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Configuration Consensus of Two Underactuated Planar Rigid Bodies

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Abstract

A consensus control framework for configuration of multiple underactuated planar rigid bodies is developed. Following the results by Bullo *et al.* (2000), we propose a control law that achieves asymptotic consensus between the planar rigid bodies. Finally, we present a numerical example to show efficacy of the proposed approach.

1. Introduction

There are increasingly many researches in the field of underactuated systems, i.e., systems with fewer actuators than the system's degrees of freedom. Several methods have been successful to solve certain classes of underactuated systems' problem, e.g., [1–6].

One example of underactuated systems is underactuated planar rigid body, e.g., hovercraft. Pettersen and Egeland [7] proposed a continuous periodic time-varying feedback control law to exponentially stabilize the configuration, both the position and orientation, of an underactuated surface vessel. The proposed feedback law is robust to model parameter uncertainty since it does not depend on exact knowledge of the model parameters.

While the result in [7] only provided a continuous periodic time-varying feedback law, Mazenc *et al.* [8] provided a smooth time-varying state feedbacks. The control law guaranteed global uniform asymptotic stability of an underactuated surface vessel. The design of this control law relies on the backstepping approach.

While the last two papers mentioned above considered the case where the underactuated surface vessel needs to reach a certain configuration point, in [9–11] they considered the case for the underactuated surface vessel to track any desired trajectory.

Aguiar *et al.* [9] used nonlinear Lyapunov-based control algorithm and backstepping approach to do the position tracking of an underactuated hovercraft. The control law yields global stability and exponential convergence of the position tracking error to a neighborhood of the origin that can be made arbitrarily small. They did the experiments using Caltech Multi-Vehicle Wireless Testbed (MVWT). The results showed stability and reasonable performance in spite of large modelling errors.

Ghommam *et al.* [10] derived a discontinuous feedback control law using backstepping approach to solve the control problem of uniform global stabilization and tracking of an underactuated surface vessel.

Aguiar *et al.* [11] proposed a solution to the trajectory tracking and path following for an underactuated autonomous vehicle in the presence of parametric modeling uncertainty. They combined backstepping technique, adaptive switching supervisory control and a nonlinear Lyapunov-based tracking control law to design a hybrid controller. The controller yields global boundedness and convergence of the position tracking error to a small neighborhood, and robustness to parametric modeling uncertainty.

In the last decade, a number of researches about multiple underactuated vehicles has increased, e.g. [12, 13]. The problem they tried to solve is the coordinated path-following problem of multiple underactuated vehicles along any given paths while keeping a desired inter-vehicle formation pattern. They combined Lyapunov technique and graph theory to design the control law. To the best of our knowledge, there is no result of planar rigid bodies' configuration consensus for the underactuated case.

In this paper, we adopt a method in [2] to achieve configuration consensus of two underactuated planar rigid bodies without referring to any leader or external reference. Our idea is to use approximate evolution and adopt the control law from [2] to bring the configurations and velocities of two underactuated planar rigid bodies close to each other. From that point, we use another control law from [2] to drive the bodies such that the differences of configurations and velocities go to zero as time goes to infinity.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{N}_0$ denotes the set of nonnegative integers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose.

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2. Problem Setting

Consider the nonlinear dynamical system representing underactuated planar rigid bodies given by [14]

$$M\dot{v}_i(t) = \begin{bmatrix} 0 \\ m\omega_i(t)v_{i_y}(t) \\ -m\omega_i(t)v_{i_x}(t) \end{bmatrix} + \mathbf{e}_2 u_{i_1}(t) + (-h\mathbf{e}_1 + \mathbf{e}_3)u_{i_2}(t),$$
$$v_i(0) = v_{i0}, \quad t \geq 0, \quad i = 1, 2, \quad (1)$$

and the kinematic equations of the planar rigid bodies given by

$$\dot{q}_i(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_i(t) & -\sin\theta_i(t) \\ 0 & \sin\theta_i(t) & \cos\theta_i(t) \end{bmatrix} v_i(t),$$
$$q_i(0) = q_{i0}, \quad t \geq 0, \quad i = 1, 2, \quad (2)$$

where $M = \mathrm{diag}[J, m, m] \in \mathbb{R}^{3\times3}$, J is the moment of inertia of the rigid bodies about the center of mass, m is the mass of the bodies, $v_i(t) \triangleq [\omega_i(t), v_{i_x}(t), v_{i_y}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ is the velocity of the body i with respect to the body-fixed frame, ω_i is the angular velocity of the body i, v_{i_x}, v_{i_y} are the translational velocity of the body i, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes the standard basis on $\mathbb{R}^3$, $h \in \mathbb{R}$ is the distance between the center of mass B and the point P (see Figure 2.1) to which the two forces u_{i_1} and u_{i_2} are applied on body i, $u_i(t) \triangleq [u_{i_1}(t), u_{i_2}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ is the force input of the body i, $q_i(t) \triangleq [\theta_i(t), x_i(t), y_i(t)]^{\mathrm{T}} \in \mathbb{R}^3$ denotes the configuration of the planar rigid body i with respect to the inertial frame, the coordinate $(x_i(t), y_i(t))$ denotes the position of the center of mass of the planar rigid body i, and $\theta_i(t)$ is the angle between the positive x-axis of the inertial frame and the body i frame.

In the synchronization problem our control objective is to drive the two planar rigid bodies such that their velocities $v_1(t)$, $v_2(t)$ go to 0 and their configurations $q_1(t)$, $q_2(t)$ converge to each other without referring to any leader or external reference.

3. Mathematical Preliminaries

In this section, we give some definitions used in this paper. Furthermore, we recall our system's local controllability and the system trajectory's approximate evolution under small-amplitude forcing [2]. In the last subsection, we also recall the inversion algorithm [2] to control the velocity of the body.

3.1. Definitions

First, $q_i = [\theta_i, x_i, y_i]^{\mathrm{T}}$ be equivalently expressed in the matrix form $Q_i \in \mathrm{SE}(2)$ as

$$Q_i \triangleq \begin{bmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{bmatrix},$$

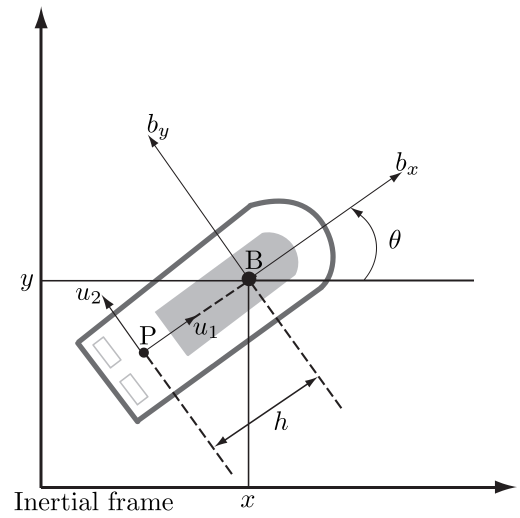

Figure 2.1: A planar rigid body in the inertial frame with two forces applied at the point P with distance h from the center of mass B. The configuration of this planar rigid body in the inertial frame is given by (θ, x, y).

to simplify the computation in the following sections. Furthermore, define $\widehat{v} \in \mathfrak{se}(2)$ as

$$\widehat{v} = l(v) \triangleq \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix},$$

for $v \triangleq [\omega, v_x, v_y]^{\mathrm{T}}$, where $l(\cdot)$ maps an element in $\mathbb{R}^3 \to \mathfrak{se}(2)$. The inverse map $l^{-1}(\cdot) \triangleq (\cdot)^\vee$ maps an element in $\mathfrak{se}(2) \to \mathbb{R}^3$. Hence, we may write $v = (\widehat{v})^\vee$. The Lie bracket on $\mathfrak{se}(2)$ of the two vectors $v, w \in \mathbb{R}^3$ is given by the matrix commutation

$$[v, w] = \left(\widehat{v}\widehat{w} - \widehat{w}\widehat{v}\right)^\vee.$$

The *symmetric product* $\langle v : w \rangle$ of the two vectors v, $w \in \mathbb{R}^3$ with respect to $M \in \mathbb{R}^{3\times3}$ is given by

$$\langle v : w \rangle \triangleq -M^{-1}(\mathrm{ad}_v^{\mathrm{T}} M w + \mathrm{ad}_w^{\mathrm{T}} M v),$$

where the adjoint operator on $\mathfrak{se}(2)$ is given by

$$\mathrm{ad}_{(\omega, v_x, v_y)} = \begin{bmatrix} 0 & 0 & 0 \\ v_y & 0 & -\omega \\ -v_x & \omega & 0 \end{bmatrix}.$$

We can define the *exponential map* $\exp(\cdot)$ by the usual power series [15] as

$$\exp(\widehat{v}) \triangleq \sum_{k=0}^{\infty} \frac{\widehat{v}^k}{k!},$$

and the *inverse map* $\log(\cdot)$ of the exponential map $Q \triangleq \exp(\widehat{v})$ [15] as

$$\log(Q) \triangleq \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(Q - I_3)^k}{k}.$$

Note that $\exp(\cdot)$ maps an element in $\mathfrak{se}(2) \to \mathrm{SE}(2)$ and $\log(\cdot)$ maps an element in $\mathrm{SE}(2) \to \mathfrak{se}(2)$. Using the symmetric product we can reformulate (1) as

$$\dot{v}_i(t) = -\frac{1}{2}\langle v_i(t) : v_i(t)\rangle + b_1 u_{i_1}(t) + b_2 u_{i_2}(t),$$
$$v_i(0) = v_{i0}, \quad t \geq 0, \quad i = 1, 2, \quad (3)$$

where $b_1 = \frac{1}{m}\mathbf{e}_2$, and $b_2 = -\frac{h}{J}\mathbf{e}_1 + \frac{1}{m}\mathbf{e}_3$.

3.2. Local Controllability

This section recalls a nonlinear controllability condition of the individual planar rigid body. The controllability condition presented here allows us to use the inversion algorithm in Section 3.4 below. Specifically, consider the kinematic and the kinetic equations of individual planar rigid body as

$$\dot{q}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & -\sin\theta(t) \\ 0 & \sin\theta(t) & \cos\theta(t) \end{bmatrix} v(t),$$
$$q(0) = q_0, \quad t \geq 0, \quad (4)$$
$$\dot{v}(t) = -\frac{1}{2}\langle v(t) : v(t)\rangle + b_1 u_1(t) + b_2 u_2(t), \ v(0) = v_0. \quad (5)$$

The relevant symmetric products for the system (4), (5) are given by

$$\langle b_1 : b_1\rangle = 0,$$
$$\langle b_1 : b_2\rangle = \frac{-h}{Jm}\mathbf{e}_3,$$
$$\langle b_2 : b_2\rangle = \frac{2h}{Jm}\mathbf{e}_2 = \frac{2h}{J}b_1.$$

Note that the matrix $[b_1, b_2, \langle b_1 : b_2\rangle]$ has full rank. Furthermore, the bad symmetric products $\langle b_1 : b_1\rangle$ and $\langle b_2 : b_2\rangle$ are given by linear combinations of b_1 and b_2 and hence it follows from [2] that the system (4), (5) is *small-time locally controllable (STLC) at zero velocity with second-order symmetric products*. Using this fact, we follow the input design procedures in [2] for each of the planar rigid body.

3.3. Approximate Evolution Under Small-Amplitude Forcing

In this section we recall the behavior of the system (4), (5) under small-amplitude forcing. Let for any given quantity $y(\epsilon)$, y^k denote the kth term in the Taylor series expansion of $y(\epsilon)$ about $\epsilon = 0$. For example, we write $v(t,\epsilon) = \epsilon v^1(t) + \epsilon^2 v^2(t) + O(\epsilon^3)$.

Now consider the force inputs for the individual planar rigid body of the form

$$u_j(t,\epsilon) = \epsilon u_j^1(t) + \epsilon^2 u_j^2(t), \quad j = 1, 2,$$

where $0 < \epsilon \ll 1$ and $u_j^1(t), u_j^2(t)$ are $O(1)$. Furthermore, define $c^1(t), c^2(t)$ as

$$c^1(t) \triangleq \sum_{j=1}^{2} b_j u_j^1(t), \quad (6)$$
$$c^2(t) \triangleq \sum_{j=1}^{2} b_j u_j^2(t), \quad (7)$$

so that $c^1(t)$ and $c^2(t)$ are $O(1)$. Then, it follows that

$$\sum_{j=1}^{2} b_j u_j(t,\epsilon) = \sum_{j=1}^{2} b_j(\epsilon u_j^1(t) + \epsilon^2 u_j^2(t)) = \epsilon c^1(t) + \epsilon^2 c^2(t). \quad (8)$$

Next, define the first integral function of $h(t) \in \mathbb{R}^3$, as

$$\bar{h}(t) \triangleq \int_{2k\pi}^{t} h(\tau)\mathrm{d}\tau, \quad 2k\pi < t \leq 2(k+1)\pi, \quad k \in \mathbb{N}_0,$$

with $\bar{h}(0) \triangleq 0$. Likewise, higher order integrals, as for example $\overline{\overline{g}h}(t) = \int_{2k\pi}^{t}\int_{2k\pi}^{s} g(\tau)\mathrm{d}\tau h(s)\mathrm{d}s$, are defined recursively.

The following proposition describes the behavior of the system (4), (5) when forced by small (order of ϵ and order of ϵ^2) amplitude inputs as defined in (8).

Proposition 3.1 Approximate Evolution [2]. For $0 < \epsilon \ll 1$ and for inputs of the form (8), let $(Q(t), v(t))$ be the solution of the system (4), (5). Assume that the initial condition for $v(\cdot)$ satisfies $v_0 = \epsilon v_0^1 + \epsilon^2 v_0^2$, where v_0^1 and v_0^2 are constants. Then it follows that

$$v(t,\epsilon) = \epsilon v^1(t) + \epsilon^2 v^2(t) + \epsilon^3 v^3(t) + O(\epsilon^4), \quad (9)$$
$$(\log Q_0^{-1}Q(t,\epsilon))^\vee = x(t,\epsilon) = \epsilon x^1(t) + \epsilon^2 x^2(t) + O(\epsilon^3), \quad (10)$$

where

$$v^1(t) = v_0^1 + \overline{c^1}(t),$$
$$v^2(t) = v_0^2 - \langle v_0^1 : v_0^1\rangle\frac{t}{2} - \left\langle v_0^1 : \overline{\overline{c^1}}(t)\right\rangle + \overline{\left(c^2 - \frac{1}{2}\langle\overline{c^1} : \overline{c^1}\rangle\right)}(t),$$
$$v^3(t) = -\langle v_0^1 : v_0^2\rangle t + \langle v_0^1 : \langle v_0^1 : v_0^1\rangle\rangle\frac{t^2}{4} + \left\langle v_0^1 : \left\langle v_0^1 : \overline{\overline{\overline{c^1}}}(t)\right\rangle\right\rangle - \left\langle v_0^1 : \overline{\overline{\left(c^2 - \frac{1}{2}\langle\overline{c^1} : \overline{c^1}\rangle\right)}}(t)\right\rangle - \left\langle\overline{\overline{c^1}}(t) : v_0^2\right\rangle + \frac{1}{2}\overline{\left\langle\langle v_0^1 : v_0^1\rangle t : \overline{c^1}(t)\right\rangle} + \overline{\left\langle\overline{c^1} : \left\langle v_0^1 : \overline{\overline{c^1}}\right\rangle\right\rangle}(t)$$

$$-\overline{\left\langle \overline{c^1} : \left(\overline{c^2 - \frac{1}{2}\langle \overline{c^1} : \overline{c^1} \rangle} \right) \right\rangle}(t),$$

and

$$\begin{aligned} x^1(t) &= v_0^1 t + \overline{\overline{c^1}}(t), \\ x^2(t) &= v_0^2 t - \langle v_0^1 : v_0^1 \rangle \frac{t^2}{4} + \left(\overline{\overline{c^2 - \frac{1}{2}\langle \overline{c^1} : \overline{c^1} \rangle}} \right)(t) \\ &\quad - \left\langle v_0^1 : \overline{\overline{\overline{c^1}}}(t) \right\rangle - \frac{1}{2} \overline{\left[v_0^1 + \overline{c^1}, v_0^1 t + \overline{\overline{c^1}} \right]}(t). \end{aligned}$$

This approximate evolution (9), (10) is valid for either t being small enough, or t bounded and $\epsilon \ll 1$ [4]. In the following sections, we assume that ϵ is small enough so that (9), (10) are valid for $0 \le t \le 2\pi$.

3.4. Inversion Algorithm

As we saw in the preceding section, the form of approximate evolution (9), (10) of (4), (5) is very complex. Since the system (4), (5) is STLC at zero velocity with second-order symmetric products, we follow the approach in [2] and use the following inversion algorithm. This inversion algorithm finds the force inputs $u_1(t)$, $u_2(t)$, $0 \le t \le 2\pi$, to control the body velocity. Note that a special periodic funtion (11) below in the inversion algorithm allow us to simplify the form of approximate evolution (9), (10).

Inversion Algorithm [2]. Let $\varphi = (v_\mathrm{d} - v_0)/\epsilon^2 \in \mathbb{R}^3$, where $v_\mathrm{d} \in \mathbb{R}^3$ is any desired velocity at time 2π, and derive $u_1(t), u_2(t)$ by taking the following steps:

i) Define the scalar function

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \big(\alpha \sin \alpha t - \beta \sin \beta t \big), \tag{11}$$

where α, β, $\alpha \neq \beta$, are natural numbers.

ii) Let $z_1, z_2, z_{12} \in \mathbb{R}$ be constants such that

$$\varphi = z_1 b_1 + z_2 b_2 + z_{12} \langle b_1 : b_2 \rangle. \tag{12}$$

Note that since $b_1, b_2, \langle b_1 : b_2 \rangle$ are mutually independent, z_1, z_2, z_{12} can be determined uniquely.

iii) Set

$$c^1(t) = b_1 u_1^1(t) + b_2 u_2^1(t), \tag{13}$$

$$c^2(t) = b_1 u_1^2(t) + b_2 u_2^2(t), \tag{14}$$

where

$$u_1^1(t) = \sqrt{|z_{12}|}\psi(t), \tag{15}$$

$$u_1^2(t) = \frac{1}{2\pi} z_1 + \frac{1}{4\pi} \frac{2h}{J} |z_{12}|, \tag{16}$$

$$u_2^1(t) = -\sqrt{|z_{12}|}\,\mathrm{sgn}(z_{12}) \psi(t), \tag{17}$$

$$u_2^2(t) = \frac{1}{2\pi} z_2. \tag{18}$$

iv) Finally, set

$$u_1(t) = \epsilon u_1^1(t) + \epsilon^2 u_1^2(t), \tag{19}$$

$$u_2(t) = \epsilon u_2^1(t) + \epsilon^2 u_2^2(t). \tag{20}$$

Using (19), (20), we are able to control the body velocity from v_0 at time 0 to $v_\mathrm{d} + O(\epsilon^4)$ at time 2π.

4. Control Law

In this section we design control laws to drive the underactuated vehicles given by (2), (3) such that $\|v_1(t)\| \to 0$, $\|v_2(t)\| \to 0$, and $\|(\log Q_1^{-1}(t) Q_2(t))^\vee\| \to 0$, as $t \to \infty$. To achieve this objective, we need two steps. The first step is to make two bodies close to each other from arbitrary positions and the second step is to make two bodies converge to each other. For the first step, we use the control law in Theorem 4.1 and then we use the control law in Theorem 4.2 for the second step. While Theorem 4.1 requires the initial velocities being 0, Theorem 4.2 requires the initial relative errors of two bodies to be already small. Because of the requirement of Theorem 4.2, we cannot use this theorem from the beginning if the initial relative configuration of both bodies is large even if their initial velocities are 0. On the other hand, Theorem 4.1 cannot guarantee to make two bodies converge to each other.

Our theorems work as follows. First in Theorem 4.1, using $v_1(0)$, $v_2(0)$, and $(\log Q_1^{-1}(0) Q_2(0))^\vee$, the inversion algorithm finds $u_1(t)$, $u_2(t)$ as in (19), (20). This control input will last for 2π units of time. At $t = 2\pi$, using the approximated values $v_1(2\pi)$, $v_2(2\pi)$, and $(\log Q_1^{-1}(2\pi) Q_2(2\pi))^\vee$, the inversion algorithm finds new $u_1(t)$, $u_2(t)$ that last for another 2π units of time. We repeat this procedure until the relative error between the two bodies is sufficiently small. From this point, we switch to another control law given in Theorem 4.2 to make the relative error converge to 0.

Theorem 4.1. Consider the nonlinear dynamical system given by (2), (3) with initial conditions $v_{i0} = 0$, $i = 1, 2$. Let $0 < \epsilon \ll 1$. Then the control law for the time interval $2k\pi \le t < 2(k+1)\pi$, $k \in \mathbb{N}_0$, given by

$$u_{i_1}(t) = \epsilon \sqrt{|z_{i_{12}}(t)|}\psi(t) + \frac{\epsilon^2}{2\pi} z_{i_1}(t) + \frac{\epsilon^2}{4\pi} \frac{2h}{J} |z_{i_{12}}(t)|, \tag{21}$$

$$u_{i_2}(t) = -\epsilon \sqrt{|z_{i_{12}}(t)|}\,\mathrm{sgn}(z_{i_{12}}(t)) \psi(t) + \frac{\epsilon^2}{2\pi} z_{i_2}(t), \tag{22}$$

where $\psi(\cdot)$ is given by (11),

$$z_{i_1}(t) = m \varphi_{i_2}(2k\pi), \quad 2k\pi \le t < 2(k+1)\pi, \tag{23}$$

$$z_{i_2}(t) = -\frac{J}{h} \varphi_{i_1}(2k\pi), \tag{24}$$

$$z_{i_{12}}(t) = -\frac{J^2}{h^2} \varphi_{i_1}(2k\pi) - \frac{Jm}{h} \varphi_{i_3}(2k\pi), \tag{25}$$

and

$$\varphi_1(2k\pi) = \begin{bmatrix} \varphi_{1_1}(2k\pi) \\ \varphi_{1_2}(2k\pi) \\ \varphi_{1_3}(2k\pi) \end{bmatrix} = \frac{v_2(2k\pi) - v_1(2k\pi)}{\epsilon^2} + (\log Q_1^{-1}(2k\pi)Q_2(2k\pi))^\vee, \quad (26)$$

$$\varphi_2(2k\pi) = \begin{bmatrix} \varphi_{2_1}(2k\pi) \\ \varphi_{2_2}(2k\pi) \\ \varphi_{2_3}(2k\pi) \end{bmatrix} = \frac{v_1(2k\pi) - v_2(2k\pi)}{\epsilon^2} + (\log Q_2^{-1}(2k\pi)Q_1(2k\pi))^\vee, \quad (27)$$

guarantees $v_1(2k\pi) = O(\epsilon^4)$, $v_2(2k\pi) = O(\epsilon^4)$, and $(\log Q_1^{-1}(2k\pi)Q_2(2k\pi))^\vee = O(\epsilon^3)$ for sufficiently large k.

Proof. The proof is omitted due to space limitations. □

Remark 4.1. Although it seems that the magnitude of the control inputs (21), (22) might be large because of the division by ϵ^2 in (26), (27), the magnitude of control inputs is in fact always at most order of ϵ. This is because the velocities $v_1(t)$, $v_2(t)$ are always at most order of ϵ^2.

The control law in Theorem 4.1 can drive both planar rigid bodies such that their orientations and velocities at $t = 2k\pi$ are close to each other for k sufficiently large. This control law cannot guarantee asymptotic zero velocity for both bodies and asymptotic zero relative configuration errors. Instead, there are errors of orders $O(\epsilon^4)$ and $O(\epsilon^3)$ in the velocities and the relative configuration, respectively. Using the control law in Theorem 4.2 below, we show that we can drive both planar rigid bodies such that their velocities and the differences of their orientation go to 0 as t goes to infinity, given their initial differences are small enough.

There are two differences between Theorems 4.1 and 4.2. The first difference is instead of using fixed ϵ in the control law as in Theorem 4.1, Theorem 4.2 uses time-varying ϵ which is getting smaller as k increases. This affects the magnitude of control inputs which also become smaller to control the body velocities to zero. The second difference is the fact that the way of specifying the desired velocity $\varphi_i(2k\pi), i = 1, 2$, in Theorem 4.1 is not uniform over the switching instants when new $u_1(t)$ and $u_2(t)$ are recalculated.

Theorem 4.2. Consider the nonlinear dynamical system given by (2), (3) with $\|[[(\log Q_1^{-1}(0)Q_2(0))^\vee]^\mathrm{T}, [v_1(0) - v_2(0)]^\mathrm{T}]^\mathrm{T}\| \ll 1$. Then the control law for the time interval $2k\pi \le t < 2(k+1)\pi$, $k \in \mathbb{N}_0$, given by

$$u_{i_1}(t) = \epsilon_k\sqrt{|z_{i_{12}}(t)|}\psi(t) + \frac{\epsilon_k^2}{2\pi}z_{i_1}(t) + \frac{\epsilon_k^2}{4\pi}\frac{2h}{J}|z_{i_{12}}(t)|, \quad (28)$$

$$u_{i_2}(t) = -\epsilon_k\sqrt{|z_{i_{12}}(t)|}\mathrm{sgn}(z_{i_{12}}(t))\psi(t) + \frac{\epsilon_k^2}{2\pi}z_{i_2}(t), \quad (29)$$

where $\psi(\cdot), z_{i_1}(\cdot), z_{i_2}(\cdot), z_{i_{12}}(\cdot)$ are given by (11), (23)–(25),

$$\begin{bmatrix} \varphi_{1_1}(2k\pi) \\ \varphi_{1_2}(2k\pi) \\ \varphi_{1_3}(2k\pi) \end{bmatrix} = \begin{cases} \frac{v_2(2k\pi)}{2\epsilon_k^2} - \frac{v_1(2k\pi)}{\epsilon_k^2} \\ \quad + \frac{(\log Q_1^{-1}(2k\pi)Q_2(2k\pi))^\vee}{4\pi\epsilon_k^2}, & k \text{ even}, \\ -\frac{v_1(2k\pi)}{\epsilon_k^2}, & k \text{ odd}, \end{cases}$$

$$\begin{bmatrix} \varphi_{2_1}(2k\pi) \\ \varphi_{2_2}(2k\pi) \\ \varphi_{2_3}(2k\pi) \end{bmatrix} = \begin{cases} \frac{v_1(2k\pi)}{2\epsilon_k^2} - \frac{v_2(2k\pi)}{\epsilon_k^2} \\ \quad + \frac{(\log Q_2^{-1}(2k\pi)Q_1(2k\pi))^\vee}{4\pi\epsilon_k^2}, & k \text{ even}, \\ -\frac{v_2(2k\pi)}{\epsilon_k^2}, & k \text{ odd}, \end{cases}$$

and

$$\epsilon_k = \begin{cases} \|[[(\log Q_1^{-1}(2k\pi)Q_2(2k\pi))^\vee]^\mathrm{T}, \\ \quad [v_1(2k\pi) - v_2(2k\pi)]^\mathrm{T}]^\mathrm{T}\|^{\frac{1}{2}}, & k \text{ even}, \\ \epsilon_{k-1}, & k \text{ odd}, \end{cases}$$

$\epsilon_k > 0$, guarantees that $\|v_1(t)\| \to 0$, $\|v_2(t)\| \to 0$, and $\|(\log Q_1^{-1}(t)Q_2(t))^\vee\| \to 0$ as $t \to \infty$.

Proof. The proof is omitted due to space limitations. □

Remark 4.2. The difference of $\varphi_i(2k\pi), i = 1, 2$, for k even and k odd in Theorem 4.2 is to make the bodies closer to each other while at the same time make the body velocities closer to 0.

5. Illustrative Numerical Example

We present a simulation to illustrate the usefulness of the theory. We use $J = 10$ and $m = 20$ for the matrix M. The initial configuration for body 1 and body 2 are $q_{10} = [\pi/2, 0, 0]^\mathrm{T}$ and $q_{20} = [\pi/2, 2, 2]^\mathrm{T}$ respectively. The initial velocities are 0 for both bodies.

In this simulation, we use the control law given by Theorem 4.1 with $\epsilon = 0.1$ for $t \in [0, 6\pi)$. At $t = 6\pi$, we change our control law with the one in Theorem 4.2. We kept using this control law until $t = 24\pi$. We show the location of both bodies in Figure 5.1. The body 1 and the body 2 are plot in yellow and in purple respectively. We use color gradation to make the figure clearer. The gradation rule is as k increases, the color become more vivid. We see that both bodies are converge to each other. In Figure 5.2, we see that the error decreases exponentially.

6. Conclusion

In this paper, we proposed consensus control framework to synchronize two underactuated planar rigid

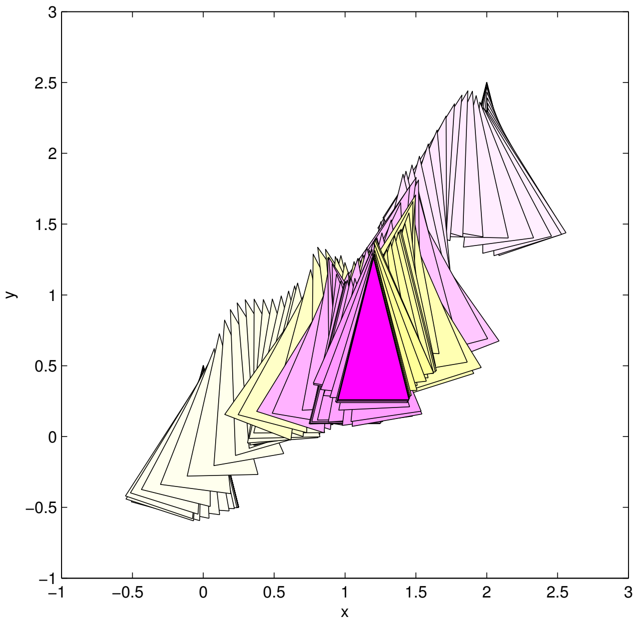

Figure 5.1: Location of the two planar rigid bodies

bodies. The control laws are based on approximate evolution under small-amplitude forcing. The control laws have been applied numerically to illustrate efficacy of the proposed approach.

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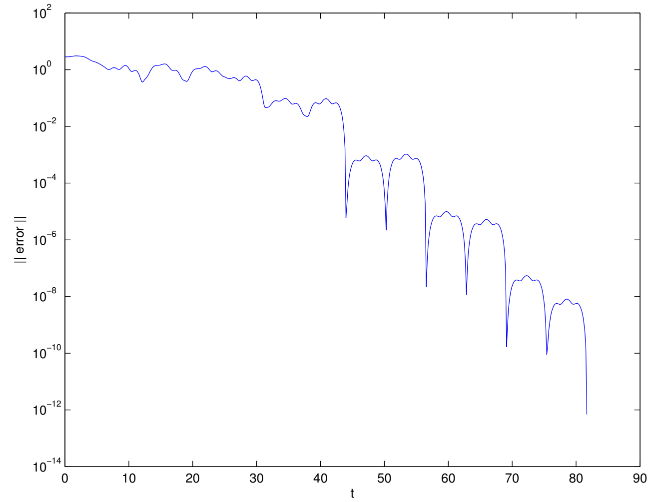

Figure 5.2: Error norm versus time

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