{(k,q)} = \begin{cases} 1, & \text{if } \mathcal{Q}(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)) = \mathcal{Q}(\tilde{\Lambda}_{s,n}^{(k,q)}(\mathbf{x}_n)), \\ & \qquad \forall s \in \mathbf{N}(r), \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

If $\iota_{r,n}^{(k,q)} = 1, \forall r$ (for a fixed k), the transitivity of the equality operator assures that $\mathcal{Q}(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)) = \mathcal{Q}(\tilde{\Lambda}_{s,n}^{(k,q)}(\mathbf{x}_n))$, $\forall r \neq s$. Therefore, if this condition is verified, one could replace $\Lambda_n^{(q)}(\mathbf{x}_n)$ in (13)-(14) with the quantized version of $\tilde{\Lambda}_{s,n}^{(k,q)}(\mathbf{x}_n)$ for any chosen s, yielding the same particle set and weights at all nodes.

The latter convergence condition can be verified by running a parallel *minimum consensus* protocol [7]. Namely, it suffices to verify whether $\min\limits_r\{\iota_{r,n}^{(k,q)}\} = 1$ for each particle q. In turn, $\min\limits_r\{\iota_{r,n}^{(k,q)}\}$ can be computed iteratively at each node r using the recursion

$$\tilde{\iota}_{r,n}^{(l+1,q)} = \min\{\tilde{\iota}_{s,n}^{(l,q)}\}, \quad s \in \{\mathbf{N}(r) \cup r\}, \tag{19}$$

where $\tilde{\iota}_{r,n}^{(0,q)} = \iota_{r,n}^{(k,q)}$ and l is a separate iteration index. Minimum (or Maximum) consensus iterations are guaranteed to converge in at most D steps, where D is the diameter of the network graph, i.e., the longest shortest path between any two pair of nodes [7]. Therefore, if $\tilde{\iota}_{r,n}^{(D,q)} = 1$ at any node, one can assure that consensus on the quantized sum of log densities has been achieved.

To avoid the need to perform D minimum consensus steps for each average consensus step k, one can use the technique introduced in [7], which consists of making $l = k$ and reseting the minimum consensus protocol at every D steps, i.e., making $\tilde{\iota}_{r,n}^{(k,q)} = \iota_{r,n}^{(k,q)}$, if $\mod(k, D) = 0$. Note that this procedure only allows one to check whether consensus has been reached for k such that $\mod(k-1, D) = 0$.

4.2. CB-II Algorithm

The algorithm CB-I suffers from the fact that there is no obvious way to quantize $\Lambda_{r,n}^{(k,q)}$, since these quantities are unbounded and their distribution is hard to determine a priori. Therefore, if an excessively fine quantization step is employed, average consensus may take too long to converge. On the other hand, if an overly coarse step is used, particle filter performance may be affected.

To sidestep this limitation, we propose that consensus be checked instead for the quantities

$$\tilde{p}_r^k(\mathbf{x}_n|\mathbf{x}_{0:n-1}^{(q)}, y_{1:R,0:n}) = \mathcal{Q}_1\left(\frac{\exp\left(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)\right)}{\sum\limits_{\mathbf{x}_n}\exp\left(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)\right)}\right), \tag{20}$$

$$\tilde{w}_{r,n}^{(k,q)} = \mathcal{Q}_2\left(\frac{\tilde{w}_{r,n-1}^{(q)}\sum\limits_{\mathbf{x}_n}\exp\left(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n) - \tilde{\Lambda}_{r,n-1}^{(q)}(\mathbf{x}_{n-1}^{(q)})\right)}{\sum\limits_q\sum\limits_{\mathbf{x}_n}\exp\left(\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n) - \tilde{\Lambda}_{r,n-1}^{(q)}(\mathbf{x}_{n-1}^{(q)})\right)}\right), \tag{21}$$

which, unlike $\tilde{\Lambda}_{r,n}^{(k,q)}$, are bounded real-valued numbers in the interval $[0, 1]$. In (20) and (21), $\tilde{\Lambda}_{r,n}^{(k,q)}(\mathbf{x}_n)$ is propagated via consensus averaging as in the CB-I algorithm. On the other hand, $\tilde{w}_{r,n-1}^{(q)}$ and $\tilde{\Lambda}_{r,n-1}^{(q)}(\mathbf{x}_{n-1}^{(q)})$ denote the last unquantized approximations obtained via consensus averaging at instant $n-1$.

5. SIMULATION RESULTS

The steady state performance of the proposed algorithm was assessed via simulations consisting of 200 independent Monte Carlo runs. In each realization, we computed the mean bit error rate (BER) as a function of E_B/N_0, transmitting a random sequence of 300 i.i.d bits, with the first 150 bits discarded to allow for convergence. For comparison, we ran with the same setup the DcPF-II algorithm [5]. The simulated system has $R = 3$ receiving nodes and the filters employed $Q = 300$ particles. All algorithms perform synchronized residual resampling [3] at all iterations. The transmission channels $\mathbf{h}_r$

have $L = 3$ coefficients, and were obtained by sampling independently in each realization and for each receiver from a complex Gaussian p.d.f. $\mathcal{N}(\mathbf{0}; \Lambda)$, $\Lambda = \text{diag}(2, 1, 0.5)$, and normalized so that $\|\mathbf{h}_r\|^2 = 1$. The noise variances were determined as $\sigma^2 = \|\mathbf{h}_r\|^2 N_0/E_B$. The model hyperparameters were set to $\alpha = 3$, $\beta = 0.1$ and $\epsilon = 1$. In the following results, for all values of σ^2, we assumed for the CB-I algorithm $\mathcal{Q}(x) = [x]$, where $[\cdot]$ denotes rounding to the nearest integer. For the CB-II algorithm we adopted $\mathcal{Q}_1(x) = \mathcal{Q}_2(x) = [Qx]/Q$. The average consensus weights $\{a_{rs}\}$ employed were

$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The results are shown in Fig. 1. For comparison, we also show in Fig. 1 the mean BER when the receivers operate independently and do not cooperate to improve their local signal estimate. The mean BER for the centralized Forward-Backward algorithm with perfect knowledge of the channel parameters is shown as a lower bound to performance. As one may observe, the cooperative algorithms (DcPF-II, CB-I and CB-II) outperform the isolated receivers. As expected, the consensus-based algorithms exhibit a performance gap in relation to the DcPF-II due to quantization.

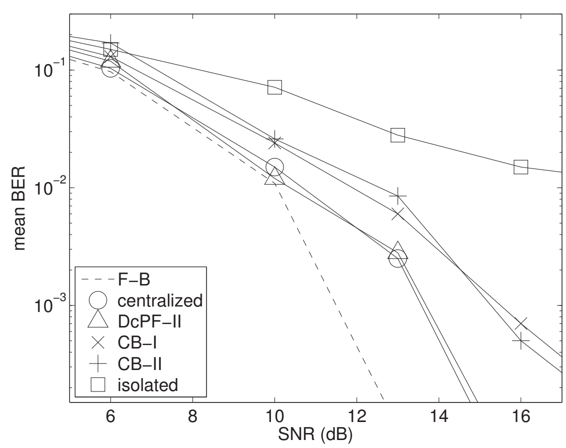

Fig. 1. Mean bit error rate (BER) estimated in 200 independent runs.

6. CONCLUSIONS

We introduced in this paper two new consensus-based distributed particle filtering algorithms. The techniques described in the paper can be applied to any filtering problem with conditionally independent linear Gaussian observations and discrete-valued variables. The two proposed algorithms (CB-I and CB-II) quantize distinct particle-filter-related variables, leading therefore to different convergence behaviors. In a cooperative blind equalization problem with multiple distributed receivers, the CB-I and CB-II algorithms exhibited similar BER performances in simulated Monte Carlo experiments. Both algorithms outperformed the non-cooperative isolated receivers and showed a small performance loss compared both to the centralized blind equalizer, which employs a fusion center, and the optimal decentralized DcPF-II algorithm from [5], which requires message broadcasting to the entire network.

7. REFERENCES

[1] A. Ribeiro, G. B. Giannakis, and S. I. Roumeliotis, "SOI-KF: Distributed Kalman Filtering With Low-Cost Communications Using the Sign of Innovations," *IEEE Trans. on Sig. Proc.*, vol. 54, no. 12, pp. 4782–4795, 2006.

[2] C. G. Lopes and A. H. Sayed, "Incremental Adaptive Strategies Over Distributed Networks," *IEEE Trans. on Sig. Proc.*, vol. 55, no. 8, pp. 4064–4077, Aug. 2007.

[3] M. J. Coates, "Distributed particle filtering for sensor networks," in *3rd Intl. Symp. on Inf. Proc. in Sensor Networks*, pp. 99–107, Berkeley - CA, April 2004.

[4] H. Sangjin, M. Hong and P. M. Djurić, "An efficient fixed-point implementation of residual resampling scheme for high-speed particle filters," *IEEE Sig. Proc. Lett.*, vol. 11, no. 5, pp. 482–485, May 2004.

[5] C. J. Bordin Jr. and M. G. S. Bruno, "Cooperative blind equalization of frequency-selective channels in sensor networks using decentralized particle filtering," in *42nd Asilomar Conf. on Sign., Syst. and Comp.*, pp. 1198–1201, Pacific Grove - CA, Oct. 2008.

[6] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Syst. Control Lett.*, vol. 53, no. 1, pp. 65-78, Sep. 2004.

[7] V. Yadav and M. V. Salapaka, "Distributed protocol for determining when averaging consensus is reached," in *45th Annual Allerton Conf.*, pp. 715–720, Allerton House - UIUC, Sep. 2007.

[8] S. Farahmand, S. I. Roumeliotis and G. B. Giannakis, "Particle filter adaptation for distributed sensors via set membership," in *IEEE Intl. Conf. on Acoustics, Speech, and Sig. Proc.*, pp. 3374–3377, Dallas - TX, Mar. 2010.

[9] C. J. Bordin Jr. and M. G. S. Bruno, "Particle Filters for Joint Blind Equalization and Decoding in Frequency-Selective Channels," *IEEE Trans. on Sig. Proc.*, vol. 56, no. 6, pp. 2395–2405, June 2008.

[10] J. R. Norris, *Markov Chains*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 1997.