



# Model reduction for nonlinear systems with incremental gain or passivity properties<sup>☆</sup>



B. Besselink, N. van de Wouw<sup>1</sup>, H. Nijmeijer

Department of Mechanical Engineering, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

## ARTICLE INFO

### Article history:

Received 28 October 2011  
Received in revised form  
3 May 2012  
Accepted 23 October 2012  
Available online 12 February 2013

### Keywords:

Model reduction  
Controller reduction  
Nonlinear systems  
Stability analysis  
Error analysis

## ABSTRACT

In this paper, model reduction techniques for a class of nonlinear systems are proposed. Specifically, nonlinear systems are considered that can be decomposed as the feedback interconnection of a high-order linear subsystem and a nonlinear subsystem of relatively low order, allowing for the application of well-developed reduction techniques for linear systems. In this setting, conditions are given under which internal stability, as well as passivity or a bound on the  $\mathcal{L}_2$  gain are preserved for the reduced-order nonlinear model. Additionally, a priori error bounds are given. In the derivation of the error bound, an incremental gain (or incremental passivity) property of the nonlinear subsystem is shown to be instrumental. Additionally, the techniques developed in this paper are applied in the scope of controller reduction, as is illustrated by means of an industrial temperature control benchmark example.

© 2013 Elsevier Ltd. All rights reserved.

## 1. Introduction

Model reduction is a tool for the approximation of complex high-order systems, leading to reduced-order systems that allow for efficient analysis or facilitate control design and implementation. Since the reduced-order model is used as a substitute for the original high-order model, it is of importance to preserve key system properties during the reduction process. Herein, stability is the most crucial one. However, the preservation of gain properties or passivity is relevant in many (control) applications as well. Additionally, the availability of an error bound is highly instrumental in determining the quality of the approximation. In this paper, techniques are proposed for the model and controller reduction for a class of nonlinear systems that, firstly, preserve stability, gain or passivity properties and, secondly, allow for the computation of an error bound.

For linear systems, model reduction techniques addressing these aspects exist in the literature. Herein, balanced truncation

(Moore, 1981) is amongst the most popular methods for the reduction of asymptotically stable linear systems, since it preserves stability (Pernebo & Silverman, 1982) and an error bound exists (Enns, 1984; Glover, 1984). Extensions of balanced truncation such as positive real balancing (Desai & Pal, 1984; Harshvardhana, Jonckheere, & Silverman, 1984) and bounded real balancing (Opdenacker & Jonckheere, 1988) provide methods that preserve passivity and contractivity, respectively, where the latter basically preserves a certain bound on the gain. For an overview, see Antoulas (2005). For nonlinear systems, existing model reduction techniques typically do not satisfy the properties of preservation of stability and the existence of an error bound. Here, it is remarked that both internal stability (i.e. stability of an equilibrium point for zero input) as well as input–output stability (i.e. a bounded gain) have to be considered separately. This is contrary to linear systems, for which internal stability implies input–output stability. Nonetheless, balanced truncation for nonlinear systems (Fujimoto & Scherpen, 2010; Scherpen, 1993) provides a method preserving (local) stability of the equilibrium point. Here, results on input–output stability are not available and (as a result) no error bound exists. Additionally, the approach is computationally challenging. The same properties hold for moment matching for nonlinear systems (Astolfi, 2010). Recently, an extension of balanced truncation for nonlinear systems has been developed aiming at the preservation of dissipativity, including the cases of gain and passivity preservation (Ionescu, Fujimoto, & Scherpen, 2010). Similarly, recently obtained results on moment matching for nonlinear systems can be exploited for the preservation of passivity (Ionescu & Astolfi, 2010). However, no error bounds are available

<sup>☆</sup> The material in this paper was partially presented at the 50th IEEE Conference on Decision and Control (CDC) and European Control Conference (ECC), December 12–15, 2011, Orlando, Florida, USA. This paper was recommended for publication in revised form by Associate Editor Alessandro Astolfi under the direction of Editor Andrew R. Teel. This research is financially supported by the Dutch Technology Foundation STW.

E-mail addresses: [b.besselink@tue.nl](mailto:b.besselink@tue.nl) (B. Besselink), [n.v.d.wouw@tue.nl](mailto:n.v.d.wouw@tue.nl) (N. van de Wouw), [h.nijmeijer@tue.nl](mailto:h.nijmeijer@tue.nl) (H. Nijmeijer).

<sup>1</sup> Tel.: +31 40 247 3358; fax: +31 40 246 1418.

for these methods. A computationally attractive approach for the reduction of nonlinear systems is given by trajectory piecewise linear approximation (Rewieński & White, 2003), where model reduction techniques for linear systems are exploited. Results on input–output stability are available for a subclass of nonlinear systems (Bond & Daniel, 2009), but no error bound exists. Moreover, data-based reduction methods such as proper orthogonal decomposition (Sirovich, 1987) or balancing using empirical gramians (Lall, Marsden, & Glavaški, 2002) do generally not preserve stability nor exhibit an error bound. In Prescott and Papachristodoulou (2012), an approach towards model reduction for biochemical networks is presented, where bounds on the output error are given for given initial conditions. However, inputs are not considered.

Thus, model reduction techniques for nonlinear systems generally lack a guarantee on the preservation of internal and/or input–output stability, as well as an error bound. In this paper, a model reduction procedure is presented addressing these properties, for a class of nonlinear systems. In addition, the preservation of contractivity (an input–output gain smaller than 1) and passivity is discussed. Herein, nonlinear systems are considered that can be decomposed into the feedback interconnection of a high-order linear subsystem and a nonlinear subsystem of relatively low order. This is motivated by the observation that nonlinearities act only locally in many engineering applications. Examples include mechanical systems with friction or hysteresis, electrical circuits with nonlinear components and linear systems with nonlinear actuator dynamics.

In this setting, model reduction is performed on the linear subsystem only. This allows for the use of well-developed existing model reduction techniques for linear systems, making the approach computationally attractive. Here, the nonlinear subsystem is assumed to satisfy either an incremental input–output gain property or to be incrementally passive. These incremental properties prove to be crucial in the derivation of an error bound, since they characterize the evolution of errors introduced by model reduction of the linear subsystem. Similar ideas were used for systems with static nonlinearities in Besselink, van de Wouw, and Nijmeijer (2009) and Reis and Heinkenschloss (2009), where the latter focuses on passivity preservation in the scope of electrical circuits. The current paper extends these results in three ways. First, the results are extended to dynamic nonlinearities and both internal and input–output stability properties are explicitly addressed (see also Besselink, van de Wouw, & Nijmeijer, 2011). Secondly, the case of the preservation of contractivity is addressed and a relation with passivity preservation is derived. Thirdly, the results are applied in the scope of controller reduction, hereby building upon existing techniques for controlled linear systems. Additionally, the results on controller reduction are applied to a temperature-control benchmark example of a lab-on-a-chip, as is of interest in many biomedical applications.

Summarizing, the main contributions of the paper are the development of model (and controller) reduction procedures for a class of nonlinear systems that, firstly, preserve key system properties (such as stability, bounded  $\mathcal{L}_2$  gain and passivity), and, secondly, provide a computable error bound.

The remainder of this paper is organized as follows. The research problem is stated in Section 2 and preliminaries regarding incremental gain properties are discussed in Section 3. Model reduction techniques for linear systems are reviewed in Section 4 before discussing the main results on reduction of nonlinear systems in Section 5, hereby addressing the preservation of stability and input–output gain or passivity. Here, error bounds are derived as well. In Section 6, these results are used in the scope of controller reduction. The model and controller reduction procedures are illustrated in Section 7, where the latter is applied to a temperature control benchmark example representing a lab-on-a-chip. Finally, conclusions are stated in Section 8.

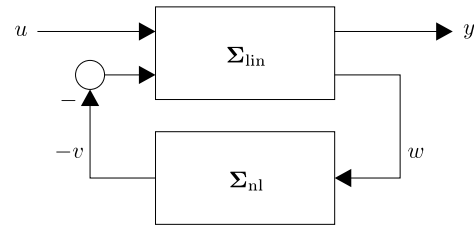


Fig. 1. Nonlinear system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$ .

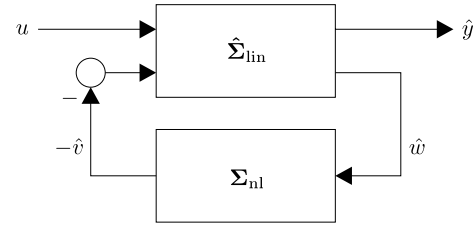


Fig. 2. Reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$ .

**Notation.** The field of real (complex) numbers is denoted by  $\mathbb{R}$  ( $\mathbb{C}$ ). For a vector  $x \in \mathbb{R}^n$ ,  $|x|^2 = x^T x$ . The space  $\mathcal{L}_2^n$  consists of all functions  $x : [0, \infty) \rightarrow \mathbb{R}^n$  which are bounded using the norm  $\|x\|_2^2 = \int_0^\infty |x(t)|^2 dt$ .

## 2. Problem setting

In this paper, nonlinear systems as depicted in Fig. 1 are considered. Here, the system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  consists of a feedback configuration of a high-order linear subsystem  $\Sigma_{\text{lin}}$  and a nonlinear subsystem  $\Sigma_{\text{nl}}$  of relatively low order, where  $\mathcal{I}(\cdot, \cdot)$  denotes the interconnection as in Fig. 1. The linear subsystem  $\Sigma_{\text{lin}}$  is given as

$$\Sigma_{\text{lin}} : \begin{cases} \dot{x} = Ax + B_u u + B_v v, \\ y = C_y x + D_{yu} u + D_{yv} v, \\ w = C_w x + D_{wu} u + D_{wv} v, \end{cases} \quad (1)$$

with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$ . The linear subsystem is coupled to the nonlinear subsystem via  $v \in \mathbb{R}^{n_v}$  and  $w \in \mathbb{R}^{n_w}$ , where the nonlinear dynamics is given as

$$\Sigma_{\text{nl}} : \begin{cases} \dot{z} = g(z, w), \\ -v = h(z, w), \end{cases} \quad (2)$$

with  $z \in \mathbb{R}^{n_z}$  and  $g$  locally Lipschitz continuous in  $z$ , continuous in  $w$  and satisfying  $g(0, 0) = 0$ . Moreover,  $h$  is continuous with  $h(0, 0) = 0$ .

Since only the linear subsystem  $\Sigma_{\text{lin}}$  in  $\Sigma$  is assumed to be of high order, a model reduction procedure is proposed in which only  $\Sigma_{\text{lin}}$  is reduced. This leads to a reduced-order linear subsystem  $\hat{\Sigma}_{\text{lin}}$  of the form

$$\hat{\Sigma}_{\text{lin}} : \begin{cases} \dot{\hat{x}} = \hat{A} \hat{x} + \hat{B}_u u + \hat{B}_v \hat{v}, \\ \hat{y} = \hat{C}_y \hat{x} + \hat{D}_{yu} u + \hat{D}_{yv} \hat{v}, \\ \hat{w} = \hat{C}_w \hat{x} + \hat{D}_{wu} u + \hat{D}_{wv} \hat{v}, \end{cases} \quad (3)$$

with  $\hat{x} \in \mathbb{R}^k$ ,  $k < n$  and where the dimensions of the inputs and outputs remain unchanged (i.e.  $\hat{y} \in \mathbb{R}^p$ ,  $\hat{v} \in \mathbb{R}^{n_v}$  and  $\hat{w} \in \mathbb{R}^{n_w}$ ). Such a reduction approach allows for the application of well-developed existing model reduction techniques for linear systems, making the approach computationally attractive. Finally, the interconnection of the reduced-order linear subsystem and the original nonlinear subsystem leads to the reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$  as in Fig. 2.

This paper deals with stability properties of the reduced-order nonlinear system  $\hat{\Sigma}$  when the high-order system  $\Sigma$  exhibits a

bounded incremental  $\mathcal{L}_2$  gain. As a special case, the preservation of contractivity is discussed, which is closely related to the case of passivity preservation. In all cases, error bounds for the reduced-order nonlinear system are provided.

### 3. Incremental $\mathcal{L}_2$ gain and incremental passivity

The property of an incremental  $\mathcal{L}_2$  gain will be exploited in the scope of model reduction and is discussed in this section. Nonlinear systems of the form

$$\dot{x} = f(x, u), \quad y = h(x, u), \quad (4)$$

are considered, where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$  and  $f(0, 0) = 0$ ,  $h(0, 0) = 0$ . Input–output properties such as  $\mathcal{L}_2$  gain (or passivity) can be conveniently characterized using the theory of dissipative systems (see Willems, 1972).

**Definition 1.** A system (4) is said to be dissipative with respect to the supply rate  $s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$  if there exists a storage function  $S : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $S \geq 0$  and

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t)) dt \quad (5)$$

for all  $t_1 \geq t_0$ . Here,  $x$  and  $y$  are the solutions of (4) for the state and output, respectively, for input function  $u$ .

Throughout this paper, it is assumed that the storage functions  $S$  are differentiable, such that (5) is equivalent to the differential dissipation inequality  $\dot{S} \leq s(u, y)$ . In order to define incremental input–output properties of (4), an auxiliary system is defined as

$$\begin{aligned} \dot{x}_1 &= f(x_1, u_1), & y_1 &= h(x_1, u_1), \\ \dot{x}_2 &= f(x_2, u_2), & y_2 &= h(x_2, u_2). \end{aligned} \quad (6)$$

Then, the property of a bounded incremental  $\mathcal{L}_2$  gain can be defined as follows (see e.g. Romanchuk & James, 1996).

**Definition 2.** A system (4) is said to have a bounded incremental  $\mathcal{L}_2$  gain  $\gamma$  if the corresponding auxiliary system (6) is dissipative with respect to the supply rate

$$s(u_1, u_2, y_1, y_2) = \gamma^2 |u_1 - u_2|^2 - |y_1 - y_2|^2. \quad (7)$$

If, in (7),  $\gamma \leq 1$  ( $\gamma < 1$ ), the system (4) is said to be incrementally (strictly) contractive.

Incremental passivity can be defined similarly, hereby using (Pavlov & Marconi, 2008) and the terminology from Brogliato, Lozano, Maschke, and Egeland (2007).

**Definition 3.** A system (4) satisfying  $m = p$  is said to be incrementally passive if there exist parameters  $\delta \geq 0$ ,  $\epsilon \geq 0$  such that the corresponding auxiliary system (6) is dissipative with respect to the supply rate

$$\begin{aligned} s(u_1, u_2, y_1, y_2) &= (u_1 - u_2)^T (y_1 - y_2) - \delta |u_1 - u_2|^2 \\ &\quad - \epsilon |y_1 - y_2|^2. \end{aligned} \quad (8)$$

If, in (8),  $\delta > 0$  and  $\epsilon > 0$ , the system (4) is said to be incrementally very strictly passive.

**Remark 4.** Definitions 2 and 3 deal with incremental input–output properties for nonlinear dynamical systems as in (4). However, these properties can also be defined for static nonlinearities. Namely, a nonlinearity  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^p$  has a bounded incremental  $\mathcal{L}_2$  gain (is incrementally passive) if  $s(u_1, u_2, \phi(u_1), \phi(u_2)) \geq 0$  for all  $u_1, u_2 \in \mathbb{R}^m$ , with  $s$  the supply rate as in (7) (as in (8)). Consequently, the results in this paper also hold when the subsystem  $\Sigma_{nl}$  is replaced by a static nonlinearity  $\varphi : \mathbb{R}^{n_w} \rightarrow \mathbb{R}^{n_v}$ .

Finally, dissipativity can be linked to internal stability properties (i.e. stability of  $x = 0$  for  $u = 0$ ). Thereto, the following definitions are useful (see e.g. Hill & Moylan, 1976; van der Schaft, 2000).

**Definition 5.** The system (4) is zero-state observable if  $u(t) = 0$ ,  $y(t) = 0$ ,  $\forall t \geq 0$  implies  $x(t) = 0$ ,  $\forall t \geq 0$ .

**Definition 6.** The system (4) is reachable from 0 if for all  $x^*$ , there exists an input  $u$  and time  $T$  such that  $u$  steers the system from  $x(0) = 0$  to  $x(T) = x^*$ .

For nonlinear systems that are zero-state observable and reachable from 0, the storage function  $S$  showing a bounded  $\mathcal{L}_2$  gain or passivity is positive definite (see Hill & Moylan, 1976) and can act as a candidate Lyapunov function.

### 4. Model reduction for linear systems

The model reduction procedures for nonlinear systems presented in this paper rely on the reduction of the linear subsystem. Therefore, relevant model reduction procedures for linear systems are briefly reviewed in this section. Minimal, asymptotically stable, linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \quad (9)$$

are considered, with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$ . The transfer function associated with (9) is given as  $G(s) = C(sI - A)^{-1}B + D$ ,  $s \in \mathbb{C}$ .

Balanced truncation (Moore, 1981) is the most popular method for the reduction of asymptotically stable linear systems, since it preserves stability (Pernebo & Silverman, 1982) and satisfies a bound on the error (Enns, 1984). Optimal Hankel norm approximation (Glover, 1984) is an alternative with the same properties.

An extension of balanced truncation focusing on the preservation of contractivity is given in Opdenacker and Jonckheere (1988) and is known as bounded real balancing.

**Definition 7.** The transfer function  $G$  of an asymptotically stable system (9) is said to be bounded real if  $I - G^T(-j\omega)G(j\omega) \geq 0$  for all  $\omega \in \mathbb{R}$  and  $I - D^T D \geq 0$ . It is strictly bounded real when the inequalities are strict.

The following well-known result forms the basis for bounded real balancing.

**Lemma 8** (Opdenacker & Jonckheere, 1988). *Let (9) be a minimal and asymptotically stable system. Then, the transfer function  $G$  is bounded real if and only if (9) is contractive. If  $R_c := I - D^T D > 0$ , these statements are equivalent to the existence of a real matrix  $P = P^T > 0$  satisfying*

$$A^T P + PA + C^T C + (PB + C^T D)R_c^{-1}(PB + C^T D)^T = 0. \quad (10)$$

*Then, in particular, (10) admits two extremal solutions such that  $0 < P_{\min} \leq P \leq P_{\max}$ .*

In Lemma 8, the extremal solutions  $P_{\min}$  and  $P_{\max}$  of (10) define the available storage and required supply, characterizing the maximum amount of energy that can be extracted from the system for a given initial condition and the least amount of energy needed to reach a certain state, respectively. This motivates a balancing procedure on the basis of  $P_{\min}$  and  $P_{\max}$ . Thus, a realization is pursued in which  $P_{\min} = P_{\max}^{-1} = \text{diag}\{\xi_1, \xi_2, \dots, \xi_n\}$ , with  $\xi_i^2 = \lambda_i(P_{\min}P_{\max}^{-1})$  the so-called bounded real singular values satisfying  $1 \geq \xi_1 \geq \xi_2 \geq \dots \geq \xi_n > 0$  and  $\lambda_i(X)$  the  $i$ -th eigenvalue of  $X$ .

**Theorem 9** (Opdenacker & Jonckheere, 1988). Let (9) be a minimal, asymptotically stable system with strictly bounded real transfer function, and in bounded real balanced coordinates. Then, the reduced-order system obtained by truncation is minimal, asymptotically stable, has a strictly bounded real transfer function, and is in bounded real balanced coordinates. Moreover, the error bound

$$\|y - \hat{y}\|_2 \leq 2 \left( \sum_{i=k+1}^n \xi_i \right) \|u\|_2 \quad (11)$$

holds for distinct bounded real singular values  $\xi_i$ .

A model reduction procedure for passive systems has been developed using similar ideas. This method, known as positive real balancing, can be found in Desai and Pal (1984) and Harshvardhana et al. (1984).

## 5. Model reduction for nonlinear systems

In this section, model reduction for nonlinear systems with a bounded incremental  $\mathcal{L}_2$  gain is considered, where the case of preservation of contractivity is derived as a special case. Moreover, a relation to the preservation of passivity is discussed.

### 5.1. Systems with bounded incremental $\mathcal{L}_2$ gain

As discussed in Section 2, nonlinear systems  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  as in Fig. 1 are considered. Here, the following assumption is adopted throughout the paper.

**Assumption 10.** Let the system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  satisfy the following assumptions:

- (1) The feedback interconnection  $\mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  is well-posed, i.e. for each  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$ ,  $z \in \mathbb{R}^{n_z}$ , the equation  $w = C_w x + D_w u - D_{wv} h(z, w)$  can uniquely be solved for  $w$ ;
- (2) The linear subsystem  $\Sigma_{\text{lin}}$  is asymptotically stable and (1) is a minimal realization;
- (3) The nonlinear subsystem  $\Sigma_{\text{nl}}$  is reachable from 0 and zero-state observable.

Moreover, it is assumed that the nonlinear system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  has a bounded incremental  $\mathcal{L}_2$  gain.

**Assumption 11.** Let the system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  satisfy the following assumptions:

- (1) The nonlinear subsystem  $\Sigma_{\text{nl}}$  has a bounded incremental  $\mathcal{L}_2$  gain  $\mu$ ;
- (2) The small-gain condition  $\gamma_{wv}\mu < 1$  holds, with  $\gamma_{wv}$  the (incremental)  $\mathcal{L}_2$  gain of  $\Sigma_{\text{lin}}$  with respect to input  $v$  and output  $w$ .

By Assumption 10, the linear subsystem is asymptotically stable. This allows for the introduction of the input–output operators  $F_y : \mathcal{L}_2^m \times \mathcal{L}_2^{n_v} \rightarrow \mathcal{L}_2^p$  and  $F_w : \mathcal{L}_2^m \times \mathcal{L}_2^{n_v} \rightarrow \mathcal{L}_2^{n_w}$  defined as  $y = F_y(u, v)$  and  $w = F_w(u, v)$ , respectively. These operators characterize the outputs  $y$  and  $w$  of the linear subsystem for given inputs  $u$  and  $v$  and zero initial condition  $x(0) = 0$ . For linear systems, asymptotic stability implies a bounded incremental  $\mathcal{L}_2$  gain, such that the input–output operators satisfy

$$\|F_i(u_1, v_1) - F_i(u_2, v_2)\|_2 \leq \gamma_{iu} \|u_1 - u_2\|_2 + \gamma_{iv} \|v_1 - v_2\|_2, \quad (12)$$

for all  $u_1, u_2 \in \mathcal{L}_2^m$ ,  $v_1, v_2 \in \mathcal{L}_2^{n_v}$  and some bounded  $\gamma_{iu}, \gamma_{iv} \geq 0$  with  $i \in \{y, w\}$ . Due to linearity, the incremental  $\mathcal{L}_2$  gain is equivalent to the (non-incremental)  $\mathcal{L}_2$  gain, such that the gains  $\gamma_{ij}$  in (12) can be chosen as the  $\mathcal{H}_\infty$  norm of the corresponding transfer function  $C_i(sI - A)^{-1}B_j + D_{ij}$  with  $i \in \{y, w\}$  and  $j \in \{u, v\}$ .

For the nonlinear subsystem  $\Sigma_{\text{nl}}$ , it is recalled that Assumption 11 holds. Herein, the first item implies that the outputs  $-v$  of the nonlinear subsystem remain bounded for bounded inputs  $w$ , as follows from the properties  $g(0, 0) = 0$ ,  $h(0, 0) = 0$ . This allows for the definition of the input–output operator  $G : \mathcal{L}_2^{n_w} \rightarrow \mathcal{L}_2^{n_v}$  as  $-v = Gw$ , where  $-v$  is the solution of  $\Sigma_{\text{nl}}$  for input  $w$  and zero initial condition  $z(0) = 0$ . Then, the property of a bounded incremental  $\mathcal{L}_2$  gain in Assumption 11 implies the bound

$$\|v_1 - v_2\|_2 = \|Gw_1 - Gw_2\|_2 \leq \mu \|w_1 - w_2\|_2, \quad (13)$$

for all  $w_1, w_2 \in \mathcal{L}_2^{n_w}$ .

Under these assumptions, the total nonlinear system also has a bounded incremental  $\mathcal{L}_2$  gain, as stated next.

**Lemma 12.** Let  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  satisfy Assumptions 10 and 11. Then,  $\Sigma$  has a bounded incremental  $\mathcal{L}_2$  gain (from input  $u$  to output  $y$ ), where the gain is bounded by

$$\gamma = \gamma_{yu} + \frac{\gamma_{yv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu}. \quad (14)$$

Additionally, the origin is an asymptotically stable equilibrium point of  $\Sigma$  for  $u = 0$ .

**Proof.** Substitution of (13) in (12) for  $i = w$  yields

$$\|w_1 - w_2\|_2 \leq \gamma_{wu} \|u_1 - u_2\|_2 + \gamma_{wv}\mu \|w_1 - w_2\|_2, \quad (15)$$

where it is noted that the small-gain condition  $\gamma_{wv}\mu < 1$  guarantees boundedness of  $\|w_1 - w_2\|_2$ . Namely,

$$\|w_1 - w_2\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv}\mu} \|u_1 - u_2\|_2 \quad (16)$$

holds. Substitution of (16) in (12) for  $i = y$  by using (14).

To prove stability of the origin, it is recalled that the property of a bounded incremental  $\mathcal{L}_2$  gain implies a bounded  $\mathcal{L}_2$  gain. Thus, for the linear subsystem, there exists a storage function  $S_{\text{lin}}(x)$  that satisfies (for  $u = 0$ )

$$\dot{S}_{\text{lin}} \leq \gamma_{wv}^2 |v|^2 - |w|^2. \quad (17)$$

Here, it is remarked that  $S_{\text{lin}}$  is positive semi-definite. It is not guaranteed to be positive definite since the linear subsystem  $\Sigma_{\text{lin}}$  is minimal with respect to the combined inputs  $(u, v)$  and outputs  $(y, w)$ , which does not imply minimality with respect to the input  $v$  and output  $w$ . Next, if  $S_{\text{nl}}(z_1, z_2)$  denotes the storage function that characterizes the incremental  $\mathcal{L}_2$  gain for the nonlinear subsystem, then  $S_{\text{nl}}(z, 0)$  is a storage function for the supply rate  $\mu^2 |w|^2 - |v|^2$ . By reachability from 0 and zero-state observability of  $\Sigma_{\text{nl}}$ ,  $S_{\text{nl}}(z, 0)$  is positive definite. Differentiation of the composed storage function  $S(x, z) = S_{\text{lin}}(x) + \alpha^2 S_{\text{nl}}(z, 0)$  with respect to time gives

$$\dot{S} \leq (\gamma_{wv}^2 - \alpha^2) |v|^2 + (\alpha^2 \mu^2 - 1) |w|^2, \quad (18)$$

where the small-gain condition  $\gamma_{wv}\mu < 1$  implies that  $\alpha$  can be chosen as  $\gamma_{wv} < \alpha < \mu^{-1}$ , such that the right-hand side of (18) is negative semi-definite. In order to prove stability using the semi-definite Lyapunov function candidate  $S$ , let  $\mathcal{M}$  denote the largest positively invariant set contained in  $\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^{n_z} \mid S(x, z) = 0\}$ . Since  $S$  is positive semi-definite,  $S(x, z) = 0$  implies  $v = 0$ ,  $w = 0$ . As a result,  $\mathcal{M}$  is also contained in  $\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^{n_z} \mid C_w x = 0, h(z, 0) = 0\}$ . Thus, by asymptotic stability of  $\Sigma_{\text{lin}}$  and zero-state observability of  $\Sigma_{\text{nl}}$ , the origin is asymptotically stable for all initial conditions  $(x_0, z_0) \in \mathcal{M}$ . Application of van der Schaft (2000, Theorem 3.2.9) (see also Iggidr, Kalitine, & Outbib, 1996) proves stability of the origin of  $\mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  for  $u = 0$ .

To prove asymptotic stability, LaSalle's invariance principle (see van der Schaft, 2000, Theorem 3.2.3) is used. By stability, there

exists, for each initial condition  $x(0)$  in a neighborhood of the origin, a compact set  $\mathcal{B}$  such that  $x(t) \in \mathcal{B}$  for all  $t \geq 0$ . As a result of (18), any solution approaches the largest invariant set in  $\mathcal{B}$  that satisfies  $v = 0$  and  $w = 0$ . For the nonlinear subsystem, by zero-state observability, this invariant set is given by  $z = 0$ . By asymptotic stability of the linear subsystem,  $x = 0$  is the only invariant set. Hence, the origin of the nonlinear system  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  is asymptotically stable.  $\square$

It is recalled that reduction is applied to the linear subsystem only, leading to the reduced-order linear subsystem  $\hat{\Sigma}_{\text{lin}}$ . Herein, the following assumption is adopted.

**Assumption 13.** Let the reduced-order linear subsystem  $\hat{\Sigma}_{\text{lin}}$  as in (3) satisfy the following assumptions:

- (1)  $\hat{\Sigma}_{\text{lin}}$  is asymptotically stable;
- (2) An error bound on reduction of the linear subsystem exists of the form

$$\|E_i(u_1, v_1) - E_i(u_2, v_2)\|_2 \leq \varepsilon_{iu} \|u_1 - u_2\|_2 + \varepsilon_{iv} \|v_1 - v_2\|_2 \quad (19)$$

with  $\varepsilon_{iu}, \varepsilon_{iv} > 0$  and  $i \in \{y, w\}$ . Here,  $E_i = F_i - \hat{F}_i$ ,  $i \in \{y, w\}$  denotes the error operator with  $\hat{F}_y : \mathcal{L}_2^m \times \mathcal{L}_2^{n_v} \rightarrow \mathcal{L}_2^p$  and  $\hat{F}_w : \mathcal{L}_2^m \times \mathcal{L}_2^{n_v} \rightarrow \mathcal{L}_2^{n_w}$  the input–output operators of the reduced-order linear subsystem  $\hat{\Sigma}_{\text{lin}}$  for zero initial condition;

- (3) The feedback interconnection  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$  is well-posed.

It is noted that the input–output operators  $\hat{F}_i$ ,  $i \in \{y, w\}$ , indeed exist due to asymptotic stability of  $\hat{\Sigma}_{\text{lin}}$ . Furthermore, even though the assumption on the error bound as in (19) might seem restrictive at first sight, it is remarked that this incremental form is directly implied by an ordinary (i.e. non-incremental) error bound, due to linearity. Actually, model reduction techniques for linear systems satisfying the first two items of Assumption 13 exist. Balanced truncation and optimal Hankel norm approximation (Glover, 1984) are two such examples. Here, it is mentioned that balanced truncation leaves the direct-feedthrough matrix unchanged, such that the third item of Assumption 13 is also automatically satisfied. This does not hold for optimal Hankel norm approximation. Finally, it is noted that these methods yield a single error bound  $\varepsilon_{\text{lin}}$ , such that no distinction is made between different input–output combinations as in (19). In this case, the relation  $\varepsilon_{ij} \leq \varepsilon_{\text{lin}}$  holds with  $i \in \{y, w\}, j \in \{u, v\}$ .

The next result formalizes the conditions under which, firstly, the reduced-order nonlinear system inherits stability properties from the original system, and, secondly, an error bound can be guaranteed.

**Theorem 14.** Let  $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  satisfy Assumptions 10 and 11. Furthermore, let  $\hat{\Sigma}_{\text{lin}}$  be a reduced-order linear subsystem satisfying Assumption 13. Then, the following statements hold:

- (1) The reduced-order system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$  has a bounded incremental  $\mathcal{L}_2$  gain and the origin is an asymptotically stable equilibrium for  $u = 0$  when

$$(\gamma_{wv} + \varepsilon_{wv})\mu < 1; \quad (20)$$

- (2) Let (20) be satisfied. Then, the output error  $\delta y = y - \hat{y}$  is bounded as  $\|\delta y\|_2 \leq \varepsilon \|u\|_2$ , with

$$\varepsilon = \varepsilon_{yu} + \frac{\varepsilon_{yv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu} + \frac{(\gamma_{yv} + \varepsilon_{yv})\mu}{1 - (\gamma_{wv} + \varepsilon_{wv})\mu} \left( \varepsilon_{wu} + \frac{\varepsilon_{wv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu} \right). \quad (21)$$

**Proof.** The two statements are proven separately.

1. *Input–output stability and internal stability.* Lemma 12 directly guarantees a bounded incremental  $\mathcal{L}_2$  gain and asymptotic stability of the origin when the small-gain condition  $\hat{\gamma}_{wv}\mu < 1$  holds. However, the incremental gain  $\hat{\gamma}_{wv}$  of the reduced-order linear subsystem is not known a priori. Nonetheless, an upper bound for  $\hat{\gamma}_{wv}$  can be obtained by considering the equality

$$\hat{F}_w(u_1, v_1) - \hat{F}_w(u_2, v_2) = F_w(u_1, v_1) - F_w(u_2, v_2) - E_w(u_1, v_1) + E_w(u_2, v_2), \quad (22)$$

as follows from the definition of the error operator. Then,

$$\|\hat{F}_w(u, v_1) - \hat{F}_w(u, v_2)\|_2 \leq \|F_w(u, v_1) - F_w(u, v_2)\|_2 + \|E_w(u, v_1) - E_w(u, v_2)\|_2 \leq (\gamma_{wv} + \varepsilon_{wv})\|v_1 - v_2\|_2. \quad (23)$$

Here, the latter inequality follows from the incremental bound on the high-order linear subsystem (12) and the error bound (19). Clearly,  $\gamma_{wv} + \varepsilon_{wv}$  provides an upper bound to the incremental  $\mathcal{L}_2$  gain  $\hat{\gamma}_{wv}$  of the reduced-order linear subsystem. Hence, (20) implies  $\hat{\gamma}_{wv}\mu < 1$ , which proves the first statement via Lemma 12.

2. *Error bound.* As a first step in error analysis, bounds on the magnitude of the signals  $w$  and  $v$  will be derived. Here, by using the fact that  $w = 0$  is the unique solution of  $\Sigma = (\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$  to  $u = 0$  (for zero initial condition), (16) in the proof of Lemma 12 directly leads to

$$\|w\|_2 \leq \frac{\gamma_{wu}}{1 - \gamma_{wv}\mu} \|u\|_2. \quad (24)$$

Substitution of (24) in (13), hereby using  $G0 = 0$ , gives

$$\|v\|_2 \leq \frac{\mu\gamma_{wu}}{1 - \gamma_{wv}\mu} \|u\|_2. \quad (25)$$

Next, the error  $\delta w = w - \hat{w}$  is considered, which gives

$$\begin{aligned} \delta w &= F_w(u, v) - \hat{F}_w(u, \hat{v}) \\ &= F_w(u, v) - \hat{F}_w(u, v) + \hat{F}_w(u, v) - \hat{F}_w(u, \hat{v}), \end{aligned} \quad (26)$$

such that  $\|\delta w\|_2$  can be bounded as

$$\|\delta w\|_2 \leq \|F_w(u, v) - \hat{F}_w(u, v)\|_2 + \|\hat{F}_w(u, v) - \hat{F}_w(u, \hat{v})\|_2. \quad (27)$$

In (27), the first term is related to the error bound on the linear subsystem, which is bounded by (19). The second term can be related to the incremental  $\mathcal{L}_2$  gain of the reduced-order linear subsystem, which yields

$$\|\delta w\|_2 \leq \varepsilon_{wu}\|u\|_2 + \varepsilon_{wv}\|v\|_2 + \hat{\gamma}_{wv}\|\delta v\|_2. \quad (28)$$

The gain  $\hat{\gamma}_{wv}$  is unknown a priori, but can be bounded as  $\hat{\gamma}_{wv} \leq \gamma_{wv} + \varepsilon_{wv}$ , as is shown in the proof of the first part of this theorem. Furthermore, (13) implies  $\|\delta v\|_2 \leq \mu\|\delta w\|_2$ . Exploiting this in (28) leads to

$$\|\delta w\|_2 \leq \frac{\varepsilon_{wu}\|u\|_2 + \varepsilon_{wv}\|v\|_2}{1 - (\gamma_{wv} + \varepsilon_{wv})\mu}, \quad (29)$$

where it is noted that the small-gain condition (20) guarantees boundedness of (29). Substitution of (25) in (29) and the use of this result in (13) leads to a bound on the error  $\delta v = v - \hat{v}$  as

$$\|\delta v\|_2 \leq \frac{\mu}{1 - (\gamma_{wv} + \varepsilon_{wv})\mu} \left( \varepsilon_{wu} + \frac{\varepsilon_{wv}\mu\gamma_{wu}}{1 - \gamma_{wv}\mu} \right) \|u\|_2. \quad (30)$$

By construction, (30) provides a bound on  $\delta v$  in the coupled configuration. This result will be exploited to obtain the final error

bound. Hereto, the output error  $\delta y = y - \hat{y}$  is introduced, which is given by

$$\begin{aligned} \delta y &= F_y(u, v) - \hat{F}_y(u, \hat{v}) \\ &= F_y(u, v) - \hat{F}_y(u, v) + \hat{F}_y(u, v) - \hat{F}_y(u, \hat{v}). \end{aligned} \quad (31)$$

Here, the introduction of the term  $\hat{F}_y(u, v)$  leads to

$$\|\delta y\|_2 \leq \|F_y(u, v) - \hat{F}_y(u, v)\|_2 + \|\hat{F}_y(u, v) - \hat{F}_y(u, \hat{v})\|_2, \quad (32)$$

where the first term is related to the error introduced by reduction of the linear subsystem, which is bounded by (19). The second term can be related to the incremental gain of the reduced-order linear subsystem, such that

$$\|\delta y\|_2 \leq \varepsilon_{yu} \|u\|_2 + \varepsilon_{yv} \|v\|_2 + \hat{\gamma}_{yv} \|\delta v\|_2. \quad (33)$$

Again, the gain  $\hat{\gamma}_{yv}$  is unknown a priori, but can be bounded as  $\hat{\gamma}_{yv} \leq \gamma_{yv} + \varepsilon_{yv}$ . Then, substitution of the bound on  $v$  (25) and the bound on  $\delta v$  (30) in (33) leads to the output error bound as in (21), which proves the second statement of the theorem.  $\square$

In the proof of the error bound in Theorem 14, the incremental gain property of the input–output operators plays a crucial role. Namely, the incremental gains characterize the amplification of perturbations going through the subsystems, where these perturbations are introduced by model reduction of the linear subsystem. The small-gain theorem then guarantees boundedness of the perturbations in the bidirectionally coupled configuration as in Fig. 2.

The result in Theorem 14 is based on the availability of the error bounds  $\varepsilon_{ij}$ ,  $i \in \{y, w\}$ ,  $j \in \{u, v\}$ , for the linear subsystem, providing bounds on all relevant input–output pairs. However, as mentioned before, existing model reduction techniques for linear systems generally provide a single error bound  $\varepsilon_{lin}$ . When this error bound is exploited as  $\varepsilon_{ij} \leq \varepsilon_{lin}$  for  $i \in \{y, w\}$ ,  $j \in \{u, v\}$ , the error bound (21) reduces to

$$\varepsilon = \varepsilon_{lin} \left( 1 + \frac{\mu \gamma_{wu}}{1 - \gamma_{wv} \mu} \right) \left( 1 + \frac{(\gamma_{yv} + \varepsilon_{lin}) \mu}{1 - (\gamma_{wv} + \varepsilon_{lin}) \mu} \right). \quad (34)$$

**Remark 15.** The condition for stability (20) and the error bound (21) depend only on properties of the high-order system and the error bound on the linear subsystems and can therefore be evaluated a priori. However, a tighter error bound can be obtained when the gains  $\hat{\gamma}_{wv}$  and  $\hat{\gamma}_{yv}$  of the reduced-order linear subsystem are computed a posteriori (i.e. after the reduction has been employed). These gains can directly be used in (28) and (33), respectively, instead of using their bounds  $\gamma_{iv} + \varepsilon_{iv}$ ,  $i \in \{y, w\}$ . Additionally, the availability of  $\hat{\gamma}_{wv}$  will allow for the direct evaluation of stability via  $\hat{\gamma}_{wv} \mu < 1$  instead of via (20).

**Remark 16.** The availability of the explicit expression (21) allows for a reduction procedure in which the error bound is minimized. Namely, in the reduction of the linear subsystem, emphasis can be placed on the input–output combination that has the largest contribution to the overall error in (21).

## 5.2. Incrementally contractive systems

As a special case of the results in the Section 5.1, systems will be discussed that are (incrementally) contractive. Then, the following assumption replaces Assumption 11.

**Assumption 17.** Let the system  $\Sigma = \mathcal{L}(\Sigma_{lin}, \Sigma_{nl})$  satisfy the following assumptions:

- (1) The linear subsystem  $\Sigma_{lin}$  has a strictly bounded real transfer function, i.e. is (incrementally) contractive;
- (2) The nonlinear subsystem  $\Sigma_{nl}$  is incrementally strictly contractive with gain  $\mu$  ( $\mu < 1$ ).

Under Assumptions 10 and 17, the total nonlinear system  $\Sigma$  is contractive, as formalized in the following lemma.

**Lemma 18.** Let  $\Sigma = \mathcal{L}(\Sigma_{lin}, \Sigma_{nl})$  satisfy Assumptions 10 and 17. Then,  $\Sigma$  is incrementally contractive. Additionally, the origin is an asymptotically stable equilibrium point of  $\Sigma$  for  $u = 0$ .

**Proof.** By Assumption 17, there exist nonnegative storage functions  $S_{lin}(x_1, x_2)$  and  $S_{nl}(z_1, z_2)$  such that

$$\dot{S}_{lin} \leq |u_1 - u_2|^2 + |v_1 - v_2|^2 - |y_1 - y_2|^2 - |w_1 - w_2|^2, \quad (35)$$

$$\dot{S}_{nl} \leq \mu^2 |w_1 - w_2|^2 - |v_1 - v_2|^2. \quad (36)$$

Then, using the storage function  $S(x_1, x_2, z_1, z_2) := S_{lin}(x_1, x_2) + S_{nl}(z_1, z_2)$ , it is easily shown that  $\Sigma$  is incrementally contractive.

The proof of stability of the origin (for  $u = 0$ ) directly follows from Lemma 12 by noting that Assumption 17 can be interpreted as a special case of Assumption 11.  $\square$

In the reduction of the linear subsystem for nonlinear systems satisfying Assumption 17, the bounded real balancing procedure as discussed in Section 4 is applied. Then, the reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{L}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  is incrementally contractive and satisfies an error bound, as formalized in the next theorem.

**Theorem 19.** Let  $\Sigma = \mathcal{L}(\Sigma_{lin}, \Sigma_{nl})$  satisfy Assumptions 10 and 17 and let  $\hat{\Sigma}_{lin}$  be the reduced-order linear subsystem obtained by bounded real balancing. Then, the reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{L}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  is incrementally contractive, the origin of  $\hat{\Sigma}$  is asymptotically stable for  $u = 0$  and the output error is bounded as

$$\|y - \hat{y}\|_2 \leq 2 \left( \frac{1}{1 - \mu} \right)^2 \left( \sum_{i=k+1}^n \xi_i \right) \|u\|_2, \quad (37)$$

with  $\xi_i$  distinct bounded real singular values.

**Proof.** The properties of bounded real balancing, as stated in Theorem 9, guarantee that the reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{L}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  satisfies Assumptions 10 and 17. Here, it is remarked that bounded real balancing leaves the direct feedthrough matrix  $D$  unchanged, such that the interconnection  $\mathcal{L}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  is indeed well-posed. Then, incremental contractivity and asymptotic stability of the origin follow from Lemma 18.

The error bound can be proven along the same lines as in Theorem 14. By (incremental) contractivity of the linear subsystem, the gains  $\gamma_{ij}$  in the proof of Theorem 14 satisfy  $\gamma_{ij} \leq 1$ ,  $i \in \{y, w\}$ ,  $j \in \{u, v\}$ . Moreover, by the application of bounded real balancing, the gains  $\hat{\gamma}_{ij}$  of the reduced-order linear systems can also be a priori bounded as  $\hat{\gamma}_{ij} \leq 1$ . Finally, the single error bound  $\varepsilon_{lin} = 2 \sum_{i=k+1}^n \xi_i$  (see Theorem 19) is used to provide the bounds  $\varepsilon_{ij} \leq \varepsilon_{lin}$ . Then, the application of these bounds in the proof of Theorem 14 gives the error bound (37).  $\square$

## 5.3. Incrementally passive systems

The ideas as used in Sections 5.1 and 5.2 can be exploited in the development of a model reduction procedure for the preservation of passivity.

In particular, if the linear subsystem  $\Sigma_{lin}$  has a strictly positive real transfer function (i.e. is incrementally passive) and the nonlinear subsystem  $\Sigma_{nl}$  is incrementally very strictly passive (see Definition 3), then the interconnection  $\Sigma = \mathcal{L}(\Sigma_{lin}, \Sigma_{nl})$  is

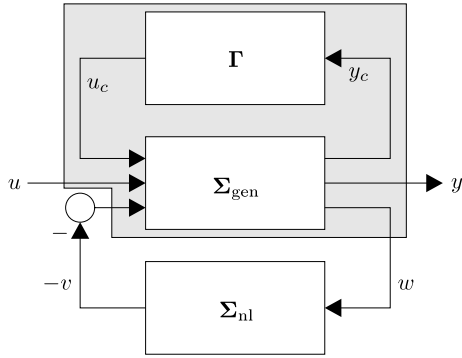


Fig. 3. Controlled nonlinear system. The gray box represents the controlled linear subsystem  $\Sigma_{lin} = \mathcal{F}_u(\Sigma_{gen}, \Gamma)$ .

incrementally passive and the origin is an asymptotically stable equilibrium for  $u = 0$ . If the reduction procedure of positive real balancing is applied to  $\Sigma_{lin}$  to obtain a reduced-order linear subsystem  $\hat{\Sigma}_{lin}$ , these properties are preserved in the reduced-order nonlinear system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{lin}, \Sigma_{nl})$ . Finally, under some additional conditions, an error bound can be derived. Further details on passivity preservation for systems of the form as in Fig. 1 can be found in Besselink (2012).

**Remark 20.** Contractive and passive nonlinear systems as in Fig. 1 can be related via a scattering transformation (see e.g. van der Schaft, 2000 for a definition). For the contractivity and passivity preserving reduction procedures discussed in Sections 5.2 and 5.3, it can be shown that this scattering relation is preserved after reduction. In fact, this relation is crucial in the derivation of an error bound for the reduction of (incrementally) passive nonlinear systems. Details can be found in Besselink (2012).

### 6. Controller reduction for closed-loop nonlinear systems

The model reduction procedures for nonlinear systems that can be decomposed as in Fig. 1 exploit existing linear model reduction techniques. As a consequence, the results in Section 5 can be extended towards procedures for controller reduction by exploiting existing controller reduction techniques for linear systems.

Controlled nonlinear systems as in Fig. 3 are considered. Here,  $\Sigma_{gen}$  is a generalized system, which contains linear dynamics and linear weighting filters defining the control problem. Specifically, the generalized system is chosen such that the minimization of the  $\mathcal{L}_2$  gain from the performance input  $u$  to the performance output  $y$  gives increased performance. The linear controller is denoted by  $\Gamma$ , which has input  $y_c$  and output  $u_c$ . When combining the controller and generalized system into a (controlled) linear system as  $\Sigma_{lin} = \mathcal{F}_u(\Sigma_{gen}, \Gamma)$ , with  $\Sigma_{lin}$  as in (1) and  $\mathcal{F}_u$  the upper linear fractional transformation, it is clear that the controlled linear system in Fig. 3 is a special case of the linear subsystem in Fig. 1. Finally,  $\Sigma_{nl}$  is a nonlinear dynamic system of the form (2). It is remarked that the configuration in Fig. 3 does not necessarily represent the control of a system with local nonlinearity by means of a linear controller. Namely, some nonlinear controllers can be cast in the same framework, where an example is given by variable-gain controllers (see e.g. Heertjes & Steinbuch, 2004; van de Wouw, Pastink, Heertjes, Pavlov, & Nijmeijer, 2008).

In the configuration in Fig. 3, the controller  $\Gamma$  may have been designed on the basis of the linear generalized system only, where stability properties of the full nonlinear system are guaranteed by means of a small-gain argument in the loop connecting the nonlinear subsystem  $\Sigma_{nl}$ . In this case, controller design is

performed with respect to the input  $d = [y^T \ v^T]^T$  and output  $e = [y^T \ w^T]^T$  and standard controller synthesis techniques such as  $\mathcal{H}_\infty$  control design can be exploited. However, controller synthesis is not within the scope of this paper. Instead, the controller  $\Gamma$  is assumed to be given and the reduction of this controller is pursued. Next, it is assumed that the controller is designed such that the nonlinear controlled system satisfies Assumption 10. Additionally, the following assumption is adopted.

**Assumption 21.** Let the controlled system  $\Sigma = \mathcal{I}(\Sigma_{lin}, \Sigma_{nl})$  satisfy the following assumptions:

- (1) The controller  $\Gamma$  ensures that  $\Sigma_{lin} = \mathcal{F}_u(\Sigma_{gen}, \Gamma)$  satisfies  $\|e\|_2 \leq \gamma_{con} \|d\|_2$  for some  $\gamma_{con} > 0$ ;
- (2) The nonlinear subsystem  $\Sigma_{nl}$  has a bounded incremental  $\mathcal{L}_2$  gain with gain  $\mu$ ;
- (3) The small-gain condition  $\gamma_{con}\mu < 1$  holds.

For such systems, a reduced-order linear controller  $\hat{\Gamma}$  will be pursued, leading to a reduced-order controlled system  $\Sigma = \mathcal{I}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  as in Fig. 2 with  $\hat{\Sigma}_{lin} = \mathcal{F}_u(\Sigma_{gen}, \hat{\Gamma})$  of the form (3). Two distinct objectives for controller reduction are stated in the next sections.

#### 6.1. Approximation of closed-loop behavior

Here, the objective is to find a reduced-order controller  $\hat{\Gamma}$  such that the reduced-order closed-loop dynamics resembles the original high-order closed-loop dynamics, from the external input  $u$  to the external output  $y$ . This problem is addressed in the following corollary.

**Corollary 22.** Let  $\Sigma = \mathcal{I}(\Sigma_{lin}, \Sigma_{nl})$  with  $\Sigma_{lin} = \mathcal{F}_u(\Sigma_{gen}, \Gamma)$  satisfy Assumptions 10 and 21. Furthermore, let  $\hat{\Gamma}$  be a reduced-order controller such that  $\hat{\Sigma}_{lin} = \mathcal{F}_u(\Sigma_{gen}, \hat{\Gamma})$  is asymptotically stable and the error bound  $\|e - \hat{e}\|_2 \leq \varepsilon_{con} \|d\|_2$  on reduction of the (controlled) linear subsystem  $\Sigma_{lin}$  holds for some  $\varepsilon_{con} > 0$ , with  $\hat{e} = [\hat{y}^T \ \hat{w}^T]^T$ . Then, the following holds:

- (1) The controlled nonlinear system  $\Sigma = \mathcal{I}(\Sigma_{lin}, \Sigma_{nl})$  has a bounded incremental  $\mathcal{L}_2$  gain and the origin is an asymptotically stable equilibrium point for  $u = 0$ ;
- (2) The reduced-order controlled nonlinear system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  has a bounded incremental  $\mathcal{L}_2$  gain and the origin is an asymptotically stable equilibrium point of  $\hat{\Sigma}$  for  $u = 0$  when
 
$$(\gamma_{con} + \varepsilon_{con})\mu < 1; \tag{38}$$
- (3) If (38) holds, then the error  $\delta y = y - \hat{y}$  between the closed-loop outputs of the high-order and reduced-order nonlinear system  $\Sigma$  and  $\hat{\Sigma}$  is bounded as  $\|\delta y\|_2 \leq \varepsilon \|u\|_2$ , with

$$\varepsilon = \varepsilon_{con} \left( \frac{1}{1 - \gamma_{con}\mu} \right) \left( \frac{1}{1 - (\gamma_{con} + \varepsilon_{con})\mu} \right). \tag{39}$$

**Proof.** To prove the corollary, it is remarked that Assumption 21 is basically a restatement of Assumption 11. Then, the first item directly follows from Lemma 12. Similarly, the assumptions on the reduced-order controller are equivalent to Assumption 13, with  $\varepsilon_{ij} \leq \varepsilon_{con}$ . Then, the second and third item follow from Theorem 14, with  $\gamma_{ij} \leq \gamma_{con}$ ,  $i \in \{u, v\}$ ,  $j \in \{y, w\}$ .  $\square$

The results in Corollary 22 rely on the application of controller reduction techniques for linear systems that guarantee stability of the reduced-order closed-loop linear subsystem as well as an error bound of the form  $\|e - \hat{e}\|_2 \leq \varepsilon_{con} \|d\|_2$ . Methods satisfying these assumptions and guaranteeing an a priori error bound are given in Gao, Lam, and Wang (2006) and Zhou, D'Souza, and Cloutier (1995). Alternatively, stability and an error bound can be evaluated a posteriori (i.e. after the reduction has been employed), allowing for the application of other controller reduction techniques (e.g. those in Ceton, Wortelboer, & Bosgra, 1993).

## 6.2. Performance preservation

For performance preservation, the reduced-order controller  $\hat{\Gamma}$  is required to guarantee the same performance as the high-order controller  $\Gamma$ , i.e. the implication  $\|y\|_2 \leq \gamma_{cl}\|u\|_2 \Rightarrow \|\hat{y}\|_2 \leq \gamma_{cl}\|u\|_2$  should hold for some  $\gamma_{cl}$ . After scaling to  $\gamma_{cl} = 1$ , this represents the preservation of contractivity, leading to the following corollary.

**Corollary 23.** Let  $\Sigma = \mathcal{I}(\Sigma_{lin}, \Sigma_{nl})$  with  $\Sigma_{lin} = \mathcal{F}_u(\Sigma_{gen}, \Gamma)$  satisfy Assumptions 10 and 21 with  $\gamma_{con} = 1$  and  $\mu < 1$ . Furthermore, let  $\hat{\Gamma}$  be a reduced-order controller such that  $\hat{\Sigma}_{lin} = \mathcal{F}_u(\Sigma_{gen}, \hat{\Gamma})$  is asymptotically stable and the bound  $\|\hat{e}\|_2 \leq \|d\|_2$  holds, with  $\hat{e} = [\hat{y}^T \hat{w}^T]^T$ . Then, the following statements hold:

- (1) The controlled nonlinear system  $\Sigma = \mathcal{I}(\Sigma_{lin}, \Sigma_{nl})$  is incrementally contractive and the origin is an asymptotically stable equilibrium point for  $u = 0$ ;
- (2) The reduced-order controlled nonlinear system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{lin}, \Sigma_{nl})$  is incrementally contractive and the origin is an asymptotically stable equilibrium for  $u = 0$ .

If, in addition, the error bound  $\|e - \hat{e}\|_2 \leq \varepsilon_{con}\|d\|_2$  is available on reduction of the (controlled) linear subsystem  $\Sigma_{lin}$ , the following statement holds:

- (3) The error  $\delta y = y - \hat{y}$  between the closed-loop outputs of  $\Sigma$  and  $\hat{\Sigma}$  is bounded as  $\|\delta y\|_2 \leq \varepsilon\|u\|_2$ , with  $\varepsilon = \varepsilon_{con}(1 - \mu)^{-2}$ .

**Proof.** Assumption 21 with  $\gamma_{con} = 1$  and  $\mu < 1$  implies that the linear subsystem is (incrementally) contractive, whereas the nonlinear subsystem is incrementally strictly contractive. Thus, Assumption 17 holds, such that the first item follows from Lemma 18. Additionally, the assumptions on the reduced-order controller resemble the properties of bounded real balancing, such that the remaining items follow from Theorem 19.  $\square$

As before, the result in Corollary 23 is based on properties of the reduced-order controlled linear subsystem. Specifically, the reduced-order controller is assumed to be performance preserving in the sense that the reduced-order controlled linear subsystem satisfies the performance criterion  $\|\hat{e}\|_2 \leq \|d\|_2$ . A controller reduction technique that satisfies this assumption has been proposed in Goddard and Glover (1998). Alternatives following the same ideas are given by Wang and Huang (2003) and Wang, Sreeram, and Liu (2001). Here, it is noted that these methods do not give an a priori error bound of the form  $\|e - \hat{e}\|_2 \leq \varepsilon_{con}\|d\|_2$ , such that the third item can only be obtained by the a posteriori computation of the required error bound on the reduced-order linear subsystem. Also, this is the only item that requires the property of incremental strict contractivity of the nonlinear subsystem  $\Sigma_{nl}$ . Thus, the first two items also hold when the nonlinear subsystem satisfies the less restrictive assumption of strict contractivity.

## 7. Examples

The model and controller reduction procedures are illustrated by means of examples in Sections 7.1 and 7.2, respectively. Here, the first example gives a detailed illustration of the theoretical developments of this paper, whereas the second example illustrates the applicability of the proposed techniques to an engineering case study.

### 7.1. Flexible beam example

To illustrate the model reduction procedure for nonlinear systems satisfying a bounded incremental  $\mathcal{L}_2$  gain in Section 5, the

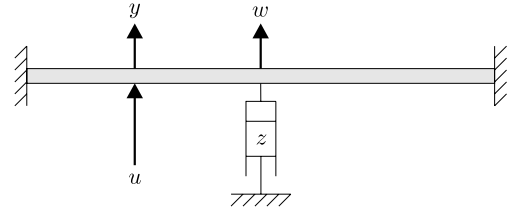


Fig. 4. Flexible beam with nonlinear damping.

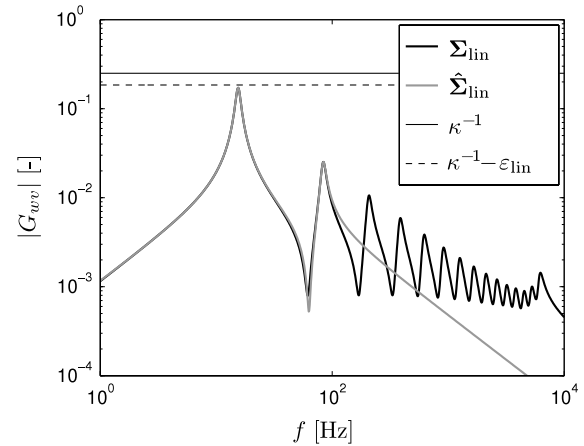


Fig. 5. Frequency response functions for the high-order and reduced-order linear subsystems.

flexible beam system in Fig. 4 is considered. The beam (without the damper) is modeled using Euler beam elements, which yields a minimal asymptotically stable linear model of the form (1) with  $x \in \mathbb{R}^{60}$ . The input  $u \in \mathbb{R}$  is a force acting on the beam, whereas the vertical deflection  $y \in \mathbb{R}$  is the output. In its center, the beam is supported by a nonlinear damping element, which is described by

$$\Sigma_{nl} : \dot{z} = -z - \sigma(z) + \kappa w, \quad -v = z, \quad (40)$$

with  $z \in \mathbb{R}$  the internal state. In (40),  $w \in \mathbb{R}$  is the vertical velocity of the beam center,  $v \in \mathbb{R}$  is the damping force and it is noted that the interconnection satisfies Assumption 10. Moreover,  $\sigma$  is an arbitrary nondecreasing continuous function, such that the storage function  $S(z_1, z_2) = \frac{1}{2}(z_1 - z_2)^2$  shows that the incremental  $\mathcal{L}_2$  gain of (40) is bounded by  $\kappa$ , i.e.  $\mu = \kappa$ . After choosing  $\kappa$  such that  $\gamma_{wv}\kappa < 1$ , Assumption 11 holds.

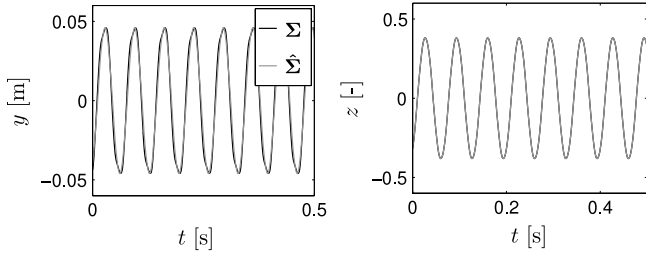
Balanced truncation is applied to the linear beam model to obtain an asymptotically stable reduced-order subsystem  $\hat{\Sigma}_{lin}$  for  $k = 4$  satisfying Assumption 13. The frequency response functions  $G_{wv}$  of  $\Sigma_{lin}$  and  $\hat{G}_{wv}$  of  $\hat{\Sigma}_{lin}$  are depicted in Fig. 5, where the line  $\mu^{-1}$  (for  $\mu = \kappa = 4$ ) shows that both the high-order and reduced-order nonlinear system have a bounded incremental  $\mathcal{L}_2$  gain and an asymptotically stable equilibrium for  $u = 0$  (see Lemma 12). In fact, stability of the reduced-order nonlinear system can be guaranteed a priori (i.e. without computing the frequency response function  $\hat{G}_{wv}$ ) since the a priori error bound for balanced truncation  $\varepsilon_{lin} = 6.522 \cdot 10^{-2}$  satisfies  $\gamma_{wv} < \mu^{-1} - \varepsilon_{lin}$  (see Fig. 4). Thus, condition (20) (with  $\varepsilon_{wv} \leq \varepsilon_{lin}$ ) in Theorem 14 holds and stability is guaranteed.

Error bounds on the full nonlinear system are computed using Theorem 14 and can be found in Table 1, for several values of  $\kappa$ . Here, an a priori error bound is computed using the bounds  $\varepsilon_{ij} \leq \varepsilon_{lin}$ , leading to (34). Moreover, an a posteriori error bound is obtained by computing the error bounds  $\varepsilon_{ij}$  and the evaluation of (21). Clearly, the latter leads to less conservative results, even though it is remarked that the conservatism in a priori error bound

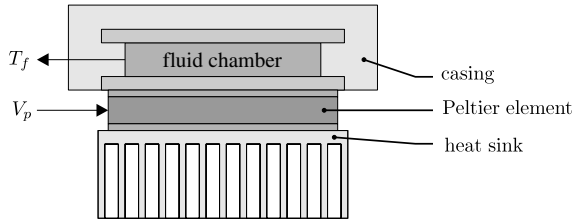


**Table 1**  
A priori (using (34)) and a posteriori (using (21)) error bounds.

$\kappa$	$\varepsilon$ (a priori)	$\varepsilon$ (a posteriori)
3	$1.214 \cdot 10^{-1}$	$2.157 \cdot 10^{-4}$
4	$3.060 \cdot 10^{-1}$	$2.694 \cdot 10^{-4}$
5	–	$6.449 \cdot 10^{-4}$



**Fig. 6.** Comparison of the high-order system  $\Sigma$  and reduced-order system  $\hat{\Sigma}$  for  $k = 4$  and with input signal  $u(t) = 10 \text{ sign}(\sin(2\pi 15t))$ : output  $y$  (left) and state  $z$  (right).



**Fig. 7.** Lab-on-a-chip benchmark system.

is largely due to the large value of  $\varepsilon_{\text{lin}}$  (compared to  $\varepsilon_{ij}$ ) rather than the results of [Theorem 14](#).

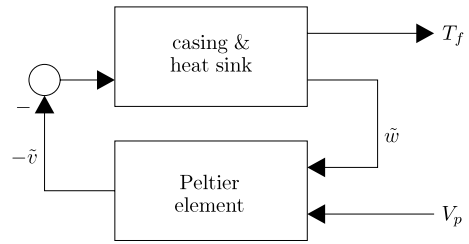
**Remark 24.** For  $\kappa = 5$ , stability cannot be guaranteed a priori as  $\gamma_{wv} > \mu^{-1} - \varepsilon_{\text{lin}}$  and the satisfaction of (20) (with  $\varepsilon_{wv} \leq \varepsilon_{\text{lin}}$ ) is unknown. As a result, the a priori error bound in [Table 1](#) cannot be computed. Of course, a less conservative guarantee on stability properties is obtained by the a posteriori computation of  $\varepsilon_{wv}$ .

[Fig. 6](#) shows a comparison of the response of the high-order and reduced-order nonlinear systems for  $\sigma(z) = \arctan(10z)$  and  $\kappa = 4$ , indicating an accurate approximation.

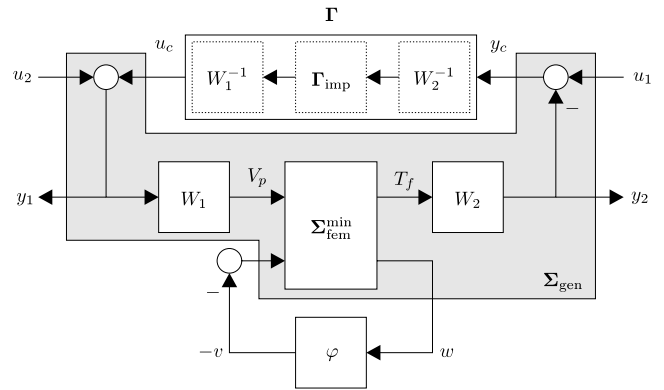
### 7.2. Lab-on-a-chip benchmark example

The controller reduction techniques developed in [Section 6](#) are applied in the design of a low-order controller for the temperature control in an industrial lab-on-a-chip. An important application of a lab-on-a-chip is in the field of molecular biology for disease diagnostics. Herein, biochemical analysis techniques are applied on small volumes of fluid, such as e.g. blood. These analysis techniques rely on an accurate control of the fluid temperature. Since a lab-on-a-chip is typically (part of) a disposable product, the on-board computational power is limited and the controller is required to be relatively simple (and thus of low order). Herein, controller reduction provides a means for obtaining this controller on the basis of a complex (high-order) model, hereby using  $\mathcal{H}_\infty$  controller design techniques.

The lab-on-a-chip benchmark system as depicted in the top graph in [Fig. 7](#) basically consists out of three parts. First, a casing forms a fluid chamber, which contains the fluid of which the temperature needs to be controlled. To cool or heat the fluid, a Peltier element is used. A Peltier element is an electric heat pump and transfers heat between its two sides. Therefore, a heat sink is needed to exchange heat with the environment. In this setting,



**Fig. 8.** Lab-on-a-chip benchmark system model.



**Fig. 9.** Control configuration for the lab-on-a-chip benchmark system.

the fluid temperature needs to track a reference trajectory, such that the objective is the design of a tracking controller. Herein, the Peltier element input voltage  $V_p$  is prescribed by the controller, hereby using a measurement of the fluid temperature  $T_f$ , where  $T_f$  represents the deviation from the environmental temperature.

A model of the lab-on-a-chip benchmark system leads to the block diagram as shown in [Fig. 8](#). The casing and heat sink typically exhibit temperature nonuniformities and are therefore modeled using a finite element approach, leading to a (single) high-order model for the thermal dynamics. Herein, the nonlinear effect of radiation is not taken into account. Besides the fluid temperature  $T_f$ , the temperatures of the casing and heat sink at the boundary with the Peltier element form the outputs  $\tilde{w}$  of the finite element model, whereas the inputs  $\tilde{v}$  represent the heat flows resulting from the heat pumping action of the Peltier element. The Peltier element represents the actuator and is modeled as a static nonlinearity, which describes these heat flows as a function of the casing and heat sink temperatures and the input voltage  $V_p$ , where the latter is the control input. The model in [Fig. 8](#) thus consists out of the feedback interconnection of a high-order linear system and a static nonlinearity, such that it falls in the scope of earlier sections. However, in order to reduce the conservatism in the computation of the gain of the Peltier element, a loop transformation is applied. This transforms the linear system describing the casing and heat sink in a linear system  $\Sigma_{\text{fem}}$ , where removing the uncontrollable and unobservable modes leads to the minimal realization  $\Sigma_{\text{fem}}^{\text{min}}$  of order 44. This system is depicted in [Fig. 9](#), where  $\varphi$  represents the static nonlinearity describing the Peltier element (after loop transformation). Furthermore,  $v \in \mathbb{R}^2$  denote the (transformed) heat flows between the Peltier element and the casing and heat sink, whereas  $w \in \mathbb{R}^3$  combines the temperatures at the top and bottom boundary of the Peltier element and the input voltage. Since only a limited range in temperatures is of (practical) interest, the incremental  $\mathcal{L}_2$  gain of the static nonlinearity is bounded and can be obtained as  $\mu = \sup_{w \in \mathcal{W}} \bar{\sigma}((\partial\varphi/\partial w)(w))$ , where  $\mathcal{W}$  represents the compact set (in the variable  $w$ ) of interest and  $\bar{\sigma}(\cdot)$  denotes the largest singular value.

As stated before, the tracking control of the fluid temperature is of interest. Thus, a tracking controller is designed which, firstly, ensures stability by enforcing a small-gain condition in the loop connecting the static nonlinearity  $\varphi$  to the linear closed-loop dynamics and, secondly, enforces the desired tracking performance. Thereto, the controller uses the measurement of the (scaled) fluid temperature  $T_f$  and prescribes the Peltier element voltage  $V_p$ . For controller design, a  $\mathcal{H}_\infty$  loopshaping procedure is used (McFarlane & Glover, 1990), leading to the control configuration in Fig. 9. Here,  $W_1$  and  $W_2$  are the weighting filters employed in the loopshaping procedure, which are included in the feedback loop. Furthermore,  $u = [u_1 \ u_2]^T$  and  $y = [y_1 \ y_2]^T$  form the performance inputs and performance outputs, respectively, leading to the so-called four-block problem. When the generalized system  $\Sigma_{\text{gen}}$  is defined as in Fig. 9, it is clear that the closed-loop system is of the form as in Fig. 3, where the nonlinear subsystem  $\Sigma_{\text{nl}}$  is replaced by the static nonlinearity (see Remark 4). By inclusion of the weighting filters, the generalized system is of order 48, such that controller design leads to a controller  $\Gamma$  of order  $n_c = 48$ . Then, the controlled linear subsystem  $\Sigma_{\text{lin}} = \mathcal{F}_u(\Sigma_{\text{gen}}, \Gamma)$  (see Fig. 3) is of the form (1) with  $n = 96$ . The control problem is scaled such that the static nonlinearity  $\varphi$  is incrementally strictly contractive, with gain  $\mu = 0.95$ . Then, the controller is designed such that the controlled linear subsystem  $\Sigma_{\text{lin}} = \mathcal{F}_u(\Sigma_{\text{gen}}, \Gamma)$  is (incrementally) contractive with gain  $\gamma_{\text{con}} = 0.945$  (from input  $d = [u^T \ v^T]^T$  to output  $e = [y^T \ w^T]^T$ ). Hence, the controlled nonlinear system satisfies Assumption 21. Thus, by application of item 1 of Corollary 22, the controlled nonlinear system has a bounded incremental  $\mathcal{L}_2$  gain and the origin is an asymptotically stable equilibrium point for  $u = 0$ . It is remarked that, due to scaling, this can also be concluded from item 1 of Corollary 23, such that the controlled nonlinear system is in fact (incrementally) contractive.

The controller  $\Gamma$  is reduced by applying closed-loop balanced truncation (Ceton et al., 1993) to the controlled linear subsystem  $\Sigma_{\text{lin}} = \mathcal{F}_u(\Sigma_{\text{gen}}, \Gamma)$ , leading to a reduced-order controller  $\hat{\Gamma}$  of order  $n_c = 4$ . Since closed-loop balanced truncation does not give an a priori guarantee on stability of the reduced-order linear subsystem  $\hat{\Sigma}_{\text{lin}} = \mathcal{F}_u(\Sigma_{\text{gen}}, \hat{\Gamma})$ , stability is verified after the reduction. Additionally, an error bound on reduction of the controlled linear subsystem is computed as  $\|e - \hat{e}\|_2 \leq \varepsilon_{\text{con}} \|d\|_2$ , with  $\varepsilon_{\text{con}} = 0.053$ .

The reduced-order controlled linear subsystem satisfies the assumptions in the statement of Corollary 22 and it is easily checked that the condition (38) holds. Hence, the reduced-order controlled nonlinear system has a bounded incremental  $\mathcal{L}_2$  gain and the origin is an asymptotically stable equilibrium point for  $u = 0$ . Furthermore, evaluation of the error bound (39) yields  $\varepsilon = 10.2$ , providing a bound on the difference between the closed-loop behavior of the high-order and reduced-order controlled nonlinear systems  $\Sigma$  and  $\hat{\Sigma}$ , respectively. Here, it is remarked that this bound is likely to be conservative as a result of the fact that the small-gain condition (38) is only satisfied with a small margin.

Even though it has been concluded that stability properties are preserved after controller reduction, the problem can also be considered from a different perspective. Namely, due to scaling, the conditions of Corollary 23 hold as well. In fact, an upper bound on the gain of the reduced-order controlled linear subsystem is given by  $\gamma_{\text{con}} + \varepsilon_{\text{con}} < 1$ , such that  $\hat{\Sigma}_{\text{lin}}$  is (incrementally) contractive. Then, by item 2 of Corollary 23, the reduced-order controlled nonlinear system  $\hat{\Sigma}$  is (incrementally) contractive as well, indicating performance preservation.

The magnitude of the frequency response function of the high-order and reduced-order controllers are compared in Fig. 10, from which it is clear that, despite the large reduction, the reduced-order controller matches the original high-order controller well. Finally,

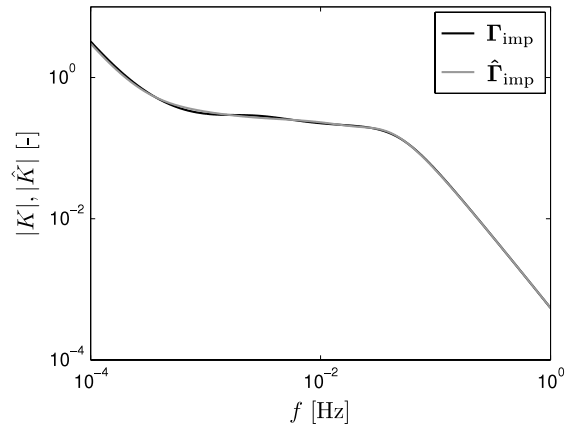


Fig. 10. Bode magnitude graphs of the transfer functions  $K$  and  $\hat{K}$  of the implementations of the high-order controller  $\Gamma$  and reduced-order controller  $\hat{\Gamma}$ , respectively.

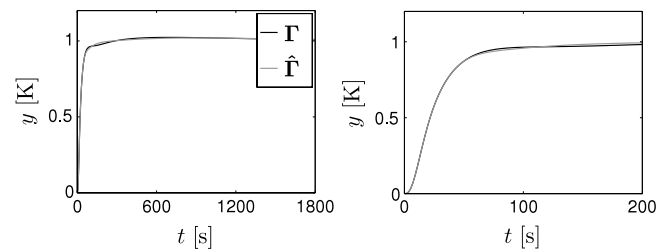


Fig. 11. Step response of the closed-loop using the high-order controller  $\Gamma$  and reduced-order controller  $\hat{\Gamma}$ . The right graph is a zoomed version of the left graph.

the closed-loop behaviors are compared in Fig. 11, which shows the fluid temperature for a 1 K step in the reference temperature. Here, it is recalled that deviations from the environmental temperature are considered. Again, it is clear that the performance obtained by the reduced-order controller is similar to that of the high-order controller.

Resuming, in this section the results developed in this paper have been applied in the scope of controller reduction for an industrial temperature control unit for a lab-on-a-chip, hereby validating the applicability of the techniques presented.

## 8. Conclusions

In this paper, a class of model reduction procedures is presented for nonlinear systems that can be decomposed into a feedback interconnection of a linear and nonlinear subsystem. Here, reduction is employed on the linear subsystem only, allowing for the use of existing model reduction techniques for linear systems and making this approach computationally attractive. First, conditions are given under which stability of the reduced-order nonlinear model can be guaranteed. Furthermore, it is shown that the application of specific reduction techniques for linear systems yields a reduction method in which contractivity or passivity is preserved. Also, a priori error bounds are derived, where the properties of a bounded incremental  $\mathcal{L}_2$  gain or incremental passivity of the nonlinear subsystem play an important role.

The techniques developed in this paper are applied in the scope of controller reduction for a class of controlled nonlinear systems, hereby again exploiting existing controller reduction procedures for linear systems and addressing the objectives of approximation of closed-loop behavior and performance preservation. This approach is illustrated by means of application to an industrial temperature control problem for a lab-on-a-chip.

## References

- Antoulas, A. C. (2005). *Approximation of large-scale dynamical systems*. Philadelphia, USA: SIAM.
- Astolfi, A. (2010). Model reduction by moment matching for linear and nonlinear systems. *IEEE Transactions on Automatic Control*, 55(10), 2321–2336.
- Besselink, B. (2012). *Model reduction for nonlinear control systems—with stability preservation and error bounds*. Ph.D. Thesis. Eindhoven University of Technology. Eindhoven, The Netherlands.
- Besselink, B., van de Wouw, N., & Nijmeijer, H. (2009). An error bound for model reduction of Lur'e-type systems. In *Proceedings of the joint 48th IEEE conference on decision and control and 28th Chinese control conference* (pp. 3264–3269). Shanghai, China.
- Besselink, B., van de Wouw, N., & Nijmeijer, H. (2011). Model reduction of nonlinear systems with bounded incremental  $\mathcal{L}_2$  gain. In *Proceedings of the joint 50th IEEE conference on decision and control and European control conference* (pp. 7170–7175). Orlando, USA.
- Bond, B. N., & Daniel, L. (2009). Stable reduced models for nonlinear descriptor systems through piecewise-linear approximation and projection. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 28(10), 1467–1480.
- Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2007). *Communications and control engineering series, Dissipative systems analysis and control: theory and applications* (2nd ed.). London, Great Britain: Springer-Verlag.
- Ceton, C., Wortelboer, P.M.R., & Bosgra, O.H. (1993). Frequency weighted closed-loop balanced reduction. In *Proceedings of the 2nd European control conference, Groningen. vol. 2* (pp. 697–701). The Netherlands.
- Desai, U., & Pal, D. (1984). A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, AC-29(12), 1097–1100.
- Enns, D.F. (1984). Model reduction with balanced realizations: an error bound and a frequency weighted generalization. In *Proceedings of the 23rd IEEE conference on decision and control* (pp. 127–132). Las Vegas, USA.
- Fujimoto, K., & Scherpen, J. M. A. (2010). Balanced realization and model order reduction for nonlinear systems based on singular value analysis. *SIAM Journal on Control and Optimization*, 48(7), 4591–4623.
- Gao, H., Lam, J., & Wang, C. (2006). Controller reduction with error performance: continuous- and discrete-time cases. *International Journal of Control*, 79(6), 604–616.
- Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their  $L^\infty$ -error bounds. *International Journal of Control*, 39(6), 1115–1193.
- Goddard, P. J., & Glover, K. (1998). Controller approximation: approaches for preserving  $H_\infty$  performance. *IEEE Transactions on Automatic Control*, 43(7), 858–871.
- Harshavardhana, P., Jonckheere, E., & Silverman, L. (1984). Stochastic balancing and approximation—stability and minimality. *IEEE Transactions on Automatic Control*, AC-29(8), 744–746.
- Heertjes, M., & Steinbuch, M. (2004). Stability and performance of a variable gain controller with application to a DVD storage drive. *Automatica*, 40(4), 591–602.
- Hill, D., & Moylan, P. (1976). The stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control*, AC-21(5), 708–711.
- Iggidr, A., Kalitine, B., & Outbib, R. (1996). Semidefinite Lyapunov functions, stability and stabilization. *Mathematics of Control, Signals, and Systems (MCS)*, 9(2), 95–106.
- Ionescu, T.C., & Astolfi, A. (2010). On moment matching with preservation of passivity and stability. In *Proceedings of the 49th IEEE conference on decision and control* (pp. 6189–6194). Atlanta, USA.
- Ionescu, T. C., Fujimoto, K., & Scherpen, J. M. A. (2010). Dissipativity preserving balancing for nonlinear systems—a Hankel operator approach. *Systems & Control Letters*, 59(3–4), 180–194.
- Lall, S., Marsden, J. E., & Glavaški, S. (2002). A subspace approach to balanced truncation for model reduction of nonlinear control systems. *International Journal of Robust and Nonlinear Control*, 12(6), 519–535.
- McFarlane, D. C., & Glover, K. (1990). *Lecture notes in control and information sciences: vol. 138. Robust controller design using normalized coprime factor plant descriptions*. Berlin, Germany: Springer-Verlag.
- Moore, B. C. (1981). Principal component analysis in linear systems—controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, AC-26(1), 17–32.
- Opdenacker, P. C., & Jonckheere, E. A. (1988). A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Transactions on Circuits and Systems*, 35(2), 184–189.
- Pavlov, A., & Marconi, L. (2008). Incremental passivity and output regulation. *Systems & Control Letters*, 57(5), 400–409.
- Pernebo, L., & Silverman, L. M. (1982). Model reduction via balanced state space representations. *IEEE Transactions on Automatic Control*, AC-27(2), 382–387.
- Prescott, T. P., & Papachristodoulou, A. (2012). Guaranteed error bounds for structured complexity reduction of biochemical networks. *Journal of Theoretical Biology*, 304, 172–182.
- Reis, T., & Heinkenschloss, M. (2009). Model reduction with a-priori error bounds for a class of nonlinear electrical circuits. In *Proceedings of the joint 48th IEEE conference on decision and control and 28th Chinese control conference* (pp. 5376–5383). Shanghai, China.
- Rewierński, M., & White, J. (2003). A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 22(2), 155–170.
- Romanchuk, B.G., & James, M.R. (1996). Characterization of the  $L_p$  incremental gain for nonlinear systems. In *Proceedings of the 35th IEEE conference on decision and control*, vol. 3 (pp. 3270–3275). Kobe, Japan.
- Scherpen, J. M. A. (1993). Balancing for nonlinear systems. *Systems & Control Letters*, 21, 143–153.
- Sirovich, L. (1987). Turbulence and the dynamics of coherent structures, part I: coherent structures. *Quarterly of Applied Mathematics*, 45(3), 561–571.
- van der Schaft, A. J. (2000). *Communications and control engineering series,  $L_2$ -gain and passivity techniques in nonlinear control* (2nd ed.). London, Great Britain: Springer-Verlag.
- van de Wouw, N., Pastink, H. A., Heertjes, M. F., Pavlov, A. V., & Nijmeijer, H. (2008). Performance of convergence-based variable-gain control of optical storage drives. *Automatica*, 44(1), 15–27.
- Wang, J.-Z., & Huang, L. (2003). Controller order reduction with guaranteed performance via coprime factorization. *International Journal of Robust and Nonlinear Control*, 13(6), 501–517.
- Wang, G., Sreeram, V., & Liu, W. Q. (2001). Performance preserving controller reduction via additive perturbation of the closed-loop transfer function. *IEEE Transactions on Automatic Control*, 46(5), 771–775.
- Willems, J. C. (1972). Dissipative dynamical systems part I: general theory. *Archive for Rational Mechanics and Analysis*, 45(5), 321–351.
- Zhou, K., D'Souza, C., & Cloutier, J. R. (1995). Structurally balanced controller order reduction with guaranteed closed loop performance. *Systems & Control Letters*, 24(4), 235–242.



**B. Besselink** received his M.Sc. degree (cum laude) in Mechanical Engineering from Eindhoven University of Technology, Eindhoven, The Netherlands, in 2008. In 2012, he received his Ph.D. from the same university for his thesis on model reduction for nonlinear control systems. He was a short-term visiting researcher at Tokyo Institute of Technology, Tokyo, Japan in 2012 and is a postdoctoral researcher at KTH Royal Institute of Technology in Stockholm, Sweden, since October 2012. His research interests include system theory for nonlinear dynamical systems and model reduction techniques.



**N. van de Wouw** (born, 1970) obtained his M.Sc. degree (with honors) and his Ph.D. in Mechanical Engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, in 1994 and 1999, respectively. From 1999 until now he has been affiliated with the Department of Mechanical Engineering of the Eindhoven University of Technology in the group of Dynamics and Control as an assistant/associate professor. In 2000, Nathan van de Wouw was working at Philips Applied Technologies, Eindhoven, The Netherlands, and, in 2001, he was working at The Netherlands Organization for Applied Scientific Research (TNO), Delft, The Netherlands. He has held positions as a visiting professor at the University of California Santa Barbara, USA, in 2006/2007, at the University of Melbourne, Australia, in 2009/2010 and at the University of Minnesota, USA, in 2012. Nathan van de Wouw has published a large number of journal and conference papers and the books 'Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach' with A.V. Pavlov and H. Nijmeijer (Birkhauser, 2005) and 'Stability and Convergence of Mechanical Systems with Unilateral Constraints' with R.I. Leine (Springer-Verlag, 2008). He currently is an Associate Editor for the journal "Automatica". His current research interests are the analysis and control of nonlinear/non-smooth systems and networked control systems.



**H. Nijmeijer** (1955) obtained his M.Sc. degree and his Ph.D. in Mathematics from the University of Groningen, Groningen, The Netherlands, in 1979 and 1983, respectively. From 1983 until 2000 he was affiliated with the Department of Applied Mathematics of the University of Twente, Enschede, The Netherlands. Since 1997 he was also part-time affiliated with the Department of Mechanical Engineering of the Eindhoven University of Technology, Eindhoven, The Netherlands. Since 2000, he is a full professor at Eindhoven, and chairs the Dynamics and Control group. He has published a large number of journal and conference papers, and several books, including the 'classical' *Nonlinear Dynamical Control Systems* (Springer, 1990, co-author A.J. van der Schaft), with A. Rodriguez, *Synchronization of Mechanical Systems* (World Scientific, 2003) with R.I. Leine, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems* (Springer-Verlag, 2004), and with A. Pavlov and N. van de Wouw, *Uniform Output Regulation of Nonlinear Systems* (Birkhauser 2005). Recently the book Thor I. Fossen and H. Nijmeijer (eds) *Parametric Resonance in Dynamical Systems*, Springer (2012) has been published. Henk Nijmeijer was editor in chief of the Journal of Applied

Mathematics until 2009, corresponding editor of the SIAM Journal on Control and Optimization, and board member of the International Journal of Control, Automatica, Journal of Dynamical Control Systems, International Journal of Bifurcation and Chaos, International Journal of Robust and Nonlinear Control, Journal of Nonlinear Dynamics and the Journal of Applied Mathematics and Computer Science, and Hybrid Systems. He is a fellow of the IEEE since 2000 and was awarded in

1990 the IEE Heaviside premium. He is an honorary knight of the 'golden feedback loop' (NTNU), since 2011. In the 2008 research evaluation of the Dutch Mechanical Engineering Departments the Dynamics and Control group was evaluated as excellent regarding all aspects (quality, productivity, relevance and viability). His research interests are in the broad scope of dynamics and control and their applications.