# An adaptive test for the two-sample location problem based on U-statistics

Wolfgang Kössler and Narinder Kumar
Humboldt-Universität zu Berlin Panjab University Chandigarh

**Abstract**

For the two-sample location problem we consider a general class of tests, all members of it are based on U-statistics. The asymptotic efficicacies are investigated in detail. We construct an adaptive test where all statistics involved are suitably chosen U-statistics. It is shown the adaptive test proposed has good asymptotic and finite power properties.

*Keywords: Wilcoxon-Mann-Whitney test, asymptotic efficacy, asymptotic power, simulation study, tailweight, skewness.*

## 1 Introduction

Let $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ be independent random samples from populations with absolutely continuous distribution functions $F(x)$ and $F(x-\vartheta)$, $\vartheta \in \mathbf{R}$, respectively. We wish to test

$$H_0 : \quad \vartheta = 0$$

against

$$H_1 : \quad \vartheta > 0.$$

The Wilcoxon-Mann-Whitney test is the most familiar nonparametric test for this problem. It was generalized to linear rank tests with various other scores, such as the Median test, the normal scores test and the Savage test are proposed, see e.g. Hájek, Šidak and Sen (1999).

Another generalization is to consider a class of tests based on U-statistics. This interesting class of tests has been attached considerable attraction in the literature (cf. e.g. Deshpande and Kochar (1982), Shetty and Govindarajulu (1988), Kumar (1997), Xie and Priebe (2000)).

Following Kumar, Singh and Öztürk (2003) a general class $U_{k,i}$ of U-statistics is defined in Section 2. Local alternatives of the form $\vartheta = \theta_N = \theta/\sqrt{N}$, $N = n_1 + n_2$, are considered and the asymptotic efficacies of the tests based on $U_{k,i}$ are compared detailly in Section 3. It is shown that there are different tests of this type which are efficient for densities with short, medium or long (right or left) tails, respectively. For example, the test based on $U_{5,1}$ is efficient for densities with short tails, and that based on $U_{5,3}$ is efficient for densities with long tails. However, the practising statistician has generally no clear idea on the underlying density, thus he/she should apply an adaptive test which takes into account the given data set. In Sections 4 and 5 two versions of such an adaptive test are proposed, one of them is distribution-free. The adaptive tests first classify the underlying distribution with respect to some measures like (right and left) tailweight and skewness and then select an appropriate test based on U-statistics. In Section 6 a simulation study is performed and the finite power is compared with the asymptotic power. It is shown that one of the adaptive tests behaves well also for moderate sample sizes.

# 2 Test statistics

We consider the class of U-statistics, which was proposed by Kumar, Singh and Öztürk (2003). Let $k$ and $i$ be fixed integers such that $1 \le i \le k \le \min(n_1, n_2)$. Define

$$\Phi_i(x_1, \ldots, x_k, y_1, \ldots, y_k) = \begin{cases} 2 & \text{if} \quad x_{(i)k} < y_{(i)k} \text{ and } x_{(k-i+1)k} < y_{(k-i+1)k} \\ 1 & \text{if either} \quad x_{(i)k} < y_{(i)k} \text{ or } x_{(k-i+1)k} < y_{(k-i+1)k} \\ 0 & \text{otherwise,} \end{cases}$$

where $x_{(i)k}$ is the $i$th order statistic in a subsample of size $k$ from the $X$-sample (and likewise for $y$'s). Let $U_{k,i}$ be the U-statistic associated with kernel $\Phi_i$, i.e.

$$U_{k,i} = \frac{n_1 n_2}{\binom{n_1}{k} \cdot \binom{n_2}{k}} \sum \Phi_i(X_{r_1}, \ldots, X_{r_k}, Y_{s_1}, \ldots, Y_{s_k}) - n_1 n_2,$$

where the summation extends over all possible combinations $(r_1, \ldots, r_k)$ of $k$ integers from $\{1, \ldots, n_1\}$ and all possible combinations $(s_1, \ldots, s_k)$ of $k$ integers from $\{1, \ldots, n_2\}$. The null hypothesis $H_0$ is rejected in favour of $H_1$ for large values of $U_{k,i}$.

**Remark:** The following special cases are of particular interest.

For $i = 1$ and $k = 1$ we have the Wilcoxon-Mann-Whitney test.

For $i = 1$ or $i = k$ we have the Despande-Kochar test (cf. Despande and Kochar, 1982).

For $i = (k+1)/2$ we have the Kumar-test (cf. Kumar, 1997).

Let

$$\begin{aligned}
\varphi_{1,0}^{(i)}(x) &= \mathbf{E}\Phi_i(x, X_2, \ldots, X_k, Y_1, \ldots, Y_k) \\
\varphi_{0,1}^{(i)}(y) &= \mathbf{E}\Phi_i(X_1, \ldots, X_k, y, Y_2, \ldots, Y_k) \\
\zeta_{1,0}^{(i)} &= \mathbf{Var}(\varphi_{1,0}^{(i)}(X)) \\
\zeta_{0,1}^{(i)} &= \mathbf{Var}(\varphi_{0,1}^{(i)}(Y)),
\end{aligned}$$

where $\mathbf{E}$ and $\mathbf{Var}$ denote the expectation and variance rescpectively. Moreover, let $F_{(i)k}(.)$ be the cummulative distribution function of the $i$th order statistics of a sample of size $k$.

**Proposition 2.1 (cf. Kumar, Singh and Öztürk, 2003)** *Under assumptions $N \to \infty$, $n_1/N \to \lambda$, $0 < \lambda < 1$ the limiting distribution of $N^{1/2}(U_{k,i} - \eta_{k,i})/\sigma_{k,i}$ is standard normal, where expectation $\eta_{k,i} = \mathbf{E}U_{k,i}$ and variance $\sigma_{k,i}^2 =$*__var__ *$(U_{k,i})$ have the forms*

$$\begin{aligned}
\eta_{k,i} &= n_1 n_2 \left( \int_{-\infty}^{\infty} F_{(i)k}(y)\, dF_{(i)k}(y-\theta) + \int_{-\infty}^{\infty} F_{(k-i+1)k}(y)\, dF_{(k-i+1)k}(y-\theta) \right) - n_1 n_2 \\
\sigma_{k,i}^2 &= n_1^2 n_2^2 \Big( \frac{k^2 \zeta_{10}^{(i)}}{\lambda} + \frac{k^2 \zeta_{01}^{(i)}}{1-\lambda} \Big).
\end{aligned}$$

**Remark:** Under $H_0$ we have $\eta_{k,i} = 0$ and

$$\sigma_{k,i}^2 = n_1^2 n_2^2 k^2 \frac{\rho_{k,i}}{\lambda(1-\lambda)},$$

where $\rho_{k,i}$ depends on $k$ and $i$ only. The expression for $\rho_{k,i}$ is rather long, that is why we do not write it out. It can be found in Kumar, Singh and Öztürk (2003).

# 3 The asymptotic efficacies

The asymptotic Pitman efficacies AE of the statistics $U_{k,i}$ under the alternative $\theta_N = N^{-1/2} \cdot \theta$ are given by

$$AE(U_{k,i}|f) = \lambda(1-\lambda) \cdot C^2_{k,i}(f),$$

where $f(\cdot)$ denotes the probability density function belonging to the c.d.f. $F(\cdot)$ and

$$C_{k,i}(f) = \frac{(\binom{k}{i} i)^2}{(k^2 \rho_{k,i})^{1/2}} \cdot \Big( \int_{-\infty}^{\infty} (F(x))^{2i-2}(1-F(x))^{2k-2i} f^2(x)\, dx + \int_{-\infty}^{\infty} (F(x))^{2k-2i}(1-F(x))^{2i-2} f^2(x)\, dx \Big)$$

(cf. Kumar, Singh and Öztürk, 2003).

(Note that the asymptotic efficacy is defined by the limit of $\eta^2_{k,i}/\sigma^2_{k,i}$.)

To obtain procedures that are practically important also for moderate to small sample sizes we restrict the further investigations to the case of small $k$, $k \leq 5$.

We compute the asymptotic Pitman efficacies for all tests $U_{k,i}$ with $1 \leq i \leq k$, $k \leq 5$. Values of the factors $C^2_{k,i}(f)$ for various densities are presented in Table 1. The L-DE density was proposed by Policello and Hettmansperger (1976), the U-L by Gastwirth (1965), the RST is named after Ramberg, Schmeiser and Tukey, cf. Ramberg and Schmeiser (1972, 1974), CN$(\epsilon, \sigma)$ is the scale contaminated normal with contaminating proportion $\epsilon$, Mielke denotes the Mielke (1972) density and BT is the density of Box and Tiao (1962).

The bolded entries denote the, for the given density, asymptotically best test among the considered tests. On the first view we see that the columns for $U_{3,1}, U_{5,1}$ and $U_{5,3}$ have the most bolded entries. This observation gives rise to the idea to base an adaptive test on these few statistics. (The classical test $U_{1,1}$ is also included in the adaptive test.)

For various densities asymptotic power functions (together with finite power functions) are given in Figures 3 and 4.

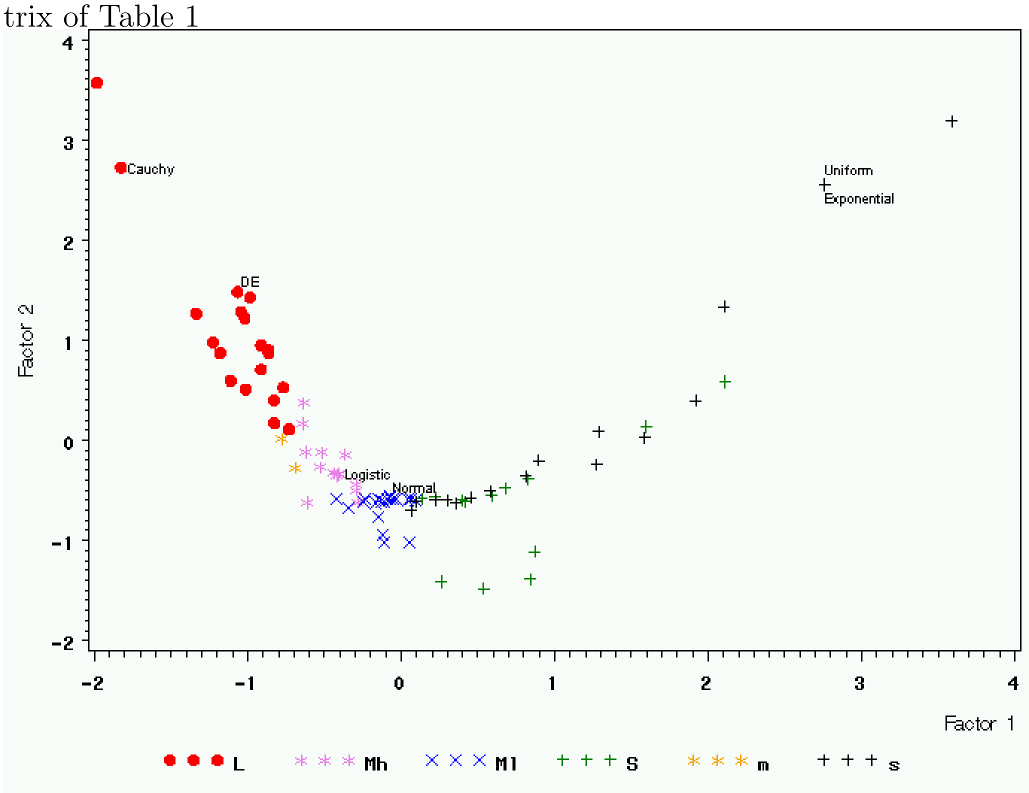
The blue dotted line is for $U_{1,1}$, the violet short-dashed line for $U_{3,1}$, the green long-dashed line for $U_{5,1}$, the red dashed-dotted line for $U_{5,3}$ (and the black continuous line for the adaptive test, see below).

Applying a similar idea of Hall and Joiner (1982) the content of information in the asymptotic efficacy matrix is analysed by a principal component analysis where the densities are the observations and the efficacies of the $U_{i,j}$ are the variables. The first principal component explains already 96% of the variabilty (Figure 1). For better visibility we preferred a two-dimensional plot. In Figure 1 nearly symmetric densities with short tails (S) are denoted by a green plus, that with medium tails (Ml and Mh) by a violet star and a blue X and that with long tails (L) with a red dot. (Ml and Mh stand for medium to light tails and medium to heavy tails respectively.) Skew densities are denoted by a black plus (short tails, s) and a yellow star (medium tails, m) respectively. The dots (L) denote densities with long tails, the stars (M and m) denote densities with medium tails, and the plus (S and s) denote densities with short tails. The small letters (m and s) are for skew densities. On the left side we have densities with long tails, in the centre that with medium tails, and on the right that with short tails. For an exact definition what we understand by long, medium and short tails see below. On the first view we see that the $AE(U_{k,i})$ classify the densities according to their tailweight. The skewness seems to play a marginal role only.

# 4 Selector statistics

There are some proposals for adaptive tests for the two-sample location problem in the literature, see e.g. Hogg (1974, 1982), Hogg, Fisher and Randles (1975), Ruberg (1986) and Büning (1994). We apply the concept of Hogg (1974), that is, to classify at first the type of the underlying density with respect to one measure of skewness $\hat{s}$ and to three measures of tailweight $\hat{t}$,

Figure 1: The first two principal components of the asymptotic efficacy matrix of Table 1



$\hat{t}_r$ and $\hat{t}_l$, which are defined by

$$\hat{s} = \frac{\hat{Q}(0.95) + \hat{Q}(0.05) - 2 \cdot \hat{Q}(0.5)}{\hat{Q}(0.95) - \hat{Q}(0.05)} \tag{1}$$

$$\hat{t} = \frac{\hat{Q}(0.95) - \hat{Q}(0.05)}{\hat{Q}(0.85) - \hat{Q}(0.15)} \tag{2}$$

$$\hat{t}_l = \frac{\hat{Q}(0.5) - \hat{Q}(0.05)}{\hat{Q}(0.5) - \hat{Q}(0.15)} \qquad \hat{t}_r = \frac{\hat{Q}(0.95) - \hat{Q}(0.5)}{\hat{Q}(0.85) - \hat{Q}(0.5)} \tag{3}$$

where $\hat{Q}(u)$ is the so-called classical quantile estimate of $F^{-1}(u)$,

$$\hat{Q}(u) = \begin{cases} X_{(1)} - (1-\delta) \cdot (X_{(2)} - X_{(1)}) & \text{if} \quad u < 1/(2 \cdot N) \\ (1-\delta) \cdot X_{(j)} + \delta \cdot X_{(j+1)} & \text{if} \quad \frac{1}{2 \cdot N} \leq u \leq \frac{2 \cdot N - 1}{2 \cdot N} \\ X_{(N)} + \delta(X_{(N)} - X_{(N-1)}) & \text{if} \quad u > (2 \cdot N - 1)/(2 \cdot N), \end{cases} \tag{4}$$

where $\delta = N \cdot u + 1/2 - j$ and $j = \lfloor N \cdot u + 1/2 \rfloor$. Note that $\hat{t}_l$ and $\hat{t}_r$ are measures of left tailweight and right tailweight, respectively.

In Tables 2 and 3 the values of the corresponding theoretical measures $s$, $t$, $t_r$ and $t_l$, for various selected densities are presented. (For symmetric densities we have $s = 0$ and $t = t_r = t_l$.)

Comparing Table 1 with Tables 2 and 3 roughly we see that $U_{5,1}$ is the asymptotically best test for symmetric densities with small tailweight, $U_{3,1}$ for symmetric densities with small to medium tailweight, $U_{1,1}$ for symmetric densities with medium to larger tailweight, and $U_{5,3}$ for symmetric densities with large tailweight. The tests $U_{3,1}$ and $U_{1,1}$ should be included in a adaptive test since they are the (asymptotically) best for the normal and for the logistic density, respectively (at least among the considered tests).

The measure of skewness gives no clear classification idea. That is why we consider left tailweight $t_l$ and right tailweight $t_r$, and classify densities as densities with partially short tails if $t_l < 1.55$ or $t_r < 1.55$. They are classified to have partially medium tails if $t_l < 1.8$ or $t_r < 1.8$ and if they have not partially short tails.

# 5 Presentation of the adaptive test

The reasoning of the last two sections gives rise to the following adaptive test.

Define regions $E_1, \ldots, E_7$ of $\mathbb{R}^4$ which are based on the so called selector statistic $\hat{S} = (\hat{s}, \hat{t}, \hat{t}_l, \hat{t}_r)$

| | |
|---|---|
| $E_1 = \{\hat{t} < 1.55, \lvert\hat{s}\rvert \leq 0.2\}$ | "nearly symmetric, short tails" (S) |
| $E_2 = \{1.55 \leq \hat{t} < 1.65, \lvert\hat{s}\rvert \leq 0.2\}$ | "nearly symmetric, light medium tails" (Ml) |
| $E_3 = \{1.65 \leq \hat{t} \leq 1.8, \lvert\hat{s}\rvert \leq 0.2\}$ | "nearly symmetric, heavy medium tails" (Mh) |
| $E_4 = \{\hat{t} > 1.8, \lvert\hat{s}\rvert \leq 0.2\}$ | "nearly symmetric, long tails" (L) |
| $E_5 = \{(\hat{t}_l < 1.55 \vee \hat{t}_r < 1.55), \lvert\hat{s}\rvert > 0.2\}$ | "skew, partially short tails" (s) |
| $E_6 = \{(\hat{t}_l > 1.65 \wedge \hat{t}_r > 1.65), \lvert\hat{s}\rvert > 0.2\}$ | "skew, long tails" (l) |
| $E_7 = \{\hat{t}_l \geq 1.55,\ \hat{t}_r \geq 1.55, \lvert\hat{s}\rvert > 0.2\} \setminus E_6$ | "skew, partially medium tails" (m) |

where $\hat{s}, \hat{t}, \hat{t}_l$ and $\hat{t}_r$ are given by (1) to (3). Note that there was no density which belongs to class $E_6$.

Looking again at Tables 2 and 3 we see that the vast majority of densities is classified correctly, i.e. they fall in that class that has the highest asymptotic power. If the classification is not correct, then the efficacy loss is very small in almost all cases. In Table 1 the chosen test is underlined if it is not already the (bolded) best.

Now, we propose the Adaptive test $A$ which is based on the four U-statistics $U_{5,1}$, $U_{3,1}$, $U_{1,1}$, and $U_{5,3}$. We denote the tests by (5,1), (3,1), (1,1) and (5,3), respectively.

$$A = A(S) = \begin{cases} (5,1) & \text{if} \quad S \in E_1 \cup E_5 \\ (3,1) & \text{if} \quad S \in E_2 \cup E_7 \\ (1,1) & \text{if} \quad S \in E_3 \\ (5,3) & \text{if} \quad S \in E_4 \cup E_6 \end{cases} \tag{5}$$

In Figure 2 the corresponding adaptive scheme is given. As indicated above the skewness plays only a marginal role. It is included in the adaptive scheme implicitely by left and right tailweight.

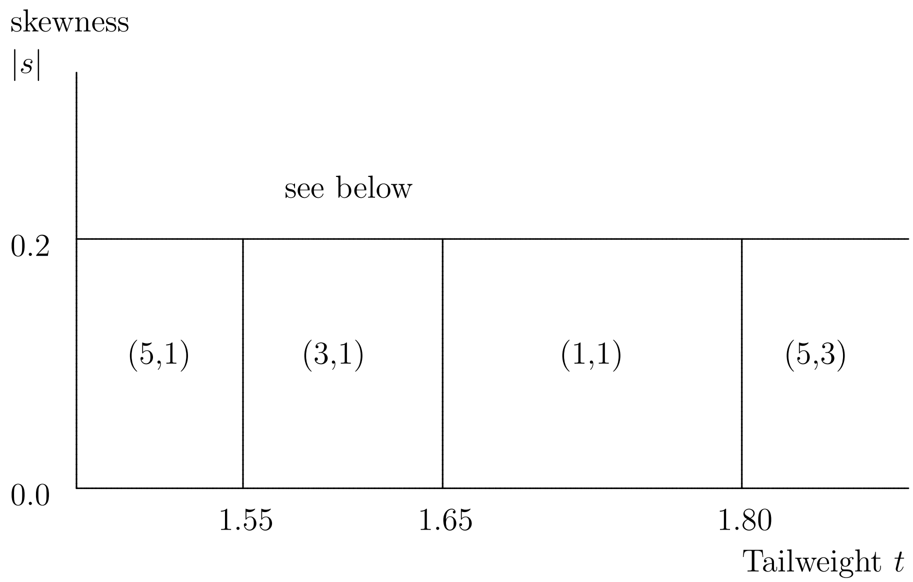
The two-stage procedure defined above is asymptoticaly distribution-free since the selector statistic $S$ is based on the order statistic only and the U-statistics are based on the ranks only.

The Adaptive test A is only asymptotically distribution-free because asymptotic critical values are used in the adaptive scheme.

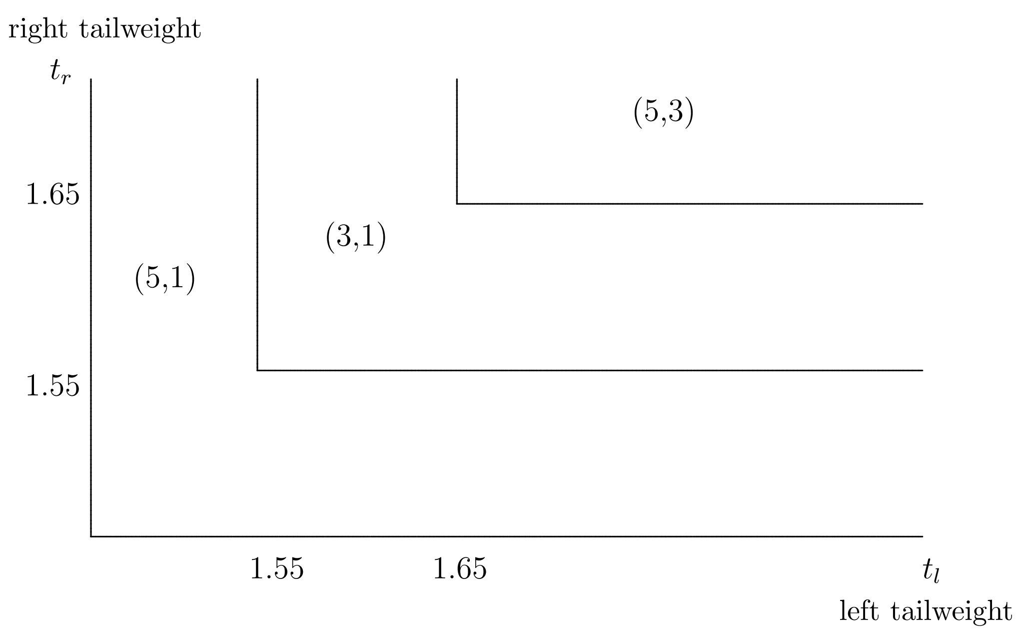
**Proposition 5.1** *Let $\sigma_F$ be the standard deviation of the underlying cdf. $F$, if it exists and let $\{\theta_N\}$ be a sequence of 'near' alternatives with $\sqrt{N}\theta_N \rightarrow$*

Figure 2: Adaptive scheme

a) symmetric or moderately skew densities




b) very skew densities ($|\hat{s}| > 0.2$)



$\sigma_F\theta$. *The asymptotic power function of the Adaptive test A equals*

$$\beta(\theta) = \begin{cases} 1 - \Phi(z_{1-\alpha} - \sqrt{AE(U_{5,1}|f)} \cdot \sigma_F \cdot \theta) & \textit{if} \quad f \in E_1 \cup E_5 \\ 1 - \Phi(z_{1-\alpha} - \sqrt{AE(U_{3,1}|f)} \cdot \sigma_F \cdot \theta) & \textit{if} \quad f \in E_2 \cup E_7 \\ 1 - \Phi(z_{1-\alpha} - \sqrt{AE(U_{1,1}|f)} \cdot \sigma_F \cdot \theta) & \textit{if} \quad f \in E_3 \\ 1 - \Phi(z_{1-\alpha} - \sqrt{AE(U_{5,3}|f)} \cdot \sigma_F \cdot \theta) & \textit{if} \quad f \in E_4 \cup E_6 \end{cases}$$

*Proof:* Let be $h = 1$ if $(k,i) = (5,1)$, $h = 2$ if $(k,i) = (3,1)$, $h = 3$ if $(k,i) = (1,1)$, $h = 4$ if $(k,i) = (5,3)$ Let be $T_1 = U_{5,1}$, $T_2 = U_{3,1}$, $T_3 = U_{1,1}$, $T_4 = U_{5,3}$ and $D_1 = E_1 \cup E_5$, $D_2 = E_2 \cup E_7$, $D_3 = E_3$, $D_4 = E_4 \cup E_6$. The proposition follows from the total probability theorem and from the consistency of the selector statistics, i.e.

$$\begin{aligned} \beta(\theta) &= \sum_{h=1}^{4} P_{\sigma_F\theta}(T_h > c_{\alpha h}|T_h \text{ chosen}) \cdot P_{\sigma_F\theta}(T_h \text{ chosen}) \\ &= \sum_{h=1}^{4} \big(1 - \Phi(z_{1-\alpha} - \sqrt{AE(T_h|f)} \cdot \sigma_F \cdot \theta) + o(1)\big) \cdot \begin{cases} 1 + o(1) & \text{if} \quad f \in D_h \\ o(1) & \text{else} \end{cases} \\ &\sim 1 - \Phi(z_{1-\alpha} - \sqrt{AE(T_h|f)} \cdot \sigma_F \cdot \theta) \quad \text{if} \quad f \in D_h, \end{aligned}$$

where $c_{\alpha h}$ is the $(1-\alpha)$-quantile of the asymptotic null distribution of $T_h$. ■

**Remark:** For the Cauchy the factor $\sigma_F$ is given by $\sigma_F = \sigma_{Cau} = F^{-1}(\Phi(1)) = 1.8373$. This factor is introduced to have similar power values for the various distributions.

The Adaptive test $A(\hat{S})$ is based on selector statistics computed from the pooled sample. However, location differences may effect the estimates of tailweight and skewness. That is why we consider also a modification $A(\hat{S}')$ of the adaptive test, where tailweights and skewness are estimated from the single samples. Let $\hat{s}_i$ and $\hat{t}_i$, $\hat{t}_{l,i}$ and $\hat{t}_{r,i}$, $i = 1, 2$ be statistics of the form (1) to (3) for skewness and tailweight, left tailweight and right tailweight, respectively. Applying the $A(\hat{S}')$-test the statistics

$$\hat{s}^* = \frac{m}{N}\hat{s}_1 + \frac{n}{N}\hat{s}_2 \qquad \hat{t}^* = \frac{m}{N}\hat{t}_1 + \frac{n}{N}\hat{t}_2$$

$$\hat{t}_l^* = \frac{m}{N}\hat{t}_{l,1} + \frac{n}{N}\hat{t}_{l,2} \qquad \hat{t_r}^* = \frac{m}{N}\hat{t}_{r,1} + \frac{n}{N}\hat{t}_{r,2}$$

are used instead of $\hat{s}$, $\hat{t}$, $\hat{t_l}$ and $\hat{t_r}$. This procedure is also asymptotically distribution-free. However, it is not distribution-free also if the exact critical values are used. This property is due to the fact that the selector statistic is no longer based on the pure order statistic.

**Remark:** Another adaptive test which is also based on U-statistics, but with another classes of U-statistics (and in another context) is proposed in Kössler (2005). It turns out, that the adaptive test presented here is better for the majority of considered densities, especially for short tail densities (uniform, exponential).

**Remark:** Other candidates for an adaptive test are linear combinations of U-statistics which are proposed by Xie and Priebe (2002). The authors define two families, there denoted by $C_{r,s}$ and $WC_{r,s}$, of generalized Mann-Whitney-Wilcoxon (GMWW) statistics and of weighted generalized Mann-Whitney-Wilcoxon (WGMWW) statistics and investigate the case $r = s$ ($r$ and $s$ are the subsample sizes, $r = s = k$ in our notation). For unimodal densities the possible gain in asymptotic power is very small if linear combinations of U-statistics are admissed (except perhaps for the normal). An idea may be to use the optimal linear combination of U-statistics if the density is normal or nearly normal. However, this will complicate the adaptive procedure and, perhaps, decrease the power for other densities (that are classified in the same class as the normal).

For multimodal densities the consideration of tests of the class WGMWW seems to be useful.

# 6 Comparison to adaptive tests based on linear rank statistics

Restrictive adaptive tests for the two-sample location problem based on linear rank tests are proposed by Hogg (1974, 1982), Hogg, Fisher and Randles (1975), Handl (1986) and Büning (1994). All of them are based on the concept of Hogg (1974), and they use few linear rank statistics, with the following scores: Gastwirth (GA, for short tails), Wilcoxon (WI, for medium tails), Median (Hogg, 1974, and Hogg et al. (1975), Long-tail (LT, Handl (1986) and Büning (1994), both for long tails) and Hogg-Fisher-Randles (HFR, for right-skew densities). Since the Median test is known to be bad for most densities (except for the doubleexponential), we restrict to the scores GA, WI, LT and HFR, and call the corresponding adaptive scheme $B(S)$ (where the form of $S$ is not of interest here). In Table 4 we take, for some densities, that tests that are, asymptotically, the two best among the tests in the Adaptive schemes $A(S)$ and $B(S)$, respectively.

For the classical densities considered the U-statistics based test $A(S)$ has slightly higher asymptotic power than the test $B(S)$. (For the logistic they are, of course, the same.) For the densities U-L (0.75) and L-D (0.75) that are 'optimal' densities for the tests GA and LT (cf. Büning and Kössler, 1999) it is vice versa.

# 7 Simulation study

In order to assess whether the asymptotic theory can also be applied for medium to small sample sizes a simulation study (10,000 replications each) is performed. We choose the following six distributions:

- Uniform distribution (density with small tailweight),
- Normal distribution (density with medium tailweight),
- Logistic distribution (density with medium tailweight),
- Double exponential distribution (density with large tailweight),
- Cauchy distribution (density with very large tailweight),
- Exponential distribution (very skew density)

We consider the four single tests $U_{5,1}, U_{3,1}, U_{1,1}, U_{5,3}$ and the two Adaptive tests $A(S)$ and $A(S')$. The sample sizes $n_1 = n_2 = 10$, $n_1 = n_2 = 40$ and the alternatives $\theta_N = N^{-1/2}\theta\sigma_F$ with various $\theta$ are considered. Recall that the factor $\sigma_F$ denotes the standard deviation of the underlying distribution function $F$. The factor $\sigma_F$ is introduced to have similar values of the power function. (For the Cauchy density the factor is set to $\sigma_F = \sigma_{Cau} = F^{-1}(\Phi(1)) = 1.8373$.)

The Adaptive test $A(S')$ is slightly anticonservative, also for $n_1 = n_2 = 40$, with correspondingly slightly higher power than the Adaptive test $A(S)$ (except for the Cauchy). This test is not considered here. For $n_1 = n_2 = 40$ the results of the simulation study are summarized in Figures 3 and 4. The blue dotted line is for $U_{1,1}$, the violet short-dashed line for $U_{3,1}$, the green long-dashed line for $U_{5,1}$, the red dashed-dotted line for $U_{5,3}$ and the continuous line is for the Adaptive test $A(S)$. At first we see that, for $n_1 = n_2 = 40$ the finite power is well approximated by the asymptotic power (except for the Cauchy). Moreover, it can be seen that, for a given density, there is always, sometimes together with another test, a single test which is the best. The test $U_{5,1}$ is the best for the uniform and for the exponential (together with the Adaptive test $A(S)$), the test $U_{1,1}$ is the best for the normal and for the logistic density, and the test $U_{5,3}$ is the best for the doubleexponential and for the Cauchy. All these facts are not surprising. Also, not surprisingly, the tests $U_{5,1}$ and $U_{5,3}$ may be bad for some densities. The tests $U_{1,1}$ and $A(S)$ are, over all densities, the best. However, for the uniform and for the exponential densities the adaptive test is clearly better.

For the Cauchy density, somewhat surprisingly, the test $U_{1,1}$ is clearly better than the adaptive tests. The Adaptive test $A(S')$ is better than $A(S)$. The reason for these facts is, that for small and moderate sample sizes, the misclassification rate into the class $E_1$ is relatively large.

# 8 Conclusions

What are the results of our study? At first, we see that the finite power of the considered tests based on U-statistics can be well approximated by their asymptotic power. Second, there are modifications of the "classical" Mann-Whitney test MW that may have (considerably) higher power than MW for symmetric as well asymmetric densities. Third, the Adaptive test $A(S)$ is a serious concurrent for the Wilcoxon-Mann-Whitney test $U_{1,1}$ for moderate to large sample sizes.

## References

Box, G.E.P., Tiao, G.C. (1962). A further look at robustness via Bayes's theorem. *Biometrika*, 49, 419-432.

Büning, H. (1994). Robust and adaptive tests for the two-sample location problem. *OR Spektrum*, **16**, 33-39.

Deshpande, J.V., Kochar, S.C. (1982). Some competitors of Wilcoxon-Mann-Whitney test for location alternatives. *Journal of Indian Statistical Association*, **19**, 9-18.

Gastwirth, J.L. (1965). Percentile modifications of two-sample rank tests. *Journal of the American Statistical Association*, **60**, 1127-1141.

Hájek, J., Šidák, Z., Sen, P.K. (1999). Theory of Rank Tests, Academic Press, San Diego.

Hall, D.L., Joiner, B.L. (1983). Asymptotic Relative Efficiency of $R$-estimators of Location. *Communications in statistics, Theory and Methods*, **12**, 739-763.

Handl, A (1986). *Masszahlen zur Klassifizierung von Verteilungen bei der Konstruktion adaptiver verteilungsfreier Tests im unverbundenen Zweistichprobenproblem*, unpublished dissertation, Freie Universität Berlin.

Hogg, R.V. (1974). Adaptive robust procedures: partial review and some suggestions for future applications and theory. *Journal of the American Statistical Association*, **69**, 909-923.

Hogg, R.V. (1982). On adaptive statistical inference. *Communications in Statistics - Theory and Methods*, **11**, 2531-2542.

Hogg, R.V., Fisher, D.M., Randles, R.H. (1975). A two-sample adaptive distribution-free test. *Journal of the American Statistical Association*, **70**, 656-661.

Kössler, W. (2005). Some c-Sample Rank Tests of Homogeneity Against Ordered Alternatives based on U-statistics. *Journal of Nonparametric Statistics*, **17**, 777-796.

Kumar, N. (1997). A Class of Two-Sample tests for Location Based on Sub-Sample Medians. *Communications in Statistics, Theory and Methods*, **26** (4), 943-951.

Kumar, N., Singh, R.S., Öztürk, Ö. (2003). A New Class of Distribution-Free Tests for Location parameters. *Sequential Analysis*, **22**, No. 1, 107-128.

Mielke, P.W.Jr. (1972). Asymptotic behaviour of two-sample tests based on powers of ranks for detecting scale and location alternatives. *Journal of the American Statistical Association*, **67**, 850-854.

Policello, G.E., Hettmansperger, T.P. (1976). Adaptive robust procedures for the one-sample location problem. *Journal of the American Statistical Association*, **71**, 624-633.

Ramberg, J.S., Schmeiser, B.W. (1972). An approximate method for generating symmetric random variables, *Communications of the ACM*, **11**, 987-990.

Ramberg, J.S., Schmeiser, B.W. (1974). An approximate method for generating asymmetric random variables, *Communications of the ACM*, **17**, 78-82.

Ruberg, S.J. (1986). A continuously adaptive nonparametric two-sample test. *Communications in Statistics, Theory and Methods*, **15**, 2899-2920.

Shetty, I.D., Govindarajulu, Z. (1988). A Two-Sample Test for Location. *Communications in Statistics, Theory and Methods*, **27** (7), 2389-2401.

Xie, J., Priebe, C.E. (2000). Generalizing the Mann-Whitney-Wilcoxon Statistic. *Nonparametric Statistics*, **12**, 661-682.

Xie, J., Priebe, C.E. (2002). A weighted ggeneralization of the Mann-Whitney-Wilcoxon Statistic. *Journal of Statistical Planning and Inference*, **102**, 441-466.

Figure 3: The asymptotic and finite ($n_1 = n_2 = 40$) power functions of the tests $U_{1,1}, U_{3,1}, U_{5,1}, U_{5,3}$ and $A(S)$; densities: uniform, normal and logistic.

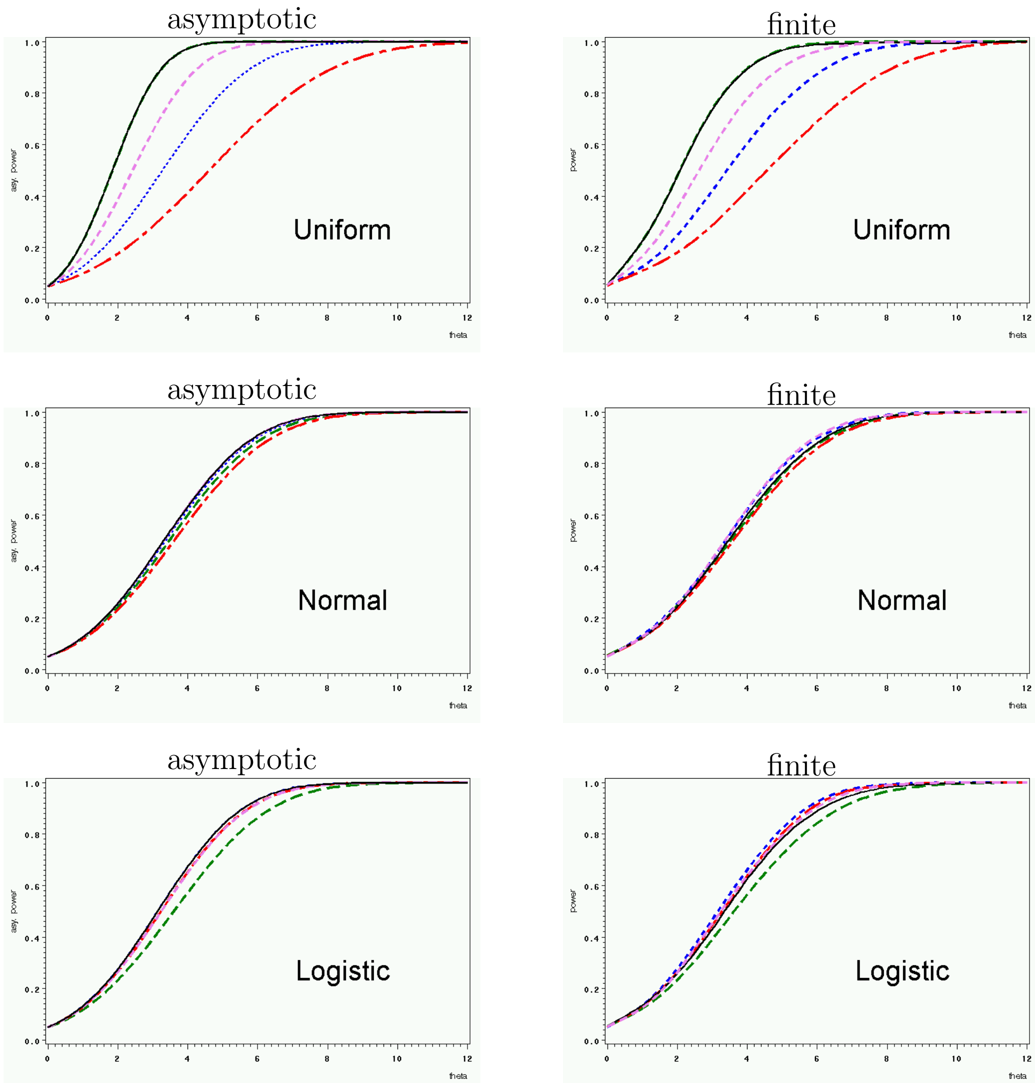


Figure 4: The asymptotic and finite ($n_1 = n_2 = 40$) power functions of the tests $U_{1,1}, U_{3,1}, U_{5,1}, U_{5,3}$ and $A(S)$, densities: doubleexponential, Cauchy and exponential (Continuation from Figure 3).

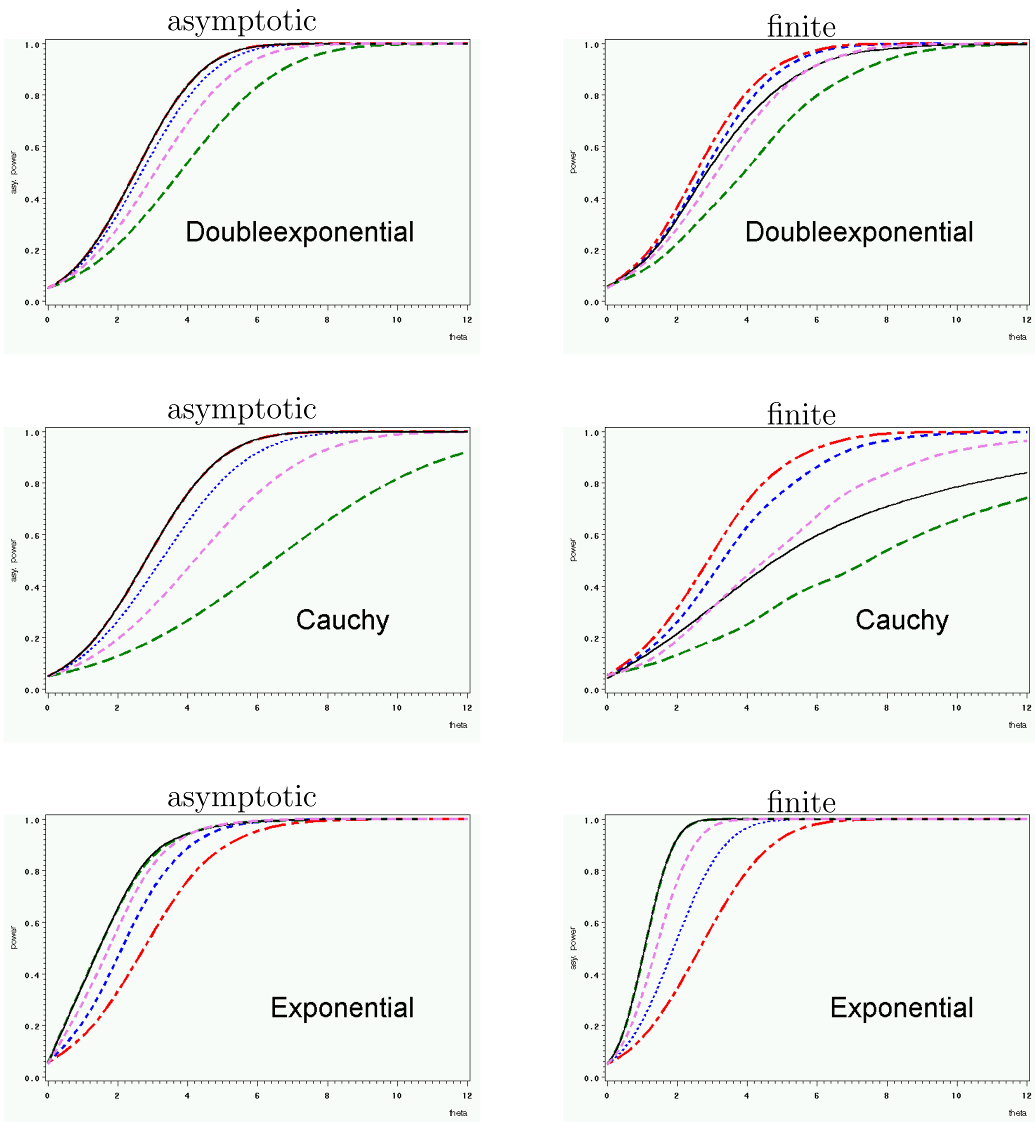


Table 1: Values of the factors $C^2_{k,i}(f)$, $k \leq 5$, for various densities $f$.

| Density | $U_{1,1}$ | $U_{2,1}$ | $U_{3,1}$ | $U_{3,2}$ | $U_{4,1}$ | $U_{4,2}$ | $U_{5,1}$ | $U_{5,2}$ | $U_{5,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| Uniform | 12.00 | 14.74 | 22.09 | 7.053 | 30.01 | 7.535 | **38.00** | 9.109 | 6.095 |
| Normal | 0.954 | 0.976 | **0.985** | 0.873 | 0.950 | 0.889 | 0.900 | 0.931 | 0.832 |
| Logistic | **0.333** | 0.332 | 0.313 | 0.324 | 0.284 | 0.327 | 0.254 | 0.332 | 0.315 |
| Doubleex | 0.750 | 0.705 | 0.576 | 0.833 | 0.462 | 0.816 | 0.379 | 0.749 | **0.866** |
| Cauchy | 0.304 | 0.268 | 0.182 | 0.382 | 0.118 | 0.368 | 0.077 | 0.318 | **0.409** |
| $t_2$ | 0.520 | 0.489 | 0.398 | 0.572 | 0.311 | 0.566 | 0.244 | 0.538 | **0.579** |
| $t_{10}$ | 0.843 | **0.846** | 0.816 | 0.802 | 0.755 | 0.813 | 0.688 | 0.835 | 0.772 |
| L-DE 0.55 | 0.745 | 0.701 | 0.574 | **0.826** | 0.461 | 0.809 | 0.378 | 0.728 | 0.822 |
| L-DE 0.61 | 0.726 | 0.686 | 0.567 | 0.798 | 0.458 | 0.783 | 0.377 | 0.728 | **0.822** |
| L-DE 0.70 | 0.672 | 0.641 | 0.544 | 0.720 | 0.448 | 0.712 | 0.373 | 0.678 | **0.730** |
| L-DE 0.75 | 0.630 | 0.606 | 0.524 | 0.663 | 0.438 | 0.659 | 0.368 | 0.637 | **0.665** |
| L-DE 0.80 | 0.581 | 0.563 | 0.497 | **0.599** | 0.423 | **0.599** | 0.360 | 0.587 | 0.594 |
| L-DE 0.90 | **0.464** | 0.457 | 0.422 | 0.460 | 0.373 | 0.463 | 0.328 | 0.466 | 0.450 |
| L-DE 0.95 | **0.400** | 0.396 | 0.372 | 0.391 | 0.334 | 0.394 | 0.300 | 0.399 | 0.380 |
| L-DE 0.97 | **0.373** | 0.371 | 0.349 | 0.363 | 0.315 | 0.367 | 0.282 | 0.372 | 0.353 |
| L-DE 0.99 | **0.347** | 0.345 | 0.325 | 0.337 | 0.295 | 0.340 | 0.264 | 0.345 | 0.328 |
| U-L 0.55 | 0.298 | **0.299** | 0.289 | 0.282 | 0.267 | 0.287 | 0.242 | 0.296 | 0.271 |
| U-L 0.61 | 0.250 | 0.254 | **0.255** | 0.228 | 0.242 | 0.233 | 0.224 | 0.248 | 0.215 |
| U-L 0.70 | 0.173 | 0.181 | 0.194 | 0.147 | **0.195** | 0.152 | 0.188 | 0.169 | 0.134 |
| U-L 0.75 | 0.130 | 0.139 | 0.156 | 0.105 | **0.163** | 0.110 | 0.161 | 0.125 | 0.094 |
| U-L 0.80 | 0.090 | 0.099 | 0.116 | 0.069 | 0.126 | 0.073 | **0.130** | 0.085 | 0.061 |
| U-L 0.90 | 0.026 | 0.029 | 0.038 | 0.018 | 0.048 | 0.019 | **0.054** | 0.022 | 0.015 |
| RST (-1) | 0.042 | 0.037 | 0.024 | 0.055 | 0.015 | 0.052 | 0.009 | 0.044 | **0.060** |
| RST (-0.5) | 0.464 | 0.427 | 0.328 | 0.533 | 0.243 | 0.522 | 0.182 | 0.480 | **0.550** |
| RST (-0.4) | 0.891 | 0.832 | 0.665 | 0.994 | 0.511 | 0.980 | 0.396 | 0.919 | **1.016** |
| RST (-0.3) | 1.952 | 1.850 | 1.537 | 2.113 | 1.230 | 2.095 | 0.987 | 2.003 | **2.132** |
| RST (-0.2) | 5.426 | 5.220 | 4.518 | **5.678** | 3.765 | 5.664 | 3.135 | 5.524 | 5.662 |
| RST (-0.1) | 26.87 | 26.27 | 23.73 | 27.13 | 20.61 | **27.22** | 17.81 | 27.09 | 26.71 |
| RST (0.05) | 148.6 | **149.2** | 144.1 | 141.5 | 133.4 | 143.3 | 121.9 | 147.0 | 136.7 |
| RST (0.14) | 23.08 | 23.56 | **23.73** | 21.15 | 22.84 | 21.55 | 21.59 | 22.50 | 20.19 |
| RST (0.2) | 12.91 | 13.33 | **13.81** | 11.52 | 13.65 | 11.78 | 13.20 | 12.45 | 10.91 |
| RST (0.4) | 5.036 | 5.425 | 6.196 | 4.078 | 6.686 | 4.222 | **6.978** | 4.643 | 3.762 |
| CN (.01,2) | 0.941 | 0.960 | **0.966** | 0.863 | 0.928 | 0.879 | 0.876 | 0.918 | 0.823 |
| CN (.02,2) | 0.927 | 0.944 | **0.947** | 0.853 | 0.907 | 0.869 | 0.853 | 0.907 | 0.814 |
| CN (.03,2) | 0.887 | **0.900** | 0.893 | 0.823 | 0.846 | 0.837 | 0.788 | 0.871 | 0.787 |
| CN (.05,2) | 0.824 | **0.831** | 0.811 | 0.775 | 0.757 | 0.787 | 0.694 | 0.815 | 0.744 |
| CN (.01,3) | 0.934 | 0.952 | **0.954** | 0.859 | 0.913 | 0.875 | 0.859 | 0.913 | 0.820 |
| CN (.02,3) | 0.914 | **0.928** | 0.924 | 0.845 | 0.878 | 0.860 | 0.819 | 0.896 | 0.807 |
| CN (.03,3) | 0.854 | **0.861** | 0.839 | 0.804 | 0.780 | 0.817 | 0.713 | 0.846 | 0.771 |
| CN (.05,3) | 0.761 | 0.759 | 0.716 | 0.738 | 0.644 | 0.747 | 0.569 | **0.766** | 0.712 |
| CN (.01,5) | 0.928 | **0.944** | 0.943 | 0.855 | 0.899 | 0.871 | 0.841 | 0.909 | 0.817 |
| CN (.02,5) | 0.901 | **0.913** | 0.902 | 0.838 | 0.850 | 0.853 | 0.786 | 0.887 | 0.802 |
| CN (.03,5) | **0.825** | **0.825** | 0.789 | 0.788 | 0.718 | 0.799 | 0.642 | **0.825** | 0.757 |
| CN (.05,5) | 0.709 | 0.696 | 0.631 | 0.707 | 0.544 | 0.713 | 0.461 | **0.726** | 0.686 |

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| Density | $U_{1,1}$ | $U_{2,1}$ | $U_{3,1}$ | $U_{3,2}$ | $U_{4,1}$ | $U_{4,2}$ | $U_{5,1}$ | $U_{5,2}$ | $U_{5,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| Mielke (0.2) | 0.620 | 0.591 | 0.500 | 0.667 | 0.413 | 0.658 | 0.347 | 0.620 | **0.682** |
| Mielke (0.4) | 0.521 | 0.503 | 0.439 | 0.545 | 0.373 | 0.541 | 0.319 | 0.521 | **0.548** |
| Mielke (0.6) | <u>0.444</u> | 0.433 | 0.389 | **0.452** | 0.339 | **0.452** | 0.295 | 0.444 | 0.449 |
| Mielke (0.8) | **0.383** | 0.377 | 0.348 | 0.380 | 0.309 | 0.382 | 0.273 | 0.382 | 0.374 |
| Mielke (1.5) | 0.245 | **0.249** | <u>0.246</u> | 0.227 | 0.233 | 0.231 | 0.215 | 0.242 | 0.215 |
| Mielke (2.0) | 0.188 | 0.193 | **0.200** | 0.167 | 0.195 | 0.171 | 0.185 | 0.184 | 0.156 |
| Mielke (5.0) | 0.061 | 0.067 | 0.079 | 0.047 | 0.087 | 0.049 | **0.090** | 0.057 | 0.042 |
| Mielke (20) | 0.006 | 0.007 | 0.010 | 0.004 | 0.013 | 0.004 | **0.015** | 0.005 | 0.003 |
| BT (0.25) | **0.660** | 0.658 | 0.624 | 0.640 | 0.572 | 0.645 | 0.520 | 0.650 | 0.626 |
| BT (0.5) | <u>0.444</u> | 0.433 | 0.389 | **0.454** | 0.340 | 0.452 | 0.298 | 0.441 | **0.454** |
| BT (0.75) | 0.292 | 0.279 | 0.239 | 0.311 | 0.200 | 0.308 | 0.170 | 0.291 | **0.318** |
| BT (1.25) | 0.118 | 0.116 | 0.098 | 0.131 | 0.081 | 0.132 | 0.068 | 0.123 | **0.136** |
| BT (1.5) | 0.073 | 0.077 | 0.070 | 0.080 | 0.061 | 0.083 | 0.055 | 0.080 | **0.081** |
| Exponential | 3.000 | 3.684 | 5.522 | 1.763 | 7.502 | 1.884 | **9.500** | 2.277 | 1.524 |
| Gamma (1.5) | 1.216 | 1.365 | 1.705 | 0.895 | 1.994 | 0.936 | **2.230** | 1.062 | 0.808 |
| Gamma (2.0) | 0.750 | 0.815 | 0.951 | 0.595 | 1.047 | 0.617 | **1.112** | 0.683 | 0.546 |
| Gamma (2.5) | 0.540 | 0.578 | 0.651 | 0.445 | 0.694 | 0.459 | **0.717** | 0.502 | 0.412 |
| Gamma (3.0) | 0.422 | 0.447 | 0.493 | 0.355 | 0.515 | 0.365 | **0.523** | 0.396 | 0.330 |
| Gamma (4.0) | 0.293 | 0.307 | 0.330 | 0.252 | **0.337** | 0.259 | <u>0.336</u> | 0.278 | 0.237 |
| Gamma (5.0) | 0.224 | 0.234 | <u>0.248</u> | 0.196 | **0.250** | 0.201 | 0.246 | 0.214 | 0.184 |
| Gamma (10) | 0.103 | 0.106 | <u>0.110</u> | 0.092 | **0.108** | 0.094 | 0.105 | 0.100 | 0.087 |
| Weibull (1.1) | 2.919 | 3.448 | 4.793 | 1.898 | 6.135 | 2.008 | **7.407** | 2.360 | 1.673 |
| Weibull (1.5) | 3.391 | 3.706 | 4.375 | 2.662 | 4.877 | 2.762 | **5.249** | 3.058 | 2.446 |
| Weibull (2.0) | 4.712 | 4.980 | 5.447 | 3.996 | **5.665** | 4.110 | <u>5.470</u> | 4.428 | 3.738 |
| Weibull (2.5) | 6.516 | 6.795 | 7.203 | 5.693 | **7.276** | 5.835 | <u>7.182</u> | 6.216 | 5.364 |
| Weibull (3.0) | 8.732 | 9.050 | **9.450** | 7.736 | 9.413 | 7.916 | <u>9.174</u> | 8.387 | 7.315 |
| Weibull (4.0) | 14.33 | 14.78 | **15.25** | 12.84 | 15.01 | 13.12 | <u>14.48</u> | 13.84 | 12.18 |
| Weibull (5.0) | 21.46 | 22.09 | **22.70** | 19.30 | 22.26 | 19.71 | <u>21.39</u> | 20.76 | 18.32 |
| LogNor (0.05) | 382.5 | 391.0 | **395.0** | 349.3 | 281.3 | 356.0 | 361.2 | 372.6 | 332.9 |
| LogNor (0.1) | 95.97 | 98.21 | **99.44** | 87.47 | 96.18 | 89.18 | <u>91.29</u> | 93.41 | 83.31 |
| LogNor (0.3) | 11.10 | 11.47 | **11.89** | 9.893 | 11.75 | 10.12 | <u>11.37</u> | 10.70 | 9.363 |
| LogNor (0.5) | 4.328 | 4.559 | 4.946 | 3.690 | 5.094 | 3.797 | **5.104** | 4.095 | 3.448 |
| LogNor (0.9) | 1.768 | 1.984 | 2.478 | 1.290 | 2.874 | 1.356 | **3.173** | 1.561 | 1.153 |
| LogNor (1.0) | 1.574 | 1.804 | 2.350 | 1.090 | 2.819 | 1.154 | **3.195** | 1.356 | 0.959 |
| LogNor (1.1) | 1.445 | 1.692 | 2.304 | 0.944 | 2.860 | 1.007 | **3.328** | 1.211 | 0.817 |
| LogNor (1.5) | 1.307 | 1.690 | 2.765 | 0.640 | 3.924 | 0.708 | **5.045** | 0.947 | 0.509 |
| RST (.2,.4) | 7.599 | 8.040 | 8.812 | 6.413 | 9.161 | 6.608 | **9.256** | 7.156 | 5.976 |
| RST (.2,.49) | 6.638 | 7.107 | 8.005 | 5.449 | 8.528 | 5.634 | **8.800** | 6.170 | 5.040 |
| RST (.4.,49) | 4.532 | 4.933 | 5.767 | 3.582 | 6.353 | 3.720 | **6.750** | 4.127 | 3.284 |
| RST (-.2,-.4) | 2.061 | 1.991 | <u>1.743</u> | **2.135** | 1.465 | 2.134 | 1.227 | 2.099 | 2.116 |
| RST (-.2,-.49) | 1.473 | 1.435 | <u>1.283</u> | **1.496** | 1.010 | 1.502 | 0.935 | 1.495 | 1.471 |
| RST (.4-,.49) | 0.659 | 0.613 | 0.485 | 0.742 | 0.369 | 0.730 | 0.282 | 0.681 | **0.760** |

Table 2: Values of the tailweight $t$ for various symmetric densities

| Density | $t$ | Density | $t$ | Density | $t$ | Density | $t$ |
|---|---|---|---|---|---|---|---|
| Uniform | 1.286 | Normal | 1.587 | Logistic | 1.697 | Doubleexp | 1.912 |
| Cauchy | 3.217 | $t_2$ | 2.107 | $t_{10}$ | 1.672 | | |
| L-DE 0.55 | 1.911 | L-DE 0.61 | 1.905 | L-DE 0.70 | 1.884 | L-DE 0.75 | 1.864 |
| L-DE 0.80 | 1.836 | L-DE 0.90 | 1.753 | L-DE 0.95 | 1.712 | L-DE 0.97 | 1.703 |
| L-DE 0.99 | 1.698 | | | | | | |
| U-L 0.55 | 1.668 | U-L 0.61 | 1.623 | U-L 0.70 | 1.534 | U-L 0.75 | 1.474 |
| U-L 0.80 | 1.409 | U-L 0.90 | 1.300 | | | | |
| RST (-1) | 3.451 | RST (-0.5) | 2.302 | RST (-0.4) | 2.146 | RST (-0.3) | 2.010 |
| RST (-0.2) | 1.891 | RST (-0.1) | 1.788 | RST (0.05) | 1.657 | RST (0.14) | 1.591 |
| RST (0.2) | 1.552 | RST (0.4) | 1.446 | | | | |
| CN (.01,2) | 1.592 | CN (.02,2) | 1.597 | CN (.03,2) | 1.601 | CN (.05,2) | 1.611 |
| CN (.01,3) | 1.596 | CN (.02,3) | 1.605 | CN (.03,3) | 1.615 | CN (.05,3) | 1.635 |
| CN (.01,5) | 1.600 | CN (.02,5) | 1.614 | CN (.03,5) | 1.629 | CN (.05,5) | 1.665 |
| Mielke 0.2 | 1.860 | Mielke 0.4 | 1.812 | Mielke 0.6 | 1.770 | Mielke 0.8 | 1.732 |
| Mielke 1.5 | 1.626 | Mielke 2.0 | 1.571 | Mielke 5.0 | 1.404 | Mielke 20.0 | 1.292 |
| BT (0.25) | 1.670 | BT(0.5) | 1.752 | BT(0.75) | 1.833 | BT(1.25) | 1.992 |
| BT (1.5) | 2.071 | | | | | | |

Table 3: Values of skewness $s$, tailweight $t$, left tailweight $t_l$ and right tailweight $t_r$ for various skew densities

| Density | Skewness | Tailweight | Left tailweight | Right tailweight |
|---|---|---|---|---|
| Exponential | 0.564 | 1.697 | 1.210 | 1.912 |
| Gamma (1.5) | 0.460 | 1.651 | 1.284 | 1.846 |
| Gamma (2.0) | 0.401 | 1.668 | 1.329 | 1.810 |
| Gamma (2.5) | 0.354 | 1.621 | 1.360 | 1.785 |
| Gamma (3.0) | 0.338 | 1.653 | 1.382 | 1.768 |
| Gamma (4.0) | 0.298 | 1.643 | 1.412 | 1.743 |
| Gamma (5.0) | 0.271 | 1.636 | 1.432 | 1.727 |
| Gamma (10.0) | 0.202 | 1.619 | 1.480 | 1.686 |
| Weibull (1.1) | 0.508 | 1.654 | 1.237 | 1.859 |
| Weibull (1.5) | 0.335 | 1.571 | 1.329 | 1.728 |
| Weibull (2.0) | 0.194 | 1.544 | 1.411 | 1.649 |
| Weibull (2.5) | 0.103 | 1.541 | 1.470 | 1.605 |
| Weibull (3.0) | 0.040 | 1.546 | 1.513 | 1.577 |
| Weibull (4.0) | -0.04 | 1.559 | 1.573 | 1.544 |
| Weibull (5.0) | -0.088 | 1.571 | 1.612 | 1.525 |
| LogNormal (0.05) | 0.041 | 1.588 | 1.563 | 1.611 |
| LogNormal (0.1) | 0.082 | 1.591 | 1.541 | 1.634 |
| LogNormal (0.3) | 0.242 | 1.626 | 1.457 | 1.749 |
| LogNormal (0.5) | 0.390 | 1.695 | 1.386 | 1.879 |
| LogNormal (0.9) | 0.629 | 1.940 | 1.274 | 2.202 |
| LogNormal (1.0) | 0.676 | 2.024 | 1.251 | 2.298 |
| LogNormal (1.1) | 0.719 | 2.117 | 1.229 | 2.401 |
| LogNormal (1.5) | 0.844 | 2.588 | 1.160 | 2.890 |
| RST (0.2,0.4) | 0.029 | 1.486 | 1.486 | 1.486 |
| RST (0.2,0.49) | 0.014 | 1.459 | 1.469 | 1.449 |
| RST (0.4,0.49) | -0.011 | 1.426 | 1.436 | 1.417 |
| RST (-0.2-,0.4) | 0.375 | 2.073 | 1.718 | 2.289 |
| RST (-0.2,-0.49) | 0.490 | 2.195 | 1.657 | 2.470 |
| RST (-0.4,-0.49) | 0.153 | 2.226 | 2.086 | 2.342 |