Reducibility of Joint Relay Positioning and Flow Optimization Problem

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***Abstract*—This paper shows how to reduce the otherwise hard joint relay positioning and flow optimization problem into a sequence a two simpler decoupled problems. We consider a class of wireless multicast hypergraphs not limited by interference and are mainly characterized by their hyperarc rate functions, that are increasing and convex in power, and decreasing in distance between the transmit node and the farthest end node of the hyperarc. The set-up consists of a single multicast flow session involving a source, multiple destinations and a relay that can be positioned freely. The first problem formulates the relay positioning problem in a purely geometric sense, and once the optimal relay position is obtained the second problem addresses the flow optimization. Furthermore, simple and efficient algorithms are presented that solve these problems.**

I. INTRODUCTION

We consider the relay placement problem under a relatively simple construct of a single multicast session consisting of a source s, a destination set T and an arbitrarily positionable relay r, all on a 2-D Euclidean plane. We capture the broadcast nature of wireless networks but assume no interference for the multiple access. This is not true in general, but it can be realized through the multiplexing of time, frequency or code. Given such a scenario, the problem can be stated as: *What is the optimal relay position that maximizes the multicast flow from s to T?* Similarly, we can also ask: *What is the optimal relay position that minimizes the cost (in terms of total network power) for a feasible target multicast flow F?*

A fairly general class of acyclic hypergraphs are considered. Given a finite set of nodes $\mathcal{N}$ positioned on a 2-D Euclidean plane the hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ is characterized by the following rules of construction of the hyperarcs in the set $\mathcal{A}$:

1) A transmit node $i \in \{s, r\}$ has a radio range and the set of nodes lying inside it are given by $T_i \subset T$. A hyperarc (i, V_i) emanating from i connects to a set of receivers (or end nodes) $V_i \subset \mathcal{N} \backslash \{i\}$ in its radio range.
2) Each hyperarc is associated with a rate function that is convex and increasing in transmit node power and decreasing in distance between the transmit node and the farthest end node of the hyperarc. Each end node $v \in V_i$ spanned by the hyperarc (i, V_i) can decode the information sent over the hyperarc equally reliably, i.e. all the end nodes in V_i get the same rate.
3) There exists at least one path from s to any $t \in T$.

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The construction rules capture the distance-based broadcast approach, e.g. time sharing for broadcasting is a special case of the above construction rules that is also capacity optimal. In relation to the special case of our hypergraph model, the authors addressed the first question (of max-flow) in the context of the Low-SNR Broadcast Relay Channel in [1], [2]. This paper generalizes it in two major directions. First, we solve the joint relay positioning and max-flow optimization problem for the hypergraph model characterized above involving a fairly general class of rate functions and present efficient algorithms that solve the problem. Second, we address the min-cost flow problem and establish a duality relation between the max-flow and min-cost problems in terms of optimal relay positioning and present an algorithm that solves an important special case of the min-cost problem.

The relay positioning problem has been studied in various settings [3]–[5]. In most cases, the problem is either heuristically solved due to inherent complexity, or approximately solved using simpler methods but compromising accuracy. We reduce the joint non-convex problem into an easily solvable sequence of two decoupled problems. The first problem solves for optimal relay position in a purely geometric sense with no flow optimization involved. Upon obtaining the optimal relay position the second problem addresses the flow optimization.

Section II develops the wireless network model. Section III presents the key multicast flow concentration ideas for max-flow and min-cost flow that are central to the reducibility of the joint problem. In Section IV, we present algorithms solving the problems, and finally we conclude in Section V.

II. PRELIMINARIES AND MODEL

Consider a hypergraph $\mathcal{G}(\mathcal{N}, \mathcal{A})$ consisting of $|\mathcal{N}|$ nodes placed on a 2-D Euclidean plane with $|\mathcal{A}|$ hyperarcs. The set $\mathcal{N} = \{s, r, t_1, .., t_n\}$ consists of a source node s, a relay r and an ordered destination set $T = \{t_1, .., t_n\}$ (in increasing distance from s). The node positions on the 2-D Euclidean plane are denoted by the set of two-tuple vectors $\mathcal{Z} = \{z_i = (x_j, y_j) | \forall j \in \mathcal{N}\}$. The only positionable node is the relay r.

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To motivate our model, we review the Low-SNR Achievable Hypergraph Model that was developed and presented in [1].

A. Low-SNR (Wideband) Broadcast and MAC Channel Model

Consider the AWGN Low-SNR (wideband) Broadcast Channel with a single source s and multiple destinations $T = \{t_1, .., t_n\}$. From [6] and [7], we know that for broadcasting superposition coding is equivalent to time sharing, and is optimal. Implying that the broadcast communication from a single source to multiple receivers can be decomposed into communication over n hyperarcs sharing the common source power. Therefore, we get the set of hyperarcs $\mathcal{A}_{bc} = \{(s, t_1), (s, t_1 t_2), .., (s, t_1 t_2 .. t_n)\}$ (refer Figure 1).

Furthermore, in the Low-SNR (wideband) regime interference becomes negligible with respect to noise, and all sources can achieve their point-to-point capacities analogous to Frequency Division Multiple Access (FDMA). In general, the MAC Channel consisting of n sources $s_1, ..., s_n$ transmitting to a common destination t can be interpreted as n point-to-point arcs. Thus, we get $\mathcal{A}_{mac} = \{(s_1, t), .., (s_n, t)\}$. Due to the linearity of hyperarc capacity in SNR, each hyperarc $(s, t_1..t_j) \in \mathcal{A}_{bc} \cup \mathcal{A}_{mac}$ is associated with the rate function

$$R^s_{t_j} = \frac{P^s_{t_j}}{N_0 D^{\alpha}_{st_j}} \text{ bits/sec}, \tag{1}$$

where t_j is the farthest end node of the hyperarc from s, $\alpha \geq 2$ is the path loss exponent and N_0 is the channel noise variance.

B. Low-SNR (Wideband) Achievable Hypergraph Model

By concatenating the Low-SNR Broadcast Channel and MAC Channel models we obtain an Achievable Hypergraph Model, for instance the Broadcast Relay Channel consisting of a single source, n destinations and a relay. Although, time sharing and FDMA are capacity achieving schemes, the Achievable Hypergraph Model is not necessarily capacity achieving. In contrast, it is easy to scale to larger and complex networks which is practically important.

For simplicity, throughout this paper we will assume that all the nodes lie in the radio range of s and r for any arrangement of the nodes $\mathcal{N}$ on the plane, and that both transmitters use time-sharing based broadcasting. As there are only two transmitters in the system, i.e. s and r, for a given position of r and assuming (without loss of generality) the arrangements of node sets $\{t_1, .., r, .., t_n\}$ and $\{t'_1, .., t'_n\}$ in increasing order of distance from s and r, respectively, the sets $\mathcal{A}_s = \{(s, t_1), (s, t_1 t_2), .., (s, t_1..r), .., (s, t_1..r..t_n)\}$ and $\mathcal{A}_r = \{(r, t'_1), (r, t'_1 t'_2), .., (s, t'_1..t'_n)\}$ of hyperarcs emanating from s and r can be constructed easily by the using the simple construction rules stated in the previous section. Here, the set $\{t'_1, .., t'_n\}$ is simply the destination set T arranged in increasing order of distance from r. Consequently, the set of hyperarcs is given by $\mathcal{A} = \mathcal{A}_s \cup \mathcal{A}_r$ (refer Figure 1).

Any hyperarc $(u, V_{k_u}) \in \mathcal{A}$ is associated with a rate function $R^u_{v_{k_u}} = f(P^u_{v_{k_u}}, D_{uv_{k_u}})$, where $P^u_{k_u}$ and $D_{uv_{k_u}}$ denotes the fraction of the total transmit node power allocated for the hyperarc and the Euclidean distance between transmit node u and the farthest end node $v_{k_u} \in V_{k_u}$ from u, respectively.

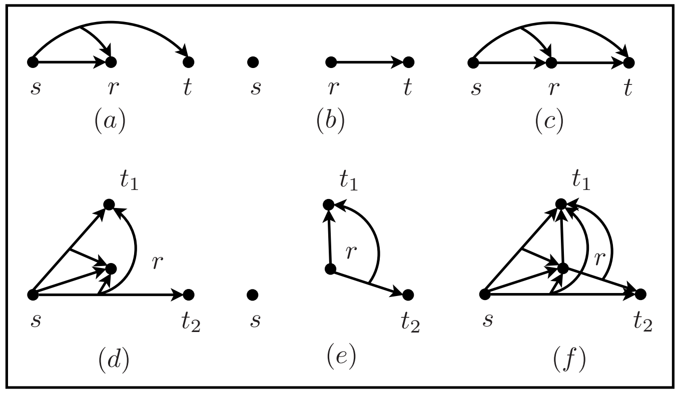

Fig. 1. Hyperarcs are constructed in increasing order of distance from the transmitter. (a)-(c): 3 node system. (a): $\mathcal{A}_s = \{(s, r), (s, rt)\}$. (b): $\mathcal{A}_r = \{(r, t)\}$. (c): $\mathcal{G}(\mathcal{N}, \mathcal{A})$, where $\mathcal{A} = \mathcal{A}_s \cup \mathcal{A}_r$. (d)-(f): 4 node system with $T = \{t_1, t_2\}$ such that $D_{sr} < D_{st_1} < D_{st_2}$ and $D_{rt_1} < D_{rt_2}$. (d): $\mathcal{A}_s = \{(s, r), (s, rt_1), (s, rt_1 t_2)\}$. (e): $\mathcal{A}_r = \{(r, t_1), (r, t_1 t_2)\}$. (f): $\mathcal{G}(\mathcal{N}, \mathcal{A})$.

We assume that the hyperarc rate function $R^u_{v_{k_u}}$ is increasing and convex in power $P^u_{v_{k_u}}$ and decreasing in $D_{uv_{k_u}}$. Furthermore, we write the hyperarc rate function into two separable functions of power and distance

$$R^u_{v_{k_u}} = \frac{g(P^u_{v_{k_u}})}{h(D_{uv_{k_u}})} \tag{2}$$

where $g : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is increasing and convex and $h : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is increasing. This separation can be assumed, without loss of generality as the results in this paper are valid for any rate function $R^u_{v_{k_u}} = f(P^u_{v_{k_u}}, D_{uv_{k_u}})$, where f is increasing and convex in power and decreasing in distance and in general is not separable and if the inverse function f^{-1} (mapping the range of f to its domain) exists. Moreover, to comply with standard physical wireless channel models we assume that

$$\frac{\partial g(P^u_{v_{k_u}})}{\partial P^u_{v_{k_u}}} \leq \frac{\partial h(D_{uv_{k_u}})}{\partial D_{uv_{k_u}}}, \tag{3}$$

$\forall (P^u_{v_{k_u}} = D_{uv_{k_u}}) \in \mathbf{dom}(P^u_{v_{k_u}}, D_{uv_{k_u}})$. Inequality (3) implies that if the variables $P^u_{v_{k_u}}$ and $D_{uv_{k_u}}$ approach infinity at the same rate, then the rate function $R^u_{v_u} \longrightarrow 0$. Moreover, if the functions g and h are not differentiable entirely in $\mathbf{dom}(P^u_{v_{k_u}})$ and $\mathbf{dom}(D_{uv_{k_u}})$, respectively, then Inequality 3 can be rewritten with partial sub-derivatives, which implies that differentiability is not imperative for the results presented.

Denote the convex hull of the nodes in $\mathcal{N} \backslash \{r\}$ by $\mathcal{C}$. For a given relay position $z_r \in \mathcal{C}$, let $L_i = \{l^i_1, .., l^i_{\tau_i}\}$ be the set of paths from s to a destination $t_i \in T$ and let $L = \{l_1, .., l_\tau\}$ be the set of paths from s that span all the destination set T, therefore $L \subset \bigcup_{i \in [1,n]} L_i$. Moreover, any path in the system consists of either a single hyperarc or at most two hyperarcs as there are only two transmitters in the system and $\mathcal{G}(\mathcal{N}, \mathcal{A})$ is acyclic. Let μ and ν denote the total given power of the source and relay, respectively, and $\gamma = \frac{\nu}{\mu}$ denote their ratio, where $\gamma \in (0, \infty)$. Denote with $F_{l^i_j}$ and F_i the flow over the path l^i_j (for $j \in [1, \tau_i]$) and the total flow to the destination

$t_i \in T$, respectively, such that $F_i = \sum_{j\in[1,\tau_i]} F_{l^i_j}$. Define F to be the the multicast flow from s to the destination set T as the minimum among the total flows to each destination, then for a given relay position $z_r \in \mathcal{C}$ the multicast max-flow problem can be written as,

$$\text{Maximize} \qquad \left(F = \min_{i\in[1,n]} F_i\right) \tag{A}$$

$$\text{subject to: } F_i \leq \sum_{j=1}^{\tau_i} F_{l^i_j}, \forall i \in [1,n], \tag{4}$$

$$0 \leq F_{l^i_j} \in \mathfrak{C}(P,D), \quad \forall j \in [1,\tau_i], \forall i \in [1,n]. \tag{5}$$

The hyperarc rate constraints and node sum-power constraints are denoted by the set $\mathfrak{C}(P,D)$ in Program (A) for simplicity. Program (A) in general is non-convex.

Now, let us define the notion of cost for a given hyperarc rate $R^u_{v_{k_u}} = \frac{g(P^u_{v_{k_u}})}{h(D_{uv_{k_u}})} \geq 0$. The cost of rate $R^u_{v_{k_u}}$ is given by the total power consumed by the hyperarc to achieve $R^u_{v_{k_u}}$

$$P^u_{v_{k_u}} = g^{-1}\left(R^u_{v_{k_u}} h(D_{uv_{k_u}})\right), \tag{6}$$

where $g^{-1} : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is the inverse function of g that maps its range to its domain. Therefore, the total cost of multicast flow F is simply the sum of powers of all the hypearcs in the system. Note that the function g^{-1} is increasing and concave, and if h is convex then from Inequality (3) $g^{-1} \circ h$ increasing and convex. So for a given relay position $z_r \in \mathcal{C}$, the min-cost problem minimizing the total cost for setting up the multicast session (s,T) with a target flow F can be written as,

$$\text{Minimize} \qquad \left(P = \sum_{(u,V_{k_u})\in\mathcal{A}} P^u_{v_{k_u}}\right) \tag{B}$$

$$\text{subject to: } F \leq F_i \leq \sum_{j=1}^{\tau_i} F_{l^i_j}, \quad \forall i \in [1,n], \tag{7}$$

$$\mathfrak{C}(P,D) \ni F_{l^i_j} \geq 0, \forall j \in [1,\tau_i], \forall i \in [1,n]. \tag{8}$$

Constraint (7) makes sure that any destination $t_i \in T$ receives a minimum of flow F. Like in Program (A), we denote with the set $\mathfrak{C}(P,D)$ the hyperarc rate and power constraints.

Finally, define the point p^*, that will be crucial in developing algorithms in later sections,

$$z_{p^*} = \arg\min_{z_p}(\max(\nu^* h(D_{z_p s}), \mu^* \max_{t_i\in T}(h(D_{z_p t_i})))), \tag{9}$$

where, $\mu^* = g(\mu)$ and $\nu^* = g(\nu)$. An intuitive way to understand p^* is that if $\mu^* = \nu^* = 1$ then p^* is the circumcenter of two or more nodes in the set $\mathcal{N}\backslash\{r\}$. Note that the program in Equation (9) is a convex program, and finally D_{p^*} denotes the optimal value of the objective function.

Hereafter, we represent with $(s,T,\mathcal{Z},\gamma)$ and $(s,T,\mathcal{Z},\gamma,F)$ the joint relay positioning and flow optimization problem instances that maximizes the multicast flow and minimizes the total cost for a the target flow F, and with $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$ denote the optimal relay positions, respectively. Note that the optimal relay position $z^*_{\gamma\uparrow}$ is characterized by the power ratio γ and the optimal relay position $z^*_{F\downarrow}$ is characterized by the target flow F.

III. Multicast Flow Properties And Reduction

In this section we develop fundamental multicast flow properties that govern the multicast flow in the wireless network hypergraphs that we consider in this paper. First, we briefly note the main hurdles in jointly optimizing the problem. For a given problem instance different relay positions can result in different hypergraphs, which makes the use of standard graph-based flow optimization algorithms difficult. Moreover, the hyperarc rate function can be non-convex.

We will show that the joint problems $(s,T,\mathcal{Z},\gamma)$ and $(s,T,\mathcal{Z},\gamma,F)$ can be reduced to solving a sequence of two decoupled problems. The reduced problems are decoupled in the sense that the first problem is purely a geometric optimization problem and involves no flow optimization and vice versa for the second problem. At the same time, they are not entirely decoupled because the two problems cannot be solved separately. Now we present a series of results that are fundamental to the reducibility of the joint problem.

Proposition 1: The optimal relay positions $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$ lie inside the convex hull $\mathcal{C}$.

Refer to Appendix A in [8] for the proof. Proposition 1 tells us that only the points inside the polygon $\mathcal{C}$ need to be considered. This brings us to the next fundamental theorem.

Theorem 1 (Flow Concentration): Given the problem instance $(s,T,\mathcal{Z})$ and $z_r \in \mathcal{C}$:

(i) the maximized multicast flow F^* concentrates over at most two paths from s to the destination set T.

(ii) for any target flow $F \in [0,F^*]$ the min-cost multicast flow concentrates over at most two paths from s to T.

The proof of Theorem 1 is detailed in Appendix B of [8]. Essentially, Theorem 1 tells that for a given relay position $z_r \in \mathcal{C}$, the multicast flow F goes only over the paths that span all the destination set T, i.e. set L. Furthermore, among the paths in L, the maximized multicast flow F^* goes over only two paths, namely the path $L \ni \hat{l}_1 = \{(s,T_1),(r,T_2)\}$ that has the highest min-cut among all the paths through the relay r, and path $L \ni \hat{l}_2 = \{(s,t_1,..,t_n) = (s,T)\}$, which is the biggest hyperarc from s spanning all the destination set T, here $r \in T_1$ and $T_1 \cup T_2 = T$. The same holds for the min-cost case for a given relay position $z_r \in \mathcal{C}$. Consequently, it is also true for the optimal relay positions $z^*_{\gamma\uparrow}$ and $z^*_{F\downarrow}$. Hereafter, we only need to consider the flow over paths $\hat{l}_1$ and $\hat{l}_2$ (corresponding to the relay position in consideration).

A. Max-flow Problem - $(s,T,\mathcal{Z},\gamma)$

Assuming that the transmitted signal propagates omnidirectionally, we can geometrically represent the hyperarcs of the path $\hat{l}_1 = \{(s,T_1),(r,T_2)\}$ by circles $C^s_{T_1}$ and $C^r_{T_2}$ centered at s and r with radii $\pi_s = D_{st_k}$ and $\pi_r = D_{rt_{k'}}$ (where $D_{st_k} = \max_{t_i\in T_1}(D_{st_i})$ and $D_{rt_{k'}} = \max_{t_j\in T_2}(D_{rt_j})$), respectively. Similarly, the path $\hat{l}_2 = \{(s,T)\}$ can be represented by the circle C^s_T with radius D_{st_n}. Also, $C_\cup = C^s_{T_1} \cup C^r_{T_2}$ denotes the union region of the two circles. Then using

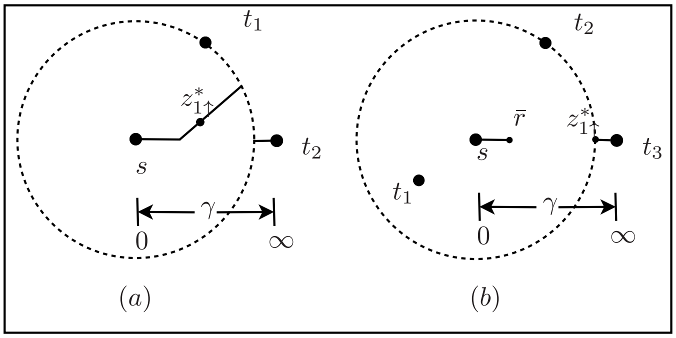

Fig. 2. The solid piecewise linear segment in examples (a) and (b) marks the set $\widehat{\mathcal{R}}$ for different values of $\pi_s \in (0, D_{st_2})$. Each point $\widehat{r}$ corresponds to $z^*_{\gamma\uparrow}$ for some $\gamma \in (0, \infty)$. (a): E.g. C^s_r with $0 < \pi_s < D_{st_1}$, $z_{\widehat{r}} = \arg\min_{\hat{z_r} \in C^s_r} \max(D_{\hat{z_r} t_1}, D_{\hat{z_r} t_2})$. Same goes for the example in (b).

Theorem 1, Program (A) can be re-written as,

$$\underset{\substack{P^s_{T_1}+P^s_T \le \mu,\\ P^r_{T_2} \le \nu, \pi_s, \pi_r}}{\text{Maximize}} \left(\min\left(\frac{g(P^s_{T_1})}{h(\pi_s)}, \frac{g(P^r_{T_2})}{h(\pi_r)} \right) + \frac{g(P^s_T)}{h(D_{st_n})} \right) \quad \text{(C)}$$

where, $P^s_{T_1}, P^r_{T_2}$ and P^s_T are the powers for hyperarcs of the paths $\hat{l}_1 = \{C^s_{T_1}, C^r_{T_2}\}$ and $\hat{l}_2 = \{C^s_T\}$, respectively. The radii variables π_s and π_r correspond to path $\hat{l}_1$ for the relay position $z_r \in \mathcal{C}$ such that $z_r \in C^s_{T_1}$ and $\mathcal{Z} \in C_\cup$. Although Program (C) is substantially reduced using Theorem 1, it is still a non-convex optimization problem. The objective function is non-convex and different positions of the relay $z_r \in \mathcal{C}$ result in different end node sets T_1 and T_2 for the hyperarcs of path $\hat{l}_1$.

On the other hand, we know that the relay position is sensitive only to the flow over path $\hat{l}_1$. Therefore, the optimal relay position maximizing the flow only over the path $\hat{l}_1$ results in global optimal relay position solving $(s, T, \mathcal{Z}, \gamma)$. This fact combined with the independent power constraints for the flow over the path $\hat{l}_1$ motivates the decoupling optimal relay position from the flow maximization.

Proposition 2: For a given problem instance $(s, T, \mathcal{Z}, \gamma)$, if for a close enough point p_ϵ on the perpendicular bisector $\perp_{jn}$ to p^* the following holds $R^*_{p_\epsilon} < R^*_{p^*}$, then $z^*_{\gamma\uparrow} = z_{p^*}$.

Refer to Appendix C in [8] for the detailed proof. $\perp_{jn}$ denotes the perpendicular bisector of the points t_j and t_n, where t_j is the farthest node in T from t_n. At point p^*, in general the following holds $\frac{g(\mu)}{h(\pi^{p^*}_s)} \ge \frac{g(\nu)}{h(\pi^{p^*}_r)}$ (from Equation (9)), thus making it naturally a good candidate for $z^*_{\gamma\uparrow}$. Proposition 2, shows that the point p^* is essentially a good lower bound on the maximized multicast rate $R^*_{p^*} \le R^*$. In addition, if positioning the relay close enough to p^* at point p_ϵ (in the direction closer to the source s) on the perpendicular bisector $\perp_{jn}$ decreases the maximized multicast rate, i.e. $R^*_{p_\epsilon} < R^*_{p^*}$, then it can be proven that $z^*_{\gamma\uparrow} = z_{p^*}$. The general case is covered in the next section.

Let us now see the problem in a different way. Consider $\pi_s \in (0, D_{st_n})$ and construct the hyperarc $C^s_{\pi_s}$. Denote by $T' = \{t_j \in T | D_{st_j} > \pi_s\}$, the set of destination nodes that lie outside the hyperarc circle $C^s_{\pi_s}$. Then compute $\widehat{r}$ such that

$$z_{\widehat{r}} = \underset{z_p \in C^s_{\pi_s}}{\arg\min} (\max_{t_j \in T'} (D_{r' t_j})),$$

and position the relay at $\widehat{r}$ (here $\widehat{r}$ is the point in $C^s_{\pi_s}$ such that the maximum among the distances to the nodes in the set T' from $\widehat{r}$ is minimized). If $D_{s\widehat{r}} < \pi_s$, then we contract the hyperarc $C^s_{\pi_s}$ to $C^s_{\widehat{r}}$, else we simply re-denote it with $C^s_{\widehat{r}}$. Finally, we can construct the hyperarc $C^{\widehat{r}}_{t_n}$ (note that $\mathcal{Z} \in C'_\cup = C^s_{\widehat{r}} \cup C^{\widehat{r}}_{t_n}$). Interestingly, the set $\mathcal{R}'$ of points $\widehat{r}$ computed in this way for different values of $\pi_s \in (0, D_{st_n})$ are the optimal relay positions $z^*_{\gamma\uparrow}$ solving $(s, T, \mathcal{Z}, \gamma)$ for some $\gamma \in (0, \infty)$. Figure 2(a) captures this interesting insight of the relationship between the points $\widehat{r}$ and $z^*_{\gamma\uparrow}$. Note that the set $\widehat{\mathcal{R}}$ of points $\widehat{r}$ is a discontinuous piecewise linear segment.

B. Min-cost Problem $(s, T, \mathcal{Z}, \gamma, F)$ And Duality

The min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ can be re-written as

$$\text{Minimize} \quad (P^s_{T_1} + P^r_{T_2} + P^s_T) \quad \text{(D)}$$

$$\text{subject to: } F \le \min\left(\frac{g(P^s_{T_1})}{h(\pi_s)}, \frac{g(Pr_{T_2})}{h(\pi_r)} \right) + \frac{g(P^s_T)}{h(D_{st_n})}, \quad (10)$$

$$P^s_{T_1} + P^s_T \le \mu, \quad P^r_{T_2} \le \nu. \quad (11)$$

Here, $F \in [0, F^*]$. From Theorem 1, we know that paths $\hat{l}_1$ and $\hat{l}_2$ carry all the min-cost target multicast flow F. For convenience, we will refer the path $\hat{l}_1$ as the cheapest path for a unit flow among all the paths through r in L for given position of relay. We claim that $z^*_{F\downarrow} \in \widehat{\mathcal{R}}$, to see this let $\hat{l}_1 = \{C^s_{T_1}, C^r_{T_2}\}$ be the two hyperarcs of path $\hat{l}_1$ corresponding to the relay position $z^*_{F\downarrow}$. Then, $z^*_{F\downarrow}$ must satisfy

$$z^*_{F\downarrow} = \underset{z_r \in C^s_{T_1}}{\arg\min} (\max_{t \in T_2} (D_{rt})), \quad (12)$$

Otherwise, we can position the relay at the point $z^*_r \in C^s_{T_1}$ solving the right hand side of Equation (12), further reducing the total cost for the target flow F contradicting the optimality of $z^*_{F\downarrow}$. Therefore, $z^*_{F\downarrow}$ (like $z^*_{\gamma\uparrow}$) always lie on on the curve $\widehat{\mathcal{R}}$. This observation motivates an interesting fundamental relationship between $z^*_{F\downarrow}$ and $z^*_{\gamma\uparrow}$.

Theorem 2 (Max-flow/Min-cost Duality): For $F \in [0, F^*]$,

$$z^*_{F\downarrow} = z^*_{\widehat{\gamma}\uparrow}, \quad (13)$$

where $\widehat{\gamma} \in [\min(\overline{\gamma}, \gamma), \max(\overline{\gamma}, \gamma)]$ and $z^*_{1\downarrow} = z^*_{\overline{\gamma}\uparrow}$.

Theorem 2 establishes the underlying duality relation between the max-flow and min-cost problem in terms of optimal relay position. Implying that the optimal relay position $z^*_{F\downarrow}$ solving the problem $(s, T, \mathcal{Z}, \gamma, F)$ is also the optimal relay position $z^*_{\widehat{\gamma}\uparrow}$ solving the problem $(s, T, \mathcal{Z}, \gamma)$ for some $\widehat{\gamma}$. The proof of Theorem 2 is presented in Appendix D of [8].

On the other hand, the min-cost problem $(s, T, \mathcal{Z}, \gamma, F)$ is not always reducible to a sequence of decoupled problems. This is beacause the path $\hat{l}_2$ can be cheaper than path $\hat{l}_1$ for a unit flow corresponding to the optimal position $z^*_{F\downarrow}$, i.e.

$$g^{-1}(h(\pi^*_s)) + g^{-1}(h(\pi^*_r)) \ge g^{-1}(h(D_{st_n})).$$

This information is not easy to get a priori as π^*_s and π^*_r are unknown. In contrast, we can safely assume that

$$g^{-1}(h(\pi^*_s)) + g^{-1}(h(\pi^*_r)) \le g^{-1}(h(D_{st_n})), \quad (14)$$

as almost all wireless network models that comply with our model result in the hyperarc cost function $g^{-1}(h(D_{uv_{k_u}}))$ being the increasing convex function of distance $D_{uv_{k_u}}$ that satisfy Inequality (14). If Inequality (14) holds, then similar to the Max-flow problem the joint optimal relay positioning and min-cost flow optimization problem in Program (D) can be reduced to a sequence of decoupled problems of computing the optimal relay position and then optimizing the hyperarc powers to achieve the min-cost flow F in the network using the similar arguments as in previous subsection.

IV. ALGORITHMS

In this section we present the general max-flow and min-cost (for problem instances satisfying Equation (14)) algorithms that solve the sequence of decoupled problems.

A. Max-flow Algorithm

Input: Problem instance $(s, T, \mathcal{Z}, \gamma)$.

1: Compute p^*, if $R^*_{p^\epsilon} < R^*_{p^*}$ (for $p_\epsilon \in \perp_{jn}$), output $z^*_{\gamma\uparrow} = z_{p^*}$, $F^* = g(\nu)h(D_{sp^*})$ and quit, else go to 2.

2: Construct the set $T' = \{t'_j \in T | D_{st'_j} < D_{p^* t'_j}\} = \{t'_1, .., t'_{j'}\}$ (ordered in increasing distance from s) and compute $p^*_{T\setminus T'}$. If $D_{st'_{j'}} \leq D_{sp^*_{T\setminus T'}}$, declare $z^*_{\gamma\uparrow} = z_{p^*_{T\setminus T'}}$ and $F^* = g(\nu)h(D_{sp^*_{T\setminus T'}})$ and quit, else go to Step 3.

3: Compute the points z^*_1 and z^*_2, and maximized multicast flow F^*_1 and F^*_2, respectively. Declare before quitting,

$$z^*_{\gamma\uparrow} = \begin{cases} z^*_1 & \text{if } F^*_1 > F^*_2, \\ z^*_2 & \text{if } F^*_1 < F^*_2. \end{cases}$$

Output: $z^*_{\gamma\uparrow}$ and F^*.

Fig. 3. Max-flow Algorithm.

The Max-flow Algorithm in Figure 3 is a simple 3 step (non-iterative) algorithm that outputs the optimal relay position and the maximized multicast flow. The first step is essentially Proposition 2, if it does not hold the second step filters the redundant nodes that are too close to the source and can be ignored. If the first and second step are not satisfied, then the third step divides the computation of $z^*_{\gamma\uparrow}$ into two regions of $\mathcal{C}$ and outputs the better one. The proof of optimality is provided in Appendix E of [8].

B. Min-cost Algorithm

We present the Min-cost Algorithm for a special case of $(s, T, \mathcal{Z}, \gamma, F)$, that is when the Inequality (14) holds and at the optimal relay position $z^*_{F\downarrow}$ all the target flow goes over path $\hat{l}_1$. This special case corresponds to satisfying Proposition 2 for the Max-flow problem. Min-cost Algorithm in Figure 4, unlike the Max-flow algorithm, is an iterative algorithm. In the first step the geometric feasibility region is constructed and in the second step this region is divided into at most $n-1$ sub-regions. Refer Appendix F of [8] for the proof of optimality.

Input: Problem instance $(s, T, \mathcal{Z}, \gamma, F)$ and $C'_\cap = C'^s \cap C'^r$, where $\pi'_s = h^{-1}\left(\frac{g(\mu)}{F}\right)$ and $\pi'_{t_n} = h^{-1}\left(\frac{g(\nu)}{F}\right)$.

1: Compute $\widehat{p} = \arg\min_{p \in C'_\cap}(h(D_{sp}) + \max_{i\in[1,n]}(h(D_{pt_i})))$, and build the set $\widehat{T} = \{\widehat{t} \in T | D_{s\widehat{t}} \leq D_{s\widehat{p}}\}$. If $\widehat{T} \neq \{\emptyset\}$, then recompute $\widehat{p} = \arg\min_{p\in C'_\cap}(h(D_{sp}) + \max_{t\in T\setminus\{\widehat{T}\}}(h(D_{pt})))$, calculate $\Psi_{\widehat{p}} = h(D_{s\widehat{p}}) + h(D_{\widehat{p}t_n})$ and to go to Step 2.

2: Build the set $\overline{T} = \{t \in T\setminus\{\widehat{T}, t_n\} | D_{st} > \pi^{\widehat{p}}_s, D_{st} \leq \pi'_s\} = \{\bar{t}_1, .., \bar{t}_l\}$ (ordered in increasing distance from s), compute the points

$$\widehat{p}_j = \arg\min_{p\in\overline{C}^s_j}(\max(h(D_{sp}), h(D_{s\bar{t}_{j-1}})) + \max_{t\in\overline{T}_j}(h(D_{pt}))),$$

and calculate the cost of unit flow $\overline{\Psi}_j = h(D_{s\widehat{p}}) + h(\max_{t\in\overline{T}_j}(D_{\widehat{p}t}))$ over the path $\hat{l}_2$ corresponding to the relay position $\widehat{p}_j$, $\forall j \in [1, l]$. Declare

$$z^*_{F\downarrow} = \begin{cases} z_{\widehat{p}} & \text{if } \Psi_{\widehat{p}} \leq \overline{\Psi}_m, \\ z_{\overline{p}_m} & \text{if } \Psi_{\widehat{p}} \geq \overline{\Psi}_m, \end{cases}$$

${P^s_{T_1}}^* = g^{-1}(h(\pi^*_s)F)$ and ${P^r_{T_2}}^* = g^{-1}(h(\pi^*_r)F)$ (where $\overline{\Psi}_m = \min_{j\in[1,n]}(\overline{\Psi}_j))$ and quit.

Output: $z^*_{F\downarrow}$, ${P^s_{T_1}}^*$ and ${P^r_{T_2}}^*$.

Fig. 4. Min-Cost Algorithm.

V. CONCLUSION

We analyze and solve the optimal relay positioning problem for a fairly general class of Broadcast Relay Hypergraphs characterized by rate functions that are increasing and convex in power, and decreasing in distance. We reduce the hard joint relay positioning and flow (max-flow/min-cost) optimization problem to a sequence of simple decoupled problems.

As a part of future work it would be of interest to extend the work presented here to the general multicommodity setting where multiple sessions use a set of relays.

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