Algorithmic and analytic framework for optimization of multi-user performance in wireless networks with interference

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Pomyśl czy przyszło ci kiedy do głowy
że błękit jest czasem siny czasem granatowy
bywa jak lazur lub jak kraska modry
cieszą się święci w niebie
na dole pies z pieskiem
że nawet niebo nie bywa niebieskie

“*Pomyśl*“, ks. Jan Twardowski

List of Publications

0.1 Books and book chapters

1. H. Boche, M. Wiczanowski, and T. Haustein, “Scheduling in Multiple Antenna Multiple Access Channel," Chapter 28 in “Smart Antennas - State of the Art," *EURASIP Book Series on Signal Processing and Communications, Hindawi Publishing Corporation*, 2005

2. S. Stanczak, M. Wiczanowski, and H. Boche, “Resource Allocation in Wireless Networks - Theory and Algorithms," *Lecture Notes in Computer Science (LNCS 4000), Springer-Verlag*, 2006

0.2 Journal articles

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2. H. Boche, and M. Wiczanowski, “The Interplay of Link Layer and Physical Layer under MIMO enhancement - Benefits and Challenges," *IEEE Wireless Communications*, Vol. 13, No. 4, pp. 48-55, Aug 2006, invited.

3. S. Stanczak, M. Wiczanowski, and H. Boche, “Distributed Fair Power Control for Wireless Networks: Objectives and Algorithms," *IEEE Transactions on Signal Processing*, 2007, to appear

4. H. Boche, M. Wiczanowski, and S. Stanczak, “On Optimal Resource Allocation in Cellular Networks with Best-Effort Traffic," *IEEE Transactions on Wireless Communications*, 2007, to appear

5. H. Boche, M. Wiczanowski, and S. Stanczak, “Unifying View on Min-Max Fairness, Max-Min Fairness, and Utility Optimization in Cellular Networks," *EURASIP Journal on Wireless Communications and Networking*, 2007, to appear

6. H. Boche, and M. Wiczanowski, “Optimization-Theoretic Analysis of Stability-Optimal Transmission Policy for Multiple Antenna Multiple Access Channel," *IEEE Transactions on Signal Processing*, 2007, to appear

7. M. Wiczanowski, S. Stanczak and H. Boche, “Performance and interference control in wireless ad-hoc and mesh networks - the generalized Lagrangian approach," *IEEE Transactions on Signal Processing*, 2007, submitted

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2. H. Boche, and M. Wiczanowski, "Stability Region of Arrival Rates and Optimal Scheduling for MIMO-MAC - A Cross-Layer Approach," *Proc. IEEE Intern. Zurich Seminar (IZS 2004)*, Feb 2004, pp. 18-21, invited
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12. H. Boche, M. Wiczanowski, and S. Stanczak, "Unifying View on Min-Max Fairness and Utility Optimization in Cellular Networks," *Proc. IEEE Wireless Communications and Networking Conference (WCNC 2005)*, Mar 2005, pp. 1280-1285

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23. H. Boche, M. Schubert, and M. Wiczanowski, "An Algebra for Log-Convex Interference Functions," *Proc. IEEE International Symposium on Information Theory and its Applications (ISITA)*, Oct 2006

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Zusammenfassung

In dieser Arbeit stellen wir einen theoretischen und algorithmischen Rahmen für die Gewährleistug von Min-Max-Fairness und Optimierung der gewichteten Summenperformanz in einem Ein-Hop-Netzwerk mit Interferenz. Die Haupteigenschaft der vorgestellten analytischen Resultate und Algorithme ist ihre, im zweierlei Sinne, große Allgemeinheit. Erstens, sie sind anwendbar auf alle Netzwerke die eine Beschreibung der Interferenz durch eine nichtnegative Matrix zulassen. Zweitens, sie sind anwendbar auf alle QoS-Funktionen der einzelnen Links die monotone Funktionen des entsprechenden Link-SIR darstellen.

Die SIR-Funktion des Links und die Interferenzmatrix, die paarweise Interferenz zwischen den Links beschreibt, stellen die Schlüsselelemente der Resultate dieser Arbeit dar (Netzwerkmodell im Kapitel 2). Im Kapitel 2 wird gezeigt, dass die konvex-analytischen Eigenschaften der QoS-Funktion des Links, als Funktion des entsprechenden Link-SIR, einen entscheidenden Einfluss auf die Existenz von lokalen/ globalen Lösungen des Leistungsallokationsproblems haben. Ebenfalls charakterisieren wir die Relation zwischen den Eigenschaften der QoS-Funktion des Links und den Eigenschaften der QoS-/ Performanzregion, definiert als die Menge aller erreichbaren Tupel von QoS-Funktionswerten der Links.

Die im Kapitel 3 vorgestellten Algorithmen berechnen eine Leistungsallokation die die (gewichtete) Summenperformanz des Netzwerkes optimiert und basieren auf konvex-analytischen Eigenschaften der QoS-Region. Der Hauptvorteil beider Algorithmen, die entsprechend für die Fälle der Summenleistungsbeschränkung und Leistungsbeschränkungen pro Link entwickelt wurden, ist ein nach unserer Ansicht günstiger Abtausch zwischen Rechenkomplexität und Konvergenzverhalten.

Die Algorithmen und Feedback-Schemata im Kapitel 4 sind gemeinsam mit dem Ziel entwickelt worden, eine verteilte Berechnung einer Leistungsallokation die die Summenperformanz optimiert zu gewährleisten. Ein spezifisches Feedback-Schema das die Interferenz schätzt is hierbei das Hauptelement, das die dezentralisierte Berechnung ermöglicht. Die dazugehörigen algorithmischen Konzepte zielen auf eine bestmögliche Ausnutzung des Feedback-Schemas, im Sinne der ermöglichten dezentralisierten Berechnung, einer niedrigen Rechenkomplexität und eines guten Konvergenzverhaltens. Die Algorithmen basieren auf dem Konzept einer nichtlinearen, bzw. verallgemeinerten, Lagrange-Funktion und auf einem spezifischen Ansatz der Aufspaltung von Variablen.

Wegen erhöhter potentieller Performanz die unter Verwendung von mehreren Antennen pro Link erreichbar ist, gilt ein spezielles Interesse dem Problem der räumlichen Leistungsallokation in MIMO-Netzwerken (*Multiple-Input Multiple-Output*). Im Kapitel 5 beschäftigen wir uns mit einem speziellen Problem der räumlichen Leistungsallokation die die gewichtete Summenperformanz im MIMO-Vielfachzugriffskanal optimiert. Das betrachtete Problem entspricht der Berechnung der sogenannten Stabilitätsoptimalen Strategie, bestehend aus räumlicher Leistungsallokation und SIC-Reihenfolge (*Successive Interference Cancellation*). Basierend auf konvex-analytischen Eigenschaften der QoS-Region (in dem speziellen Fall, der Kapazitätsregion), charakterisieren wir einige nützliche Eigenschaften der Stabilitätsoptimalen Strategie. Der entsprechende Algorithmus der die Strategie berechnet benutzt einen Ansatz der Aufspaltung des ursprünglichen Problems in ein Ensemble von gekoppelten Ein-Link Problemen.

Das Problem der Charakterisierung und Berechnung einer min-max-fairen Leistungsallokation wird im Kapitel 6 behandelt. Dort beweisen wir, dass der Abtausch zwischen Min-Max-Fairness und Optimalität der gewichteten Summenperformanz als ein Sattelpunkt der Summenperformanz, als Funktion der Link-Gewichte und Link-Leistungen, aufgefasst werden kann. Im Kapitel 6 erhalten wir ebenfalls Einsichten in die Relation zwischen Gewährleistung der Min-Max-Fairness und einem gegensätzlichen Ansatz der maximalen Degradation des besten Wertes der QoS-Funktion des Links. Wir zeigen die generelle Verschiedenheit beider Ansätze und ihre Abhängigkeit von kombinatorischen und spektralen Eigenschaften der Interferenzmatrix.

Abstract

The contribution of this work is an analytic and algorithmic framework for achieving min-max fairness and optimization of weighted aggregated performance in single-hop networks with interference. The key feature of the analytic results and algorithms within the framework is their great generality in the two-fold sense. First, they apply to any network which allows the description of the interference in the form of a nonnegative matrix. Second, they apply to any link QoS function being a monotone function of the corresponding link SIR.

The key ingredients of all results of the work are the link SIR function and the interference matrix, which describes the pairwise interference across the links (network model in Chapter 2). The convex-analytic properties of the link QoS function, understood as a function of the corresponding link SIR, are shown in Chapter 2 to have crucial influence on the existence of local/ global solutions to the power allocation problem. We also characterize a relation between properties of the QoS function and the properties of the QoS/ performance region, which is understood as the set of all achievable tuples of link QoS values.

The algorithms proposed in Chapter 3 compute a power allocation optimizing the (weighted) aggregated performance of the network and rely strongly on the convex-analytic properties of the QoS region. The key advantage of the two algorithms, proposed for the cases of sum-power constraint and per-link power constraints, respectively, is an in our view advantageous trade-off of computational complexity and convergence behavior.

The algorithms and feedback schemes in Chapter 4 are designed jointly for the purpose of distributed computation of a power allocation optimizing the aggregated performance. The key ingredient allowing for decentralized conduction is hereby a specific feedback scheme estimating the interference. The corresponding algorithmic concepts aim at best possible utilization of the scheme in the sense of ensured decentralized conduction, low computational complexity and good convergence behavior. The algorithms rely on the concept of nonlinear, or generalized, Lagrangean function and on a specific approach of variable splitting.

Due to increased performance potential achieved under incorporation of multiple antennas per link, particular interest is in the problem of spatial power allocation in MIMO (*Multiple-Input Multiple-Output*) networks. In Chapter 5 we deal with a particular problem of spatial power allocation optimizing weighted aggregated performance in the MIMO multiple access channel. The considered problem corresponds precisely to finding the so-called stability-optimal policy consisting of spatial power allocation and order of *Successive Interference Cancellation* (SIC). Relying on convex-analytic features of the QoS region (in this particular case, the capacity region), we provide several useful characterizations of the stability-optimal policy. The corresponding algorithm computing the policy makes use of the splitting of the original problem into a set of coupled single-link problems.

The problem of characterization and computation of a min-max fair power allocation is addressed in Chapter 6. We prove that the trade-off of min-max fairness and optimality of weighted aggregated performance has the interpretation of a saddle point of the weighted aggregated performance regarded as a function of link weights and link powers. In Chapter 6 we also obtain insights in the relation between the approach of ensuring min-max fairness and an opposite approach of maximally degrading the best link QoS. We show the general nonequivalence of both approaches and their dependence on the combinatorial and spectral properties of the interference matrix.

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List of Symbols

$\mathbb{R}$	the space of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{R}_-$	the set of nonpositive real numbers
$\mathbb{R}^N$	N-dimensional real Euclidean space
$\mathbb{R}^N_+$	nonnegative orthant of N-dimensional real Euclidean space
$\mathbb{R}^N_{++}$	positive orthant of N-dimensional real Euclidean space
$\mathbb{C}^N$	N-dimensional complex Euclidean space
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_+$	the set of positive natural numbers
$\mathbb{S}^N$	the space of N-dimensional Hermitian matrices
$\mathbb{S}^N_+$	the cone of N-dimensional (Hermitian) positive semidefinite matrices
$\lambda_i(\cdot)$	i-th eigenvalue of a square matrix
$\lambda_{max}(\cdot)$	maximum eigenvalue of a Hermitian matrix
$\cdot'$	(with respect to matrices/ vectors) Hermitian transpose of a matrix/ vector
$\cdot'$	(with respect to functions) the first-derivative function of a function with scalar domain
$\cdot''$	the second-derivative function of a function with scalar domain
$\cdot'''$	the third-derivative function of a function with scalar domain
$\boldsymbol{e}_i$	i-th vector from the canonical orthonormal base; $\boldsymbol{e}_i = (0, \ldots, 0, \underset{i\text{-th}}{1}, 0, \ldots, 0)$
$\nabla \cdot$	gradient of a function
$\nabla_{\boldsymbol{a}} \cdot$	gradient of a function with respect to its subdomain of variables $\boldsymbol{a}$; has the i-th element $\frac{\partial}{\partial a_i} \cdot$
$\nabla^2 \cdot$	Hessian matrix of a function
$\nabla^2_{\boldsymbol{a}} \cdot$	Hessian matrix of a function with respect to its subdomain of variables $\boldsymbol{a}$; has the ij-th element $\frac{\partial^2}{\partial a_i \partial a_j} \cdot$
$\nabla^2_{\boldsymbol{a},\boldsymbol{b}} \cdot$	Hessian matrix of a function with respect to its subdomains of variables $\boldsymbol{a}$ and $\boldsymbol{b}$; has the ij-th element $\frac{\partial^2}{\partial a_i \partial b_j} \cdot$
$\mathbf{1}$	the all-one vector; $\mathbf{1} = (1, \ldots, 1)$
$\mathbf{0}$	the all-zero matrix/ vector
$\boldsymbol{I}$	the identity matrix
$\boldsymbol{I}_N$	the identity matrix of explicitly given size $N \times N$
$(\cdot)_k$	k-th element of a vector
$(\cdot)_{kj}$	kj-th element of a matrix
$(\cdot)_{k\cdot}$	k-th row of a matrix
$(\cdot)_{\cdot k}$	k-th column of a matrix
$\rho(\cdot)$	spectral radius of a square matrix

$\lvert \cdot \rvert$	magnitude/ absolute value of a scalar
$\lVert \cdot \rVert$	norm of a vector
$\lVert \cdot \rVert_i$	i-norm of a vector
$\lvert\lvert\lvert \cdot \rvert\rvert\rvert$	matrix-norm of a matrix
$\lvert\lvert\lvert \cdot \rvert\rvert\rvert_i$	i-matrix-norm of a matrix
$tr(\cdot)$	trace of a square matrix
$\det(\cdot)$	determinant of a square matrix
$diag(\cdot)$	matrix obtained by setting the nondiagonal elements of a square matrix to zero
$\delta\cdot$	boundary of a set
$conv(\cdot)$	convex hull of a set
$cl(\cdot)$	closure of a set
$Re\cdot$	real-valued part of a matrix/ vector/ scalar
$Im\cdot$	imaginary part of a matrix/ vector/ scalar
$Pr(\cdot)$	probability of an event

Contents

1
Introduction

In the year 1956 in Sweden, the Ericsson company set up the system of telephony MTA (Mobile Telephone system A), the worlds first autonomous system of wireless telephony for public use. This date can be seen as the formal origin of the world-wide proliferation of wireless communications services. The first non-voice wireless digital services came to their own in the 1990s with the expansion of the second-generation mobile telephony systems GSM (Groupe Spécial Mobile), IS-136, iDEN, IS-95 and the introduction of the first Wireless LAN (Local Area Network) standard 802.11 b [1]. In currently existing heterogeneous wireless networks, different kinds of wideband services are the dominating traffic part. Moreover, further improvement of availability and quality of wideband real-time services is one of the key issues in standardization work for future wireless communications systems.

1.1 State of the art and related works

The heterogeneity of the wideband traffic in current and future networks in combination with the time-variant and unreliable nature of wireless communications channels enforce a need for increased efficiency and improved adaptivity of resource allocation algorithms. Such need for better algorithms for the allocation of power, bandwidth, time and antennas initiated a lively research.

The pioneering contributions were concerned with the power-efficient operation of a cellular network with fixed per-link requirements on the value of some *Quality of Service* (QoS) function, such as delay or rate, see [2], [3], [4], [5] for the deterministic view and [6] for the incorporation of stochasticity of wireless channels. In the interesting case of achievable per-link requirements with respect to a given QoS function, a power allocation efficient in the above sense represents so-called min-max fair power allocation [7] (in the references called rather a max-min fair power allocation). Further development of the framework of min-max fair power allocation was pursued e.g. in [8] [9]. In [10] the authors developed further the stochastic view from [6], while in [11] the aspects of computational efficiency of min-max fair power allocation were addressed. More specialized theory and algorithms for min-max fair power allocation in CDMA (*Code Division Multiple Access*) networks can be found e.g. in [12], [13], [14], [15].

The approach of power allocation optimizing the (weighted) aggregated performance/ QoS of the entire network was adapted from the wired network context [16], [17] and occurred in later works as an alternative to min-max fair power allocation. A great deal of the corresponding works is concerned with the optimization of (weighted) network throughput, understood as the (weighted) sum of link capacities, under single antenna or multiple antennas per link, see e.g. [18], [19] [20], [21] and references therein. A more general algorithmic theory not restricted to link capacity as a QoS function can be found e.g. in [22], [23], [24] (see also references therein), where in the two latter works an attractive game-theoretic view of the power allocation problem is utilized.

Currently, the research on algorithmic power and bandwidth allocation, both in the sense of min-max fairness and optimization of aggregated performance, incorporates usually the cross-layer view of network layers and aims at the extension of the algorithmic concepts towards multi-hop ad hoc networks [25], [26], [27]. Particularly interesting appear here the approaches utilizing a specific splitting of the multi-hop power and bandwidth allocation problem relying on Lagrangean duality [28], [29], [30]. Certain interest is also in the redesign/ adaption of the algorithms to arising new network topologies, such as mesh(ed) networks, and their improvement in terms of scalability [31], [32].

1.2 THE SCOPE OF THE WORK

The contribution of this work is an analytic and algorithmic framework for achieving min-max fairness and optimization of weighted aggregated performance, in the sense described above, in single-hop networks with interference. The key feature of the provided framework is its two-fold generality.

First, the provided analytic results and algorithms are applicable to arbitrary networks with interfering links as long as the pairwise interference across the links can be described by a nonnegative matrix. Due to this feature, all results of the work are applicable, in particular, to networks with multiple antennas at either link transmitter or link receiver, or to CDMA networks provided that the channels are flat fading and the link receivers are linear. The provided framework covers also the typical case of single-hop communication within a multi-hop ad hoc network, that is, the case of separated links sharing the same resource (bandwidth slot, time slot, spreading sequence, etc.).

Second, the results of this work are general in the sense that, except monotonicity in link SIR (*Signal-to-Interference-and-noise-Ratio*), no further assumptions on the link QoS function are required. Thus, the provided results are applicable, in particular, when link capacity, link symbol error rate or link MMSE (*Minimum Mean Square Error*) is the link QoS function of interest.

The generality of the results of this work stands, in our view, in contrast to numerous works referenced above, which are restricted to particular physical layer designs (e.g. single-antenna per link) and/ or particular medium access policies (e.g. CDMA) and/ or particular link QoS functions (usually link capacity).

The key ingredients of all results of the work are the link SIR function and the interference matrix, which describes the pairwise interference across the links (network model in Chapter 2). The convex-analytic properties of the link QoS function, understood as a function of the corresponding link SIR, are shown in Chapter 2 to have crucial influence on the existence of local/ global solutions to the power allocation problem. We also characterize a relation between properties of the QoS function and the properties of the QoS/ performance region, which is understood as the set of all achievable tuples of link QoS values (the capacity region is a prominent example of a QoS region when link capacity is taken as link QoS function [21], [19]).

The algorithms proposed in Chapter 3 compute a power allocation optimizing the (weighted) aggregated performance of the network and rely strongly on the convex-analytic properties of the

QoS region. The key advantage of the two algorithms, proposed for the cases of sum-power constraint and per-link power constraints, respectively, is an in our view advantageous trade-off of computational complexity and convergence behavior.

While we suggest centralized conduction of the algorithms from Chapter 3, the algorithms and feedback schemes in Chapter 4 are designed jointly for the purpose of distributed computation of a power allocation optimizing the aggregated performance. The key ingredient allowing for decentralized conduction is hereby a specific feedback scheme estimating the interference. The corresponding algorithmic concepts aim at best possible utilization of the scheme in the sense of ensured decentralized conduction, low computational complexity and good convergence behavior. The algorithms rely on the concept of nonlinear, or generalized, Lagrangean function and on a specific approach of variable splitting.

Due to increased performance potential achieved under incorporation of multiple antennas per link, particular interest is in the problem of spatial power allocation in MIMO (*Multiple-Input Multiple-Output*) networks. In Chapter 5 we deal with a particular problem of spatial power allocation optimizing weighted aggregated performance in the MIMO multiple access channel. The considered problem corresponds precisely to finding the so-called stability-optimal policy consisting of spatial power allocation and order of *Successive Interference Cancellation* (SIC). Relying on convex-analytic features of the QoS region (in this particular case, the capacity region), we provide several useful characterizations of the stability-optimal policy. The corresponding algorithm computing the policy makes use of the splitting of the original problem into a set of coupled single-link problems.

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Appendix A includes specialized notions and concepts (from nonnegative matrix theory, optimization theory, convex analysis and geometry), which the reader might be not familiar with and which are used in the results of the work. On the other side, some notions/ concepts in the appendix are fundamental and well-established, but are included in the appendix due to their frequent use and importance.

1.3 NOTATION

Any vector is understood as a column vector and $'$ denotes the transpose of a vector/ matrix. We denote vectors of dimension $N \in \mathbb{N}$ by small-type bold letters, e.g. $\boldsymbol{a} = (a_1, \dots, a_N)$, and matrices of dimension $N \times M$, $N, M \in \mathbb{N}$, by capital bold letters, e.g.

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NM} \end{pmatrix}.$$

Such matrix is sometimes written simply as $\boldsymbol{A} = (A_{kl})$. We define $(\boldsymbol{A})_{kl} = A_{kl}$ and $(\boldsymbol{a})_k = a_k$, $1 \leq k, \leq N$, $1 \leq l \leq M$. If the dimension of a vector/ matrix is not given explicitly, then it always follows from the context with no ambiguity. If the dimension $N \in \mathbb{N}$ of a vector $\boldsymbol{a}$ is clear, then we sometimes simplify the notation by writing $\boldsymbol{a} \geq 0$ instead $\boldsymbol{a} \in \mathbb{R}_+^N$ or $\boldsymbol{a} > 0$ instead $\boldsymbol{a} \in \mathbb{R}_{++}^N$. Similarly, if the dimension $N \in \mathbb{N}$ of a Hermitian matrix $\boldsymbol{A}$ is clear, we write simply $\boldsymbol{A} \succeq 0$ instead of $\boldsymbol{A} \in \mathbb{S}_+^N$ or $\boldsymbol{A} \preceq 0$ instead of $-\boldsymbol{A} \in \mathbb{S}_+^N$. Complying with the convention, if $\boldsymbol{A} - \boldsymbol{B} \in \mathbb{S}_+^N$, we write simply $\boldsymbol{A} \succeq \boldsymbol{B}$ instead.

The logarithm function and exponential function defined on vectorial domain are understood as componentwise logarithm and componentwise exponential function, respectively; given $\boldsymbol{a} \in \mathbb{R}_+^N$ we have $\log \boldsymbol{a} = (\log a_1, \ldots, \log a_N)$ and $e^{\boldsymbol{a}} = (e^{a_1}, \ldots, e^{a_N})$. Hereby, we define $\log 0 = -\infty$ complying with the convention.

In functional expressions we identify, without introducing ambiguity, vector pairs, say $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^N \times \mathbb{R}^M$, with stacked column vectors $(\boldsymbol{a}' \ \boldsymbol{b}')' \in \mathbb{R}^{N+M}$. Thus, the operator $\nabla_{(\boldsymbol{a},\boldsymbol{b})} \cdot$ is equivalent to $\nabla_{(\boldsymbol{a}' \ \boldsymbol{b}')'} \cdot$ and represents the gradient with respect to $(\boldsymbol{a}, \boldsymbol{b})$ (precisely, $(\boldsymbol{a}' \ \boldsymbol{b}')'$). Similarly, when $\boldsymbol{c} \in \mathbb{R}^L$, the operator $\nabla^2_{(\boldsymbol{a},\boldsymbol{b}),\boldsymbol{c}} \cdot$ is equivalent to $\nabla^2_{(\boldsymbol{a}' \ \boldsymbol{b}')',\boldsymbol{c}} \cdot$ and is defined as $\left(\nabla^2_{(\boldsymbol{a},\boldsymbol{b}),\boldsymbol{c}} \cdot\right)_{km} = \frac{\partial^2}{\partial a_k \partial c_m} \cdot$, for $1 \leq k \leq N$, $1 \leq m \leq L$, and $\left(\nabla^2_{(\boldsymbol{a},\boldsymbol{b}),\boldsymbol{c}} \cdot\right)_{lm} = \frac{\partial^2}{\partial b_l \partial c_m} \cdot$, for $N + 1 \leq l \leq N + M$, $1 \leq m \leq L$. In the case of an iterate argument, say $\boldsymbol{a}(n) \in \mathbb{R}^N$, $n \in \mathbb{N}$, we simplify the notation of a derivative by writing $\frac{\partial}{\partial a_k} f(\boldsymbol{a}(n))$ instead of $\frac{\partial}{\partial a_k} f(\boldsymbol{a})|_{\boldsymbol{a}=\boldsymbol{a}(n)}$, for any Frechet-differentiable function $\boldsymbol{a} \mapsto f(\boldsymbol{a})$, $\boldsymbol{a} \in \mathbb{R}^N$ (and analogously for the second derivative under twice Frechet-differentiable function f). Using a bar sign, we sometimes implicitly distinguish a particular argument of f, say $\bar{\boldsymbol{a}} \in \mathbb{R}^N$ (resp., $\boldsymbol{a} \in \mathbb{R}^N$), from a general argument $\boldsymbol{a} \in \mathbb{R}^N$ (resp., $\bar{\boldsymbol{a}} \in \mathbb{R}^N$) from the domain of f, so that $\frac{\partial}{\partial a_k} f(\bar{\boldsymbol{a}}) = \frac{\partial}{\partial a_k} f(\boldsymbol{a})|_{\boldsymbol{a}=\bar{\boldsymbol{a}}}$ (resp., $\frac{\partial}{\partial \bar{a}_k} f(\boldsymbol{a}) = \frac{\partial}{\partial \bar{a}_k} f(\bar{\boldsymbol{a}})|_{\bar{\boldsymbol{a}}=\boldsymbol{a}}$).

Given $\boldsymbol{a} \in \mathbb{R}^N$, we denote by $S(\boldsymbol{a}) = S_\epsilon(\boldsymbol{a})$ an ϵ-neighborhood of $\boldsymbol{a}$, where $\epsilon > 0$ is assumed to be chosen appropriately small in each considered case.

2

Optimization of aggregated performance and achieving min-max fairness in the view of (log-) convexity

In this chapter we first introduce the network model in Section 2.1 and then state the optimization problems considered in this work. The network model and notation introduced below is valid throughout the work. The first problem of interest is the so-called optimization of weighted aggregated performance and is introduced in Section 2.3. This problem is later in the focus of Chapters 3-5. In Section 2.3 we characterize the solvability of the problem of the aggregated performance optimization problem and propose possible problem reformulation and interpretation. The solvability results exhibit the importance of the feature of log-convexity of the SIR function as a function of the link performance value.

The second considered problem of achieving so-called min-max fairness is introduced in Section 2.4 and is later in the scope of Chapter 6.

Besides the model and problem introduction, in this chapter we also provide some general results on convexity of the performance region of the network (Section 2.2), which originate from [33], [34], [35]. Similarly to the solvability issues of the problem of aggregated performance optimization, the convexity property of the performance region is in strong relation with the crucial feature of log-convexity of the SIR function (as a function of the link performance value). Basic notions of Lagrangian optimization theory and convex analysis used in this chapter are explained in Appendices A.3, A.4.

2.1 Preliminaries on link power, SIR and link performance

We consider a network with the set of nonorthogonal links $\mathcal{K} = \{1, \ldots, K\}$. The presented results hold in particular for the cellular uplink (multiple access) and the cellular downlink (broadcast). Link transmit powers p_k, $1 \leq k \leq K$, are grouped into the vector $\boldsymbol{p} = (p_1, \ldots, p_K)$. We focus mostly on two kinds of power constraints; individual (per-transmitter) transmit power limits $\hat{\boldsymbol{p}} = (p_1, \ldots, p_K)$, as in the uplink, and the limitation of the transmit sum-power budget by P, as in the downlink. In the first case, the set of available power vectors, the *power region*, is

$$\mathcal{P}_{\hat{\boldsymbol{p}}} = \{\boldsymbol{p} \geq 0 : \boldsymbol{p} \leq \hat{\boldsymbol{p}}\}.$$

In the latter case, the power region takes the form

$$\mathcal{P}_P = \{\boldsymbol{p} \geq 0 : \mathbf{1}'\boldsymbol{p} \leq P\}.$$

We usually universally denote the power region as $\mathcal{P} \in \{\mathcal{P}_{\hat{\boldsymbol{p}}}, \mathcal{P}_P\}$.

We assume linear receivers for all links. We denote the SIR (*Signal to (Noise and) Interference Ratio*) function of the k-th link as $\boldsymbol{p} \mapsto \gamma_k(\boldsymbol{p})$, which can be written as (see also [36])

$$\gamma_k(\boldsymbol{p}) = \frac{p_k}{\sum_{\substack{l=1 \\ l\neq k}}^{K} V_{kl} p_l + \sigma_k^2}, \quad \boldsymbol{p} \in \mathcal{P}, \quad 1 \leq k \leq K. \tag{2.1}$$

Each interference coefficient, or cross-talk coefficient, V_{kl} models the interference influence of the l-th link signal on the k-th receiver, $k \neq l$. By $\sigma_k^2 \geq 0$ we denote the variance of Additive White Gaussian Noise (AWGN) at the output of the k-th receiver. In the context of weighted aggregated performance optimization (Chapters 3-5) we require $\sigma_k^2 > 0$, $1 \leq k \leq K$, while for the analysis of min-max fairness in Chapter 6 we set $\sigma_k^2 = 0$, $1 \leq k \leq K$.

Independently of the network realization, each interference coefficient V_{kl} depends on the coefficient h_{kl} of the channel from the l-th link transmitter to the k-th link receiver (throughout the work all antenna-to-antenna channels are assumed to be reciprocal and flat-fading, and thus described by scalar coefficients [37]). In general, we set

$$\begin{aligned} V_{kl} &= \frac{|h_{kl}|^2}{|h_{kk}|^2}, \quad k \neq l, \\ V_{kk} &= 0, \end{aligned} \qquad 1 \leq k, l \leq K. \tag{2.2}$$

In precise terms, the cross-talk coefficients V_{kl} and the noise variances σ_k^2 depend additionally on other factors depending on particular network realization, e.g. on aperiodic cross- and auto-correlations of sequences in the CDMA (Code Division Multiple Access) case [36]. For simplicity of presentation, this influence is assumed throughout the work to be included in the (squared magnitudes of) channel coefficients $|h_{kl}|^2$, $1 \leq k, l \leq K$.

Writing all SIR expressions (2.1) in matrix form we get

$$(\boldsymbol{I} - \Gamma(\boldsymbol{p})\boldsymbol{V})\boldsymbol{p} = \Gamma(\boldsymbol{p})\boldsymbol{\sigma}^2, \tag{2.3}$$

with the function $\boldsymbol{p} \mapsto \Gamma(\boldsymbol{p}) = diag(\gamma_1(\boldsymbol{p}), \ldots, \gamma_K(\boldsymbol{p}))$, $\boldsymbol{p} \in \mathcal{P}$, vector $\boldsymbol{\sigma}^2 = (\sigma_1^2, \ldots, \sigma_K^2)$ and the nonnegative *interference matrix*, or *cross-talk matrix*, $\boldsymbol{V}$, defined as

$$(\boldsymbol{V})_{kl} = V_{kl}, \quad 1 \leq k, l \leq K.$$

Throughout the work, we denote the left and right Perron-Frobenius eigenvectors (in short, PF eigenvectors) of the nonnegative interference matrix as $\boldsymbol{l} = \boldsymbol{l}(\boldsymbol{V})$ and $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{V})$, respectively. We do not assume here the normalization of the PF eigenvectors to $\|\boldsymbol{r}\|_2 = \|\boldsymbol{l}\|_2 = 1$ in general. Vectors $\boldsymbol{l}$, $\boldsymbol{r}$ are included in the left and right PF eigenmanifolds of the interference matrix, which we denote as

$$\mathcal{L} = \mathcal{L}(\boldsymbol{V}) = \{\boldsymbol{x} \neq 0 : \boldsymbol{V}'\boldsymbol{x} = \rho(\boldsymbol{V})\boldsymbol{x}\}$$

and

$$\mathcal{R} = \mathcal{R}(\boldsymbol{V}) = \{\boldsymbol{x} \neq 0 : \boldsymbol{V}\boldsymbol{x} = \rho(\boldsymbol{V})\boldsymbol{x}\}$$

respectively, where $\mathcal{L}, \mathcal{R} \subseteq \mathbb{R}_+^K$ is obvious from the nonnegativity of $\boldsymbol{V}$ and $\rho(\cdot)$ denotes the spectral radius [38].

For presentation purposes (in particular, to comply with the framework of Perron-Frobenius Theory applied widely in this work) it is sometimes useful to make the SIR a separate notion by writing

$$\gamma_k = \gamma_k(\boldsymbol{p}), \quad 1 \leq k \leq K, \quad \text{and} \quad \boldsymbol{\Gamma} = diag(\gamma_1, \ldots, \gamma_K) = \Gamma(\boldsymbol{p}), \quad \boldsymbol{p} \in \mathcal{P}. \tag{2.4}$$

From the Perron-Frobenius Theory is known that the *SIR matrix* $\boldsymbol{\Gamma}$ is generated by the unique nonnegative power vector

$$\boldsymbol{p} = (\boldsymbol{I} - \boldsymbol{\Gamma}\boldsymbol{V})^{-1}\boldsymbol{\Gamma}\boldsymbol{\sigma}^2 \tag{2.5}$$

(that is, (2.3) is uniquely solvable for $\boldsymbol{p} \geq 0$) if and only if $\rho(\boldsymbol{\Gamma}\boldsymbol{V}) < 1$ [39], [40]. In other words, as long as the spectral radius of the matrix $\boldsymbol{\Gamma}\boldsymbol{V}$ is smaller than unity, there exists a continuous one-to-one mapping (2.5) from the space of SIR matrices to the space of power vectors.

Our interest is in functions characterizing the link quality in terms of the desired QoS (*Quality-of-Service*) or simply some performance measurement. We group such link-specific QoS values q_k in the *QoS/ performance vector* $\boldsymbol{q} = (q_1, \ldots, q_K)$. For each link $1 \leq k \leq K$ we assume a one-to-one twice differentiable dependence $q_k \mapsto \Phi(q_k) = \gamma_k$, $1 \leq k \leq K$. Thus, there exists an inverse mapping $\Psi = \Phi^{-1}$ such that

$$\gamma_k \mapsto \Psi(\gamma_k) = q_k, \quad \gamma_k \geq 0, \quad 1 \leq k \leq K.$$

Without loss of generality we assume throughout that Ψ is decreasing (if the interest is in some increasing Ψ, it has to be used simply with negative sign). For instance, for the (negative) link capacity in Gaussian channel we have

$$\Psi(\gamma) = -\log(1 + \gamma), \quad \gamma \geq 0, \tag{2.6}$$

and for the normalized symbol error rate averaged over realizations of the Rayleigh fading we have

$$\Psi(\gamma) = 1/\gamma^a, \quad \gamma \geq 0, \tag{2.7}$$

with $a > 0$ as the diversity order.

In the context of min-max fairness issues in Chapter 6, we use also a modified dependence of the link performance on the corresponding link SIR of the form

$$\gamma_k \mapsto F(\frac{1}{\gamma_k}) = q_k, \quad 0 \leq \gamma_k < \infty, \quad 1 \leq k \leq K, \tag{2.8}$$

so that the performance functions Ψ and F are related according to

$$\Psi(\gamma) = F(\frac{1}{\gamma}), \quad 0 \leq \gamma < \infty. \tag{2.9}$$

Consequently, function F follows to be twice differentiable and increasing. It is important to notice that *we assume the performance function F to be defined only for positive arguments (inverse SIR values).*

The introduced dependence (2.8) with increasing F is quite special, but applies to any QoS function being a monotone function of the SIR. In particular, to obtain the link capacity (2.6) and channel-averaged symbol error rate under Rayleigh fading (2.7) we have to set

$$F(y) = -\log(1 + y^{-1}), \quad y > 0$$

and

$$F(y) = y^a, \quad y > 0, \quad a > 0,$$

respectively.

The componentwise extensions of Φ and Ψ to matrix-/ vector-valued functions are written as $\boldsymbol{q} \mapsto \boldsymbol{\Phi}(\boldsymbol{q}) = \boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma} \mapsto \boldsymbol{\Psi}(\boldsymbol{\Gamma}) = \boldsymbol{q}$, respectively. When concatenated with the mapping inverse to (2.5), $\boldsymbol{\Psi}$ yields the QoS vector as a function of power vector,

$$\boldsymbol{p} \longmapsto \boldsymbol{\Gamma} \overset{\boldsymbol{\Psi}}{\longmapsto} \boldsymbol{q}, \quad \boldsymbol{p} \in \mathcal{P}, \tag{2.10}$$

which can be written explicitly as $\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}))$, $\boldsymbol{p} \in \mathcal{P}$. From (2.10) arises the notion of the *QoS/ performance region* as the set of all performance vectors achievable with the vectors in the power region. Precisely, in the case of sum-power constraint we have the QoS region

$$\mathcal{Q}_P = \{\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p})) : \boldsymbol{p} \in \mathcal{P}_P\},$$

while the performance region under individual power constraints is

$$\mathcal{Q}_{\hat{\boldsymbol{p}}} = \{\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p})) : \boldsymbol{p} \in \mathcal{P}_{\hat{\boldsymbol{p}}}\}$$

Sometimes we use the more general notion $\mathcal{Q} = \{\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p})) : \boldsymbol{p} \in \mathcal{P}\}$. The inverse of the dependence (2.10) is

$$\boldsymbol{q} \overset{\boldsymbol{\Phi}}{\longmapsto} \boldsymbol{\Gamma} \longmapsto \boldsymbol{p}, \quad \boldsymbol{q} \in \mathcal{Q}, \tag{2.11}$$

which can be written with (2.5) explicitly as $\boldsymbol{p} = (\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2$, $\boldsymbol{q} \in \mathcal{Q}$. For completeness we also define the QoS region of power-unconstrained networks as $\mathcal{Q}_\infty = \{\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p})) : \boldsymbol{p} \geq 0\}$.

2.1.1 Link power and link performance in multi-antenna channels

In Chapter 5 we use a network model with extended physical layer in the sense of multiple antenna, that is, *Multiple-Input Multiple-Output* (MIMO), link channels. We also restrict us there to the case of a multiple access channel in the particular form of a cellular uplink [41].

We consider a *slotted* multi-antenna multiple access channel, which means that the channel parameters are observable, and can be influenced, only in the discrete-time pattern nT, $n \in \mathbb{N}$. Each link transmitter is equipped with n_t transmit antennas and the common link receiver, the base station, has n_r receive antennas. However, all results from Chapter 5 can by straightforwardly generalized to the case with different number of transmit antennas per link. Slightly loosening the assumption of time-invariant channels in the single-antenna case, we assume the channels to remain constant within the slots $[nT, (n+1)T)$, $n \in \mathbb{N}$, but allow them to take independent values from some common distribution from slot to slot. Such assumption is commonly referred to as *iid (independently identically distributed) block fading* [42], [20], [21]. It has to be noted that the assumption of iid block fading is slightly too restrictive than necessary, but makes the results from Chapter 5 more readable.

We denote the instantaneous value of a multi-antenna channel between the transmitter of link i and the base station in slot $n \in \mathbb{N}$ as $\boldsymbol{H}_i(n) \in \mathbb{C}^{n_r \times n_t}$. We group the instantaneous channel values of all links in $\mathcal{H}(n)=\{\boldsymbol{H}_i(n)\}_{i=1}^K$, $n \in \mathbb{N}$. In the multi-antenna case we require that the transmitters know the instantaneous states of the corresponding channels. Thus, we assume sufficiently accurate channel estimation at the base station and either a reliable delayless feedback channel or also a sufficiently accurate channel estimation at all transmitters. The AWGN assumption is retained, and additionally we assume the noise to be spatially, that is, among receive antennas, uncorrelated, so that the noise covariance matrix takes the form $\boldsymbol{I}\sigma^2 \in \mathbb{R}^{n_r \times n_r}$.

Under multiple antennas per link, the link power is no more sufficient in describing the transmitter. With $\boldsymbol{x}_i(n) \in \mathbb{C}^{n_t}$ as the vector of transmit symbols of i-th link in n-th slot, we define the corresponding (instantaneous) *transmit covariance matrix* as

$$\boldsymbol{Q}_i(n) = E(\boldsymbol{x}_i(n)\boldsymbol{x}_i'(n)), \quad 1 \leq i \leq K, \quad n \in \mathbb{N}.$$

We group the instantaneous transmit covariance matrices of all links in $\mathcal{Q}(n)=\{\boldsymbol{Q}_i(n)\}_{i=1}^K$, $n \in \mathbb{N}$. The transmit covariance matrix of each link is by definition positive semidefinite, which we denote as $\boldsymbol{Q}_i \succeq 0$, $1 \leq i \leq K$, or slightly generalizing as $\mathcal{Q} \succeq 0$. Clearly, the transmit power of i-th multi-antenna link in n-th slot satisfies

$$p_i(n) = tr(\boldsymbol{Q}_i(n)), \quad 1 \leq i \leq K, \quad n \in \mathbb{N},$$

so that the definition of the power region is intuitively extendable to the multi-antenna case as the set of available transmit covariance matrices. Precisely, we have

$$\mathcal{P}_{\hat{\boldsymbol{p}}} = \{\mathcal{Q} = \{\boldsymbol{Q}_i\}_{i=1}^K \succeq 0 : tr(\boldsymbol{Q}_i) \leq \hat{p}_i, 1 \leq i \leq K\}$$

in the uplink-typical case of individual (per-transmitter) power constraints $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_K)$ and

$$\mathcal{P}_P = \{\mathcal{Q} = \{\boldsymbol{Q}_i\}_{i=1}^K \succeq 0 : \sum_{i=1}^K tr(\boldsymbol{Q}_i) \leq P\}$$

under sum-power constrained by P.

We assume the use of *Successive Interference Cancellation* (SIC) in the MIMO multiple access channel. SIC is known to be the optimal signal (post-) processing scheme in the multiple access channel in terms of information theory. Precisely, by SIC and time sharing we can achieve the boundary rate vectors in the capacity region of the MIMO multiple access channel [41], [43].

The (instantaneous) order of SIC of link signals in n-th slot is represented by a permutation $(i, n) \mapsto \pi_k(i, n)$, $(i, n) \in \mathcal{K} \times \mathbb{N}$. The subscript, used only sometimes, labels hereby the permutation (the SIC order) as the k-th one from the ordered set of $K!$ possible permutations (SIC orders), say Π_K. Given SIC order π_k, we have precisely $\pi_k(1, n)$ as the last decoded link signal, ..., and $\pi_k(K, n)$ as the first decoded link signal in n-th slot. Thus, π_k denotes actually the inverse SIC order. In figures we also use a more intuitive notation of the SIC order in the form $\pi_k(n) = \pi_k(1, n) \leftarrow \ldots \leftarrow \pi_k(K, n)$, $n \in \mathbb{N}$.

The achievable (instantaneous) data rate function on i-th link in n-th slot takes the form

$$(\mathcal{Q}(n), \mathcal{H}(n), \pi_k(n)) \mapsto R_i(\mathcal{Q}(n), \mathcal{H}(n)), \quad (\mathcal{Q}(n), \pi_k(n)) \in \mathcal{P} \times \Pi_K, \quad n \in \mathbb{N}. \tag{2.12}$$

The differences and similarities between the link data rate and link capacity are addressed later in Section 5.2.3. We group the (values of) link data rates $R_i(\mathcal{Q}(n), \mathcal{H}(n)) = R_i(n)$, in n-th slot in the *rate vector* $\boldsymbol{R}(n) = (R_1(n), \ldots, R_K(n))$, $n \in \mathbb{N}$. Intuitively, a pair $(\mathcal{Q}(n), \pi_k(n)) \in \mathcal{P} \times \Pi_K$, $n \in \mathbb{N}$, can be referred to as a *transmission policy* of the multiple access channel in n-th slot since it defines the instantaneous transmit and receive strategy in the multiple access channel. Clearly, according to our model, a transmission policy in n-th slot is in general dependent on the parameters observable up to time instant nT.

The system bandwidth is denoted as W. In the context of multi-antenna multiple access channel in Chapter 5, we regard the link capacity and capacity region of the channel as the maximum achievable rate in [bit/s] and the set of all achievable rate vectors in [bit/s], respectively.

Since the results of Chapter 5 concern a one slot-view, the indication by (respectively, dependence on) the slot index will be sometimes dropped (respectively, neglected) there.

It has to be noted that in the context of most multi-antenna considerations, and in particular in Chapter 5, the dependence (2.12) assumes the role of the relation (2.10) in the general network with interference (the definition of the transmission policy as the argument in (2.12) is here however specific for the MIMO multiple access channel). This is caused by the dominant interest in the link data rate and link capacity as QoS functions in multi-antenna networks, see e.g. the variety of works [44], [45], [42], [20], [46] and references therein. Thus, in Chapter 5 we also concentrate on the rate vector as the only QoS vector of interest. In contrast to the general relation of power vector and QoS vector (2.10), the MIMO-specific relation (2.12) is usually not represented as a concatenation of the SIR function and a QoS function (here, the data rate function). The lack of such structure is caused by the simple fact that an established and meaningful notion of SIR function which gives rise to useful QoS vectors according to (2.10) is nonexistent under multiple antennas per link.

2.2 Convexity of the performance region

Convexity of the QoS region is a desired property from the point of view of design of resource allocation policies. For instance, for any two achievable QoS vectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}$ (i.e. $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)} \in \mathcal{Q}$), it is known in such case that any their convex combination $\boldsymbol{q}(t) = (1-t)\boldsymbol{q}^{(1)} + t\boldsymbol{q}^{(2)}$, $t \in (0,1)$, can be achieved by a power vector from the power region as well. Thus, if for some $t \in (0,1)$ the combined QoS vector $\boldsymbol{q}(t)$ is favorable compared to $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)}$, known algorithms can be applied to achieve the performance corresponding to $\boldsymbol{q}(t)$ (see e.g. [39] for the case $q = \Psi(\gamma) = 1/\gamma$ in CDMA networks). Furthermore, special algorithmic resource allocation schemes, relying strongly on the convexity property of $\mathcal{Q}$, are applicable when convexity of $\mathcal{Q}$ is ensured (see e.g. [28] for the approach of optimization of aggregated QoS performance with $\Psi(\gamma) = \log(\gamma)$).

In [39] and [40] the authors proved convexity of the QoS regions $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ and $\mathcal{Q}_\infty$ for some particular QoS functions, such as $q = \Psi(\gamma) = 1/\gamma$. In [36], the following convexity condition for the downlink performance region $\mathcal{Q}_P$ for general performance functions was provided.

Proposition 1 *If Φ is log-convex, then the QoS region $\mathcal{Q}_P$ is a convex set.*

As a new result, which parallels Proposition 1, we provide a similar convexity condition for the uplink performance region $\mathcal{Q}_{\hat{\boldsymbol{p}}}$.

Proposition 2 *If Φ is log-convex, then the QoS region $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ is a convex set.*

Proof Let function $\boldsymbol{q} \mapsto L_{\boldsymbol{\alpha}}(\boldsymbol{q}) = \boldsymbol{\alpha}'(\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2$ be defined for $\boldsymbol{q} \in \mathbb{R}^K$ such that $\rho(\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V}) < 1$. Utilizing the Neumann series expansion we can write further

$$L_{\boldsymbol{\alpha}}(\boldsymbol{q}) = \sum_{k=0}^{\infty} \boldsymbol{\alpha}'(\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^k \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2, \quad \boldsymbol{q} \in \mathbb{R}^K \text{ such that } \rho(\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V}) < 1.$$

Since, by assumption, Φ is log-convex and the product and the sum of log-convex functions are log-convex [47], $L_{\boldsymbol{\alpha}}$ is log-convex as well for $\boldsymbol{\alpha} \geq 0$.

By (2.5) and (2.10), this implies that a linear combination of transmit powers, with nonnegative weights, is a log-convex function of the QoS vector. By setting $\boldsymbol{\alpha} = \boldsymbol{e}_k$, for some $k \in \mathcal{K}$, the same holds for any single link transmit power. Choose now two power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} \in \mathcal{P}_{\hat{\boldsymbol{p}}}$. Then, for the corresponding vectors $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)} \in \mathbb{R}^K$ obtained from $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}$ by (2.10), respectively, we have $\boldsymbol{q}^{(1)}, \boldsymbol{q}^{(2)} \in \mathcal{Q}_{\hat{\boldsymbol{p}}}$. Let now a QoS vector $\boldsymbol{q}(t) = (1-t)\boldsymbol{q}^{(1)} + t\boldsymbol{q}^{(2)}$, $t \in (0,1)$ be defined and let $\boldsymbol{p}(t) = (p_1(t), \ldots, p_K(t))$ be the power vector associated with $\boldsymbol{q}(t)$ through (2.11). Now, by the shown log-convexity of p_k, $k \in \mathcal{K}$, as a function of the QoS vector, we have (by definition of log-convexity and power constraints)

$$p_k(t) \leq (p_k^{(1)})^{(1-t)} (p_k^{(2)})^t \leq (\hat{p}_k)^{(1-t)} (\hat{p}_k)^t = \hat{p}_k, \quad t \in (0,1), \quad k \in \mathcal{K}. \tag{2.13}$$

Thus, it is implied by (2.10) again that $\boldsymbol{q}(t) \in \mathcal{Q}_{\hat{\boldsymbol{p}}}$, $t \in (0,1)$, which completes the proof. □

Fortunately, there is a number of useful QoS functions that correspond to log-convex QoS-SIR dependences Φ and thus ensure convexity of the QoS region. Some examples are the following.

- $q = \Psi(\gamma) = -\log \frac{\gamma}{1+\gamma}$ as the logarithmically (e.g. in dB) expressed effective bandwidth for linear MMSE (*Minimum Mean Square Error*) receivers. In fact, $\gamma = \Phi(q) = \frac{\exp(-q)}{1-\exp(-q)}$ is log-convex.
- $q = \Psi(\gamma) = \frac{1}{\gamma^a}$ as the channel-averaged normalized symbol error rate (under receiver diversity $a > 0$ and Rayleigh fading) or as the effective spreading factor in CDMA ($a = 1$). Then, $\gamma = \Phi(q) = \frac{1}{q^{1/a}}$ is log-convex.
- $q = \Psi(\gamma) = -\log \gamma$ as the logarithmically (e.g. in dB) expressed SIR, or high-SIR approximation of the link capacity. In fact, $\gamma = \Phi(q) = \exp(-q)$ is log-convex.

The following Lemma shows further that the log-convexity property of Φ is equivalent to convexity of the function

$$x \mapsto \Psi_e(x) = \Psi(e^x), \quad x \in \mathbb{R}.$$

The latter characterization might sometimes appear to be favorable.

Lemma 1 *An inverse performance function $\Phi = \Psi^{-1}$ is log-convex if and only if function Ψ_e is convex.*

Proof Let $\Phi(q) = \Psi^{-1}(q)$, $q \in \mathbb{R}$, be log-convex, which means

$$\Phi(q_1)^{(1-t)} \Phi(q_2)^t \geq \Phi((1-t)q_1 + tq_2), \quad t \in (0,1), \quad q_1, q_2 \in \mathbb{R}. \tag{2.14}$$

Thus, by decreasingness of Φ (due to decreasingness of Ψ), we have by (2.14) also $\Psi(\Phi(q_1)^{(1-t)}\Phi(q_2)^t) \leq (1-t)q_1 + tq_2$, $t \in (0,1)$, $q_1, q_2 \in \mathbb{R}$. Consequently, by substituting

$$\Phi(q_i) = e^{x_i}, \quad i = 1, 2, \tag{2.15}$$

and reformulating we yield

$$\Psi(e^{(1-t)x_1 + tx_2}) \leq (1-t)\Psi(e^{x_1}) + t\Psi(e^{x_2}), \quad t \in (0,1), \quad x_1, x_2 \in \mathbb{R},$$

which is equivalent to convexity of $\Psi_e(x)$, $x \in \mathbb{R}$. The converse proof is a straightforward inversion, using the same substitution (2.15), the decreasingness property of Ψ_e and the inverted order of transformations. □

2.3 Optimization of weighted aggregated performance

In this section, and later in Chapters 3-5, we focus on the weighted sum of performance functions

$$\boldsymbol{q} \mapsto \boldsymbol{\alpha}'\boldsymbol{q}, \quad \boldsymbol{q} \in \mathcal{Q}, \quad \text{or, equivalently} \quad \boldsymbol{p} \mapsto \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p})), \quad \boldsymbol{p} \in \mathcal{P}, \tag{2.16}$$

with $\boldsymbol{\alpha} \in \mathcal{A}$ and

$$\mathcal{A} = \{\boldsymbol{\alpha} \geq 0 : \|\boldsymbol{\alpha}\| = 1\}, \tag{2.17}$$

as the objective in the optimization of power allocation. It is intuitive to require the norm-constraint in (2.17) to be the 1-norm constraint. However, there is no loss in generality when other norms are taken, as is the case e.g. in Chapter 3.

The optimization of weighted aggregated performance given in (2.16) is the most common optimization goal under *best-effort*, or *elastic traffic* [16], [28], [23]. In analogy to the original definition in [16] (for wired traffic), best-effort traffic comes from applications that are able to modify their QoS according to the achievable limits within the network and traffic priorities. Hereby, the link weights α_k, $1 \leq k \leq K$, in (2.16) are usually determined by the corresponding traffic priorities.

With assumed decreasingness of Ψ, the problem of weighted aggregated performance optimization takes the form

$$\min_{\boldsymbol{p} \in \mathcal{P}} \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p})). \tag{2.18}$$

From geometry it is known that the power allocation $\boldsymbol{p}_{\boldsymbol{\alpha}}$ solving (2.18) generates the *Pareto-optimal* QoS vector $\boldsymbol{q}_{\boldsymbol{\alpha}} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}_{\boldsymbol{\alpha}}))$, which is the vector at which the hyperplane with normal vector $\boldsymbol{\alpha}$ supports the set of all achievable QoS vectors, that is the performance region $\mathcal{Q}$ [47]. In other words, any solution to the problem (2.18) is one-to-one associated, by mapping (2.10), with some solution of the scalarized vector optimization of the form

$$\min_{\boldsymbol{q} \in \mathcal{Q}} \boldsymbol{\alpha}'\boldsymbol{q}, \tag{2.19}$$

Problem (2.19) and Pareto optimality is illustrated in Fig. 2.1.

2.3.1 Global optimizers

We can show that log-convexity of the QoS-SIR mapping ensures the existence of only global optimizers of problem (2.18). The result is a consequence of convexity of the performance region.

Proposition 3 *If $\Phi = \Psi^{-1}$ is log-convex, then any local minimizer of problem (2.18) is global as well, and the Kuhn-Tucker conditions are necessary and sufficient optimality conditions, provided that $\mathcal{P}$ satisfies constraint qualification.*

Proof By (2.16), one can see that any solution to the problem (2.18) is one-to-one associated, by mapping (2.10), with some solution to the problem

$$\min_{\boldsymbol{q} \in \mathcal{Q}} \boldsymbol{\alpha}'\boldsymbol{q},$$

which is convex due to convexity of $\mathcal{Q}$, implied by log-convexity of Φ (Propositions 1, 2). By contradiction, assume the existence of at least two distinct local minimizers $\tilde{\boldsymbol{p}}, \check{\boldsymbol{p}} \in \mathcal{P}$ of (2.18), with

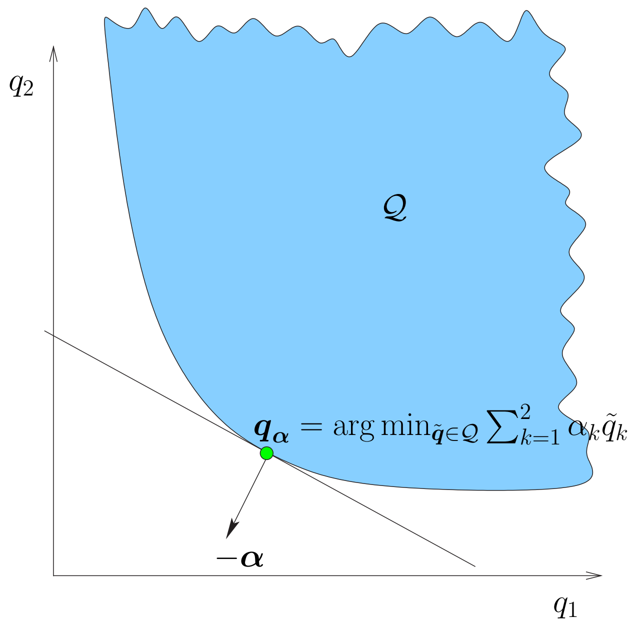

Figure 2.1: An exemplary QoS region in the two-link case with a Pareto-optimal QoS vector $\boldsymbol{q}_{\boldsymbol{\alpha}}$ for some weight vector $\boldsymbol{\alpha}$.

only one of them, say $\tilde{\boldsymbol{p}}$, global. Let the distinct local minimizers of (2.19) uniquely associated with $\tilde{\boldsymbol{p}}$ and $\check{\boldsymbol{p}}$ be $\tilde{\boldsymbol{q}} \in \mathcal{Q}$ and $\check{\boldsymbol{q}} \in \mathcal{Q}$, respectively. By convexity of the problem (2.19), the local minimizers $\tilde{\boldsymbol{q}}$ and $\check{\boldsymbol{q}}$ and all their convex combinations $\boldsymbol{q}(t) = (1-t)\tilde{\boldsymbol{q}} + t\check{\boldsymbol{q}}$, $t \in (0,1)$, are also global solutions to (2.19) [48]. Thus, $\check{\boldsymbol{p}}$ is a global minimizer of (2.18) as well, which contradicts the assumption and proves that all local minimizers of (2.18) are also global. The necessity and sufficiency of the Kuhn-Tucker conditions follows by the standard optimization theory due to satisfied constraint qualification [48]. □

Existence of only global minimizers of problem (2.18) implies that any locally converging optimization routine finds a globally optimal power allocation. Thus, Proposition 3 implies that adaptive online power (re-) allocation according to (2.18) is significantly facilitated for QoS functions with log-convex QoS-SIR dependence.

2.3.2 Matrix characterization of the solution

The constraint inequalities determining the domain in (2.18) take the form $-\boldsymbol{p} \leq 0$, $\sum_{k=1}^{K} p_k - P \leq 0$ in the downlink case ($\mathcal{P}_P$) and $-\boldsymbol{p} \leq 0$, $\boldsymbol{p} - \hat{\boldsymbol{p}} \leq 0$ in the uplink case ($\mathcal{P}_{\hat{\boldsymbol{p}}}$). With the Perron-Frobenius Theory (Section 2.1), the vectorial nonnegativity constraint on the power allocation can be replaced in both cases by the scalar inequality constraint $\rho(\Gamma(\boldsymbol{p})\boldsymbol{V}) < 1$. With this, the Lagrangian of problem (2.18) can be written as

$$L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \mu, \nu) = \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p})) + \mu(\sum_{k=1}^{K} p_k - P) + \nu(\rho(\Gamma(\boldsymbol{p})\boldsymbol{V}) - 1) \tag{2.20}$$

in the case of sum-power constraint (e.g. downlink) and

$$L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \boldsymbol{\mu}, \nu) = \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p})) + \sum_{k=1}^{K} \mu_k (p_k - \hat{p}_k) + \nu(\rho(\Gamma(\boldsymbol{p})\boldsymbol{V}) - 1) \tag{2.21}$$

in the case of individual power constraints (e.g. uplink), with μ, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$ and ν as the Lagrangean multipliers. Since the complementary slackness condition $\nu(\rho(\boldsymbol{\Gamma}(\boldsymbol{p})\boldsymbol{V}) - 1) = 0$ is a necessary optimality condition and we further have $\rho(\Gamma(\boldsymbol{p})\boldsymbol{V}) \to 1$ only if $\boldsymbol{p} \to \infty$, it follows that the optimum value of the Lagrange multiplier ν is $\nu = 0$ [49], [48]. This lets us state the Kuhn-Tucker conditions $\nabla_{\boldsymbol{p}} L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \mu, 0) = 0$ and $\nabla_{\boldsymbol{p}} L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \boldsymbol{\mu}, 0) = 0$ in the downlink and uplink, respectively, in a nice compact form. Letting function $\boldsymbol{p} \mapsto \boldsymbol{g} = (g_1(\boldsymbol{p}), \ldots, g_K(\boldsymbol{p}))$, $\boldsymbol{p} \geq 0$, with

$$g_k(\boldsymbol{p}) = \alpha_k \Psi'(\gamma_k(\boldsymbol{p})) \frac{\gamma_k(\boldsymbol{p})}{p_k}, \quad 1 \leq k \leq K, \tag{2.22}$$

we yield precisely

$$\nabla_{\boldsymbol{p}} L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \mu, 0) = \boldsymbol{g}(\boldsymbol{p}) - \boldsymbol{V}'\Gamma(\boldsymbol{p})\boldsymbol{g}(\boldsymbol{p}) + \mu\mathbf{1} = 0$$

in the downlink case and

$$\nabla_{\boldsymbol{p}} L_{\boldsymbol{\alpha}}(\boldsymbol{p}, \boldsymbol{\mu}, 0) = \boldsymbol{g}(\boldsymbol{p}) - \boldsymbol{V}'\Gamma(\boldsymbol{p})\boldsymbol{g}(\boldsymbol{p}) + \boldsymbol{\mu} = 0,$$

in the uplink.

For QoS functions with log-convex QoS-SIR dependence, this yields with the remaining Kuhn-Tucker conditions and Proposition 3 a necessary and sufficient matrix equation characterization of the optimal power allocation in (2.18) (not that the constraint qualification is satisfied in the cases of $\mathcal{P}_P$ and $\mathcal{P}_{\hat{\mathbf{p}}}$).

Proposition 4 *With Φ as a log-convex function, the power vector $\mathbf{p}$ generating the SIR matrix $\boldsymbol{\Gamma}$ solves problem (2.18) if and only if it solves*

$$\begin{cases} \boldsymbol{p} = (\boldsymbol{I} - \boldsymbol{\Gamma}\boldsymbol{V})^{-1}\boldsymbol{\Gamma}\boldsymbol{\sigma}^2 \\ \mathbf{g}(\boldsymbol{p}) = -(\boldsymbol{I} - (\boldsymbol{\Gamma}\boldsymbol{V})')^{-1}\boldsymbol{c}, \\ \rho(\boldsymbol{\Gamma}\boldsymbol{V}) < 1 \end{cases} \tag{2.23}$$

with $\boldsymbol{c} = \mu\mathbf{1} \geq 0$, $\sum_{k=1}^K p_k - P \leq 0$, $\mu(\sum_{k=1}^K p_k - P) = 0$ under sum-power constraint and $\boldsymbol{c} = \boldsymbol{\mu} \geq 0$, $\mathbf{p} - \hat{\mathbf{p}} \leq 0$, $\boldsymbol{\mu}'(\mathbf{p} - \hat{\mathbf{p}}) = 0$ under power constraints per link.

Obviously, under lack of log-convexity of Ψ, Proposition 4 provides a necessary and sufficient matrix equation characterization of a local minimizer of (2.18). In some sense, the structural similarity of the matrix equations in the optimality conditions (2.23) gives rise to efficient decentralized algorithmic solutions to problem (2.18) (Chapter 4).

2.3.3 Fairness of medium access

The links which are allocated zero transmit power are said to be *idle*. From the point of view of fairness in the network it is desirable when $\alpha_k > 0$ implies $p_k > 0$ under the optimality in terms of (2.18), that is, when nonzero link priority implies a non-idle link under optimized weighted aggregated performance. Such feature ensures medium access for any nonzero traffic priority at the optimum of weighted aggregated performance (a kind of medium access fairness). We can show that the class of QoS functions with log-convex QoS-SIR dependence provides such kind of fairness in medium access.

Proposition 5 *Given $\boldsymbol{\alpha} > 0$ and a log-convex function Φ, any solution to (2.18) is positive.*

Proof We first prove the following crucial Lemma.
Lemma *If $\Phi = \Psi^{-1}$ is log-convex, then $\Psi'(0) = -\infty$.*
By assumed decreasingness and differentiability of Φ, we have $\Psi'(\gamma) < 0,\ \gamma \geq 0$. Then, by Lemma 1, Ψ has a log-convex inverse if and only if Ψ_e is convex. Obviously, Ψ_e is convex if and only if $\Psi'_e(x) = \Psi'(e^x)e^x$ is nondecreasing. Take a series $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, with $\lim_{n\to\infty} x_n = -\infty$ and assume by contradiction $\Psi'(0) = c > -\infty$. Then, $\lim_{n\to\infty} \Psi'_e(x_n) = \lim_{n\to\infty} \Psi'(e^{x_n})e^{x_n} = c \cdot 0 = 0$, due to the continuity of Ψ (implied by differentiability [50]). Further, we have $\Psi'_e(x_n) = \Psi'(e^{x_n})e^{x_n} < 0,\ n \in \mathbb{N}$, due to $\Psi'(\gamma) < 0,\ \gamma \geq 0$. Since this holds for any series $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} x_n = -\infty$, we yield by separability of $\mathbb{R}$ that $\lim_{x\to-\infty} \Psi'_e(x) = 0$ and $\Psi'_e(x) < 0,\ x \in \mathbb{R}$. But this contradicts nondecreasingness of Ψ'_e and completes the proof of the Lemma.

Let now a series of power vectors $\{\boldsymbol{p}_n\}_{n\in\mathbb{N}}$ be convergent to $\tilde{\boldsymbol{p}} \in \mathcal{P}$ and let, by contradiction, $\tilde{\boldsymbol{p}}$ be a solution to (2.18) such that $\tilde{p}_k = 0$, for some $k \in \mathcal{K}$. Then, it is clear from (2.1) and the assumption $\sigma_k^2 > 0$ that for the k-th SIR function we have $\lim_{n\to\infty} \gamma_k(\boldsymbol{p}^{(n)}) = \gamma_k(\tilde{\boldsymbol{p}}) = 0$. Thus, with the Lemma above we have then

$$\lim_{n\to\infty} \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p}^{(n)})) = \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\tilde{\boldsymbol{p}})) = \infty,$$

which contradicts the assumption that $\tilde{\boldsymbol{p}}$ is a solution to (2.18) and completes the proof. □

2.3.4 Convex reformulation of the problem

We showed that for QoS functions with log-convex QoS-SIR dependence the online power (re-) allocation is facilitated due to the existence of only global minimizers of problem (2.18). From the point of view of online solvability of problem (2.18) an even more desirable, but more restrictive, property is convexity of the problem statement (that is, convexity of the objective and the optimization domain [47]). Under convexity of the problem, powerful tools of convex optimization, such as interior point methods, can be used in the design of iterative optimization schemes. Convexity of the problem statement ensures good global convergence behavior of applied iterative schemes.

We show that under log-convexity of Φ the optimization problem (2.18) can be translated into an equivalent convex form by logarithmic transformation of the domain.

Proposition 6 *Let Φ be log-convex and $\mathcal{X} = \{\boldsymbol{x} = \log \boldsymbol{p} : \boldsymbol{p} \in \mathcal{P}\}$. Then, the function*

$$\boldsymbol{x} \mapsto \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})), \quad \boldsymbol{x} \in \mathcal{X}, \tag{2.24}$$

is convex and the optimization problem

$$\min_{\boldsymbol{x}\in\mathcal{X}} \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})) \tag{2.25}$$

is a convex problem.

Proof With the definition of function Ψ_e, we can write for each addend in (2.24)

$$\Psi(\gamma_k(e^{\boldsymbol{x}})) = \Psi(e^{\log \gamma_k(e^{\boldsymbol{x}})}) = \Psi_e(\log \gamma_k(e^{\boldsymbol{x}})), \quad 1 \leq k \leq K. \tag{2.26}$$

By the assumption of log-convexity of Φ and by Lemma 1, Ψ_e is convex and decreasing (due to assumed decreasingness of Ψ). Further, it is known from [10] that the function $\log \gamma_k(e^{\boldsymbol{x}})$,

$1 \leq k \leq K$, is concave. Thus, it follows by the standard result from convex analysis that the concatenation $\Psi_e(\log \gamma_k(e^{\boldsymbol{x}}))$, $1 \leq k \leq K$, is a convex function [50]. Convexity of the objective (2.24) as a sum of convex functions follows then immediately. With convexity of the set $\mathcal{X}$ (precisely, sets $\mathcal{X}_P$ and $\mathcal{X}_{\hat{\boldsymbol{p}}}$), convexity of the optimization problem (2.25) is implied and the proof is completed. □

In the view of the power-QoS mapping (2.10), Proposition 6 implies that the map from logarithmic power vectors to performance vectors

$$\boldsymbol{x} \overset{\exp}{\longmapsto} \boldsymbol{p} \longmapsto \boldsymbol{\Gamma} \overset{\Psi}{\longmapsto} \boldsymbol{q}$$

is convex whenever $\Phi = \Psi^{-1}$ is log-convex.

In Fig. 2.2 a simulative comparison of convergence is provided for two different QoS parameters with log-convex QoS-SINR map. The advantage of convexity is mirrored in Fig. 2 by the fact that the gradient method applied to the convex problem form performs as well as the more efficient BFGS (Broyden-Fletcher-Goldfarb-Shanno) method applied to the nonconvex problem (2.18). In contrast to the gradient method, the BFGS method utilizes approximative second-order information [47].

It has to be underlined that the reformulation of aggregated performance optimization (2.18) in the form (2.25) is allowable under much more general conditions than under log-convexity of the QoS-SIR dependence. Precisely, the domain in problem (2.18) can be transformed logarithmically when

$$\boldsymbol{p} = \arg\min_{\tilde{\boldsymbol{p}} \in \mathcal{P}} \sum_{k=1}^{K} \alpha_k \Psi(\gamma_k(\boldsymbol{p})) > 0$$

(note, that by Proposition 5 this is satisfied in particular for QoS functions with log-convex QoS-SIR dependence). In Chapter 4 however, we use the following slightly more restrictive condition allowing us to work with problem form (2.25).

Condition 1 *Any local minimizer $\boldsymbol{p}$ of problem (2.18) satisfies $\boldsymbol{p} > 0$.*

Finally, it has to be underlined that the convexity condition and convex problem form from Proposition 6 are essentially different (although similar at first glance) from the ones used e.g. in [29] and relying on geometric programming approach. The reason is that in our case the QoS parameter is a function of link SINR, while in the multi-hop context of [29] the QoS parameters are dependent on source data-rates.

2.4 Achieving min-max fairness

The analysis of fairness issues in networks has its origin in the framework of wired networks [51], [16], [17]. Although we are free to define specialized notions of fairness, the fairness principle referred to here as *min-max fairness* is best-established. Furthermore, min-max fairness gives rise to the majority of related fairness notions applicable to different network types (wired/ wireless), different network topologies (cellular/ ad-hoc networks) and different QoS functions (e.g. the end-to-end route delay in multi-hop ad-hoc networks or link capacity in cellular networks).

In general, *the idea of min-max fairness consists in making the worst performance value* (e.g. of a route, link, etc.) *as good as possible.* In wired networks, the min-max fair equilibrium of QoS values is the one at which no QoS valuer q_i can be improved without the degradation of any QoS value q_j, $j \neq i$, which is already inferior to q_i [16], [17], [52], [53], [54], [55], [56]. The same

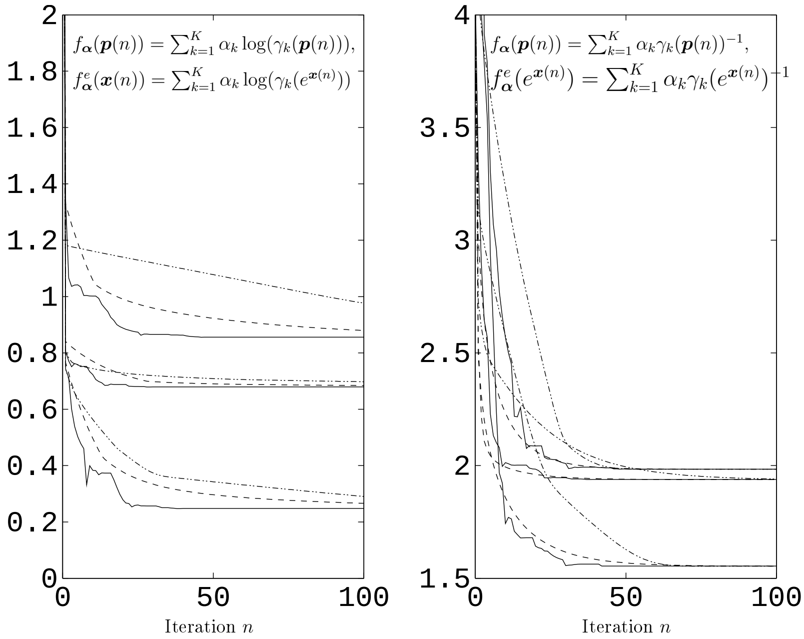

FIGURE 2.2: A comparison of convergence of different optimization methods applied to problem (6) and its convex form for two exemplary QoS parameters with log-convex QoS-SINR maps. The gradient method applied to nonconvex problem (dotted line) provides the worst convergence performance. The convergence of the gradient method applied to convex problem form (dashed line) is comparable with the convergence of the BFGS method applied to nonconvex problem form (solid line), although the latter one uses approximative second order information.

definition translates usually to the case of wireless multi-hop ad-hoc networks when the QoS values are associated with routes [57], [58], [59].

The fairness principle referred in this work as min-max fairness is equivalent to the notion of max-min fairness in the given references and in the majority of the literature on the topic. Nevertheless, we prefer here a different convention to comply with the fact that, as will be seen in Section 2.4.2, the problem of ensuring this notion of fairness takes the min-max form. The min-max problem form results from the assumption that the QoS function (2.8) is increasing in inverse SIR and thus, is decreasing in the corresponding link transmit power. Consequently, it is desired to minimize each QoS value and the worst QoS value is the maximal one. The different convention in the most references results from the increasingness of the QoS value as the function of the corresponding resource (power, bandwidth) assumed there. Hence, the problem of ensuring the same notion of fairness in the references is of max-min type.

2.4.1 Preliminaries on SIR with neglected noise

As announced in Section 2.1, we consider the problem of min-max fairness given the link SIR function

$$\gamma_k(\boldsymbol{p}) = \frac{p_k}{\sum_{j=1}^{K} V_{kj} p_j} = \frac{p_k}{(\boldsymbol{V}\boldsymbol{p})_k}, \quad \boldsymbol{p} \in \mathcal{P}, \quad 1 \leq k \leq K, \tag{2.27}$$

that is, (2.1) with $\sigma_k^2 = 0$, $1 \leq k \leq K$. To exclude "pathological" cases of interference, we also make a nonrestrictive assumption that $\sum_{j=1}^{K} V_{kj} p_j > 0$, $1 \leq k \leq K$, for some $\boldsymbol{p} \in \mathcal{P}$. The link SIR function with neglected noise (2.27) can be considered to take the role of the actual SIR function in the case when the interference power $\sum_{j=1}^{K} V_{kj}p_j$ dominates the variance σ_k^2 of the Gaussian noise perceived at the output of the link receiver, at each link receiver $k \in \mathcal{K}$. Thus, the SIR model (2.27) can correspond to an asymptotic model (2.1) in the regime of high received powers (both, the received own link powers and the interference powers). On the other side, the use of the SIR model (2.27) is justified in networks which utilize transceivers with especially low noise figures, since then the received noise variance at each receiver output is likely to be low in relation to the corresponding interference power. Low noise figure can be expected in specialized transceiver designs with high-end components. Finally, the use of SIR model (2.27) for network optimization purposes might be suitable in the case when the noise variances σ_k^2, $1 \le k \le K$, (or the noise figures of all link receivers $1 \le k \le K$) are not known to the optimizing instance, e.g. to the base station in a cellular network. In such case the assumption $\sigma_k^2 = 0$, $1 \le k \le K$, is one of the options how the optimizing instance can handle the lack of the knowledge on noise. Results relying on the SIR model (2.27) constitute a significant part within the established theory of power control, see e.g. [2], [10] and references therein.

The SIR function with neglected noise (2.27) is multiplicatively invariant in the sense that $\gamma_k(\boldsymbol{p}) = \gamma_k(c\boldsymbol{p})$, $c > 0$. Thus, as long as the power region includes some neighborhood of the origin $\mathbf{0}$, the sets of achievable link SIR values do not differ. Due to such feature, we can assume without loss of generality the unconstrained power region

$$\mathcal{P} = \mathbb{R}_+^K \tag{2.28}$$

when the noise in the SIR function is neglected (in particular, in consideration of min-max fairness).

2.4.2 The problem of min-max fairness

In wired networks, the formulation of the problem of ensuring min-max fairness as an optimization problem is prohibited by the network topology constraints, and precisely by the existence of so-called *bottleneck links* [52], [53], [56]. Similarly, in considerations of end-to-end QoS in wireless multi-hop ad-hoc networks such formulation is prohibited by the natural constraints on the routing policy [59].

In the considered (single-hop) network with performance values associated with links and link performance requirements $\boldsymbol{q}^{\text{req}} = (q_1^{\text{req}}, \ldots, q_K^{\text{req}})$, the problem of ensuring relative or weighted min-max fairness (in short, the problem of relative or weighted min-max fairness) corresponds to the optimization problem

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} \frac{F(\frac{1}{\gamma_k(\boldsymbol{p})})}{q_k^{\text{req}}} = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} \frac{F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)}{F(\frac{1}{\gamma_k^{\text{req}}})}, \tag{2.29}$$

where we defined

$$\mathcal{P}_{++} = \mathcal{P} \cap \mathbb{R}_{++}^K,$$

and where $\gamma_k^{\text{req}} = 1/F^{-1}(q_k^{\text{req}})$, $1 \le k \le K$, denotes the link SIR requirement (see [40] for the special case $q_k = \frac{1}{\gamma_k}$). The fairness notion (2.29) is referred to as relative or weighted due to normalization of absolute link QoS function by the corresponding predefined link QoS requirement.

The notion of unweighted min-max fairness neglects differences in link performance requirements and corresponds to the case $\boldsymbol{q}^{\text{req}} = c\mathbf{1}$, $c > 0$. In the behavioral and economic science, the notion of (unweighted) min-max fairness parallels ideal *social fairness* [60]. By (2.29), the problem of min-max fairness, which is in the focus of Chapter 6, follows simply as

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right). \tag{2.30}$$

Note, that as a consequence of the assumption of an unconstrained power region (2.28), we have

$$\mathcal{P}_{++} = \mathbb{R}^K_{++}$$

in (2.30). Due to this feature one can easily show that the constraint qualification condition for problem (2.30) is always satisfied [48].

3

Centralized algorithmic optimization of weighted aggregated performance

In this chapter we propose two algorithmic solutions of the problem of (weighted) aggregated performance optimization (2.18). The decentralized realization of the proposed algorithms in real-world networks appears to require a significant effort in signaling and feedback (to provide local knowledge of parameters for each network link). Due to this reasons, we claim that the algorithms proposed in this chapter are destined for centralized conduction, which is significantly facilitated, in particular, in a cellular network.

The algorithm proposed in Section 3.1 is applicable to networks with sum-power constraint, in particular in a downlink, while the algorithm from Section 3.2 works in networks with per-link power constraints, e.g. in the uplink. Both algorithmic solutions were proposed originally in [61], [62]. As we will show, both proposed solutions are attractive alternatives to the application of general iterations (for the used notions of optimization theory see Appendix A.3).

3.1 Algorithmic solution under sum-power constraint

According to Propositions 3 and 6, under a log-convex performance-SIR dependence Φ, the solutions to the problems (2.18), (2.25) can be computed by means of general locally convergent iterations, such as the gradient iteration or the Newton iteration [47], [49]. However, specialized algorithms designed for the problem (2.18) can provide certain advantages [28], [24]. Precisely, by making use of the specific problem structure of (2.18), a better convergence rate or lower computational effort can be obtained.

In this Section and in Section 3.2 we propose two algorithms, for sum-power constrained networks (e.g. downlink) and networks with per-link power constraints (e.g. uplink), respectively, which solve the problem (2.18). Our designs make use of some elements of the power-unconstrained optimization algorithm proposed in [22]. In contrast to [22], we incorporate (different kinds of) power constraints into the design.

3.1.1 Some preliminaries

We assume here that the norm-constraint in the definition (2.17) is the constraint on the 2-norm, that is, all weight vectors have unit 2-norm. Taking (2.4) into account, we can rewrite the power-SIR dependence (2.3) equivalently as $\mathbf{\Gamma V p}+\mathbf{\Gamma}\boldsymbol{\sigma}^2 = \boldsymbol{p}$. The sum-power constraint can then be expressed by

$$\mathbf{1}'\boldsymbol{p} = \mathbf{1}'\mathbf{\Gamma}\boldsymbol{V}\boldsymbol{p} + \mathbf{1}'\mathbf{\Gamma}\boldsymbol{\sigma}^2 \leq P. \tag{3.1}$$

It is clear that (3.1) is satisfied with equality for any optimizer of the problem (2.18) (that is, the sum-power constraint is tight at the optimum) [34]. We are free to scale the parameters $\boldsymbol{V}$ and $\boldsymbol{\sigma}^2$ in order to arrive at an equivalent problem form with $P = 1$. Both equations (2.3) and (3.1) can then be written in joint matrix form

$$\boldsymbol{X}(\mathbf{\Gamma})\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix}, \tag{3.2}$$

with matrix-valued function $\mathbf{\Gamma} \mapsto \boldsymbol{X}(\mathbf{\Gamma})$ defined as

$$\boldsymbol{X}(\mathbf{\Gamma}) = \begin{pmatrix} \mathbf{\Gamma}\boldsymbol{V} & \mathbf{\Gamma}\boldsymbol{\sigma}^2 \\ \mathbf{1}'\mathbf{\Gamma}\boldsymbol{V} & \mathbf{1}'\mathbf{\Gamma}\boldsymbol{\sigma}^2 \end{pmatrix}, \quad \mathbf{\Gamma} = diag(\gamma_1, \ldots, \gamma_K) \geq 0. \tag{3.3}$$

We have a simple observation.

Lemma 2 *A pair $(\boldsymbol{p}, \mathbf{\Gamma})$, with $\rho(\mathbf{\Gamma}\boldsymbol{V}) < 1$, $\mathbf{\Gamma} = \Gamma(\boldsymbol{p})$, solves the equation (3.2) if and only if*

$$\rho(\boldsymbol{X}(\mathbf{\Gamma})) = 1. \tag{3.4}$$

Proof Equation (3.2) is an eigenvalue equation for $\boldsymbol{X}(\mathbf{\Gamma})$. Condition $\rho(\mathbf{\Gamma}\boldsymbol{V}) < 1$ implies, by Proposition 1 and (2.5), that $\boldsymbol{p} \geq 0$ and thus, by (2.1), that $\boldsymbol{X}(\mathbf{\Gamma})$ is nonnegative. Let first (3.2) and assume, by contradiction, that one of the sides of (3.2) is scaled by some $c \neq 1$, which means that $\rho(\boldsymbol{X}(\mathbf{\Gamma})) \neq 1$. Then, (2.3) is violated, which contradicts the assumption $\mathbf{\Gamma} = \Gamma(\boldsymbol{p})$ and thus, implies (3.4).

Conversely, note that $\rho(\boldsymbol{X}(\mathbf{\Gamma})) = 1$ implies $\boldsymbol{X}(\mathbf{\Gamma})\boldsymbol{r} = \boldsymbol{r}$, for some eigenvector $\boldsymbol{r} \in \mathbb{R}^K$. Assuming by contradiction $\boldsymbol{r} \neq (\boldsymbol{p}'\ 1)'$ implies the violation of (2.3) again and thus, contradicts $\mathbf{\Gamma} = \Gamma(\boldsymbol{p})$. Thus, we must have $\boldsymbol{r} = (\boldsymbol{p}'\ 1)'$ so that (3.2) follows and the proof is completed. □

By Lemma 2, the equations (3.2) and (3.4) are equivalent characterizations of the manifold of SIR matrices $\{\mathbf{\Gamma} = \Gamma(\boldsymbol{p}) : \mathbf{1}'\boldsymbol{p} = 1, \boldsymbol{p} \geq 0\}$, or equivalently $\{\mathbf{\Gamma} = \Gamma(\boldsymbol{p}) : \mathbf{1}'\boldsymbol{p} = 1, \rho(\mathbf{\Gamma}\boldsymbol{V}) < 1\}$. With the bijectivity of $\mathbf{\Psi}$ (equivalently, $\mathbf{\Phi}$) and monotonicity of Ψ (equivalently, Φ), such manifold determines uniquely the manifold

$$b(\mathcal{Q}_P) = \{\boldsymbol{q} = \mathbf{\Psi}(\mathbf{\Gamma}) : \mathbf{\Gamma} = \Gamma(\boldsymbol{p}), \mathbf{1}'\boldsymbol{p} = 1, \boldsymbol{p} \geq 0\},$$

which represents the part of the boundary of the performance region $\mathcal{Q}_P$ which includes any minimizer of the problem (2.19). Thus, with Lemma 2 we can reformulate the problem (2.19) under sum-power constraint in one of the forms

$$\min_{\boldsymbol{q}\in b(\mathcal{Q}_P)} \boldsymbol{\alpha}'\boldsymbol{q} = \min_{\boldsymbol{q}\in\{\boldsymbol{q}=\mathbf{\Psi}(\mathbf{\Gamma})\,:\,\rho(\boldsymbol{X}(\mathbf{\Gamma})=1\}} \boldsymbol{\alpha}'\boldsymbol{q} = \min_{\boldsymbol{q}\in\{\boldsymbol{q}\,:\,\rho(\boldsymbol{X}(\mathbf{\Phi}(\boldsymbol{q}))=1\}} \boldsymbol{\alpha}'\boldsymbol{q}. \tag{3.5}$$

None of the problems in (3.5) is convex, but under log-convexity of Φ their local optimizers are all global due to the equivalence to (2.19) (Proposition 3). The last two problem forms in (3.5) are especially convenient, since they represent equality-constrained problems with a single scalar constraint. The corresponding Kuhn-Tucker conditions are necessary and sufficient optimality conditions under log-convexity of Φ (Proposition 3 and trivially satisfied constraint qualification) and can be easy shown to take the form [48]

$$\boldsymbol{\alpha} + \nu \nabla \rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) = 0, \quad \nu \geq 0. \tag{3.6}$$

The nonnegativity of ν follows from the fact that the equality constraint in (3.5) can be replaced by the inequality constraint $\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) \leq 1$, for which the associated Lagrange multiplier becomes nonnegative [48] (in fact, the optimizer of (3.5) can not satisfy $\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) < 1$, since then, by Lemma 2, the sum-power constraint is not tight). Condition (3.6) is equivalent to parallelism of vectors $\boldsymbol{\alpha}$ and $-\nabla(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$.

3.1.2 The algorithm

We first state the pseudo-code of the algorithm and then provide its analysis. It is assumed that some accuracy parameter $\epsilon > 0$, some step-size $s > 0$ and some start power allocation $\boldsymbol{p}(0) \in \mathcal{P}_P$ satisfying (3.1) with equality and $P = 1$ are given. By Section 3.1.1, the latter assumption implies $\boldsymbol{q}(0) = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}(0))) \in b(\mathcal{Q}_P)$.

Algorithm 1

1: ***while*** $|-\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))| \geq (1-\epsilon)\|\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))\|_2$ ***do***
2: $\boldsymbol{q}^*(n) = \boldsymbol{q}(n) + s(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n)))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n)))))\boldsymbol{\alpha})$
3: $\boldsymbol{\Gamma}^*(n) = \boldsymbol{\Phi}(\boldsymbol{q}^*(n))$
4: $\boldsymbol{p}^*(n) = (\boldsymbol{I} - \boldsymbol{\Gamma}^*(n)\boldsymbol{V})^{-1}\boldsymbol{\Gamma}^*(n)\boldsymbol{\sigma}^2$
5: $\boldsymbol{p}(n) = \frac{1}{\|\boldsymbol{p}^*(n)\|_1}\boldsymbol{p}^*(n)$
6: $n \mapsto n+1$
7: $\boldsymbol{\Gamma}(n) = \Gamma(\boldsymbol{p}(n-1))$
8: $\boldsymbol{q}(n) = \boldsymbol{\Psi}(\boldsymbol{\Gamma}(n))$
9: ***end while***

The termination condition requires precisely that the vectors $\boldsymbol{\alpha}$ and $\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))$ are at most by ϵ away from the parallelism, when parallelism is measured by the inner product value (recall the assumption $\|\boldsymbol{\alpha}\|_2 = 1$). It is easy to see that Algorithm 1 is invariant with respect to the logarithmic domain transformation, which can make the problem of aggregated performance optimization convex (Proposition 6). Under logarithmic transformation, steps 4 and 5 have to be replaced by the trivially equivalent step sequence

4': $\boldsymbol{x}^*(n) = \log(\boldsymbol{I} - \boldsymbol{\Gamma}^*(n)\boldsymbol{V})^{-1}\boldsymbol{\Gamma}^*(n)\boldsymbol{\sigma}^2$
5': $\boldsymbol{x}(n) = \boldsymbol{x}^*(n) - \log\|e^{\boldsymbol{x}^*(n)}\|_1, \quad \boldsymbol{p}(n) = e^{\boldsymbol{x}(n)}$.

In steps 1 and 2 of Algorithm 1 the computation of the gradient of the spectral radius is needed. Due to the diagonality of the SIR matrix $\boldsymbol{\Phi}(\boldsymbol{q})$, we can write for the gradient components

$$\frac{\partial}{\partial q_k}\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) = \Phi'(q_k)\frac{\partial}{\partial \gamma_k}\rho(\boldsymbol{X}(\boldsymbol{\Gamma})) = \Phi'(q_k)\frac{\partial}{\partial \gamma_k}\boldsymbol{r}'\boldsymbol{X}(\boldsymbol{\Gamma})\boldsymbol{r}, \quad 1 \leq k \leq K, \tag{3.7}$$

where $\boldsymbol{r}$ is a right Perron-Frobenius eigenvector of $\boldsymbol{X}(\boldsymbol{\Gamma})$. However, by Lemma 2 we know that $\boldsymbol{r} = (\boldsymbol{p}'\ 1)'$. Thus, after a simple calculation we yield

$$\frac{\partial}{\partial q_k}\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) = \Phi'(q_k)(p_k+1)((\boldsymbol{V}\boldsymbol{p})_k + \sigma_k^2), \quad 1 \leq k \leq K. \tag{3.8}$$

3.1.3 Analysis and convergence

Algorithm 1 generates a sequence of power vectors, such that the corresponding sequence of performance vectors terminates at a vector $\boldsymbol{q}(N)$, for which the optimality condition of parallelism of $\boldsymbol{\alpha}$ and $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(N))))$ is satisfied up to some accuracy ϵ. In the steps 7 and 8 the value of the mapping (2.10) for the given power vector iterate is computed. Step 2 is the actual update step, in which the obtained performance vector iterate $\boldsymbol{q}(n)$ is added the update term

$$\boldsymbol{\Delta}(n) = \nabla\rho(\boldsymbol{X}(n)) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(n)))\boldsymbol{\alpha} \tag{3.9}$$

scaled by the step-size. It can be easy seen, that due to assumption $\|\boldsymbol{\alpha}\|_2 = 1$ we have $\Delta'(n)\boldsymbol{\alpha} = 0$, so that the term $-(\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n)))))\boldsymbol{\alpha}$ corresponds to the component of the negative gradient $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))$ which is parallel to $\boldsymbol{\alpha}$. Consequently, the negative update term $-\Delta(n)$ corresponds to the component of $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))$ which is orthogonal to the weight vector $\boldsymbol{\alpha}$. Hence, the update step 2 provides the updated iterate $\boldsymbol{q}^*(n)$ from a shift orthogonal to the direction $\boldsymbol{\alpha}$.

It is important to notice that the updated iterate $\boldsymbol{q}^*(n)$ in general does not pertain to the manifold $b(\mathcal{Q}_P)$ or, equivalently, to the manifold $\{\boldsymbol{q} : \rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})) = 1\}$. Clearly, this is further equivalent to the feature, that for the power vector $\boldsymbol{p}^*(n)$ such that $\boldsymbol{q}^*(n) = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}^*(n)))$ we do not have, in general, tight sum-power constraint $\|\boldsymbol{p}^*(n)\|_1 = 1$.

Steps 3 and 4 are complementary to the steps 7 and 8. They compute the value of the mapping (2.11) for the updated performance vector iterate $\boldsymbol{q}^*(n)$. Finally, step 5 leads the power vector iterate back to the manifold with tight sum-power constraint by a simple rescaling. This implies that the performance vector iterate $\boldsymbol{q}(n)$ obtained in step 8 pertains, in contrast to the updated iterate $\boldsymbol{q}^*(n)$, again to the manifold $b(\mathcal{Q}_P)$.

In abstract terms, we can interpret the effect of the algorithm as a walk along the boundary part $b(\mathcal{Q}_P)$ of the performance region $\mathcal{Q}_P$ towards the point of parallelism of $\boldsymbol{\alpha}$ and $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(n))))$. The visualization of Algorithm 1 for an exemplary two-link case is provided in Fig. 3.1.

For the convergence proof we need first the following simple Lemma.

Lemma 3 *Let $\boldsymbol{q} \in b(\mathcal{Q}_P)$, such that $\boldsymbol{\alpha}'\boldsymbol{q} > \min_{\tilde{\boldsymbol{q}}\in\mathcal{Q}_P} \boldsymbol{\alpha}'\tilde{\boldsymbol{q}}$. If Φ is log-convex, then for some $s_0 > 0$ we have*

$$\boldsymbol{q} + s(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha}) \in \mathcal{Q}_P \setminus b(\mathcal{Q}_P), \quad 0 < s \leq s_0. \tag{3.10}$$

Proof By the definition of the QoS region, the manifold $b(\mathcal{Q}_P)$ follows to be a part of the boundary of $\mathcal{Q}_P$, where $\mathcal{Q}_P$ is convex due to log-convexity of Φ (Proposition 1). Since we can write $b(\mathcal{Q}_P) = \{\boldsymbol{q} \in \mathbb{R}^K : \rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})) = 1\}$, the negative gradient $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$ is orthogonal to the tangent hyperplane (supporting hyperplane) of the set $\mathcal{Q}_P$ at the vector $\boldsymbol{q} \in \mathcal{Q}_P$ [50] (equivalently, $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$ is a normal vector of such hyperplane, see visualization in Fig. 2.1). This further means that $\boldsymbol{q}$ minimizes the value of such tangent hyperplane function on $\mathcal{Q}_P$, that is, $\boldsymbol{q} = \arg\min_{\tilde{\boldsymbol{q}}\in\mathcal{Q}_P} h_{\boldsymbol{q}}(\tilde{\boldsymbol{q}})$, with

$$\tilde{\boldsymbol{q}} \mapsto h_{\boldsymbol{q}}(\tilde{\boldsymbol{q}}) = \sum_{k=1}^{K} \frac{\partial}{\partial q_k}\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))\tilde{q}_k, \quad \tilde{\boldsymbol{q}} \in \mathbb{R}^K.$$

Clearly, it follows that $\nabla h_{\boldsymbol{q}}(\tilde{\boldsymbol{q}}) = \nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$, $\tilde{\boldsymbol{q}}, \boldsymbol{q} \in \mathcal{Q}_P$ (note that the first gradient is with respect to $\tilde{\boldsymbol{q}}$ and the latter with respect to $\boldsymbol{q}$). Thus, for particular setting $\boldsymbol{q} = \tilde{\boldsymbol{q}}$ we can write

$$\begin{aligned}
&\nabla h_{\boldsymbol{q}}(\boldsymbol{q})'(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha}) = \\
&\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))'(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha}) = \\
&\|\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))\|_2^2 - (\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))'\boldsymbol{\alpha})^2 \geq \\
&\|\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))\|_2^2 - \|\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))\|_2^2\|\boldsymbol{\alpha}\|_2^2 = 0, \quad \boldsymbol{q} \in \mathcal{Q}_P,
\end{aligned} \tag{3.11}$$

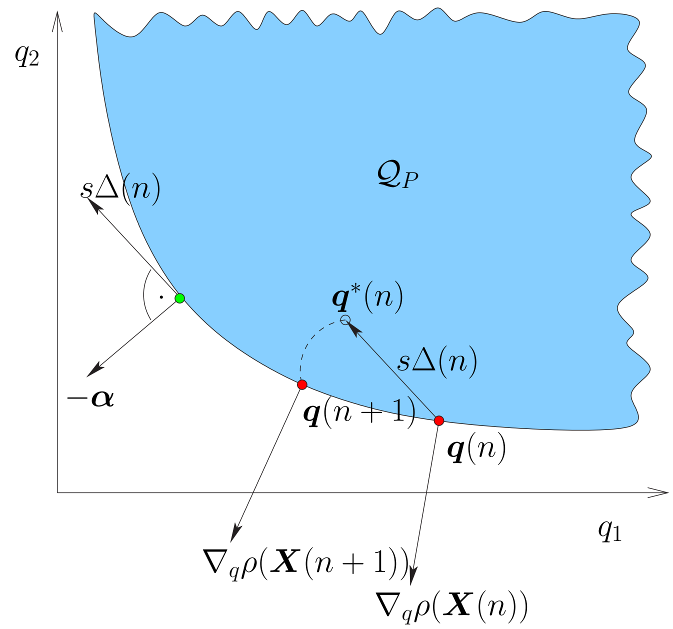

FIGURE 3.1: Visualization of the n-th iteration of Algorithm 1 for two links.

due to $\|\boldsymbol{\alpha}\|_2 = 1$. Inequality (3.11) implies that vector $\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha}$ pertains to the *cone of feasible directions*, in the sense that for some $s_0 > 0$ we have

$$\boldsymbol{q} + s(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha}) \in \mathcal{Q}_P, \quad 0 < s \leq s_0$$

(see e.g. Remark 5.1.6 in [50]). Furthermore, one can easily see that inequality (3.11) is strict whenever $\boldsymbol{q} \in \mathcal{Q}_P$ is such that $\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) \neq 0$ and the vectors $\boldsymbol{\alpha}$ and $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$ are not parallel. The first condition is satisfied for any $\boldsymbol{q} \in \mathcal{Q}_P$ due to (3.8) and decreasingness of Φ. The latter condition corresponds to the negation of condition (3.6) which is, under log-convex Φ and satisifed constraint qualification, a necessaray and sufficient optimality condition for problem (2.19). Thus, it follows that inequality (3.11) is strict if and only if $\boldsymbol{q} \in \mathcal{Q}_P$ satisfies

$$\boldsymbol{\alpha}'\boldsymbol{q} > \min_{\tilde{\boldsymbol{q}} \in \mathcal{Q}_P} \sum_{k=1}^{K} \alpha_k \tilde{q}_k.$$

This implies further that $\boldsymbol{q} + s(\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) - (\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))))\boldsymbol{\alpha})$, with $0 < s \leq s_0$, pertains to the relative interior of the cone of feasible directions and, consequently, does not pertain to the boundary of $\mathcal{Q}_P$ [50]. Thus, (3.10) is implied and the proof is completed. □

The Lemma implies, that for some step-size interval, the iterate $\boldsymbol{q}^*(n)$ obtained from the shift in step 2 of Algorithm 1 does not lie on the boundary part $b(\mathcal{Q}_P)$ of the performance region $\mathcal{Q}_P$, but lies within $\mathcal{Q}_P$, so that $\|\boldsymbol{p}^*(n)\|_1 < 1$. From the proof of Lemma 3 one can easily see that the result of the Lemma holds under the condition of log-convexity of Φ (ensuring convexity of $\mathcal{Q}_P$) loosened to merely local convexity of $\mathcal{Q}_P$ at/ around the performace vector $\boldsymbol{q}$ [50]. With Lemma 3, Algorithm 1 can be shown to be monotonically convergent in the following Proposition.

Proposition 7 *Under log-convex* Φ *and any* $0 < s \leq s_0$*, with some* $s_0 > 0$*, the iterate sequences* $\boldsymbol{q}(n), \boldsymbol{p}(n)$*,* $n \in \mathbb{N}$*, generated by Algorithm 1 converge monotonically to the global minimizers of the* *problems (2.19), (2.18), respectively, in the sense that for any* $\epsilon > 0$ *we have*

$$|-\boldsymbol{\alpha}'\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(N))))| < (1-\epsilon)\|\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}(N))))\|_2$$

for some $N = N(\epsilon)$, $N \in \mathbb{N}$, *and*

$$\boldsymbol{\alpha}'\boldsymbol{q}(n+1) - \boldsymbol{\alpha}'\boldsymbol{q}(n) < 0, \quad n \leq N, \quad n \in \mathbb{N}. \tag{3.12}$$

Proof The convergence of the algorithm is not influenced by the steps 3,4,7 and 8, since in these steps the value of the iterate is only bijectively transformed according to (2.11) and (2.10). By (3.9), the update step 2 can be written componentwise, for any $n \in \mathbb{N}$, as

$$\Psi(\gamma_k(\boldsymbol{p}^*(n))) = \Psi(\gamma_k(\boldsymbol{p}(n-1))) + s\Delta_k(n), \quad 1 \leq k \leq K. \tag{3.13}$$

For the left-hand side of (3.13) we can further write with (2.1)

$$\begin{aligned}\Psi(\gamma_k(\boldsymbol{p}^*(n))) &= \Psi\Big(\frac{p_k^*(n)}{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \sigma_k^2}\Big) = \Psi\Big(\frac{p_k^*(n)}{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \sigma_k^2} \cdot \frac{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \|\boldsymbol{p}^*(n)\|_1\sigma_k^2}{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \|\boldsymbol{p}^*(n)\|_1\sigma_k^2}\Big)\\ &= \Psi\Big(\gamma_k(\boldsymbol{p}(n))\frac{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \|\boldsymbol{p}^*(n)\|_1\sigma_k^2}{\sum_{\substack{j=1\\j\neq k}}^K V_{kj}p_j^*(n) + \sigma_k^2}\Big),\end{aligned} \tag{3.14}$$

where $\boldsymbol{p}(n)$ represents the power vector obtained from $\boldsymbol{p}^*(n)$ in step 5. By (3.13), we yield after weighted summation of both sides of (3.14) that

$$\sum_{k=1}^K \alpha_k\Psi(\gamma_k(\boldsymbol{p}(n))\delta_k(n)) = \sum_{k=1}^K \alpha_k\Psi(\gamma_k(\boldsymbol{p}(n-1))) + s\sum_{k=1}^K \alpha_k\Delta_k(n), \quad 1 \leq k \leq K, \tag{3.15}$$

where we defined

$$\delta_k(n) = \frac{\sum_{j=1,j\neq k}^K V_{kj}p_j^*(n) + \|\boldsymbol{p}^*(n)\|_1\sigma_k^2}{\sum_{j=1,j\neq k}^K V_{kj}p_j^*(n) + \sigma_k^2}, \quad 1 \leq k \leq K.$$

By the construction of Algorithm 1 (precisely, by the construction of step 2), we have for the last sum in (3.15) that $\sum_{k=1}^K \alpha_k\Delta_k(n) = \boldsymbol{\alpha}'\Delta(n) = 0$. By (2.11) and Lemma 3 follows further for steps 2,3,4 of Algorithm 1 that $\boldsymbol{p}^*(n)$ is in the interior of $\mathcal{P}_P$ and thus, $\|\boldsymbol{p}^*(n)\|_1 < 1$ whenever the step-size s does not exceed a certain threshold $s_0 \geq 0$. Consequently, we have $\delta_k(n) < 1$, $1 \leq k \leq K$, whenever $0 < s \leq s_0$.

Thus, by (3.15) and decreasingness of Ψ we yield

$$\sum_{k=1}^K \alpha_k\Psi(\gamma_k(\boldsymbol{p}(n))) - \sum_{k=1}^K \Psi(\gamma_k(\boldsymbol{p}(n-1))) < \sum_{k=1}^K \alpha_k\Psi(\gamma_k(\boldsymbol{p}(n))\delta_k) - \sum_{k=1}^K \Psi(\gamma_k(\boldsymbol{p}(n-1))) = 0, \tag{3.16}$$

whenever $0 < s \leq s_0$, which proves (3.12). Since under log-convexity of Φ any minimizer of (2.19) and (2.18) is global (Propositions 1, 3), inequality (3.16) implies monotone convergence of the bijectively related iterate sequences $\boldsymbol{q}(n), \boldsymbol{p}(n)$, $n \in \mathbb{N}$, to the global minimizers of (2.19) and (2.18),

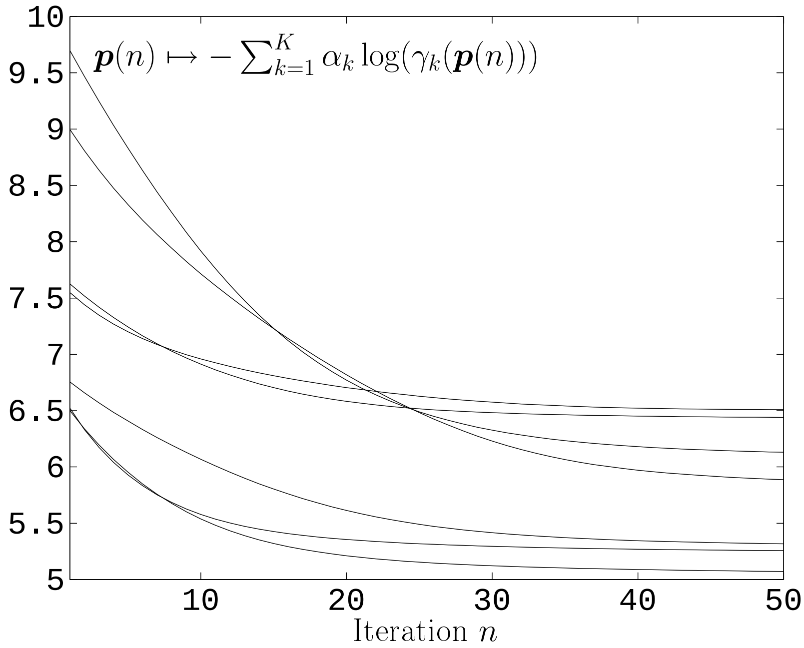

Figure 3.2: Exemplary convergence of the aggregated performance under the application of Algorithm 1. The size of the network is fixed to $K = 15$, the step-size s is fixed and the used QoS function is $\Psi(\gamma) = -\log(\gamma)$ (e.g., high-SIR link capacity approximation). The parameters $\boldsymbol{p}(0), \boldsymbol{V}, \boldsymbol{\sigma}^2, \boldsymbol{a}$ are chosen randomly from uniform distribution, for which we set $E[V_{kj}]/E[\sigma_k^2] = 10$ and $P/E[V_{kj}] = 1$, $1 \leq k, j \leq K$.

respectively. That is, given $\epsilon = 0$, the convergence is to vectors $\boldsymbol{q}, \boldsymbol{p}$ such that $\boldsymbol{q} = \boldsymbol{\Psi}(\boldsymbol{p})$ and such that $\boldsymbol{\alpha}$ is parallel to $-\nabla\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))$. Thus, due to continuity of the functionals on hand, for any $\epsilon > 0$ the termination condition in step 1 is satisfied at some finite $N = N(\epsilon)$, $N \in \mathbb{N}$, and the proof is completed. $\square$

Since Lemma 3 was mentioned to hold under loosened conditions, the same is true in the case of the resulting Proposition 7. Precisely, Algorithm 1 exhibits local convergence to some performance vector (and the associated power vector) satisfying the Kuhn-Tucker conditions of problem (2.19), if the performance region is locally convex at/ around this vector.

Exemplary simulative convergence results of Algorithm 1 under variation of network parameters $\boldsymbol{V}$ and $\boldsymbol{\sigma}^2$ are presented in Fig. 3.2 and Fig. 3.3 for different network sizes and different performance parameters. The results show that under no step-size adaptation, in moderate-size networks ($10-15$ links) a nearly-optimum is achieved after $20 - 30$ iterations. This seems to show an attractive convergence rate of the algorithm, which can be further improved by step-size adaptation. The curves seem also to imply good robustness of the algorithm with respect to parameter variations.

3.2 Algorithmic solution under per-link power constraints

This Section parallels Section 3.1, in that it contains a construction and analysis of an algorithm solving problem (2.18) under constraints on single link powers.

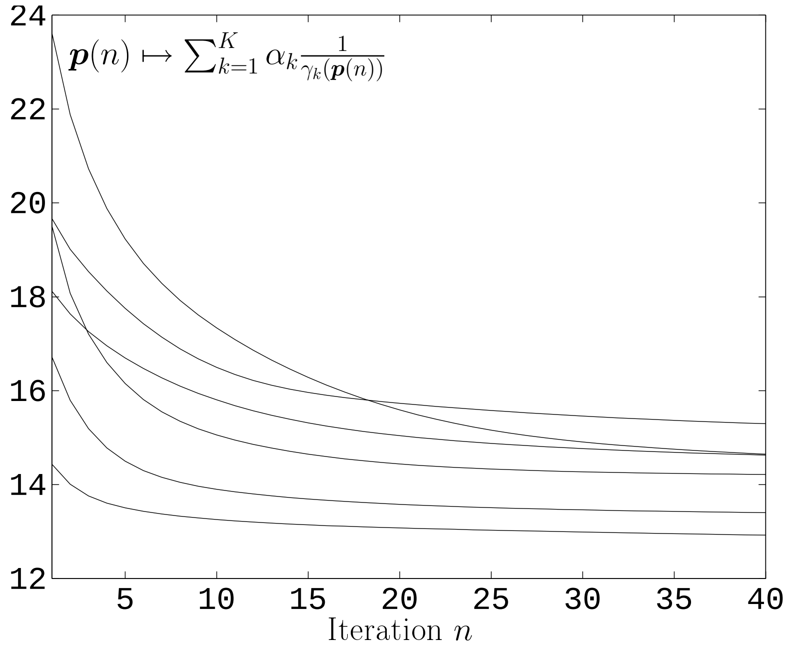

Figure 3.3: Exemplary convergence of the aggregated performance under the application of Algorithm 1. The size of the network is fixed to $K = 10$, the step-size s is fixed and the used QoS function is $\Psi(\gamma) = 1/\gamma$ (e.g., normalized channel-averaged symbol error rate under Rayleigh fading). The parameters $\boldsymbol{p}(0), \boldsymbol{V}, \boldsymbol{\sigma}^2, \boldsymbol{a}$ are chosen randomly from uniform distribution, for which we set $E[V_{kj}]/E[\sigma_k^2] = 10$ and $P/E[V_{kj}] = 1$, $1 \leq k, j \leq K$.

3.2.1 Some preliminaries

In the algorithm for networks with per-link power constraints, as e.g. in the uplink, we can not utilize the concept of the spectral radius from Section 3.1.1. However, we show that the efficient algorithm mechanism of the walk on the boundary of the performance region can be retained under power constraints per link.

The performance region $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ and the power region $\mathcal{P}_{\hat{\boldsymbol{p}}}$ are bijectively related by (2.10) and (2.11). Taking account for (2.10) and (2.11), we can write problem (2.19) in either of the forms

$$\min_{\boldsymbol{q} \in \{\boldsymbol{q} = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}))\,:\, 0 \leq \boldsymbol{p} \leq \hat{\boldsymbol{p}}\}} \boldsymbol{\alpha}'\boldsymbol{q} = \min_{\boldsymbol{q} \in \{\boldsymbol{q}\,:\, 0 \leq (\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2 \leq \hat{\boldsymbol{p}}\}} \boldsymbol{\alpha}'\boldsymbol{q}, \tag{3.17}$$

where we focus on the latter form. The corresponding Kuhn-Tucker conditions can be easily shown *not* to take the form of a parallelism condition. This is in contrast to the Kuhn-Tucker conditions of the problem formulation under sum-power constraint (3.5). Let us define an *extended power region*

$$\mathcal{P}'_{\hat{\boldsymbol{p}}} = \{\boldsymbol{p} : -\prod_{k=1}^{K} (\hat{p}_k - p_k) \leq 0\},$$

for which we have obviously $\mathcal{P}_{\hat{\boldsymbol{p}}} \subset \mathcal{P}'_{\hat{\boldsymbol{p}}}$, so that

$$\inf_{\boldsymbol{q} \in \{\boldsymbol{q}\,:\, -\prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k) \leq 0\}} \boldsymbol{\alpha}'\boldsymbol{q} \leq \min_{\boldsymbol{q} \in \{\boldsymbol{q}\,:\, 0 \leq (\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2 \leq \hat{\boldsymbol{p}}\}} \boldsymbol{\alpha}'\boldsymbol{q}. \tag{3.18}$$

Thereby, the question if the infimum on the left-hand side is achieved is not trivial, since the extended power region is an unbounded set. An exemplary comparison of $\mathcal{P}_{\hat{\boldsymbol{p}}}$ and $\mathcal{P}'_{\hat{\boldsymbol{p}}}$ for two links is depicted in Fig. 3.4.

It can be now easily deduced that the Kuhn-Tucker conditions of the problem (3.17) are equivalent to the Kuhn-Tucker conditions of the problem on the left-hand side in (3.18) subject to the additional restriction $0 \leq (\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2 \leq \hat{\boldsymbol{p}}$. This means equivalently, that the stationary points of the Lagrangian of problem (3.17) are precisely the stationary points of the Lagrangian of the problem on the left-hand side in (3.18) which satisfy the additional restriction. For the case of a log-convex QoS-SIR dependence, we have by Proposition 3, and by the satisfied constraint qualification for problem (3.17) [48], an immediate particular Lemma.

Lemma 4 *Under log-convex function* Φ, *the conditions*

$$\begin{aligned} -(\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2 &\leq 0 \\ (\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2 - \hat{\boldsymbol{p}} &\leq 0 \\ \boldsymbol{\alpha} - \nu\nabla\prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k) &= 0, \quad \nu \geq 0. \end{aligned} \tag{3.19}$$

are necessary and sufficient optimality conditions for problem (3.17).

Thus, by Lemma 4, we yield the formulation of optimality conditions of problem (3.17) as a restricted parallelism condition of vectors $\boldsymbol{\alpha}$ and $\nabla\prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)$, which is an analogy to the Kuhn-Tucker conditions (3.6) in the case of sum-power constraint. The equality in (3.19), i.e. the actual parallelism, is the consequence of the feature that for any minimizer of (3.17) some constraint on link power is always tight (satisfied with equality).

If the requirement of log-convexity of Φ in Lemma 4 is dropped, then, clearly, conditions (3.19) remain necessary optimality conditions for problem (3.17).

3.2.2 The algorithm

By the results of Section 3.2.1, it is suggested that an iteration principle similar to Algorithm 1 can be applied also to find $\boldsymbol{p} \in \mathcal{P}_{\hat{\boldsymbol{p}}}$ and $\boldsymbol{q} \in \mathcal{Q}_{\hat{\boldsymbol{p}}}$ which solve problems (2.18), (2.19), respectively, under per-link power constraints. Assuming accuracy parameter $\epsilon > 0$, step-size $s > 0$ and some start power allocation $\boldsymbol{p}(0) \in \mathcal{P}_{\hat{\boldsymbol{p}}}$ such that $\prod_{k=1}^{K}(\hat{p}_k - p_k(0)) = 0$, the resulting algorithm is as follows.

Algorithm 2

1: ***while*** $|\boldsymbol{\alpha}'\nabla\prod_{k\in\mathcal{K}}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)| >$
$(1-\epsilon)\|\nabla\prod_{k\in\mathcal{K}}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)\|_2$ ***do***
2: $\boldsymbol{q}^*(n) = \boldsymbol{q}(n) + s((\boldsymbol{\alpha}'\nabla\prod_{k\in\mathcal{K}}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k))\boldsymbol{\alpha} -$
$\nabla\prod_{k\in\mathcal{K}}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k))$
3: $\boldsymbol{\Gamma}^*(n) = \boldsymbol{\Phi}(\boldsymbol{q}^*(n))$
4: $\boldsymbol{p}^*(n) = (\boldsymbol{I} - \boldsymbol{\Gamma}^*(n)\boldsymbol{V})^{-1}\boldsymbol{\Gamma}^*(n)\boldsymbol{\sigma}^2$
5: $\boldsymbol{p}(n) = \min_{k\in\mathcal{K}} \frac{\hat{p}_k}{p_k^*(n)}\boldsymbol{p}^*(n)$
6: $n \mapsto n+1$
7: $\boldsymbol{\Gamma}(n) = \Gamma(\boldsymbol{p}(n-1))$
8: $\boldsymbol{q}(n) = \boldsymbol{\Psi}(\boldsymbol{\Gamma}(n))$
9: ***end while***

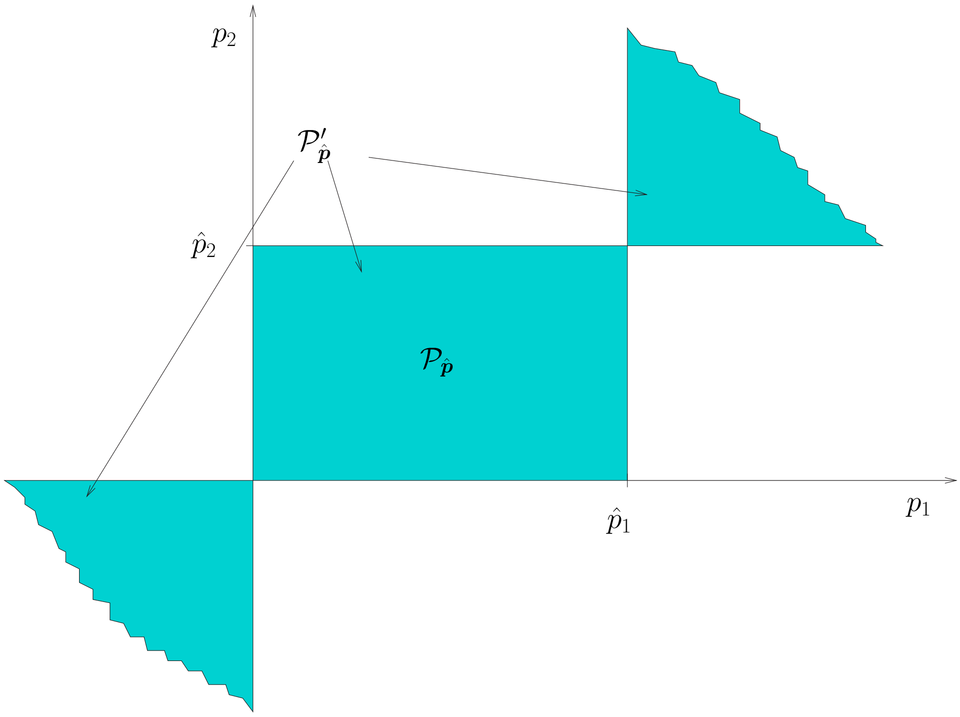

FIGURE 3.4: Comparison of the extended power region $\mathcal{P}'_{\hat{\boldsymbol{p}}}$ with the power region $\mathcal{P}_{\hat{\boldsymbol{p}}}$ in an exemplary two-link case.

Similarly to Algorithm 1, Algorithm 2 is invariant with respect to the logarithmic transformation of the domain, which is necessary to obtain a convex problem form (Proposition 6). Algorithm 2 terminates when vectors $\boldsymbol{\alpha}$ and $\nabla \prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)$ are less than ϵ away from parallelism, when parallelism is measured by the inner product value. From steps 7 and 8 follows that the performance vector iterate $\boldsymbol{q}(n)$ is the value of the mapping (2.10) for the argument $\boldsymbol{p}(n-1)$. Hence, in the termination condition and in step 2 we have the equivalence

$$\nabla \prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k) = \nabla \prod_{k=1}^{K}(\hat{p}_k - p_k(n-1)), \quad n \in \mathbb{N}.$$

In order to compute the gradient components needed in the algorithm, we first use the property $\frac{\partial}{\partial x_k}\boldsymbol{A}^{-1}(\boldsymbol{x}) = -\boldsymbol{A}^{-1}(\boldsymbol{x})(\frac{\partial}{\partial x_k}\boldsymbol{A}(\boldsymbol{x}))\boldsymbol{A}^{-1}(\boldsymbol{x})$ to get after a simple derivation [63]

$$\begin{aligned}&\frac{\partial}{\partial q_j}((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k = \\ &-\boldsymbol{e}'_k(\boldsymbol{\Phi}(\boldsymbol{q})^{-1} - \boldsymbol{V})^{-1}(\frac{\partial}{\partial q_j}\boldsymbol{\Phi}(\boldsymbol{q})^{-1})(\boldsymbol{\Phi}(\boldsymbol{q})^{-1} - \boldsymbol{V})^{-1}\boldsymbol{\sigma}^2, \quad 1 \leq k, j \leq K.\end{aligned}$$

Using the chain rule for product derivation this gives further

$$\begin{aligned}&\frac{\partial}{\partial q_j}\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)=\\&\sum_{k=1}^{K}\boldsymbol{e}_k'(\boldsymbol{\Phi}(\boldsymbol{q})^{-1}-\boldsymbol{V})^{-1}(\frac{\partial}{\partial q_j}\boldsymbol{\Phi}(\boldsymbol{q})^{-1})(\boldsymbol{\Phi}(\boldsymbol{q})^{-1}-\boldsymbol{V})^{-1}\boldsymbol{\sigma}^2\times\\&\prod_{\substack{l=1\\l\neq k}}^{K}(\hat{p}_l-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_l),\quad 1\leq k,j\leq K.\end{aligned}\tag{3.20}$$

Moreover, from $\boldsymbol{\Phi}(\boldsymbol{q})^{-1}=\boldsymbol{\Gamma}^{-1}=diag(\frac{1}{\gamma_1},\ldots,\frac{1}{\gamma_K})$ follows that the only one nonzero element of $\frac{\partial}{\partial q_j}\boldsymbol{\Phi}(\boldsymbol{q})^{-1}$ is its jj-th element, which is $\frac{\partial}{\partial q_j}\frac{1}{\Phi(q_j)}=-\frac{1}{\gamma_j^2}\Phi'(q_j)$.

3.2.3 Analysis and convergence

Algorithm 2 generates a sequence of power vectors and the corresponding sequence of performance vectors terminating at a vector $\boldsymbol{q}(N)$, for which $\boldsymbol{\alpha}$ and $\nabla_q\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{\sigma}^2)_k)$ are parallel up to some accuracy ϵ. In the steps 7 and 8 the performance vector is computed for the given power vector iterate. In the update step 2, the performance vector iterate $\boldsymbol{q}(n)$ is shifted by an update vector in the negative direction of the component of $\nabla\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)$ which is orthogonal to $\boldsymbol{\alpha}$. The updated iterate $\boldsymbol{q}^*(n)$ in general does not pertain to the boundary of the performance region $\mathcal{Q}_{\hat{\boldsymbol{p}}}$. This means that for the corresponding power vector $\boldsymbol{p}^*(n)$, obtained by steps 3 and 4, in general there may exist no component, say j, for which we have a tight constraint $p_j^*=\hat{p}_j$. This implies further that for $\boldsymbol{p}^*(n)$ in general we do not have $\prod_{k=1}^{K}(\hat{p}_k-p_k^*(n))=0$.

Thus, the power vector corresponding to performance vector $\boldsymbol{q}^*(n)$, obtained in steps 3 and 4, is lead back in step 5 to the manifold of power vectors satisfying some link power constraint with equality. Clearly, any possible power vector $\boldsymbol{p}(n)$ from such manifold satisfies $\prod_{k=1}^{K}(\hat{p}_k-p_k^*(n))=0$. Step 5 consists precisely in an orthogonal projection on the nearest boundary part of the power region $\mathcal{P}_{\hat{\boldsymbol{p}}}$.

For the convergence proof we need a Lemma analogous to Lemma 3.

Lemma 5 *Let* $\boldsymbol{q}\in\mathcal{Q}_{\hat{\boldsymbol{p}}}\cap\{\boldsymbol{q}\in\mathbb{R}^K:\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)=0\}$*, such that* $\boldsymbol{\alpha}$ *and* $\nabla\prod_{k=1}^{K}(\hat{p}_k-p_k(\boldsymbol{q}))$ *are not parallel, and such that* $\nabla\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)\neq 0$*. Then, if* Φ *is log-convex, then for some* $s_0>0$ *we have*

$$\begin{aligned}&\boldsymbol{q}+s((\boldsymbol{\alpha}'\nabla\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k))\boldsymbol{\alpha}-\nabla\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k))\in\\&\mathcal{Q}_{\hat{\boldsymbol{p}}}\setminus\{\boldsymbol{q}\in\mathbb{R}^K:\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)=0\},\quad 0<s\leq s_0.\end{aligned}\tag{3.21}$$

Proof The proof goes along exactly the same lines as the proof of Lemma 3, so that we do not duplicate it. Thereby, the role of the manifold $b(\mathcal{Q}_P)=\{\boldsymbol{q}\in\mathbb{R}^K:\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))=1\}$ is now played by the manifold $\{\boldsymbol{q}\in\mathbb{R}^K:\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)=0\}$.

(A difference between both proofs is that while the manifold $\{\boldsymbol{q}\in\mathbb{R}^K:\rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q})))=1\}$ was precisely a boundary part of $\mathcal{Q}_P$, the manifold $\{\boldsymbol{q}\in\mathbb{R}^K:\prod_{k=1}^{K}(\hat{p}_k-((\boldsymbol{I}-\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k)=$

$0\}$ includes a boundary part of $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ as a subset, but is not included in $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ itself. Such difference does not influence, however, the way of the proof, which is exactly the same.) □

Lemma 5 implies that for a sufficiently small step-size, the updated iterate $\boldsymbol{q}^*(n)$ from step 2 pertains to the performance region $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ excluding each boundary part, which corresponds to some tight constraint on link power. Equivalently, the power vector iterate $\boldsymbol{p}^*(n)$, associated with $\boldsymbol{q}^*(n)$, satisfies $0 \le \boldsymbol{p}^*(n) < \hat{\boldsymbol{p}}$. Analogously to Lemma 3, the result of Lemma 5 holds if the requirement of log-convexity of Φ (ensuring convexity of the performance region) is loosened to local convexity of $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ at/ around $\boldsymbol{q}$. With Lemma 5, the monotone convergence is now easily proven given some certain assumptions.

Proposition 8 *Let $\boldsymbol{q}(n), \boldsymbol{p}(n)$, $n \in \mathbb{N}$, be the iterate sequences generated by Algorithm 2 and let $\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k) \neq 0$, $n \in \mathbb{N}$. Then, under log-convex function Φ and any $0 < s \le s_0$, with some $s_0 > 0$, $\boldsymbol{q}(n), \boldsymbol{p}(n)$, $n \in \mathbb{N}$, converge monotonically to the global minimizers of the problems (2.19), (2.18), respectively, in the sense that (3.12) is satisfied and for any $\epsilon > 0$ we have*

$$|\boldsymbol{\alpha}'\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{\sigma}^2)_k)| \le$$
$$(1-\epsilon)\|\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(N))\boldsymbol{\sigma}^2)_k)\|_2$$

for some $N = N(\epsilon)$, $N \in \mathbb{N}$. Furthermore, we have also $0 \le \boldsymbol{p}(n) \le \hat{\boldsymbol{p}}$, $n \le N$, $n \in \mathbb{N}$.

Proof Except the last statement, the proof goes exactly along the same lines as the proof of Proposition 7, so that we do not duplicate it here. Hereby, the role of the scaling factor $\|\boldsymbol{p}^*(n)\|_1 < 1$ in now plyed by the factor $1/\min_{k\in\mathcal{K}} \hat{p}_k/p_k^*(n)$, for which we also have $1/\min_{k\in\mathcal{K}} \hat{p}_k/p_k^*(n) < 1$, by Lemma 5.

The additional feature $0 \le \boldsymbol{p}(n) \le \hat{\boldsymbol{p}}$, $n \le N$, is an immediate consequence of Lemma 5 and the orthogonal projection step 5. □

By Lemma 5 remaining true under loosened conditions, the convergence result from Proposition 8 holds under loosened conditions as well. Precisely, Algorithm 2 exhibits local convergence to some performance vector (and the associated power vector) satisfying the Kuhn-Tucker conditions of problem (2.19), if $\mathcal{Q}_{\hat{\boldsymbol{p}}}$ is locally convex at/ around this vector.

One can see that in comparison with the convergence of Algorithm 1, the convergence of Algorithm 2 needs an additional assumption of nonzero product gradient for the entire iterate sequence. While in the case of sum-power constraint we had $-\nabla_q \rho(\boldsymbol{X}(\boldsymbol{\Phi}(\boldsymbol{q}))) \neq 0$, $\boldsymbol{q} \in b(\mathcal{Q}_P)$, the intricacy of zero gradient $\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q})\boldsymbol{\sigma}^2)_k) = 0$ is not prevented under power constraints on single links. This is precisely the reason for the use of strict termination inequality in Algorithm 2. Due to strict termination inequality, Algorithm 2 does not terminate at $\boldsymbol{q}(n)$ if $\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k) = 0$, although the zero vector formally satisfies the parallelism condition with respect to $\boldsymbol{\alpha}$.

From (3.20) can be seen that $\nabla \prod_{k=1}^{K} (\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k) = 0$ is true whenever more than one power constraint is tight at the power vector iterate $\boldsymbol{p}(n)$ satisfying $\boldsymbol{q}(n) = \boldsymbol{\Psi}(\Gamma(\boldsymbol{p}(n)))$. Precisely, we have zero gradient when $\boldsymbol{p}(n)$ is a vertex of $\mathcal{P}_{\hat{\boldsymbol{p}}}$ [50]. In Algorithm 2, this occurs precisely when the power vector iterate $\boldsymbol{p}^*(n)$ corresponding to the updated iterate $\boldsymbol{q}^*(n)$ in step 2 has at least two components $1 \le j, l \le K$ satisfying $\hat{p}_j/p_j^*(n) = \hat{p}_l/p_l^*(n) = \min_{1\le k\le K} \hat{p}_k/p_k^*(n)$. To prevent such case one can add a sufficiently small correction/ perturbation term $\boldsymbol{\delta}(n) = \boldsymbol{\delta}(\boldsymbol{q}^*(n))$ to the updated iterate $\boldsymbol{q}^*(n)$ which ensures that all components of $\boldsymbol{p}^*(n)$ are different. When treating the correction term as a noise sample, Algorithm 2 can be seen as a stochastic approximation [64] (the theory of stochastic approxiations is addressed in some more detail in the context of decentralized algorithmic solutions in Chapter 4).

Under some nonrestrictive assumptions and using the techniques from [64], it is possible to prove the convergence with probability one of Algorithm 2 using the correction term. In practical applications, the occurrence of exact equality of components of the power vector iterate approaches zero, so that the correction term can be dropped. We however suggest the use of simple step-size adaptation in the uplink algorithm to compensate for too small values of $\|\nabla \prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)\|_2$.

The gradient norm becomes small whenever the power vector iterate happens to be near to some vertex of $\mathcal{P}_{\hat{\boldsymbol{p}}}$. In particular, such this may occur if the power vector of convergence of Algorithm 2 is itself a vertex of the power region. In such case we have small values of $\|\nabla \prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)\|_2$ for all performance vector iterates around the performance vector of convergence of Algorithm 2. As a consequence, the norm of the update vector $\Delta(n)$ in step 2 decreases with approaching the the vector of convergence not only due to increasing parallelism of the vectors $\boldsymbol{\alpha}$ and $\nabla \prod_{k=1}^{K}(\hat{p}_k - ((\boldsymbol{I} - \boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{V})^{-1}\boldsymbol{\Phi}(\boldsymbol{q}(n))\boldsymbol{\sigma}^2)_k)$, but also due to decreasing norm of the latter vector.

Fortunately, the incorporation of step-size adaptation (e.g. in the simplest form of adaptive upscaling of s) seems to prevent perceivable deterioration of convergence rate. This can be seen from convergence results in Fig. 3.5, for which a heuristic adaptive upscaling of step-size was utilized. The convergence rate and robustness properties of the uplink algorithm seem to be similar to those of Algorithm 1.

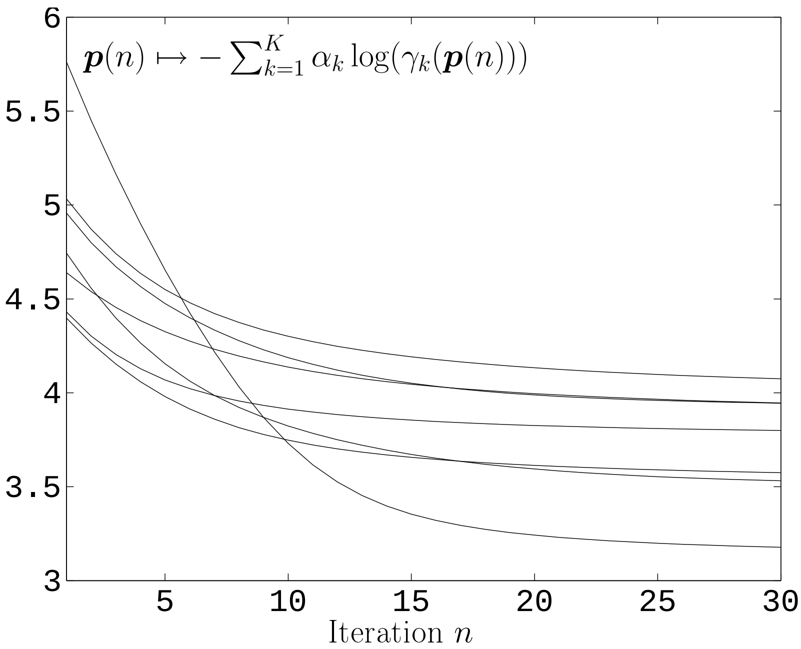

Figure 3.5: Exemplary convergence of the aggregated performance under the application of Algorithm 2. The size of the network is fixed to $K = 10$ and the QoS function is $\Psi(\gamma) = -\log(\gamma)$ (e.g., high-SIR link capacity approximation). The parameters $\boldsymbol{p}(0), \boldsymbol{V}, \boldsymbol{\sigma}^2, \boldsymbol{a}$ are chosen randomly from uniform distribution, for which we set $E[V_{kj}]/E[\sigma_k^2] = 10$ and $\hat{p}_k/E[V_{kj}] = 1/10$, $1 \leq k, j \leq K$. A simple heuristic step-size adaptation mechanism is applied.

4

Decentralized algorithmic optimization of weighted aggregated performance

In this chapter we focus on concepts of decentralized optimization of (weighted) aggregated performance (2.18). In terms of the wireless network model from Chapter 2, *we understand a solution of a network optimization problem as a decentralized, or distributed, one, if it consists of a (sequence of) decoupled actions conducted separately for each link.* Hereby, the local knowledge which is necessary to conduct the separate per-link actions is provided by means of separate signaling and feedback on each link and coarse synchronization among the links, which allows for the measurement of interference on each link.

A decentralized routine of problem solution in a wireless network consists usually of an abstract algorithm concept and a signaling and feedback scheme which are both adapted to each other in a way which results in decentralization of actions. First, in Section 4.1, we present the signaling and feedback concept relying on the idea of so-called *adjoint network*. It is shown that the concept allows for distributed conduction of the general gradient iteration [49]. In Section 4.2, we propose a concept of a specific generalized Lagrangean function and a related algorithm construction. In combination with the feedback scheme from Section 4.1, the algorithm allows for decentralized conduction with certain complexity and convergence advantages compared to the gradient iteration. Finally, in Section 4.3, we propose a specific splitting of variables and a related algorithm concept, which results (in combination with the feedback scheme from Section 4.1) in decentralized quadratic convergence.

The decentralized optimization of aggregated performance (2.18) appears to be of interest in association with separate constraints on link powers, which indicate a kind of physical decoupling of links. Otherwise, under sum-power constraint, the existence of some central control unit managing all the links is indicated, e.g. base station in the downlink, so that the application of decentralized routines is not of highest importance. For this reason, the problem of aggregated performance optimization (2.18) under per-link power constraints, that is $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$, is considered throughout this chapter.

The results of this chapter were presented originally in [65], [66], [67], [68], [69], [70]. All notions from Lagrangian optimization and the theory of numerical convergence used in this chapter are introduced and explained in Appendix A.3.

4.1 DECENTRALIZED FEEDBACK SCHEME AND DUALITY IN AGGREGATED PERFORMANCE OPTIMIZATION

We propose a signaling and feedback scheme which is a key ingredient of decentralized conduction of algorithms proposed later in Sections 4.2 and 4.3 (and of the general gradient iteration). A duality-like relation, similar to Lagrangian duality, induced by the key concept of the scheme is discussed as well.

4.1.1 THE CONCEPT OF ADJOINT NETWORK

The cross-talk matrix $\boldsymbol{V}$ characterizes the interference situation, or *interference topology*, of the network under the given geometric network topology at hand. Let us account explicitly for network interference topology by identifying a network having link set $\mathcal{K}$ and interference matrix $\boldsymbol{V}$ with a pair $(\mathcal{K}, \boldsymbol{V})$. We are interested in a specific network standing in the *adjointness* relationship with the original network.

Definition 1 *Two networks $(\mathcal{K}, \boldsymbol{V})$ and $(\mathcal{J}, \boldsymbol{U})$ are said to be adjoint (to each other) if $\boldsymbol{U} = \boldsymbol{V}'$.*

Notice that an adjoint network pair is not unique. This is due to the fact that in general $\mathcal{J} \neq \mathcal{K}$. In a pair of adjoint networks, the cross-talk factor describing the interference of j-th link at the receiver of k-th link in one network is equal to the cross-talk factor describing the interference of k-th link at the receiver of j-th link in the other network. The link SIR function in a network $(\mathcal{J}, \boldsymbol{V}')$, adjoint to $(\mathcal{K}, \boldsymbol{V})$, takes the form

$$\gamma_k^A(\boldsymbol{p}) = \frac{p_k}{\sum_{\substack{j\in\mathcal{J}\\ j\neq k}} V_{jk}p_j + \sigma_k^{2A}}, \quad \boldsymbol{p} \in \mathcal{P}, \quad k \in \mathcal{J}. \tag{4.1}$$

The issue of key importance is the fact that any given network $(\mathcal{K}, \boldsymbol{V})$ can mimic the interference topology of the adjoint network $(\mathcal{K}, \boldsymbol{V}')$ by means of a specific transmission scheme.

Proposition 9 *Given a network $(\mathcal{K}, \boldsymbol{V})$, the adjoint network $(\mathcal{K}, \boldsymbol{V}')$ can be obtained by*
i.) link-flow reversion, that is, replacing the roles of transmitters and receivers of each link $k \in \mathcal{K}$,
ii.) utilizing zero-forcing channel predistortion, in the sense of division of transmit symbols by ${h_{kk}}^{-1}$, at each link $k \in \mathcal{K}$, with the predistortion units/ blocks regarded as parts of the resulting channels.

Proof Let $\boldsymbol{p} \mapsto \gamma_k^r(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{R}_+^K$, denote the SIR function under reversed flow of link $k \in \mathcal{K}$, that is, when the k-th link transmitter assumes the role of the k-th link receiver and conversely. Since flat-fading and reciprocal channels h_{kj}, $k, j \in \mathcal{K}$, are assumed throughout, for a network $(\mathcal{K}, \boldsymbol{V})$, with interference matrix given by (2.2), we can then write

$$\gamma_k^r(\tilde{\boldsymbol{p}}) = \frac{|h_{kk}|^2 \tilde{p}_k}{\sum_{\substack{j\in\mathcal{K}\\ j\neq k}} |h_{jk}|^2 \tilde{p}_j + \sigma_k^{2r}}, \quad \tilde{\boldsymbol{p}} \in \mathbb{R}_+^K, \quad k \in \mathcal{K}, \tag{4.2}$$

with σ_k^{2r} as the noise variance at the receiver of link $k \in \mathcal{K}$ with reversed flow (i.e., at the original link transmitter). Under zero-forcing channel predistortion according to ii.), we have

$$\tilde{p}_k = \frac{p_k}{|h_{kk}|^2}, \quad \boldsymbol{p} \in \mathcal{P}, \quad k \in \mathcal{K}, \tag{4.3}$$

in (4.2), with $\boldsymbol{p}$ as the vector of input powers at the predistortion units/ blocks. Setting now (4.3) into (4.2) yields with (2.2) and (4.1) that $\gamma_k^r(\boldsymbol{p}) = \gamma_k^A(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{P}$, which completes the proof. □

4.1.2 Duality in aggregated performance optimization

In the context of problem (2.18), we recall the notion of strong Lagrangean duality and compare it with a related novel notion of network duality.

Global solvability and strong Lagrangean duality

Under general QoS functions Ψ, problem (2.18) has in general multiple local minimizers, with some of them being not global. Simplifying, one can say that problem (2.18) is solvable only locally. By Proposition 3, each local minimizer of (2.18) becomes global as well (problem (2.18) is globally solvable) if function $\Phi = \Psi^{-1}$ has the log-convexity property. Then, problem (2.18) can be rewritten in equivalent convex form (2.25) and, since constraint qualification is satisfied for problems (2.18), (2.25) with $\mathcal{P} \in \{\mathcal{P}_P, \mathcal{P}_{\hat{\boldsymbol{p}}}\}$, we have

$$\max_{\boldsymbol{\lambda}\in\mathbb{R}_+^K} \min_{\boldsymbol{x}\in\mathbb{R}^K} \bar{L}(\boldsymbol{x},\boldsymbol{\lambda}) = \min_{\boldsymbol{x}\in\mathbb{R}^K} \sup_{\boldsymbol{\lambda}\in\mathbb{R}_+^K} \bar{L}(\boldsymbol{x},\boldsymbol{\lambda}) \tag{4.4}$$

with $(\boldsymbol{x},\boldsymbol{\lambda}) \mapsto \bar{L}(\boldsymbol{x},\boldsymbol{\lambda})$, $(\boldsymbol{x},\boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^K$, as the linear Lagrangean function of problem (2.25) and with $\boldsymbol{\lambda} \in \mathbb{R}^K$ as the Lagrangean multipliers associated with the constraints of domain $\mathcal{X}$ [48]. Here, we assumed implicitly in (4.4) that the solution to (2.19) exist. The max-min min-max equality (4.4) (which however does not in general imply a saddle point [71], [72], [73], [74], [75]), is referred to as strong Lagrangean duality property of problem (2.19). With the known relation [76], [72]

$$\min_{\boldsymbol{x}\in\mathcal{X}} \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})) = \min_{\boldsymbol{x}\in\mathbb{R}^K} \sup_{\boldsymbol{\lambda}\in\mathbb{R}_+^K} \bar{L}(\boldsymbol{x},\boldsymbol{\lambda}) \tag{4.5}$$

and the definition

$$\boldsymbol{\lambda} \mapsto g(\boldsymbol{\lambda}) = \min_{\boldsymbol{x}\in\mathbb{R}^K} \bar{L}(\boldsymbol{x},\boldsymbol{\lambda}), \quad \boldsymbol{\lambda} \in \mathbb{R}_+^K,$$

of the dual (Lagrangean) function, the strong Lagrangean duality (4.4) (for problem (2.25)) can be equivalently written in a more known form

$$\min_{\boldsymbol{x}\in\mathcal{X}} \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})) = \max_{\boldsymbol{\lambda}\in\mathbb{R}_+^K} g(\boldsymbol{\lambda}), \tag{4.6}$$

with the optimization on the right-hand side known as the dual problem to problem (2.25).

Network duality

We focus now on a related duality notion, which is specific for problem (2.18) and its version (2.25) under Condition 1, and is especially interesting from the network point of view. Given Condition 1, the presentation of this idea requires the restatement of problem (2.25) in an equivalent form

$$\begin{aligned} &\min_{\boldsymbol{x},\boldsymbol{I}} \sum_{k\in\mathcal{K}} \alpha_k \Psi\Big(\frac{e^{x_k}}{I_k}\Big), \\ &\text{subject to} \quad \begin{cases} e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \le 0 \\ \sum_{\substack{j\in\mathcal{K}\\ j\neq k}} V_{kj} e^{x_j} + \sigma_k^2 - I_k \le 0, \quad k \in \mathcal{K}, \end{cases} \end{aligned} \tag{4.7}$$

where, without loss of generality, we assume $\boldsymbol{\alpha} > 0$. It implicitly assumed in (4.7) that Ψ is infinite in value outside its domain, as is the usual optimization theory convention [49], [47]. Reformulation

(4.7) differs from (2.25) merely in the incorporation of the auxiliary variable $\boldsymbol{I} \in \mathbb{R}^K$. It can be easily seen from (4.7), that for any local minimizer the second constraint holds with equality for $k \in \mathcal{K}$. This yields the following obvious Lemma.

Lemma 6 *Let Condition 1 be satisfied. Then, given* $(\boldsymbol{x}, \boldsymbol{I})$, $\boldsymbol{x}$ *and* $\boldsymbol{p} = e^{\boldsymbol{x}}$ *as corresponding local minimizers of problems (4.7), (2.25) and (2.18), respectively, we have*

$$e^{x_k}/I_k = \gamma_k(e^{\boldsymbol{x}}) = \gamma_k(\boldsymbol{p}), \quad k \in \mathcal{K}.$$

Interestingly, by considering the so-called *perturbation function* one can show that problem (4.7) looses the strong duality property of formulation (2.25) for log-convex functions $\Phi = \Psi^{-1}$ (see, e.g., [77] for a very general duality theory). However, (4.7) has the key feature that the Kuhn-Tucker conditions with respect to the associated Lagrangean function

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = &\sum_{k\in\mathcal{K}} \alpha_k \Psi(\frac{e^{x_k}}{I_k}) + \sum_{k\in\mathcal{K}} \lambda_k (e^{x_k} - \hat{p_k}) \\ &+ \sum_{k\in\mathcal{K}} \mu_k (\sum_{j\in\mathcal{K}\ ,\ j\neq k} V_{kj} e^{x_j} + \sigma_k^2 - I_k), \quad (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^{4K}, \end{aligned} \tag{4.8}$$

can be written in the form of explicit primal-dual relations. This leads directly to our network duality result. For compact presentation, we make the dependence of the SIR functions (2.1), (4.1) on noise variance explicit, by writing $\gamma_k(\boldsymbol{p}, \sigma_k^2)$ and $\gamma_k^A(\boldsymbol{p}, \sigma_k^{2A})$, $\boldsymbol{p} \in \mathcal{P}$, $\sigma_k^2, \sigma_k^{2A} \geq 0$, $k \in \mathcal{K}$, respectively.

Proposition 10 *A stationary point* $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^{2K} \times \mathbb{R}_+^{2K}$ *of the Lagrangian (4.8), such that* $(\boldsymbol{x}, \boldsymbol{I})$ *is a local minimizer of problem (4.7) represents an equilibrium of link SIR functions in the pair of the adjoint networks* $(\mathcal{K}, \boldsymbol{V})$, $(\mathcal{K}, \boldsymbol{V}')$ *in the sense*

$$\gamma_k(e^{\boldsymbol{x}}, \sigma_k^2) = \gamma_k^A(\boldsymbol{\mu}, \lambda_k), \quad k \in \mathcal{K}, \tag{4.9}$$

with dual variables/ Lagrangean multipliers $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ *determined by the primal variables according to*

$$\left\{ \begin{array}{ll} \mu_k = -\frac{\alpha_k \Psi'(e^{x_k}/I_k) e^{x_k}}{I_k^2} & \\ \lambda_k = -e^{-x_k} \frac{\partial}{\partial x_k} \sum_{j\in\mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}})), & \end{array} \right. \quad k \in \mathcal{K}. \tag{4.10}$$

Proof Since problem (4.7) satisfies constraint qualification, which is shown straightforwardly [48], any local minimizer satisfies the Kuhn-Tucker conditions together with some $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ [78]. The Kuhn-Tucker condition $\nabla_{\boldsymbol{I}} L(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0$ corresponds, written componentwise, to the first equality in (4.10). By decreasingness of Ψ and the assumption $\boldsymbol{\alpha} > 0$ this also implies $\mu > 0$. The Kuhn-Tucker condition $\nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0$ yields

$$-\frac{\alpha_k \Psi'(e^{x_k}/I_k)}{I_k} = \sum_{j\in\mathcal{K}, j\neq k} V_{jk} \mu_j + \lambda_k, \quad k \in \mathcal{K}. \tag{4.11}$$

Applying the already proven first equality in (4.10) to the left-hand side of (4.11) yields further

$$\mu_k = \frac{e^{x_k}}{I_k} (\sum_{j\in\mathcal{K}, j\neq k} V_{jk} \mu_j + \lambda_k), \quad k \in \mathcal{K}. \tag{4.12}$$

Due to $\mu > 0$ we can divide both sides of (4.12) by the positive expression in the brace which, by (4.1) and Lemma 6, is equivalent to (4.9).

Further, note that incorporating the proved first equality in (4.10) into (4.11) yields

$$\lambda_k = -\frac{\alpha_k \Psi'(e^{x_k}/I_k)}{I_k} + \sum_{j\in\mathcal{K}, j\neq k} V_{jk} \frac{\alpha_j \Psi'(e^{x_j}/I_j)e^{2x_j}}{I_j^2}, \quad k \in \mathcal{K}. \tag{4.13}$$

At the same time, we have by Lemma 6 that $\boldsymbol{x}$ is a local minimizer of the problem version (2.25) and we can write the partial derivative of the objective in (2.25) at $\boldsymbol{x}$ as

$$-\frac{\partial}{\partial x_k} \sum_{j\in\mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}})) = -\frac{\alpha_k \Psi'(e^{x_k}/I_k)e^{x_k}}{I_k} + \sum_{j\in\mathcal{K}, j\neq k} V_{jk} \frac{\alpha_j \Psi'(e^{x_j}/I_j)e^{2x_j}}{I_j^2}, \quad k \in \mathcal{K}. \tag{4.14}$$

The second equality in (4.10) follows now immediately from (4.13) and (4.14) and the proof is completed. □

In other words, Proposition 10 says that a power allocation locally optimizing the aggregated performance is characterized by the pairwise equality of link SIR values in the original network $(\mathcal{K}, \boldsymbol{V})$ and the adjoint network $(\mathcal{K}, \boldsymbol{V}')$, where the power allocation and noise in the adjoint network is determined explicitly by (4.10). The economic interpretation of the Kuhn-Tucker conditions yields an additional interpretation of the equilibrium (4.9). Precisely, in the optimal equilibrium (4.9), the transmit power of k-th link in the adjoint network corresponds to the unit Lagrange price μ_k for decrement of the interference power on k-th link in the original network [47]. Similarly, the noise variance at the receiver of link $k \in \mathcal{K}$ in the adjoint network in the optimal equilibrium corresponds to the unit Lagrange price λ_k for increment of the transmit power of link $k \in \mathcal{K}$ in the original network.

The equilibrium property (4.9) is quite similar in its nature to the strong Lagrange duality (4.4). Together with the feature that the power and noise of the adjoint network is determined in (4.9) by dual variables, this suggests referring to (4.9) as to the *network duality* associated with problem (4.7). Since (4.9) and (4.10) constitute the Kuhn-Tucker conditions of problem (4.7), the notion of network duality is applicable whenever Condition 1 is satisfied. Further, by Propositions 3, 5, Condition 1 is satisfied and all minimizers of (4.7) are global under log-convex inverse performance function $\Phi = \Psi^{-1}$. Thus, in such particular case the equilibrium (4.9), given (4.10), is a necessary and sufficient optimality condition for problem (4.7).

4.1.3 Algorithmic solution

We propose a (signaling and) feedback scheme, which realizes decentralized optimization according to (2.18), when the gradient iteration is utilized. The idea of the scheme relies on the concept of adjoint network from the last section.

Assuming Condition 1, it is convenient to focus here on the problem form (2.25). However, the feedback scheme is straightforwardly extendable for the application to problem form (2.18).

Projected gradient method

The gradient projection iteration applied to (2.25) can be written as

$$\boldsymbol{x}(n+1) = \boldsymbol{x}(n) - s(n)\boldsymbol{P}_{\hat{\mathbf{p}}} \nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}(n)})), \quad n \in \mathbb{N},$$

with $\boldsymbol{P}_{\hat{\boldsymbol{p}}}$ as the orthogonal projection matrix onto the polyhedral domain $\mathcal{X}_{\hat{\boldsymbol{p}}} = \{\boldsymbol{x} = \log \boldsymbol{p} : \boldsymbol{p} \in \mathcal{P}_{\hat{\boldsymbol{p}}}\}$ and $s(n) > 0$ as the step-size in n-th iteration. The iteration is expressible equivalently in the simple componentwise form

$$x_k(n+1) = \min\{x_k(n) - s(n)\frac{\partial}{\partial x_k}\sum_{j\in\mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}(n)})), \log \hat{p}_k\}, \quad k \in \mathcal{K}. \tag{4.15}$$

The gradient projection iteration (4.15) is known to be convergent to some local minimizer of (2.25) under an appropriate step-size sequence, see e.g. Theorems 10.5.4 and 10.5.7 in [78]. Clearly, if the inverse QoS function $\Phi = \Psi^{-1}$ is log-convex, then, by Proposition 3, the convergence is to a global minimizer. In terms of complexity reduction in real-world network optimization, the case of fixed step-size $s(n) = s$, $n \in \mathbb{N}$, appears to be of interest. For such case the interval of step-size values ensuring convergence of the projected gradient iteration can be easily characterized.

Proposition 11 *Let $s(n) = s > 0$, $n \in \mathbb{N}$. Then, iteration (4.15) is convergent to a local minimizer $\tilde{\boldsymbol{x}} \in \mathcal{X}_{\hat{\boldsymbol{p}}}$ of problem (2.25) if*

$$0 < s < \frac{2}{\rho(\nabla'(\boldsymbol{P}_{\hat{\boldsymbol{p}}}\nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\tilde{\boldsymbol{x}}}))))}. \tag{4.16}$$

Proof Under fixed step-size $s(n) = s$, $n \in \mathbb{N}$, we have the sufficient convergence condition $\rho(\boldsymbol{I} - s\nabla'(\boldsymbol{P}_{\hat{\boldsymbol{p}}}\nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\tilde{\boldsymbol{x}}})))) < 1$ for the projected gradient iteration (4.15), which follows from Ostrowski's Theorem [79]. With symmetry of the matrix $\nabla'(\boldsymbol{P}_{\hat{\boldsymbol{p}}}\nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\tilde{\boldsymbol{x}}})))$ we have $Im\lambda_k(\nabla'(\boldsymbol{P}_{\hat{\boldsymbol{p}}}\nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\tilde{\boldsymbol{x}}})))) = 0$, $k \in \mathcal{K}$, so that the sufficient convergence condition is equivalent to

$$1 - s\lambda_k(\nabla'(\boldsymbol{P}_{\hat{\boldsymbol{p}}}\nabla \sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\tilde{\boldsymbol{x}}})))) > -1, \quad k \in \mathcal{K} \tag{4.17}$$

(the eigenvalue operator λ_k should not be confused here with a Lagrange multiplier). Since the maximum magnitude of an eigenvalue corresponds to the spectral, (4.17) is equivalent to (4.16) and the proof is completed. □

The convergence obtained by the projected gradient iteration (4.15) is linear in roots and quotients, see Appendix A.3 (compare also with Section 4.2.3).

LAGRANGEAN INTERPRETATION

Since the problem formulations (2.25) and (4.7) are equivalent, we can find the equivalence of the gradient projection iteration applied to (2.25) with some type of iteration applied to the Lagrangean function of (4.7).

Proposition 12 *Let $\boldsymbol{z} = (\boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^{3K}$. The projected gradient iteration (4.15) is equivalent to the conditional iteration*

$$x_k(n+1) = \min\{x_k(n) - s(n)\frac{\partial}{\partial x_k}L(\boldsymbol{x}(n), \boldsymbol{z}), \log \hat{p}_k\}, \quad k \in \mathcal{K}, \tag{4.18}$$

with $\boldsymbol{z}$ solving

$$\begin{cases} I_k = \sum_{j\in\mathcal{K}, j\neq k} V_{kj} e^{x_j(n)} + \sigma_k^2 \\ \mu_k = -\frac{\alpha_k \Psi'(e^{x_k(n)}/I_k) e^{x_k(n)}}{I_k^2} \\ \lambda_k = 0, \end{cases} \quad k \in \mathcal{K}, \quad n \in \mathbb{N}. \tag{4.19}$$

Proof We have

$$\frac{\partial}{\partial x_k}L(\boldsymbol{x}(n),\boldsymbol{z}) = \frac{\alpha_k\Psi'(e^{x_k(n)}/I_k)e^{x_k(n)}}{I_k} + \lambda_k e^{x_k(n)} + \sum_{j\in\mathcal{K},j\neq k} V_{jk}\mu_j e^{x_k(n)}, \quad k\in\mathcal{K}. \tag{4.20}$$

Note that, by (2.1), the first equality in (4.19) is equivalent to $e^{x_k(n)}/I_k = \gamma_k(e^{\boldsymbol{x}(n)})$, $k\in\mathcal{K}$. Thus, the partial derivative of the objective $\sum_{k\in\mathcal{K}}\alpha_k\Psi(\gamma_k(e^{\boldsymbol{x}(n)}))$ can be written as

$$\frac{\partial}{\partial x_k}\sum_{j\in\mathcal{K}}\alpha_j\Psi(\gamma_j(e^{\boldsymbol{x}(n)})) = \frac{\alpha_k\Psi'(e^{x_k(n)}/I_k)e^{x_k(n)}}{I_k} - \sum_{j\in\mathcal{K},j\neq k} V_{jk}\frac{\alpha_j\Psi'(e^{x_j(n)}/I_j)e^{2x_j(n)}}{I_j^2}, \quad k\in\mathcal{K}.$$

Thus, the incorporation of the second and third equality in (4.19) into (4.20) yields $\frac{\partial}{\partial x_k}L(\boldsymbol{x}(n),\boldsymbol{z}) = \frac{\partial}{\partial x_k}\sum_{j\in\mathcal{K}}\alpha_j\Psi(\gamma_j(e^{\boldsymbol{x}(n)}))$, whenever $\boldsymbol{z}$ solves (4.19). This shows that iterations (4.15) and (4.18) are equivalent when (4.19) is satisfied, and the proof is completed. □

The iteration from Proposition 12 is not a classical primal-dual method for solving problem (4.7). It merely gives the link from (4.15) to the Lagrangian and the Kuhn-Tucker conditions of problem (4.7).

Decentralized feedback scheme

Decentralized conduction of the gradient projection iteration is possible if the knowledge of the gradient components in (4.15) can be provided to corresponding transmitters $k\in\mathcal{K}$ with no use of signaling *across the links*. This means that no transmitter/ receiver of link $k\in\mathcal{K}$ is allowed to exchange signaling information with transmitter/ receiver of link $l\in\mathcal{K}$, $l\neq k$. The latter is the case, for instance, under the use of a *flooding protocol*, so that the flooding protocol is, in our terms, not a decentralized feedback scheme [28]. With (4.10) and a telescope argument we can write for the partial derivatives in (4.15)

$$\frac{\partial}{\partial x_k}\sum_{j\in\mathcal{K}}\alpha_j\Psi(\gamma_j((e^{\boldsymbol{x}(n)}))) = -(\frac{\mu_k(n)}{\gamma_k(\boldsymbol{x}(n))} + \mu_k(n))e^{x_k(n)} + (\mu_k(n) + \sum_{j\in\mathcal{K},j\neq k} V_{jk}\mu_j(n))e^{x_k(n)}, \quad k\in\mathcal{K}. \tag{4.21}$$

It can be seen that the expression in the first brace in (4.21) is obtainable separately at each link transmitter $k\in\mathcal{K}$ through single feedback of the link SIR value from the corresponding receiver $k\in\mathcal{K}$. With Definition 1 and Proposition 9 follows further that the second-brace expression in (4.21) corresponds to the power received at the transmitter of link $k\in\mathcal{K}$ under concurrent transmission in the adjoint network mode with link transmit powers $\mu_k(n)$, $k\in\mathcal{K}$ (according to Proposition 9, measured at the inputs of predistortion units/ blocks). Hence, with the concept from Proposition 9 and (coarse) synchronization among the links, the second-brace addend is obtainable at the transmitter of each link $k\in\mathcal{K}$ in distributed manner as well.

Summarizing, the resulting feedback scheme realizing the projected gradient iteration (4.15) can be stated as follows. We assume the existence of some abstract exist condition terminating the iteration (4.15), global knowledge of the QoS function Ψ and the knowledge of weight α_k at the transmitter and receiver of the corresponding link $k\in\mathcal{K}$ (clearly, $\hat{p}_k$ is known to the link transmitter $k\in\mathcal{K}$).

Algorithm 3 (adjoint network feedback scheme)

1: *Concurrent transmission with power vector* $e^{\boldsymbol{x}(n)}$.

2: *Receiver-side estimation of transmit powers* $e^{x_k(n)}$ *and SIR values* $\gamma_k(e^{\boldsymbol{x}(n)})$, $k\in\mathcal{K}$.

3: *Per-link feedback of the SIR values* $\gamma_k(e^{\boldsymbol{x}(n)})$ *to the corresponding transmitters* $k \in \mathcal{K}$.
4: *Concurrent transmission of the adjoint network (Proposition 9) with transmit powers*

$$\mu_k(n) = -\alpha_k \Psi'(\gamma_k(\boldsymbol{e}^{\boldsymbol{x}(n)}))\gamma_k^2(\boldsymbol{e}^{\boldsymbol{x}(n)})e^{-x_k(n)}, \quad k \in \mathcal{K}.$$

5: *Transmitter-side estimation of the received powers*

$$\mu_k(n) + \sum_{j \in \mathcal{K}, j \neq k} V_{jk}\mu_j(n), \quad k \in \mathcal{K}$$

and computation of gradient components

$$\frac{\partial}{\partial x_k} \sum_{j \in \mathcal{K}} \alpha_j \Psi(\gamma_j((e^{\boldsymbol{x}(n)}))), \quad k \in \mathcal{K}$$

according to (4.21).
6: *Transmit power updates according to (4.15) and* $n \to n+1$ *if exit condition not satisfied.*

We propose to terminate the scheme after a predefined number of iterations, since this retains the decentralized manner of the scheme.

4.1.4 STOCHASTIC APPROXIMATION VIEW

The adjoint network feedback scheme is a power estimation-based scheme. For the statement of the adjoint network feedback scheme we assumed that the SIR values and the powers received under adjoint network transmission are estimated with accuracy which allows for the treatment of the algorithm in the framework of (deterministic) optimization theory. Clearly, such assumption is unrealistic for the majority of real-world networks, since the estimation time is strongly limited by virtue of the necessity of online operation. More abstractly, the estimation uncertainty which occurs in steps 2 and 5 of the scheme can be seen as the price paid for the lack of cooperation and the lack of signaling across the links. Under uncertainty, the proposed algorithm has to be analyzed in the more general framework of stochastic approximation [64], [80], [81]. In this section we characterize the behavior of the proposed feedback scheme in terms of key issues of such framework. For this aim we slightly simplify the analysis by assuming

Condition 2 *The uncertainty of the estimation of SIR values in step 2 is negligible compared to the uncertainty of the estimation of received powers in step 5 of the adjoint network feedback scheme.*

For the purposes of this section we can weaken the basic assumption on white Gaussian noise from Chapter 2. Note however, that under noise process more general than AWGN, some widely-used QoS functions, like e.g. the (negative) link capacity $\Psi(\gamma) = -\log(1+\gamma)$, $\gamma \geq 0$, loose their applicability. It is required now merely that on the discrete-time scale $l \in \mathbb{N}$ of network observation and control,

Condition 3 *The receiver noise processes* $\delta N_k(l)$, $l \in \mathbb{N}$, *are martingale-differences uncorrelated with transmit symbols and have variances* $\sigma_k^2 < \infty$, $k \in \mathcal{K}$.

Such martingale-difference property is equivalent to

$$\delta N_k(l+1) = N_k(l+1) - E(N_k(l+1)|N_k(i), i \leq l), l \in \mathbb{N}, \quad k \in \mathcal{K},$$

which is precisely the martingale property of the aggregated noise processes $N_k(l) = \sum_{i=0}^{l-1} \delta N_k(l)$, $k \in \mathcal{K}$ [82], [83]. The time scale of network operation $l \in \mathbb{N}$ should not be confused here with the larger time scale $n \in \mathbb{N}$ of iterations.

Assuming Conditions 2 and 3, the gradient components obtained in step 5 of the implementation scheme are random variables of the form

$$\Delta_k(n) = \frac{\partial}{\partial x_k} \sum_{j \in \mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}(n)})) + \delta M_k(n), \quad n \in \mathbb{N}, \quad k \in \mathcal{K}. \tag{4.22}$$

The estimation noise processes $\delta M_k(n)$, $k \in \mathcal{K}$, are dependent on the estimator type (in particular, on estimation duration) and on the noise processes in the estimation interval, say $l_s \leq l \leq l_e$, $l_s, l_e \in \mathbb{N}$. The latter dependence corresponds formally to some dependence on the stopped processes $\delta N_k(l - l_s \wedge l_e - l_s)$, $k \in \mathcal{K}$ [82]. We make a nonrestrictive assumption with respect to the estimator type by requiring that

Condition 4 *The estimation noise is zero-mean and exogenous, in the sense that $\delta M_k(n)$ is independent of the iterate value $\boldsymbol{x}(n)$ for any $n \in \mathbb{N}$, $k \in \mathcal{K}$.*

This is in particular ensured under the use of linear, unbiased estimators. Hence, with (4.22) and Conditions 3, 4 we can write

$$\delta M_k(n+1) = \Delta_k(n+1) - E(\Delta_k(n+1)|\boldsymbol{x}(0), \Delta_k(m), m \leq n), \quad n \in \mathbb{N}, \quad k \in \mathcal{K},$$

with

$$E(\Delta_k(n+1)|\boldsymbol{x}(0), \Delta_k(m), m \leq n) = \frac{\partial}{\partial x_k} \sum_{j \in \mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}(n)})), \quad n \in \mathbb{N}, \quad k \in \mathcal{K},$$

and the martingale $\Delta_k(n) = \sum_{m=0}^{n-1} \delta M_k(m)$, $k \in \mathcal{K}$. Simply said, under Conditions 3, 4, the estimation noise processes $\delta M_k(n)$, $n \in \mathbb{N}$, $k \in \mathcal{K}$, have the martingale-difference property.

Almost sure convergence

Almost sure convergence (that is, convergence with probability one) means that the event that given iteration parameters, with the start value $\boldsymbol{x}(0)$ as the most relevant one, prevent local convergence has zero probability measure. Known necessary and sufficient conditions for almost sure convergence of the projected gradient stochastic approximation

$$x_k(n+1) = \min\{x_k(n) - s(n)\Delta_k(n), \log \hat{p}_k\}, \quad n \in \mathbb{N}, \quad k \in \mathcal{K}, \tag{4.23}$$

with Δ_k, $k \in \mathcal{K}$, given in (4.22), are very technical conditions with respect to the processes [64]

$$\sum_{m=0}^{n-1} s(m) \delta M_k(m), \quad n \in \mathbb{N}, \quad k \in \mathcal{K}.$$

Regarding sufficient conditions for almost sure convergence of the stochastic approximation (4.23), either of the following conditions ensures almost sure convergence under satisfied Conditions 3, 4.

Condition 5 *i.) There exists some even $p \in \mathbb{N}$, such that $\sum_{n=0}^{\infty} s^{p/2+1}(n) < \infty$ and $E(|\Delta_k(n)|^p) < \infty$, $n \in \mathbb{N}$, $k \in \mathcal{K}$,*
ii.) for each $q > 0$, we have $\sum_{n=0}^{\infty} e^{-q/s(n)} < \infty$.

Thus, we can state the following convergence result.

Proposition 13 *Let the step-size sequence be such that*

$$\sum_{n=0}^{\infty} s(n) = \infty, \quad \lim_{n\to\infty} s(n) = 0, \tag{4.24}$$

and let Conditions 3, 4, and either of the Conditions 5 i.), ii.) be satisfied. Then, the stochastic approximation (4.23) converges almost surely to $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}_+^K$ *which satisfies the Kuhn-Tucker conditions of problem (2.25).*

Proof By decreasingness of Ψ and by satisfied Conditions 3, 4, 5 i.), it is implied that $E(|\Delta_k(n)|) < \infty$, $k \in \mathcal{K}$, $n \in \mathbb{N}$. With this, all conditions of Theorem 3.1 in Section 5 in [64] are satisfied. Since Theorem 3.1 implies almost sure convergence of the particular stochastic approximation (4.23) to a vector satisfying the Kuhn-Tucker conditions, the proof is completed. □

Note, that in opposition to the deterministic projected gradient iteration (4.15) we formally can not exclude the convergence to a primal-dual tuple not corresponding to a local minimizer, see [64] Section 5.8. Given Conditions 3, 4, it can be easily seen that when taking

$$s(n) = 1/(n+1), \quad n \in \mathbb{N}, \tag{4.25}$$

Conditions 5 i.) and (4.24) are satisfied. Thus, by Proposition 13, (4.25) is a particular step-size sequence for which the stochastic approximation (4.23) is almost surely convergent.

WEAK CONVERGENCE

Almost sure convergence is an asymptotic convergence notion, which comes into its own under $n \to \infty$ and $\lim_{n\to\infty} s(n) = 0$ [64]. Hence, the almost sure convergence property has no value in numerous real-world cases, when the iteration is truncated after some finite number of steps and the step-size sequence is bounded away from zero (in particular, the step-size is fixed positive). In such cases, the interest is in the weakest notion of probabilistic convergence referred to as *weak convergence* (convergence in distribution). The weak convergence property relies on the measure of portion of time spent by the iterates in some neighborhood of the vector of convergence.

Definition 2 *The sequence of* $\mathbb{R}^K$*-valued random variables* $X(n)$, $n \in \mathbb{N}$, *is said converge weakly to some random variable* X, *iff* $\lim_{n\to\infty} E(g(X(n))) = E(g(X))$ *for any bounded and continuous function* $g : \mathbb{R}^K \to I \subseteq \mathbb{R}$.

For relative simplicity of formulation of the generally technical conditions, we state the weak convergence result for the particular case of slowly changing step-size sequence (see e.g. [64]) for the more general case).

Proposition 14 *Let Conditions 3, 4 and (4.24) be satisfied and assume that the step-size sequence* $s(n)$, $n \in \mathbb{N}$, *is slowly changing in the sense that there exists some integer sequence* $a(n)$, $n \in \mathbb{N}$, *such that* $\lim_{n\to\infty} a(n) = \infty$ *and*

$$\lim_{n\to\infty} \sup_{0\le i\le a(n)} |s(n+1)/s(n) - 1| = 0.$$

Then, the stochastic approximation (4.23) converges weakly to $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}_+^K$ *which satisfies the Kuhn-Tucker conditions of problem (2.25).*

Proof Note that decreasingness of Ψ and satisfied Conditions 3, 4 imply so-called *uniform integrability* of the sequences $\Delta_k(n)$, $n \in \mathbb{N}$, $k \in \mathcal{K}$ (see [82]). With this, all conditions of Theorem 2.3 in Section 8 in [64] are satisfied. Since Theorem 2.3 implies weak convergence of the particular stochastic approximation (4.23) to a vector satisfying the Kuhn-Tucker conditions, the proof is completed. □

Convergence rate and averaging

The measure of convergence rate of stochastic approximation (4.23) is the (trace of the) *error covariance matrix* $\Sigma_{\boldsymbol{x}} = \lim_{n\to\infty} E(\boldsymbol{e_x}(n)\boldsymbol{e}'_{\boldsymbol{x}}(n))$. Precisely, matrix $\Sigma_{\boldsymbol{x}}$ is defined as the limit of the covariance matrix of the random vector

$$\boldsymbol{e_x}(n) = s(n)^{-1/2}(\boldsymbol{x}(n) - \tilde{\boldsymbol{x}}), \quad n \in \mathbb{N},$$

with $\tilde{\boldsymbol{x}}$ as the vector of convergence of (4.23), which represents the error process. The error process can be shown to be a Wiener process, so that the limit is a Gaussian and zero-mean random vector [80]. Since we assumed decreasing QoS function Ψ, any local minimizer of problem (2.25) is included in the boundary of the domain $\mathcal{X}_{\hat{\boldsymbol{p}}}$. For this reason, the characterization of $\Sigma_{\boldsymbol{x}}$ is extremely intricate, see e.g. [80] for an analysis approach.

To improve the convergence rate, that is, to decrease $tr(\Sigma_{\boldsymbol{x}})$, we may utilize averaging of iterates in parallel to the stochastic approximation (4.23), as was proposed in [81]. In general, the optimal averaging by [81] in (4.23) results in an averaged iterate

$$\boldsymbol{x}^E(r) = \frac{1}{R(s(r))} \sum_{n=r}^{r+R(s(r))-1} \boldsymbol{x}(n), \quad r \in \mathbb{N}, \tag{4.26}$$

with the (averaging) window-size sequence $R(s(r))$, $r \in \mathbb{N}$, determined by the step-size sequence. For instance, under $s(r) = \mathcal{O}(1/r)$ the sequence $R(s(r))$, $r \in \mathbb{N}$ increases sublinearly and under fixed step-size the window size remains fixed as well [81]. The fundamental insight from [81] is, that under step-size sequences $s(n) = \mathcal{O}(1/n)$, iterate averaging provides the optimal convergence rate (that is, minimal trace of $\Sigma_{\boldsymbol{x}}$), while for faster decreasing step-size sequences the gain from parallel averaging is negligible. Parallel averaging proved to behave good in our simulations plotted in Fig. 4.1.

4.2 Algorithmic solution based on generalized Lagrangian

We propose an alternative construction of the Lagrangean function of the problem of aggregated performance optimization. Our generalized Lagrangean function is inspired by the theory of multiplier techniques and nonlinear Lagrangians from [71], [74], [73], [72] and gives rise to an iteration which effectively trades off complexity against convergence properties. The proposed iteration is realizable in decentralized manner in combination with the adjoint network feedback scheme and has certain advantages in comparison to the projected gradient method discussed in Section 4.1.3.

4.2.1 Optimization under additional constraints

The algorithmic theory based on generalized Lagrangian allows for the treatment of an extended problem of aggregated performance optimization. Precisely, we can account in the problem (2.18)

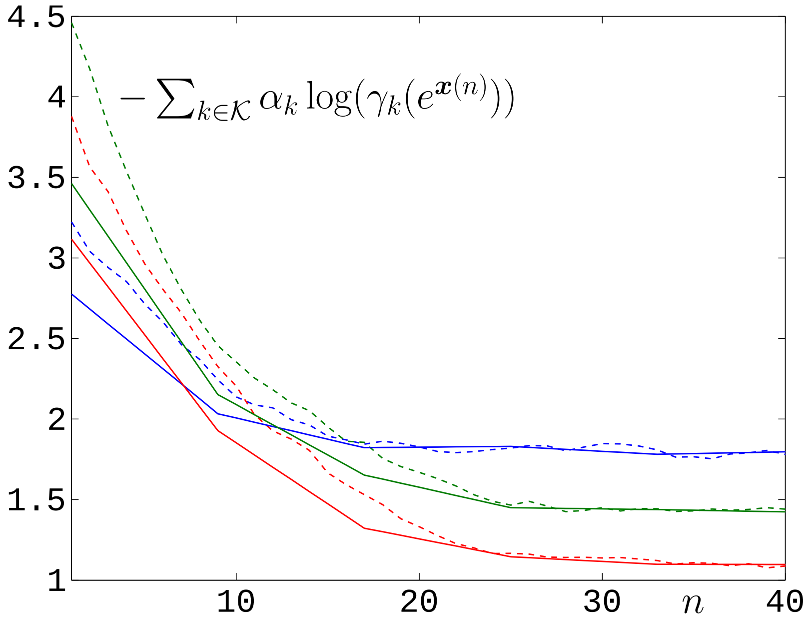

FIGURE 4.1: Exemplary non-averaged (dashed) and averaged (solid) convergence of aggregated performance obtained by stochastic approximation (4.23) with averaging (4.26). The parameter settings are $K = 8$ and $\Psi(\gamma) = -\log(\gamma)$ and $s(n) = 0.88$, $n \in \mathbb{N}$ and $R(s(r)) = 8$, $r \in \mathbb{N}$ and $E(\delta M_k(n)) = 0.175\sigma_k^2$, $k \in \mathcal{K}$.

for additional constraints on some selected links. We distinguish some subclass $\mathcal{B} \subseteq \mathcal{K}$ of constrained links and consider general constraints of the form

$$g_k(\boldsymbol{p}) - \hat{g}_k \leq 0, \quad k \in \mathcal{B}, \tag{4.27}$$

with some twice Frechet-differentiable constraint metrics $\boldsymbol{p} \mapsto g_k(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{R}_+^K$ and predefined values $\hat{g}_k$. In particular, an established power control approach consists in considering $\mathcal{B}$ as the subclass of QoS-critical links, with requirements on the values of some QoS function. If such a performance function is again monotone function of the link SIR, say $\gamma_k \mapsto \Phi(\gamma_k)$, $\gamma_k \geq 0$, then the link QoS constraints can be expressed as $\Phi(\gamma_k(\boldsymbol{p})) - \hat{\Phi}_k \leq 0$, $k \in \mathcal{B}$, with $\hat{\Phi}_k$ as the predefined QoS requirements, where Φ can be in general different from the link performance function incorporated in (2.16) (again, we implicitly change the sign of increasing QoS functions). For instance, if the links $k \in \mathcal{B}$ carry highly QoS-sensitive traffic, such as a real-time stream, it is justified to impose requirements on, e.g., the provided link rate $\Phi(\gamma_k) = -\log(1 + \gamma_k)$, $\gamma_k \geq 0$, or its approximation $\Phi(\gamma_k) = \log(\gamma_k)$, $\gamma_k \geq 0$, as explained above. By bijectivity of Φ, the link performance constraints can be equivalently written as minimum constraints on link SIR according to (4.27) with

$$g_k(\boldsymbol{p}) = -\gamma_k(\boldsymbol{p}), \quad k \in \mathcal{B}, \tag{4.28}$$

and with $-\hat{g}_k = \Phi^{-1}(\hat{\Phi}_k)$ as the resulting link SIR requirements.

A different particular kind of constraints are the constraints related to the limited power spectral density. This kind of limitations appears to be of particular importance in mesh networks, where the neighboring mesh access clusters may represent distinct systems which coexist in the same bandwidth and area [32]. To mitigate the inter-system interference in such case, it is necessary to adjust the link power constraints $\hat{\boldsymbol{p}}$ according to the topology of the mesh access clusters and provide restrictions on maximum received power for links with critically located receivers. The subclass of such links is modeled by $\mathcal{B} \subseteq \mathcal{K}$ and the corresponding received power constraints can be written as (4.27) with

$$g_k(\boldsymbol{p}) = (\boldsymbol{V}\boldsymbol{p})_k + p_k + \sigma_k^2, \quad k \in \mathcal{B}. \tag{4.29}$$

The values of $\hat{g}_k$, $k \in \mathcal{B}$, represent now the received power constraints and are adjusted according to the local power density limitations, wave propagation conditions, etc. While under received power restrictions (4.29) and our assumptions from Chapter 2 problem (2.25) has always a solution, such solution may not exist in the case of constraints on link SIR (4.28) under general values $\hat{g}_k$, $k \in \mathcal{B}$ [40], [39], [15]. Thus, hereafter we shall implicitly assume that the problem is solvable also for (4.28).

Summarizing, under Condition 1 (e.g., under log-convex inverse performance function Ψ^{-1}), the problem of aggregated performance optimization extended to account for constraints on selected critical links is obtained from problem (2.25) as

$$\begin{aligned} &\min_{\boldsymbol{x}} \sum_{k \in \mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})), \\ &\text{subject to} \quad \begin{cases} e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \leq 0 \\ g_k(e^{\boldsymbol{x}}) - \hat{g}_k \leq 0, \quad k \in \mathcal{B}. \end{cases} \end{aligned} \tag{4.30}$$

Clearly, under setting $\mathcal{B} = \varnothing$ (empty set of critical links), the extended problem form (4.30) reduces to (2.25).

It is easy to see with Proposition 6, that if the inverse QoS function Ψ^{-1} is log-convex and the functions $\boldsymbol{x} \mapsto g_k(e^{\boldsymbol{x}})$, $\boldsymbol{x} \in \mathcal{X}$, $k \in \mathcal{K}$, are convex, then the problem (4.30) is convex. The latter condition is satisfied in particular by the received power (4.29) [47], while constraints (4.28) can be easily reformulated to yield a convex form as well.

4.2.2 Generalized Lagrangian construction

For notational simplicity in the following Sections, we use a uniform formulation of all constraints in (4.30) as $h_k(\boldsymbol{x}) \leq 0$, $k \in \mathcal{L}$, with set $\mathcal{L}$ such that $L = |\mathcal{L}| = K + |\mathcal{B}|$. Let the set of tight (satisfied with equality) constraints at $\boldsymbol{x} \in \mathbb{R}^K$ be denoted as

$$\mathcal{T}(\boldsymbol{x}) = \{k \in \mathcal{L} : h_k(\boldsymbol{x}) = 0\}.$$

Complying with the optimization theory convention, we refer to $\boldsymbol{x}$ as *feasible*, if it satisfies all constraint inequalities $h_k(\boldsymbol{x}) \leq 0$, $k \in \mathcal{L}$.

The main aim of our Lagrangian construction is the property of positive definiteness of its Hessian at its stationary points which are associated with points satisfying the *Second Order Sufficiency Conditions* (SOSC). This property ensures (local) convergence of a surprisingly simple iteration we propose (Section 4.2.3). The classical linear Lagrangean function does not, in general, have such property on the points satisfying SOSC (in short, on SOSC points), so that the proposed iteration is in general divergent in combination with the linear Lagrangian.

The first modification of the Lagrangian with the desired positive semidefiniteness property falls into the framework of so-called multiplier methods and was proposed in [71]. The multiplier method proposed in [71] is applicable to equality-constrained problems and was later generalized to account for inequality constraints in [73]. More sophisticated nonlinear Lagrangians were proposed in [74] and [72]. In the context of the considered problem (4.30), the concepts from [74], [72] are applicable but their use seems to be of some intricacy. Thus, we prefer an own Lagrangian construction which shares the most properties of the concepts in [74], [72].

Definition 3 *For the optimization problem (4.30) we define the Lagrangean function*

$$L(\boldsymbol{x}, \boldsymbol{\mu}, c) = \sum_{k \in \mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})) + \sum_{k \in \mathcal{L}} \psi(\phi(\mu_k) h_k(\boldsymbol{x}) + c), \quad (\boldsymbol{x}, \boldsymbol{\mu}, c) \in \mathbb{R}^K \times \mathbb{R}^L \times \mathbb{R} \tag{4.31}$$

where the functions $\psi : \mathbb{R} \to I$, $I \subseteq \mathbb{R}$ *and* $\phi : \mathbb{R} \to \mathbb{R}_+$ *satisfy the following conditions.*

Condition 6 *Function* ψ *is twice differentiable and*
i.) $\psi'(y) > 0$, $y \in \mathbb{R}$ *(increasingness),*
ii.) $\psi''(y) > 0$, $y \in \mathbb{R}$ *(strict convexity),*
iii.) $\psi''(y)/(\psi'(y))^2$ *is strictly monotone and unbounded from above.*

Condition 7 *Function* ϕ *is twice differentiable and*
i.) $\phi(y) = \phi(-y)$, $y \in \mathbb{R}$ *(evenness),*
ii.) $\phi(y) \geq 0$, $y \in \mathbb{R}$ *(nonnegativity),*
iii.) $\phi(y) = 0$ *iff* $\phi'(y) = 0$ *iff* $y = 0$, *where* $\phi''(0) > 0$ *(has unique local extremum as a minimum with value* 0 *at* 0*).*

We sometimes group the arguments of the Lagrangian (4.31) according to $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$. One can find numerous functions which satisfy Conditions 6 and 7. Prominent examples are

$$\psi(y) = e^y \quad \text{and} \quad \phi(y) = y^{2n}, \quad n \in \mathbb{N}_+.$$

The following Proposition characterizes a connection between vectors satisfying the Kuhn-Tucker conditions (in short, Kuhn-Tucker points) of problem (4.30) and stationary points of Lagrangian (4.31).

Proposition 15 *Let* $\pm\boldsymbol{\mu} \in \mathbb{R}^L$ *denote any of the* 2^L *vectors such that* $(\pm\boldsymbol{\mu})_k = \mu_k$ *or* $(\pm\boldsymbol{\mu})_k = -\mu_k$ *independently for* $k \in \mathcal{L}$. *If* $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^L$ *is a Kuhn-Tucker point of problem (4.30), then* $(\boldsymbol{x}, \pm\boldsymbol{\mu})$, *with* $\pm\boldsymbol{\mu} = \pm\boldsymbol{\mu}(c)$ *defined by*

$$\lambda_k = \psi'(c)\phi(\pm\mu_k), \quad k \in \mathcal{L}, \tag{4.32}$$

is a stationary point of Lagrangian (4.31) for any $c \in \mathbb{R}$. *Conversely, given any* $c \in \mathbb{R}$, *if* $(\boldsymbol{x}, \pm\boldsymbol{\mu}) \in \mathbb{R}^K \times \mathbb{R}^L$, *with* $\boldsymbol{x}$ *feasible, is a stationary point of Lagrangian (4.31), then* $(\boldsymbol{x}, \boldsymbol{\lambda})$, *with* $\boldsymbol{\lambda}$ *given by (4.32), is a Kuhn-Tucker point of problem (4.30).*

Proof Let

$$\bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = F(e^{\boldsymbol{x}}) + \sum_{k \in \mathcal{L}} \lambda_k h_k(\boldsymbol{x}), \quad (\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^L, \tag{4.33}$$

be the linear Lagrangian of problem (4.30) and let $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^L$, where $\boldsymbol{x}$ is feasible, satisfy the Kuhn-Tucker conditions of problem (4.30). Then, by the complementary slackness conditions, we have

$$\lambda_k = 0, \quad k \in \mathcal{L} \setminus \mathcal{T}(\boldsymbol{x}), \tag{4.34}$$

so that $\nabla_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$ yields

$$\nabla F(e^{\boldsymbol{x}}) + \sum_{k \in \mathcal{T}(\boldsymbol{x})} \lambda_k \nabla h_k(\boldsymbol{x}) = 0. \tag{4.35}$$

By (4.32) and by Conditions 6, 7, we have further

$$\lambda_k = 0 \quad \text{iff} \quad \mu_k = 0 \quad \text{iff} \quad \phi(\mu_k) = 0 \quad \text{iff} \quad \phi'(\mu_k) = 0, \qquad k \in \mathcal{K}, \tag{4.36}$$

which implies by (4.34) and (4.35) that

$$\begin{aligned}\nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \pm\boldsymbol{\mu}, c) = \nabla F(e^{\boldsymbol{x}}) + &\sum_{k\in\mathcal{L}\setminus\mathcal{T}(\boldsymbol{x})} \phi(\pm\mu_k)\psi'(\phi(\pm\mu_k)h_k(\boldsymbol{x}) + c)\nabla h_k(\boldsymbol{x}) \\ &+ \sum_{k\in\mathcal{T}(\boldsymbol{x})} \phi(\pm\mu_k)\psi'(c)\nabla h_k(\boldsymbol{x}) = 0, \quad c \in \mathbb{R}.\end{aligned}$$

Further, note that $\frac{\partial}{\partial\tilde{\mu}_k} L(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\mu}}, c) = \psi'(\phi(\tilde{\mu}_k)h_k(\tilde{\boldsymbol{x}}) + c)\phi'(\tilde{\mu}_k)h_k(\tilde{\boldsymbol{x}})$, $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\mu}}) \in \mathbb{R}^K \times \mathbb{R}^L$, $k \in \mathcal{K}$. Thus, we have $\frac{\partial}{\partial\mu_k} L(\boldsymbol{x}, \pm\boldsymbol{\mu}, c) = 0$, $k \in \mathcal{T}(\boldsymbol{x})$, by definition of $\mathcal{T}(\boldsymbol{x})$, and

$$\frac{\partial}{\partial\mu_k} L(\boldsymbol{x}, \pm\boldsymbol{\mu}, c) = 0, \quad k \in \mathcal{L} \setminus \mathcal{T}(\boldsymbol{x}),$$

due to assumption (4.32) and (4.34), (4.36), which completes the proof of $(\boldsymbol{x}, \pm\boldsymbol{\mu})$ as a stationary point of (4.31).

Conversely, let $(\boldsymbol{x}, \pm\boldsymbol{\mu})$, with $\boldsymbol{x}$ feasible, be a stationary point of Lagrangian (4.31) for any $c \in \mathbb{R}$. Then, the Kuhn-Tucker condition $h_k(\boldsymbol{x}) \le 0$, $k \in \mathcal{K}$, is satisfied by feasibility of $\boldsymbol{x}$, and the Kuhn-Tucker condition $\boldsymbol{\lambda} \ge 0$ holds due to (4.32) and Conditions 6, 7. By $\nabla_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{x}, \pm\boldsymbol{\mu}, c) = 0$ we have also

$$\begin{aligned}\nabla F(e^{\boldsymbol{x}}) + \sum_{k\in\mathcal{L}} \phi(\pm\mu_k)\psi'(\phi(\pm\mu_k)h_k(\boldsymbol{x}) + c)\nabla h_k(\boldsymbol{x}) = 0,& \\ \psi'(\phi(\pm\mu_k)h_k(\boldsymbol{x}) + c)\phi'(\pm\mu_k)h_k(\boldsymbol{x}) = 0, \quad k \in \mathcal{L}.&\end{aligned} \tag{4.37}$$

According to Condition 6, the latter equality is satisfied if and only if either $h_k(\boldsymbol{x}) = 0$ or $\phi'(\pm\mu_k) = 0$, $k \in \mathcal{L}$. Thus, by assumption (4.32) and (4.36), the second equality in (4.37) is equivalent to $\lambda_k h_k(\boldsymbol{x}) = 0$, $k \in \mathcal{L}$, which is the complementary slackness condition. Consequently, given $k \notin \mathcal{T}(\boldsymbol{x})$, the second equality in (4.37) is satisfied if and only if $\phi'(\pm\mu_k) = 0$. Again, by assumption (4.32) and (4.36), this implies that the first equality in (4.37) is equivalent to (4.35), which is the Kuhn-Tucker condition $\nabla_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$. This completes the proof of $(\boldsymbol{x}, \boldsymbol{\lambda})$ as a vector satisfying the Kuhn-Tucker conditions of problem (4.30). □

It can be seen that due to evenness of ϕ, a single vector satisfying the Kuhn-Tucker conditions of problem (4.30) corresponds to 2^L stationary points of (4.31), which are associated with the same power vector. In terms of convergence of an algorithmic solution it is therefore not of interest which of such vectors is the vector of attraction of the algorithm.

The next results concerns the key feature of the positive semidefiniteness of the Hessian of (4.31) at any vector satisfying the SOSC of problem (4.30).

Proposition 16 *If $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^L$ satisfies strict complementarity and is an SOSC point of problem (4.30), then for the stationary point $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$ of Lagrangian (4.31), with $\boldsymbol{\mu} = \boldsymbol{\mu}(c)$ given by (4.32), there exists $c_0 = c_0(\boldsymbol{z}) \in \mathbb{R}$ such that*

$$\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}, c) \succ 0, \quad c \ge c_0 \text{ or } c \le c_0. \tag{4.38}$$

Conversely, if (4.38) is satisfied at a stationary point $\boldsymbol{z}$ of Lagrangian (4.31) and $\boldsymbol{x}$ is feasible, then $(\boldsymbol{x}, \boldsymbol{\lambda})$, with $\boldsymbol{\lambda}$ given by (4.32), is an SOSC of problem (4.30).

Proof The Hessian matrix of the Lagrangian (4.31) can be written as

$$\begin{aligned}\nabla^2_{\boldsymbol{x}} L(\tilde{\boldsymbol{z}}, c) = \nabla^2 F(e^{\tilde{\boldsymbol{x}}}) + &\sum_{k\in\mathcal{L}} \phi^2(\tilde{\mu}_k)\psi''(\phi(\tilde{\mu}_k)h_k(\tilde{\boldsymbol{x}}) + c)\nabla h_k(\tilde{\boldsymbol{x}})\nabla' h_k(\tilde{\boldsymbol{x}}) + \\ &\sum_{k\in\mathcal{L}} \phi(\tilde{\mu}_k)\psi'(\phi(\tilde{\mu}_k)h_k(\tilde{\boldsymbol{x}}) + c)\nabla^2 h_k(\tilde{\boldsymbol{x}}), \quad (\tilde{\boldsymbol{z}}, c) \in \mathbb{R}^{K+L} \times \mathbb{R}.\end{aligned} \tag{4.39}$$

By the complementary slackness condition satisfied at $(\boldsymbol{x}, \boldsymbol{\lambda})$ (since $(\boldsymbol{x}, \boldsymbol{\lambda})$ is an SOSC point) and by (4.32) we have $\lambda_k = 0$, $k \notin \mathcal{T}(\boldsymbol{x})$, which gives with (4.33) that $\nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \nabla^2 F(e^{\boldsymbol{x}}) + \sum_{k\in\mathcal{T}(\boldsymbol{x})} \phi(\mu_k)\psi'(c)\nabla^2 h_k(\boldsymbol{x})$. Thus, evaluating the Hessian (4.39) for $\tilde{\boldsymbol{z}} = \boldsymbol{z}$ and noting that, by (4.32) and Condition 6, $\lambda_k = 0$ iff $\phi(\mu_k) = 0$, we yield with (4.32) that

$$\begin{aligned}\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}, c) &= \nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) + \sum_{k\in\mathcal{T}(\boldsymbol{x})} \phi^2(\mu_k)\psi''(c)\nabla h_k(\boldsymbol{x})\nabla' h_k(\boldsymbol{x}) \\ &= \nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) + \sum_{k\in\mathcal{T}(\boldsymbol{x})} \lambda_k^2 \frac{\psi''(c)}{(\psi'(c))^2}\nabla h_k(\boldsymbol{x})\nabla' h_k(\boldsymbol{x}), \quad c \in \mathbb{R}.\end{aligned} \tag{4.40}$$

If $\mathcal{T}(\boldsymbol{x}) = \varnothing$ we have immediately by Definition 17 ii.) that $\nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \succ 0$, which gives by (4.40) that $\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}, c) \succ 0$ and completes the proof for $\mathcal{T}(\boldsymbol{x}) = \varnothing$. Thus, let now $\mathcal{T}(\boldsymbol{x}) \neq \varnothing$. Since $\lambda_k > 0$, $k \in \mathcal{T}(\boldsymbol{x})$, by strict complementarity, we have by Debreu's Theorem [84] that there exists some $\psi_0 = \psi_0(\boldsymbol{z}) \geq 0$ such that $\nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda}) + \sum_{k\in\mathcal{T}(\boldsymbol{x})} \lambda_k^2 \frac{\psi''(c)}{(\psi'(c))^2}\nabla h_k(\boldsymbol{x})\nabla' h_k(\boldsymbol{x}) \succ 0$ whenever $\psi''(c)/(\psi'(c))^2 \geq \psi_0$. But function $y \mapsto \psi''(y)/(\psi'(y))^2$, $y \in \mathbb{R}$, is by assumption strictly monotone and unbounded from above so that there exists some $c_0 = c_0(\psi_0) \in \mathbb{R}$ such that $\psi''(c)/(\psi'(c))^2 \geq \psi_0$ for either $c \geq c_0$ or $c \leq c_0$, depending on whether $\psi''(y)/(\psi'(y))^2$, $y \in \mathbb{R}$, is increasing or decreasing. This completes the sufficiency proof.

For the converse, note that with (4.40) and the definition of semidefiniteness, we obtain

$$\boldsymbol{y}'\nabla^2_{\boldsymbol{x}} \bar{L}(\boldsymbol{x}, \boldsymbol{\lambda})\boldsymbol{y} + \sum_{k\in\mathcal{T}(\boldsymbol{x})} \phi^2(\mu_k)\psi''(c)\boldsymbol{y}'\nabla h_k(\boldsymbol{x})(\boldsymbol{y}'\nabla h_k(\boldsymbol{x}))' > 0, \quad \boldsymbol{y} \neq 0, \quad c \geq c_0 \text{ or } c \leq c_0,$$

which implies the SOSC condition ii.) in Definition 17. Since further, by assumption, $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$ is a stationary point of Lagrangian (4.31) with $\boldsymbol{x}$ being feasible, the SOSC condition i.) in Definition 17 is implied by Proposition 15 and the proof of necessity is completed. □

By the proof it is readily seen that the right inequality for $c \in \mathbb{R}$ is the one which allows arbitrarily large values of $\psi''(c)/(\psi'(c))^2$, and thus depends on whether $y \mapsto \psi''(y)/(\psi'(y))^2$, $y \in \mathbb{R}$, is increasing or decreasing. For instance, under $\psi(y) = e^y$ we have $\psi''(y)/(\psi'(y))^2 = e^{-y}$, which implies that local convexity according to (4.38) is ensured for $c \leq c_0$, for some $c_0 \in \mathbb{R}$.

With our Lagrangian construction, the following feature follows now by means of elementary calculus.

Lemma 7 *If $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$ is a stationary point of (4.31), with $\boldsymbol{x}$ feasible, then $\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}, c) = diag(\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}, c))$, with*

$$(\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}, c))_{kk} = \begin{cases} 0 & k \in \mathcal{T}(\boldsymbol{x}) \\ \psi'(c)\phi''(0)h_k(\boldsymbol{x}) & k \notin \mathcal{T}(\boldsymbol{x}) \end{cases}$$

The following corollary of Proposition 16 and Lemma 7 shows the key saddle point property of the Lagrangian (4.31).

Corollary 1 *Let* $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu}) \in \mathbb{R}^K \times \mathbb{R}^L$ *be a stationary point of (4.31) such that (4.32) is satisfied for some* $\boldsymbol{\lambda} \in \mathbb{R}^L_+$. *If SOSC of problem (4.30) are satisfied at* $(\boldsymbol{x}, \boldsymbol{\lambda})$, *then we have*

$$\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}, c) \preceq 0, \quad c \geq 0$$

and there exists some $c_0 < \infty$, *such that*

$$\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}, c) \succ 0, \quad c \geq c_0 \text{ or } c \leq c_0.$$

Equivalently, $\boldsymbol{z}$ *is an isolated (local) saddle point of (4.31) for* $c \geq c_0$ *or* $c \leq c_0$, *that is, there exists some neighborhood* $S(\boldsymbol{x})$ *of* $\boldsymbol{x}$, *such that*

$$L(\boldsymbol{x}, \bar{\boldsymbol{\mu}}, c) \leq L(\boldsymbol{z}, c) < L(\bar{\boldsymbol{x}}, \boldsymbol{\mu}, c), \quad (\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) \in S(\boldsymbol{x}) \times S(\boldsymbol{\mu}), \quad c \geq c_0 \text{ or } c \leq c_0.$$

By Corollary 1, any vector satisfying the SOSC of problem (4.30) translates one-to-one to a saddle point of the generalized Lagrangian (4.31). The vector satisfying the SOSC itself is in general not a saddle point of the classical linear Lagrangian [75], [85]. In fact, recalling Section 4.1.2, a stationary point of a linear Lagrangean function is a max-min point, but in general not a saddle point of the linear Lagrangian.

4.2.3 Algorithm construction

We identify the SOSC with the property of a local minimizer, which can be done for a large class of problems [74], i.e. for a large class of constraint metrics g_k, $k \in \mathcal{B}$, in (4.30). Thus, it follows from Corollary 1 that solving the problem of aggregated performance optimization (4.30) locally consists in finding of a saddle point of the generalized Lagrangian (4.31). Clearly, if any local minimizer of problem (4.30) is global as well, then finding a saddle point of (4.31) is equivalent to finding a (global) solution to problem (4.30).

The iteration

The algorithm corresponds precisely to the primal-dual search of a saddle point of (4.31), based on the gradient iteration. Given some fixed step-size $s > 0$, the n-th iteration step can be formulated simply as

$$\boldsymbol{z}(n+1) = \boldsymbol{z}(n) + s \begin{pmatrix} -\boldsymbol{I}_K & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_L \end{pmatrix} \nabla_{\boldsymbol{z}} L(\boldsymbol{z}(n), c), \quad n \in \mathbb{N}, \tag{4.41}$$

where $\boldsymbol{I}_N$ denoted the identity matrix of size $N \in \mathbb{N}$. In iteration (4.41), the primal steps (over the logarithmic transmit powers $\boldsymbol{x} \in \mathbb{R}^K$) and the dual steps (over the vector of dual variables $\boldsymbol{\mu} \in \mathbb{R}^L$) are conducted concurrently. Interestingly, due to the Lagrangian construction (4.31) the iteration is unconstrained, both with respect to the primal variable (power vector) and the dual variable. This stands in contrast to conventional primal-dual iterations applied to the linear Lagrangian of problem (4.30) [49]. Such iterations solve the dual problem, on the right-hand side of (4.6), which is nonnegatively constrained according to $\boldsymbol{\lambda} \in \mathbb{R}^L_+$ [48]. Complying with the nature of primal-dual iterations, the iteration (4.41) is locally convergent without requiring feasibility of the consecutive primal iterates $\boldsymbol{x}(n)$, $n \in \mathbb{N}$, which stands in contrast to the simple projected gradient iteration (4.15). Thus, the proposed algorithm requires no steps or actions in addition to the iteration step (4.41) at all. The convergence is proven in the following Proposition.

Proposition 17 *Let $\boldsymbol{x} \in \mathbb{R}^K$ be a local minimizer of problem (4.30) such that the strict complementarity condition and constraint qualification from Lemma 19 are satisfied at $(\boldsymbol{x}, \boldsymbol{\nu})$ for some $\boldsymbol{\nu} \in \mathbb{R}^L$. Then, under step-size*

$$0 < s < 2 \min_{1 \le k \le K+L} \frac{|Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c))|}{|\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c))|^2}, \tag{4.42}$$

with $\lambda_k(\cdot)$ denoting the k-th eigenvalue, $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$ is a point of attraction of iteration (4.41) for $c \ge c_0$ or $c \le c_0$, with some $\boldsymbol{\mu} \in \mathbb{R}^L$ and some $c_0 \in \mathbb{R}$.

Proof By Proposition 15, if $\boldsymbol{x} \in \mathbb{R}^K$ is a local minimizer of Problem (4.30), then $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{\mu})$ is a stationary point of (4.31) for some $\boldsymbol{\mu} \in \mathbb{R}^L$. Since (4.41) is a gradient-based method applied to (4.31), an equilibrium point of the map

$$\tilde{\boldsymbol{z}} \mapsto G(\tilde{\boldsymbol{z}}) = \tilde{\boldsymbol{z}} + s\ diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla_{\tilde{\boldsymbol{z}}} L(\tilde{\boldsymbol{z}}, c), \quad \tilde{\boldsymbol{z}} \in \mathbb{R}^{K+L}, \quad c \ge 0 \text{ or } c \le c_0, \tag{4.43}$$

can be only a stationary point of (4.31) [79]. The gradient of such map can be written as

$$\nabla G(\tilde{\boldsymbol{z}}) = \boldsymbol{I} + s diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\tilde{\boldsymbol{z}}} L(\tilde{\boldsymbol{z}}, c), \tag{4.44}$$

where we can write explicitly

$$diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\tilde{\boldsymbol{z}}} L(\tilde{\boldsymbol{z}}, c) = \begin{pmatrix} -\nabla^2_{\tilde{\boldsymbol{x}}} L(\tilde{\boldsymbol{z}}, c) & -\nabla^2_{\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\mu}}} L(\tilde{\boldsymbol{z}}, c) \\ \nabla'^2_{\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\mu}}} L(\tilde{\boldsymbol{z}}, c) & \nabla^2_{\tilde{\boldsymbol{\mu}}} L(\tilde{\boldsymbol{z}}, c) \end{pmatrix}, \quad \tilde{\boldsymbol{z}} \in \mathbb{R}^{K+L}.$$

Hereby, we have $\nabla'^2_{\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\mu}}} L(\tilde{\boldsymbol{z}}, c) = \nabla^2_{\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{x}}} L(\tilde{\boldsymbol{z}}, c)$, $\tilde{\boldsymbol{z}} \in \mathbb{R}^{K+L}$, and the k-th row of the Hessian $\nabla^2_{\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{x}}} L(\tilde{\boldsymbol{z}}, c)$ can be easily shown to be of the form

$$\left(\nabla^2_{\tilde{\boldsymbol{\mu}},\tilde{\boldsymbol{x}}} L(\tilde{\boldsymbol{z}}, c)\right)_{k\cdot} = \begin{cases} \psi'(c)\phi'(\tilde{\mu}_k)\nabla' h_k(\tilde{\boldsymbol{x}}), & k \in \mathcal{T}(\tilde{\boldsymbol{x}}) \\ 0, & k \notin \mathcal{T}(\tilde{\boldsymbol{x}}), \end{cases} \quad \tilde{\boldsymbol{z}} \in \mathbb{R}^{K+L}. \tag{4.45}$$

By Ostrowski's Theorem [79], it is known that an equilibrium point $\boldsymbol{z}$ is a point of attraction of iteration (4.41) if $\rho(\nabla G(\boldsymbol{z})) = \max_{1 \le k \le K+L} |\lambda_k(G(\boldsymbol{z}))| < 1$, with $\rho(\cdot)$ denoting the spectral radius, which is by (4.44) equivalent to

$$2Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)) + s|\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c))|^2 < 0, \quad k \in \mathcal{K}. \tag{4.46}$$

Now it is easy to see that given assumption (4.42), condition (4.46) is satisfied if and only if $\max_{1 \le k \le K+L} Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)) < 0$. But due to block-skew-symmetry of the matrix $diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)$ we have [63]

$$\begin{aligned} Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)) &= Re\boldsymbol{u}'_k diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)\boldsymbol{u}_k \\ &= -Re\boldsymbol{v}'_k \nabla^2_{\tilde{\boldsymbol{x}}} L(\boldsymbol{z}, c)\boldsymbol{v}_k + Re\boldsymbol{w}'_k \nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}, c)\boldsymbol{w}_k, \quad 1 \le k \le K+L, \end{aligned} \tag{4.47}$$

with $\boldsymbol{u}_k = (\boldsymbol{v}_k, \boldsymbol{w}_k) \in \mathbb{R}^{K+L}$ as the k-th eigenvector of $diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)$. Hence, with Corollary 1 and the definition of semidefiniteness, one obtains

$$\max_{1 \le k \le K+L} Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\boldsymbol{z}} L(\boldsymbol{z}, c)) \le 0, \quad c \ge c_0 \text{ or } c \le c_0,$$

for some $c_0 \in \mathbb{R}$. Assume, by contradiction, $Re\lambda_k(diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\tilde{\boldsymbol{z}}}L(\boldsymbol{z}, c)) = 0$, for $c \geq c_0$ or $c \leq c_0$, for some $1 \leq k \leq K + L$, which implies by (4.47) and $\boldsymbol{u}_k \neq \boldsymbol{0}$ (satisfied by definition) that

$$Re\,diag(-\boldsymbol{I}_K, \boldsymbol{I}_L)\nabla^2_{\tilde{\boldsymbol{z}}}L(\boldsymbol{z}, c)\boldsymbol{u}_k = \begin{pmatrix} -\nabla^2_{\tilde{\boldsymbol{x}}}L(\boldsymbol{z}, c)\boldsymbol{v}_k - \nabla^2_{\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\mu}}}L(\boldsymbol{z}, c)\boldsymbol{w}_k \\ \nabla'^2_{\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\mu}}}L(\boldsymbol{z}, c)\boldsymbol{v}_k + \nabla^2_{\tilde{\boldsymbol{\mu}}}L(\boldsymbol{z}, c)\boldsymbol{w}_k \end{pmatrix} = \boldsymbol{0}, \quad c \geq c_0 \text{ or } c \leq c_0. \tag{4.48}$$

Since $\nabla^2_{\tilde{\boldsymbol{\mu}}}L(\boldsymbol{z}, c)$ is a diagonal matrix specified in Lemma 7, we can conclude by (4.47) that $\mathcal{T}(\boldsymbol{x}) \neq \emptyset$. Further, since we have $\nabla^2_{\tilde{\boldsymbol{x}}}L(\boldsymbol{z}, c) \succ 0$ for $c \geq c_0$ or $c \leq c_0$, by Corollary 1, it follows from (4.47) and the definition of semidefiniteness that $\boldsymbol{v}_k = 0$. Consequently, we must have $\boldsymbol{w}_k \neq \boldsymbol{0}$, so that by (4.45) and the first row in (4.48), one has $\sum_{i\in\mathcal{T}(\boldsymbol{x})} \psi'(c)\phi'(\mu_i)(\boldsymbol{w}_k)_i \nabla h_i(\boldsymbol{x}) = 0$, and there exists some $j \in \mathcal{T}(\boldsymbol{x})$ such that $\psi'(c)\phi'(\mu_j)(\boldsymbol{w}_k)_j > 0$ for $c \geq c_0$ or $c \leq c_0$. But this contradicts the constraint qualification from Lemma 19, and therefore completes the proof. □

The constraint qualification condition from Lemma 19, required in Proposition 17, seems to be nonrestrictive (see Appendix A.3 for details). Such condition is satisfied at any local minimizer of problem (4.30) under the very most "reasonably defined" constraint metrics g_k, $k \in \mathcal{K}$. In particular, condition from Lemma 19 is satisfied under constraints on received power (4.29).

Convergence behavior

Let us write iteration (4.41) in short operator form $\boldsymbol{z}(n+1) = G(\boldsymbol{z}(n))$. Consider first the convergence in absolute errors, which is expressible by the *root convergence factor (of p-th order,* $\boldsymbol{p} \geq 1$*)* $R_p(\mathcal{I}, \tilde{\boldsymbol{z}})$, with $\mathcal{I}$ as the set of all sequences of iterates convergent to a point of attraction $\tilde{\boldsymbol{z}}$ of iteration (4.41) [79]. It is an immediate consequence from Ostrowski's Theorem (it follows also from the proof of Proposition 17) that

$$R_1(\mathcal{I}, \tilde{\boldsymbol{z}}) = \rho(\nabla G(\tilde{\boldsymbol{z}})) < 1, \tag{4.49}$$

which is referred to as linear root convergence.

More interesting is the convergence of consecutive error ratios which is described by the *quotient convergence factor (of p-th order,* $p \geq 1$*)* $Q_p(\mathcal{I}, \tilde{\boldsymbol{z}})$, with $\mathcal{I}$ as the set of all sequences of iterates convergent to a point of attraction $\tilde{\boldsymbol{z}}$ of iteration (4.41) ($Q_p(\mathcal{I}, \tilde{\boldsymbol{z}})$ is defined only if $\boldsymbol{z}(n) \neq \tilde{\boldsymbol{z}}$ for all but finitely many $n \in \mathbb{N}$ [79]). For the quotient convergence factor we have the following (particular version of a) standard result [79].

Lemma 8 ([79]) *For any* $\epsilon > 0$ *there exists a norm* $\|\cdot\|$ *on* $\mathbb{R}^{K+L}$*, such that for the iteration (4.41) written as* $\boldsymbol{z}(n+1) = G(\boldsymbol{z}(n))$*,* $n \in \mathbb{N}$*, with* $\tilde{\boldsymbol{z}} \in \mathbb{R}^{K+L}$ *as its point of attraction, we have*

$$Q_1(\mathcal{I}, \tilde{\boldsymbol{z}}) \leq \rho(\nabla G(\tilde{\boldsymbol{z}})) + \epsilon.$$

Thus, similarly to the root convergence, due to $\rho(\nabla G(\tilde{\boldsymbol{z}})) < 1$ we have a linear quotient convergence of (4.41). With Proposition 17, (4.49) and Lemma 8 we can summarize the convergence behavior of iteration (4.41) as follows.

Proposition 18 *Under any step-size satisfying (4.42), the convergence of iteration (4.41) to* $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\mu}}) \in \mathbb{R}^K \times \mathbb{R}^L$ *such that* $\tilde{\boldsymbol{x}}$ *is a local minimizer of problem (4.30) is linear in roots and quotients.*

The convergence of iteration (4.41) is verified in exemplary simulations in Section 4.2.5.

4.2.4 DECENTRALIZED FEEDBACK SCHEME

The algorithmic solution (4.41) to the problem of aggregated performance optimization (4.30) was shown in Section 4.2.3 to be unconstrained (that is, of lowered complexity) and linearly locally convergent. Besides this, iteration (4.41) has the key advantage of being realizable in decentralized manner by means of (a slightly extended) adjoint network feedback scheme (Algorithm 3).

Let the dual variable argument $\boldsymbol{\mu} \in \mathbb{R}^L$ of the generalized Lagrangian (4.31) be partitioned as $\boldsymbol{\mu} = (\boldsymbol{\nu}, \boldsymbol{\eta})$, with $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_K)$ associated with power constraints $e^{x_k} - \hat{p}_k \leq 0$, $k \in \mathcal{K}$, and with $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{|\mathcal{B}|})$ associated with constraints $g_k(\boldsymbol{x}) - \hat{g}_k \leq 0$, $k \in \mathcal{B}$. In such case, the generalized Lagrangian (4.31) can be written as

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{\nu}, \boldsymbol{\eta}, c) = &\sum_{k\in\mathcal{K}} \alpha_k \Psi(\gamma_k(e^{\boldsymbol{x}})) + \sum_{k\in\mathcal{K}} \psi(\phi(\nu_k)(e^{x_k} - \hat{p}_k) + c) \\ &+ \sum_{k\in\mathcal{B}} \psi(\phi(\eta_k)(g_k(e^{\boldsymbol{x}}) - \hat{g}_k) + c), \quad (\boldsymbol{x}, \boldsymbol{\nu}, \boldsymbol{\eta}, c) \in \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{|\mathcal{B}|} \times \mathbb{R}, \end{aligned}$$

with g_k, $k \in \mathcal{B}$, given by either of (4.28), (4.29). Recall that, for iterate $\boldsymbol{x}(n)$, $n \in \mathbb{N}$, any gradient component of the first Lagrangian term, i.e. $\frac{\partial}{\partial x_k} \sum_{j\in\mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}(n)}))$, can be provided in distributed manner to any link transmitter $k \in \mathcal{K}$ by establishing an adjoint network with allocated powers

$$m_k(n) = -\alpha_k \Psi'(\gamma_k(e^{\boldsymbol{x}(n)}))\gamma_k^2(e^{\boldsymbol{x}(n)})e^{-x_k(n)}, \quad k \in \mathcal{K}, \tag{4.50}$$

as in Algorithm 3. A gradient component with respect to link power of the third term in Lagrangian (4.31) is yielded by basic calculus, for any link $k \in \mathcal{K}$, as

$$\frac{\partial}{\partial x_k} \sum_{j\in\mathcal{B}} \psi(\phi(\eta_j(n))(g_j(e^{\boldsymbol{x}}) - \hat{g}_j) + c) = (s_k(n) + \overbrace{r_k(n) + \sum_{j\in\mathcal{B}, j\neq k} V_{jk} r_j(n)}^{i.)})e^{x_k(n)}, \tag{4.51}$$

where in the case of (4.28) we have

$$r_k(n) = \psi'(\phi(\eta_k(n))(g_k(e^{\boldsymbol{x}(n)}) - \hat{g}_k) + c)\frac{g_k^2(e^{\boldsymbol{x}(n)})}{e^{x_k(n)}}\phi(\eta_k(n)), \quad k \in \mathcal{B}, \qquad r_k(n) = 0, \quad k \notin \mathcal{B}, \tag{4.52}$$

and

$$s_k(n) = r_k(n)(\frac{1}{g_k(e^{\boldsymbol{x}(n)})} - 1), \quad k \in \mathcal{K}, \tag{4.53}$$

while under (4.29) the settings are

$$r_k(n) = \psi'(\phi(\eta_k(n))(g_k(e^{\boldsymbol{x}(n)}) - \hat{g}_k) + c)\phi(\eta_k(n)), \quad k \in \mathcal{B}, \qquad r_k(n) = 0, \quad k \notin \mathcal{B}, \tag{4.54}$$

and $s_k(n) = 0$, $k \in \mathcal{K}$. In either of the cases (4.52), (4.54) we have $r_k(n) \geq 0$, $k \in \mathcal{K}$, due to Conditions 6, 7. Under separate per-link receiver-side estimation of link SIR and per-link feedback of $s_k(n)$, the knowledge of (4.53) is provided to the link transmitter $k \in \mathcal{K}$, independently on any link $k \in \mathcal{K}$. The estimation of the received power term i.) in (4.51) can be made available to any link transmitter $k \in \mathcal{K}$ by establishing, again, an adjoint network and allocation of transmit powers given by (4.52) and (4.54) in the respective cases. The feedback of $s_k(n)$ and the allocation of transmit powers (4.52) or (4.54) is possible under the knowledge of $\eta_k(n)$ at the corresponding link receiver $k \in \mathcal{K}$.

For the same iterate, the terms

$$\frac{\partial}{\partial \eta_k} \sum_{j \in \mathcal{B}} \psi(\phi(\eta_j(n))(g_j(e^{\boldsymbol{x}(n)}) - \hat{g}_j) + c) = \\ \psi'(\phi(\eta_k(n))(g_k(e^{\boldsymbol{x}(n)}) - \hat{g}_k) + c)\phi'(\eta_k(n))(g_k(e^{\boldsymbol{x}(n)}) - \hat{g}_k) \tag{4.55}$$

and

$$\frac{\partial}{\partial \nu_k} \sum_{j \in \mathcal{K}} \psi(\phi(\nu_j(n))(e^{x_j(n)} - \hat{p}_j) + c) = \psi'(\phi(\eta_k(n))(e^{x_k(n)} - \hat{p}_k) + c)\phi(\eta_k(n))e^{x_k(n)} \tag{4.56}$$

are known (can be computed) at the receiver of any link $k \in \mathcal{B}$ and the transmitter of any link $k \in \mathcal{K}$, respectively, provided the knowledge of iterates $\eta_k(n)$ and $\nu_k(n)$, respectively. The discussed features ensure decentralized conduction of (4.41) by an extension of the adjoint network feedback scheme in the following form. Hereby, we assume the existence of some abstract exit condition terminating iteration (4.41) and the knowledge of the functions Ψ, ϕ, ψ and the constant $0 < c < \infty$ at all link transmitters and link receivers. Further, the knowledge of constants $\hat{p}_k$, $\hat{g}_k$ is assumed, respectively, at the transmitter of any link $k \in \mathcal{K}$ and the receiver of any critical link $k \in \mathcal{B}$, while the knowledge of weight α_k is assumed at both the transmitter and receiver of any corresponding link $k \in \mathcal{K}$.

Algorithm 4

1: Concurrent transmission of links $k \in \mathcal{K}$ using transmit powers $e^{\boldsymbol{x}(n)}$.
2: Receiver-side estimation of transmit power $e^{x_k(n)}$, SIR $\gamma_k(e^{\boldsymbol{x}(n)})$ and received power $e^{x_k(n)}(1 + 1/\gamma_k(e^{\boldsymbol{x}(n)}))$ on any link $k \in \mathcal{K}$.
3: Receiver-side computation of component (4.55) on any link $k \in \mathcal{B}$ and transmitter-side computation of component (4.56) on any link $k \in \mathcal{K}$.
4: Per-link feedback of the SIR $\gamma_k(e^{\boldsymbol{x}(n)})$ on any link $k \in \mathcal{K}$ and, under link SIR constraints, per-link feedback of component (4.53), given (4.52), on any link $k \in \mathcal{B}$.
5: Concurrent transmission of the adjoint network using transmit powers (4.50).
6: Transmitter-side estimation of the received power $m_k(n) + \sum_{j \in \mathcal{K}, j \neq k} V_{jk} m_j(n)$ and transmitter-side computation of component $\frac{\partial}{\partial x_k} \sum_{j \in \mathcal{K}} \alpha_j \Psi(\gamma_j(e^{\boldsymbol{x}(n)}))$ on any link $k \in \mathcal{K}$.
7: Concurrent transmission of the adjoint network using either transmit powers (4.52) under link SIR constraints, or transmit powers (4.54) under received power constraints.
8: Transmitter-side estimation of the received power (4.51)-i.) and transmitter-side computation of component (4.51) on any link $k \in \mathcal{B}$.
9: Transmitter-side computation of $\frac{\partial}{\partial x_k} L(\boldsymbol{x}(n), \boldsymbol{\nu}(n), \boldsymbol{\eta}(n), c)$ and transmitter-side update

$$x_k(n+1) = x_k(n) - s\frac{\partial}{\partial x_k} L(\boldsymbol{x}(n), \boldsymbol{\nu}(n), \boldsymbol{\eta}(n), c)$$
$$\nu_k(n+1) = \nu_k(n) + s\frac{\partial}{\partial \nu_k} L(\boldsymbol{x}(n), \boldsymbol{\nu}(n), \boldsymbol{\eta}(n), c)$$

on any link $k \in \mathcal{K}$, and receiver-side update

$$\eta_k(n+1) = \eta_k(n) + s\frac{\partial}{\partial \eta_k} L(\boldsymbol{x}(n), \boldsymbol{\nu}(n), \boldsymbol{\eta}(n), c)$$

on any link $k \in \mathcal{B}$, and $n \to n+1$ if termination condition not satisfied.

Clearly, the theory from Section 4.1.4 can be straightforwardly extended to the case of noisy iterates in Algorithm 4.

4.2.5 Simulation results

We apply the simulations to the case of link capacity $\Psi(\gamma_k) = -\log(1+\gamma_k)$, $\gamma_k \geq 0$, and normalized link MMSE $\Psi(\gamma_k) = 1/(1+\gamma_k)$, $\gamma_k \geq 0$, as link QoS function. This results in nonconvex instances of problem (4.30), which satisfy strict complementarity and constraint qualification condition from Lemma 19 at any SOSC point. For the generalized Lagrangian we take $\psi(y) = e^y$ and $\phi(y) = y^2$, $y \in \mathbb{R}$.

Figures 4.2 and 4.3 show the convergence of exemplary iterate sequences obtained by the feedback scheme from Algorithm 4 in an ad-hoc network with link SIR constraints. The value of $c \in \mathbb{R}$ which ensures convergence does not pose numerical problems. Both figures seem to imply that Algorithm 4 ensures reliable linear convergence even under quite rough estimates.

Note that the slight oscillation of the performance metric in the transient phase of convergence is, besides the influence of noisy estimates, a result of the unconstrained update character: The consecutive iterates $\boldsymbol{z}(n) = (\boldsymbol{x}(n), \boldsymbol{\mu}(n))$, $n \in \mathbb{N}$, are allowed to be temporarily infeasible for several $n \in \mathbb{N}$ before reaching the point of attraction. Thus, the weighted aggregate performance happens to be superior to the actual optimal value at the point of attraction since the power vector and the values of constraint metrics may temporarily exceed the nominal constraints $\hat{p}_k$, $k \in \mathcal{K}$, and $\hat{g}_k$, $k \in \mathcal{B}$, respectively.

4.3 Algorithmic solution based on variable splitting

We propose a specific splitting of optimization variables in the problem of aggregated performance optimization. The presented variable splitting is combined with a modified Lagrangean function, related to the framework from Section 4.2.2, in order to obtain an algorithm with convergence properties improved in comparison to iteration (4.41). Similarly to iteration (4.41), the resulting algorithm is shown to be realizable in decentralized manner by means of an extended adjoint network feedback scheme from Algorithm 3.

4.3.1 Optimization under nonlinear interference

The approach of variable splitting allows for the treatment of an extended problem as well, but the extension is of different type than in Section 4.2. Precisely, in this section we rely on the generalized definition of the SIR function of the form

$$\gamma_k(\boldsymbol{p}) = \frac{p_k}{J_k(\boldsymbol{p})}, \quad \boldsymbol{p} \in \mathcal{P}, \quad k \in \mathcal{K}, \tag{4.57}$$

with function $\boldsymbol{p} \mapsto J_k(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{P}$, satisfying

$$\frac{\partial^2}{\partial x_k \partial x_j} J_k(e^{\boldsymbol{x}}) = (\nabla^2 J_k(e^{\boldsymbol{x}}))_{kj} = 0, \quad \boldsymbol{x} \in \mathcal{X}, \quad j,k \in \mathcal{K}, \quad j \neq k. \tag{4.58}$$

Condition (4.58) characterizes a class of receivers for which the interference power at their output can be expressed as

$$J_k(\boldsymbol{p}) = \sum_{l \in \mathcal{K}} j_{kl}(p_l) + c, \quad \boldsymbol{p} \in \mathcal{P}, \quad k \in \mathcal{K},$$

with a function $p \mapsto j_{kl}(p)$, $p \in I \subseteq \mathbb{R}_+$, $k, l \in \mathcal{K}$, and constant $c \in \mathbb{R}$. Thus, by (4.57) and (4.58) we extend the consideration of linear receivers, and the associated SIR functions (2.1), to receivers with interference power expressible as a sum of, in general, nonlinear functions of link transmit powers (plus a constant term). The possible nonlinearity of j_{kl}, $k, l \in \mathcal{K}$, can be a result of hardware-related nonlinear effects in transceiver signal processing.

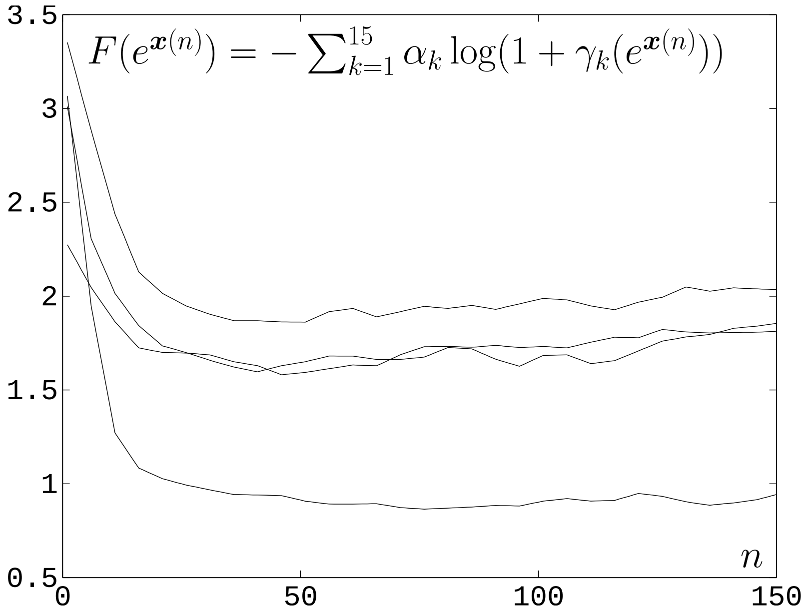

FIGURE 4.2: Exemplary convergence of aggregated performance obtained by Algorithm 4 with averaging (4.26). The settings are $K = 15$, $|\mathcal{B}| = 6$, $\Psi(\gamma) = -\log(1+\gamma)$, $c = -7.5$ and $s(n) = 0.4$, $n \in \mathbb{N}$, and $R(s(r)) = 3$, $r \in \mathbb{N}$, and the variance of estimates in steps 2, 6, 8 is $0.3\sigma_k^2$, $k \in \mathcal{K}$.

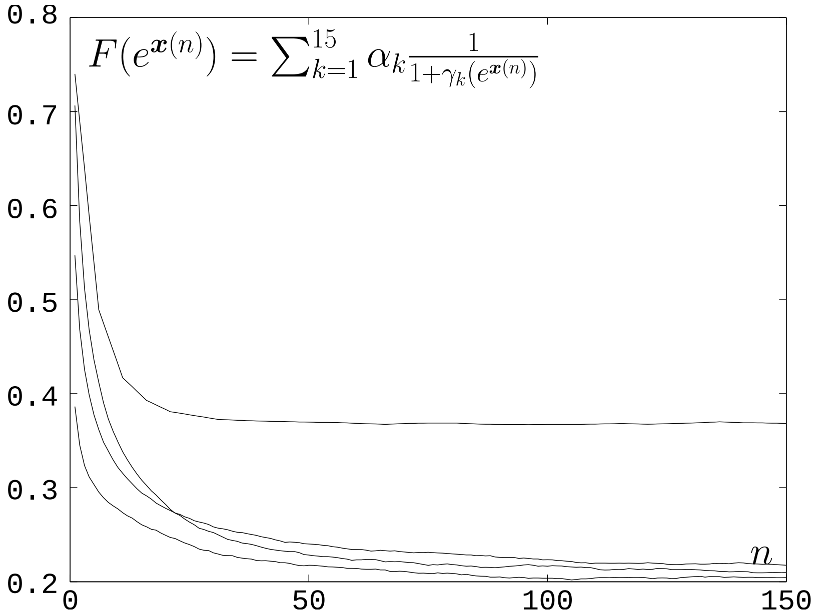

FIGURE 4.3: Exemplary convergence of aggregated performance obtained by Algorithm 4 with averaging (4.26). The settings are as in Figure 4.2 except that $\Psi(\gamma) = 1/(1+\gamma)$ and the variance of estimates in steps 2, 6, 8 is $0.15\sigma_k^2$, $k \in \mathcal{K}$.

4.3.2 Approach with modified Lagrangian

Assuming implicitly that Condition 1 is satisfied, e.g. in the case when $\Phi = \Psi^{-1}$ is log-convex (Proposition 5), we consider the version (2.25) of the problem of aggregated performance optimization. We propose a splitting of the optimization variables as used for the characterization of network duality in problem form (4.7), but using the notion of the generalized SIR function (4.57), (4.58). Precisely, we reformulate (2.25) in the form

$$\min_{\boldsymbol{x}} \max_{\boldsymbol{I}} \sum_{k\in\mathcal{K}} \alpha_k \Psi(\frac{e^{x_k}}{I_k}), \quad \text{subject to} \begin{cases} e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \le 0 \\ I_k - t_k \le 0 \\ J_k(e^{\boldsymbol{x}}) - t_k = 0, \quad k \in \mathcal{K}. \end{cases} \tag{4.59}$$

The purpose of using the telescope variable $\boldsymbol{t} = (t_1, \dots, t_K) \in \mathbb{R}^K$ is merely the separation of the constraint inequalities for the minimization variables $\boldsymbol{x}$ and the maximization variables I_k, $k \in \mathcal{K}$, and thus the necessary separation of the corresponding constraint-related addends in the Lagrangian function (see, e.g., [85], Section 4). In fact, it is readily seen that the constraint inequalities of the problem are equivalent to $e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \le 0$ and $I_k - J_k(e^{\boldsymbol{x}}) \le 0$, $k \in \mathcal{K}$. Due to the assumption of decreasing function Ψ, it is readily seen that the maximum in (4.59) is attained for the tuple of largest I_k, $k \in \mathcal{K}$, which satisfy the constraints in (4.59). Thus, for the solution of (4.59) it is necessary that $I_k = t_k$ and thus $I_k = J_k(e^{\boldsymbol{x}})$, $k \in \mathcal{K}$, which makes the equivalence of (4.59) and (2.25), with (4.57) evident. Analogously to Section 4.2.2, we refer to $\boldsymbol{x} \in \mathbb{R}^K$ as *feasible* if it satisfies the constraint inequalities in (4.59).

Definition 4 *Given problem (4.59), let the Lagrangian function*

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) = &\sum_{k\in\mathcal{K}} \alpha_k \Psi(\frac{e^{x_k}}{I_k}) + \sum_{k\in\mathcal{K}} \phi(\mu_k)(e^{x_k} - \hat{p}_k) \\ &+ \sum_{k\in\mathcal{K}} \lambda_k^J (J_k(e^{\boldsymbol{x}}) - t_k) - \sum_{k\in\mathcal{K}} \phi(\lambda_k^I)(I_k - t_k), \quad (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}, \end{aligned} \tag{4.60}$$

where the function ϕ satisfies the following condition.

Condition 8 *Function $\mu \mapsto \phi(\mu)$, $\mu \in \mathbb{R}$, is twice differentiable and*
i.) $\phi(\mu) = \phi(-\mu)$, $\mu \in \mathbb{R}$ (even),
ii.) $\phi(\mu) \ge 0$, $\mu \in \mathbb{R}$ (nonnegative),
iii.) $\phi''(\mu) > 0$, $\mu \in \mathbb{R}$ (strictly convex),
iv.) $\mu = 0$ iff $\phi(\mu) = 0$ iff $\phi'(\mu) = 0$ (unique irregularity as a minimum with value 0 at 0).

Among numerous functions satisfying Condition 7, the simplest one appears to be, again, $\phi(\mu) = \mu^2$.

The following proposition is a straightforward consequence of Condition 8 and the definition of the Kuhn-Tucker conditions (Definition 14). When Condition 6 is replaced by Condition 8, then Proposition 15 can be seen as a generalization of the following result to the case of Lagrangian (4.31)).

Proposition 19 *Let $\pm\boldsymbol{\mu} \in \mathbb{R}^K$ denote either $(\pm\boldsymbol{\mu})_k = \mu_k$ or $(\pm\boldsymbol{\mu})_k = -\mu_k$ independently for $k \in \mathcal{K}$. If $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{2K} \times \mathbb{R}_+^{2K} \times \mathbb{R}^{2K}$ is a Kuhn-Tucker point of problem (4.59), then $(\boldsymbol{x}, \boldsymbol{I}, \pm\boldsymbol{\mu}, \pm\boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t})$, with*

$$\nu_k = \phi(\pm\mu_k), \quad \eta_k = \phi(\pm\lambda_k^I) = \lambda_k^J, \qquad k \in \mathcal{K}, \tag{4.61}$$

is a stationary point of Lagrangian (4.60). Conversely, if $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$*, with* $\boldsymbol{x}$ *feasible, is a stationary point of Lagrangian (4.60), then* $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t})$*, with* $\boldsymbol{\nu}$*,* $\boldsymbol{\eta}$ *given by (4.61), is a Kuhn-Tucker point of problem (4.59).*

Proof Let $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{2K} \times \mathbb{R}_+^{2K} \times \mathbb{R}^{2K}$ be a Kuhn-Tucker point of problem (4.59), let (4.61) be satisfied, and define $\mathcal{T}(\bar{\boldsymbol{x}}) = \{k \in \mathcal{K} : e^{\bar{x}_k} - \hat{p}_k = 0\}$ and $\mathcal{L}(\bar{\boldsymbol{I}}, \bar{\boldsymbol{t}}) = \{k \in \mathcal{K} : \bar{I}_k - \bar{t}_k = 0\}$, for $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{t}}) \in \mathbb{R}^{3K}$. Then, it is evident from (4.60) and the classical linear Lagrangian

$$\begin{aligned}\bar{L}(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) = &\sum_{k \in \mathcal{K}} \alpha_k \Psi(\frac{e^{x_k}}{I_k}) + \sum_{k \in \mathcal{K}} \nu_k (e^{x_k} - \hat{p}_k) \\ &+ \sum_{k \in \mathcal{K}} \lambda_k^J (J_k(e^{\boldsymbol{x}}) - t_k) - \sum_{k \in \mathcal{K}} \eta_k (I_k - t_k), \quad (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{2K} \times \mathbb{R}_+^{2K} \times \mathbb{R}^{2K}.\end{aligned} \tag{4.62}$$

that $\nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{t}})} \bar{L}(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) = 0$ implies $\nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{t}})} L(\pm \boldsymbol{z}) = 0$, with $\pm \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \pm \boldsymbol{\mu}, \pm \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t})$. Further, by the second and fourth expression in (4.71) we have $\frac{\partial}{\partial \bar{\mu}_k} L(\pm \boldsymbol{z}) = 0$, $k \in \mathcal{T}(\boldsymbol{x})$, and $\frac{\partial}{\partial \bar{\lambda}_k^I} L(\pm \boldsymbol{z}) = 0$, $k \in \mathcal{L}(\boldsymbol{I}, \boldsymbol{t})$, by the definitions of $\mathcal{T}(\boldsymbol{x})$ and $\mathcal{L}(\boldsymbol{I}, \boldsymbol{t})$, respectively. In parallel, the second and fourth expression in (4.71) imply $\frac{\partial}{\partial \bar{\mu}_k} L(\pm \boldsymbol{z}) = 0$, $k \in \mathcal{K} \setminus \mathcal{T}(\boldsymbol{x})$, and $\frac{\partial}{\partial \bar{\lambda}_k^I} L(\pm \boldsymbol{z}) = 0$, $k \in \mathcal{K} \setminus \mathcal{L}(\boldsymbol{I}, \boldsymbol{t})$, due to the complementary slackness conditions $\nu_k = 0$, $k \in \mathcal{K} \setminus \mathcal{T}(\boldsymbol{x})$, and $\eta_k = 0$, $k \in \mathcal{K} \setminus \mathcal{T}(\boldsymbol{I}, \boldsymbol{t})$ and Condition 8 iv.). Finally, it is obvious by the constraints in (4.59) that $\nabla_{\bar{\boldsymbol{\lambda}}^J} L(\pm \boldsymbol{z}) = 0$, which completes the proof of $\pm \boldsymbol{z}$ as a stationary point of (4.60).

Conversely, let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$, with $\boldsymbol{x}$ feasible, be a stationary point of Lagrangian (4.60) and let (4.61) hold. We yield $\boldsymbol{\nu} \geq 0$, $\boldsymbol{\eta} \geq 0$ immediately by Condition 8. Further, $\nabla_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^I)} L(\boldsymbol{z}) = 0$ yields with the second and fourth expression in (4.71) and Condition 8 iv.) that

$$\nu_k (e^{x_k} - \hat{p}_k) = 0, \quad \eta_k^I (I_k - t_k) = 0, \quad k \in \mathcal{K}, \tag{4.63}$$

which are the complementary slackness conditions. By $\nabla_{\bar{\boldsymbol{I}}} L(\boldsymbol{z}) = 0$ and the third expression in (4.71), it follows that $\phi(\lambda_k^I) > 0$, $k \in \mathcal{K}$, since Ψ is decreasing and $\boldsymbol{\alpha} > 0$. This implies further $I_k - t_k = 0$, $k \in \mathcal{K}$, due to (4.63). Also, $e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \leq 0$ is obvious from feasibility and $J_k(e^{\boldsymbol{x}}) - t_k = 0$, $k \in \mathcal{K}$, is immediate from $\nabla_{\bar{\boldsymbol{\lambda}}^J} L(\boldsymbol{z}) = 0$. Finally, it is evident from (4.60), (4.62) and (4.61) that $\nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{t}})} \bar{L}(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t}) = 0$ follows from $\nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{t}})} L(\boldsymbol{z}) = 0$, which completes the proof of $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\nu}, \boldsymbol{\eta}, \boldsymbol{\lambda}^J, \boldsymbol{t})$ as a Kuhn-Tucker point. □

By the Lagrangian construction, a stationary point of (4.60), associated by (4.61) with a Kuhn-Tucker point, can be found by unconstrained iterations. This parallels the property of the generalized Lagrangian (4.31) and stands again in contrast to conventional primal-dual iterations, which solve the nonnegatively constrained dual problem (right-hand side of (4.6)).

The Lagrangean duality theory, with key elements outlined in Section 4.1.2, is straightforwardly extendable to the case of modified Lagrangian (4.60) [48], [86]. Precisely, the interest is in finding a stationary point being a max-min and a min-max point, that is, satisfying

$$\tilde{\boldsymbol{z}} = \arg \max_{(\boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^{2K}} \min_{(\boldsymbol{x}, \boldsymbol{I}) \in S(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{I}})} L(\boldsymbol{z}), \tag{4.64}$$

and

$$\tilde{\boldsymbol{z}} = \arg \min_{(\boldsymbol{x}, \boldsymbol{I}) \in S(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{I}})} \sup_{(\boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^{2K}} L(\boldsymbol{z}), \tag{4.65}$$

with $S(\boldsymbol{y})$ as a (sufficiently small) neighborhood of $\boldsymbol{y} \in \mathbb{R}^{2K}$. Analogously to the case of the classical linear Lagrangean function [48], [86], it is easily shown that if $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{I}}, \tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\lambda}}) \in \mathbb{R}^{4K}$ satisfies (4.64) and (4.65), then it is a local minimizer of problem (4.59) (see also the discussion in [72]). For finding the desired stationary point of (4.60) we construct the iteration taking the form

$$\begin{cases} \begin{pmatrix} \boldsymbol{x}(n+1) \\ \boldsymbol{\mu}(n+1) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}(n) \\ \boldsymbol{\mu}(n) \end{pmatrix} - \left(\nabla^2_{(\boldsymbol{x},\boldsymbol{\mu})} L(\boldsymbol{z}(n))\right)^{-1} \nabla_{(\boldsymbol{x},\boldsymbol{\mu})} L(\boldsymbol{z}(n)) \\ \nabla_{(\boldsymbol{I},\boldsymbol{\lambda}^I,\boldsymbol{\lambda}^J,\boldsymbol{t})} L(\boldsymbol{z}(n+1)) = 0, \end{cases} \quad \boldsymbol{z}(n) \in \mathbb{R}^{6K}, n \in \mathbb{N}. \tag{4.66}$$

Iteration (4.66) consists in applying the Newton update with respect to variables $(\boldsymbol{x}, \boldsymbol{\mu}) \in \mathbb{R}^{2K}$ and enforcing a stationary point of Lagrangian (4.60) with respect to the remaining variables $(\boldsymbol{I}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{4K}$. Thus, it can be classified as a conditional Newton iteration or Newton iteration under reduced dimensionality. By condition (4.58), it is readily seen that

$$\frac{\partial^2}{\partial x_k \partial x_j} L(\boldsymbol{z}) = 0, \quad \frac{\partial^2}{\partial \mu_k \partial \mu_j} L(\boldsymbol{z}) = 0, \quad \frac{\partial^2}{\partial \mu_k \partial x_j} L(\boldsymbol{z}) = 0, \quad \text{for} \quad k \neq j, \quad k, j \in \mathcal{K}, \tag{4.67}$$

i.e. the blocks of the Hessian matrix in (4.66) have the crucial property of being diagonal. Since, by the standard four-block inverse expression [87], we can write

$$\left(\nabla^2_{(\boldsymbol{x},\boldsymbol{\mu})} L(\boldsymbol{z})\right)^{-1} = \begin{pmatrix} \left(\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}) - \nabla^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) (\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}))^{-1} \nabla'^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z})\right)^{-1} & \left(\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z})\right)^{-1} \nabla^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) \left(\nabla'^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) (\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}))^{-1} \nabla^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) - \nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z})\right)^{-1} \\ \left(\nabla'^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) (\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}))^{-1} \nabla^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) - \nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z})\right)^{-1} \nabla'^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) (\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}))^{-1} & \left(\nabla^2_{\boldsymbol{\mu}} L(\boldsymbol{z}) - \nabla'^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z}) (\nabla^2_{\boldsymbol{x}} L(\boldsymbol{z}))^{-1} \nabla^2_{\boldsymbol{x},\boldsymbol{\mu}} L(\boldsymbol{z})\right)^{-1} \end{pmatrix}, \tag{4.68}$$

it follows further that the inverse Hessian, if existent, has the same structure as the Hessian itself: The two blocks on its block-diagonal and the two outer blocks are all diagonal matrices. Thus, the Newton update term of any k-th component of $\boldsymbol{x} \in \mathbb{R}^K$ and $\boldsymbol{\mu} \in \mathbb{R}^K$ in (4.66) is a linear combination of two gradient components; $(\nabla_{\boldsymbol{x}} L(\boldsymbol{z}))_k$ and $(\nabla_{\boldsymbol{\mu}} L(\boldsymbol{z}))_k$, $k \in \mathcal{K}$. A further property of (4.66) and Lagrangian (4.60) is that $\nabla_{(\boldsymbol{I},\boldsymbol{\lambda}^I,\boldsymbol{\lambda}^J,\boldsymbol{t})} L(\boldsymbol{z}) = 0$ can be expressed as an explicit map yielding $(\boldsymbol{I}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{4K}$ as a function of $(\boldsymbol{x}, \boldsymbol{\mu}) \in \mathbb{R}^{2K}$. These features yield the following reformulation of (4.66).

Lemma 9 *For any $n \in \mathbb{N}$, the power control iteration (4.66) can be written equivalently as*

$$\begin{cases} x_k(n+1) = & x_k(n) + \frac{(e^{x_k(n)} - \hat{p}_k)}{s_k(n)} (\phi''(\mu_k(n))(\alpha_k \Psi'_e(\log \frac{e^{x_k(n)}}{I_k(n)}) + \phi(\mu_k(n)) e^{x_k(n)} + \\ & \sum_{j \in \mathcal{K}} \lambda^J_j \frac{\partial}{\partial x_k} J_j(e^{\boldsymbol{x}(n)})) - (\phi'(\mu_k(n)))^2 e^{x_k(n)}) \\ \mu_k(n+1) = & \mu_k(n) + \frac{\phi'(\mu_k(n))}{s_k(n)} ((e^{x_k(n)} - \hat{p}_k)(\alpha_k \Psi''_e(\log \frac{e^{x_k(n)}}{I_k(n)}) + \\ & \phi(\mu_k(n)) e^{x_k(n)} + \sum_{j \in \mathcal{K}} \lambda^J_j \frac{\partial^2}{\partial x_k^2} J_j(e^{\boldsymbol{x}(n)})) - \\ & e^{x_k(n)} (\alpha_k \Psi'_e(\log \frac{e^{x_k(n)}}{I_k(n)}) + \phi(\mu_k(n)) e^{x_k(n)} + \sum_{j \in \mathcal{K}} \lambda^J_j \frac{\partial}{\partial x_k} J_j(e^{\boldsymbol{x}(n)}))) \\ I_k(n+1) = & J_k(e^{\boldsymbol{x}(n+1)}) \\ \lambda^J_k(n+1) = & -\frac{\alpha_k}{I_k} \Psi'_e(\log \frac{e^{x_k(n+1)}}{I_k(n+1)}), \end{cases} \quad k \in \mathcal{K}, \tag{4.69}$$

where additionally $\phi(\lambda_k^I(n+1)) = \lambda_k^J(n+1)$, $t_k(n+1) = I_k(n+1)$, $k \in \mathcal{K}$, *and where we defined*

$$\begin{aligned} s_k(n) =&(\phi'(\mu_k(n)))^2 e^{2x_k(n)} - \phi''(\mu_k(n))(e^{x_k(n)} - \hat{p}_k)(\alpha_k \Psi_e''(\frac{e^{x_k(n)}}{I_k(n)})+ \\ &\phi(\mu_k(n))e^{x_k(n)} + \sum_{j\in\mathcal{K}} \lambda_j^J \frac{\partial^2}{\partial x_k^2} J_j(e^{\boldsymbol{x}(n)})), \end{aligned} \quad k \in \mathcal{K}. \tag{4.70}$$

Proof The Lemma is obtained by applying elementary calculus to (4.60), (4.66) and (4.68). The main steps are the following. Given $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$, we have

$$\begin{aligned} (\nabla_{\bar{\boldsymbol{x}}} L(\boldsymbol{z}))_k &= \frac{\partial}{\partial \bar{x}_k} L(\boldsymbol{z}) = \alpha_k \Psi_e'(\log \frac{e^{x_k}}{I_k}) + \phi(\mu_k)e^{x_k} + \sum_{j\in\mathcal{K}} \lambda_j^J \frac{\partial}{\partial x_k} J_j(e^{\boldsymbol{x}}) \\ (\nabla_{\bar{\boldsymbol{\mu}}} L(\boldsymbol{z}))_k &= \frac{\partial}{\partial \bar{\mu}_k} L(\boldsymbol{z}) = \phi'(\mu_k)(e^{x_k} - \hat{p}_k) \\ (\nabla_{\bar{\boldsymbol{I}}} L(\boldsymbol{z}))_k &= \frac{\partial}{\partial \bar{I}_k} L(\boldsymbol{z}) = -\alpha_k \Psi'(\frac{e^{x_k}}{I_k})\frac{e^{x_k}}{I_k^2} - \phi(\lambda_k^I) \qquad\qquad k \in \mathcal{K}, \\ (\nabla_{\bar{\boldsymbol{\lambda}}^I} L(\boldsymbol{z}))_k &= \frac{\partial}{\partial \bar{\lambda}_k^I} L(\boldsymbol{z}) = \phi'(\lambda_k^I)(I_k - t_k), \\ (\nabla_{\bar{\boldsymbol{\lambda}}^J} L(\boldsymbol{z}))_k &= \frac{\partial}{\partial \bar{\lambda}_k^J} L(\boldsymbol{z}) = J_k(e^{\boldsymbol{x}}) - t_k, \quad (\nabla_{\bar{\boldsymbol{t}}} L(\boldsymbol{z}))_k = \frac{\partial}{\partial \bar{t}_k} L(\boldsymbol{z}) = \lambda_k^J - \phi(\lambda_k^I) \end{aligned} \tag{4.71}$$

Making use of the property $\Psi_e''(y) = \Psi''(e^y)e^{2y} + \Psi'(e^y)e^y$, $y \in \mathbb{R}$, and (4.58), we get (4.67) and

$$\begin{aligned} (\nabla_{\bar{\boldsymbol{x}}}^2 L(\boldsymbol{z}))_{kk} &= \frac{\partial^2}{\partial \bar{x}_k^2} L(\boldsymbol{z}) = \alpha_k \Psi_e''(\log \frac{e^{x_k}}{I_k}) + \phi(\mu_k)e^{x_k} + \sum_{j\in\mathcal{K}} \lambda_j^J \frac{\partial^2}{\partial x_k^2} J_j(e^{\boldsymbol{x}}) \\ (\nabla_{\bar{\boldsymbol{\mu}}}^2 L(\boldsymbol{z}))_{kk} &= \frac{\partial^2}{\partial \bar{\mu}_k^2} L(\boldsymbol{z}) = \phi''(\mu_k)(e^{x_k} - \hat{p}_k) \qquad\qquad k \in \mathcal{K}. \\ (\nabla_{\bar{\boldsymbol{x}},\bar{\boldsymbol{\mu}}}^2 L(\boldsymbol{z}))_{kk} &= \frac{\partial^2}{\partial \bar{x}_k \partial \bar{\mu}_k} L(\boldsymbol{z}) = \phi'(\mu_k)e^{x_k}, \end{aligned} \tag{4.72}$$

□

4.3.3 Local convergence and duality

We have the following general (local) quadratic convergence result. Quadratic convergence is the fastest achievable one under the use of up to second-order characteristics of the problem in the iteration.

Proposition 20 *Let* $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$ *be a stationary point of Lagrangian (4.60) which corresponds through (4.61) to a Kuhn-Tucker point of problem (4.59) and is such that* $\nabla^2_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{\mu}})} L(\bar{\boldsymbol{z}})$ *is continuous on some* $S(\boldsymbol{z})$ *and nonsingular for* $\bar{\boldsymbol{z}} = \boldsymbol{z}$. *Then,* $\boldsymbol{z}$ *is a point of attraction of iteration (4.66), and if additionally the maps* Ψ'' *and* $\bar{\boldsymbol{x}} \mapsto \frac{\partial^2}{\partial \bar{x}_k^2} J_j(e^{\bar{\boldsymbol{x}}})$, $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $k, j \in \mathcal{K}$, *are continuous, then we have quadratic quotient convergence in the sense that* $O_Q(\boldsymbol{z}) \geq 2$.

Proof Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$ be a stationary point of (4.60) associated through (4.61) with a Kuhn-Tucker point of problem (4.59) and let map $\nu \mapsto \psi(\nu)$, $\nu \in \mathbb{R}_+$, be such that $\psi(\phi(\mu)) = \mu$, for $\mu \in \mathbb{R}_+$ (inverse map of the restriction of ϕ to $\mathbb{R}_+$). Further, define maps $\bar{\boldsymbol{x}} \mapsto F(\bar{\boldsymbol{x}}) \in \mathbb{R}^K$, and $\bar{\boldsymbol{x}} \mapsto H(\bar{\boldsymbol{x}}) \in \mathbb{R}^K$, $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, such that $(F(\bar{\boldsymbol{x}}))_k = J_k(e^{\bar{\boldsymbol{x}}})$ and $(H(\bar{\boldsymbol{x}}))_k = -\alpha_k \Psi'(\frac{e^{\bar{x}_k}}{J_k(e^{\bar{\boldsymbol{x}}})})\frac{e^{\bar{x}_k}}{J_k^2(e^{\bar{\boldsymbol{x}}})}$, $k \in \mathcal{K}$. By (4.71) and Condition 7 it is now readily seen that the stationary point $\nabla L(\boldsymbol{z}) = 0$ can be expressed as

$$\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x}, \boldsymbol{\mu}) = (\boldsymbol{x}, F(\boldsymbol{x}), \boldsymbol{\mu}, \pm\psi(H(\boldsymbol{x})), H(\boldsymbol{x}), F(\boldsymbol{x})). \tag{4.73}$$

Now, let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) \mapsto G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})$, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) \in \mathbb{R}^{2K}$, be defined as

$$G(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) = (\bar{\boldsymbol{x}}' \ \bar{\boldsymbol{\mu}}')' - (\nabla^2_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\bar{\boldsymbol{x}}, F(\bar{\boldsymbol{x}}), \bar{\boldsymbol{\mu}}, \pm\psi(H(\bar{\boldsymbol{x}})), H(\bar{\boldsymbol{x}}), F(\bar{\boldsymbol{x}})))^{-1} \times \\ \nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\bar{\boldsymbol{x}}, F(\bar{\boldsymbol{x}}), \bar{\boldsymbol{\mu}}, \pm\psi(H(\bar{\boldsymbol{x}})), H(\bar{\boldsymbol{x}}), F(\bar{\boldsymbol{x}})).$$

By the nonsingularity assumption and continuity of F and H, map G is well-defined for $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) = (\boldsymbol{x}, \boldsymbol{\mu})$ and continuous on some $S(\boldsymbol{x}, \boldsymbol{\mu})$. Using the definitions and Lemma 9, precisely the last two equalities in (4.69), the Newton update in iteration (4.66) can be formulated, for any $n \in \mathbb{N}$, as

$$(\boldsymbol{x}(n+1), \boldsymbol{\mu}(n+1)) = G(\boldsymbol{x}(n), \boldsymbol{\mu}(n)), \tag{4.74}$$

while the remaining iterates are obtained by maps

$$(\boldsymbol{I}(n{+}1), \boldsymbol{\lambda}^I(n{+}1), \boldsymbol{\lambda}^J(n{+}1), \boldsymbol{t}(n{+}1)) = (F(\boldsymbol{x}(n{+}1)), \pm\psi(H(\boldsymbol{x}(n{+}1))), H(\boldsymbol{x}(n{+}1)), F(\boldsymbol{x}(n{+}1))). \tag{4.75}$$

By (4.73) and assumption $\nabla L(\boldsymbol{z}) = 0$ we have that $(\boldsymbol{x}, \boldsymbol{\mu})$ is a fixed point of map G and also

$$\nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} G(\boldsymbol{x}, \boldsymbol{\mu}) = \boldsymbol{I} - \nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} (\nabla^2_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\boldsymbol{z}))^{-1} \nabla_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\boldsymbol{z}) - (\nabla^2_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\boldsymbol{z}))^{-1} \nabla^2_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}})} L(\boldsymbol{z}) = 0.$$

($(\boldsymbol{x}, \boldsymbol{\mu})$ is a stationary point of G). This implies with 10.1.6 in [79], that $(\boldsymbol{x}, \boldsymbol{\mu})$ is a point of attraction of (4.74), which is the Newton update in (4.66). Since the remaining iterates in (4.66) are obtainable by the fixed map (4.75), it follows with (4.73) that $\boldsymbol{z}$ is a point of attraction of iteration (4.66).

From the formulation of iteration (4.66) as (4.74), (4.75) it is evident that the map (4.75) has no influence on the quotient convergence (Appendix A.3.2). Thus, for the order of quotient convergence of iteration (4.66) to $\boldsymbol{z}$ we have $O_Q(\boldsymbol{z}) = O_Q^G(\boldsymbol{x}, \boldsymbol{\mu})$, where $O_Q^G(\boldsymbol{x}, \boldsymbol{\mu})$ denotes the order of quotient convergence of (4.74) to $(\boldsymbol{x}, \boldsymbol{\mu})$. Iteration (4.74) is the conventional Newton iteration applied to the function

$$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) \mapsto L(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) = (\bar{\boldsymbol{x}}, F(\bar{\boldsymbol{x}}), \bar{\boldsymbol{\mu}}, \pm\psi(H(\bar{\boldsymbol{x}})), H(\bar{\boldsymbol{x}}), F(\bar{\boldsymbol{x}})), \quad (\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}) \in \mathbb{R}^{2K}. \tag{4.76}$$

Under continuity of Ψ'' and $\bar{\boldsymbol{x}} \mapsto \frac{\partial^2}{\partial \bar{x}_k^2} J_j(e^{\bar{\boldsymbol{x}}})$, $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $k, j \in \mathcal{K}$, it is evident from (4.60) and the definitions of F and H that the Hessian of (4.76) is continuous as well, and thus, also Lipschitz continuous on some $S(\boldsymbol{x}, \boldsymbol{\mu})$ [88]. With the general Newton Attraction Theorem (10.2.2 in [79]) this implies finally $O_Q^G(\boldsymbol{x}, \boldsymbol{\mu}) \geq 2$, which completes the proof due to $O_Q(\boldsymbol{z}) = O_Q^G(\boldsymbol{x}, \boldsymbol{\mu})$. □

Note that the *unconstrained* stationary point property is the central element of the proof of Proposition 20, which underlines the role of the modified Lagrangian (Proposition 19). An analogous application of iteration (4.66) to the conventional Lagrangian (4.62) would guarantee local convergence only under additional mapping/ projection mechanisms and such convergence would be no more quadratic, in general.

Algorithm (4.66) does not, in general, ensure a monotone descent of the aggregated performance value. Monotone descent may be, however, enforced, under retained quadratic convergence, by introducing in the iteration over $(\boldsymbol{x},\boldsymbol{\mu}) \in \mathbb{R}^{2K}$ in (4.66) a damping factor sequence $a(n)$, $n \in \mathbb{N}$, such that for some $n_0 \in \mathbb{N}$ we have $a(n) = 1$, $n \geq n_0$ (for details we refer to [79]).

In order to find a local minimizer

$$(\boldsymbol{x},\boldsymbol{I}) = \arg \min_{\bar{\boldsymbol{x}} \in S(\boldsymbol{x})} \max_{\bar{\boldsymbol{I}} \in S(\boldsymbol{I})} \sum_{k \in \mathcal{K}} \alpha_k \Psi(\frac{e^{\bar{x}_k}}{\bar{I}_k}), \quad \text{such that} \begin{cases} e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \leq 0 \\ I_k - J_k(e^{\boldsymbol{x}}) \leq 0, \quad k \in \mathcal{K}, \end{cases} \tag{4.77}$$

by means of (4.66), the point of attraction of (4.66) is desired to be a max-min and min-max point satisfying (4.64), (4.65). This case is characterized in the following result.

Proposition 21 *A stationary point $(\boldsymbol{x},\boldsymbol{I},\boldsymbol{\mu},\boldsymbol{\lambda}^I,\boldsymbol{\lambda}^J,\boldsymbol{t}) \in \mathbb{R}^{6K}$ of Lagrangian (4.60) corresponds to a local problem solution (4.77) if either Lagrangian (4.60) is a min-max function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on some $S(\boldsymbol{x},\boldsymbol{I})$ with feasible $\boldsymbol{x}$, which is equivalent to*

$$\alpha_k \varphi_e''(\log \frac{e^{x_k}}{I_k}) \geq -\phi(\mu_k) e^{x_k} - \sum_{j \in \mathcal{K}} \lambda_j^J \frac{\partial^2}{\partial \bar{x}_k^2} J_j(e^{\boldsymbol{x}}), \qquad \Psi_e''(\log \frac{e^{x_k}}{I_k}) + \Psi_e'(\log \frac{e^{x_k}}{I_k}) < 0, \tag{4.78}$$

for feasible $\boldsymbol{x}$, or Lagrangian (4.60) is a convex-concave function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on some $S(\boldsymbol{x},\boldsymbol{I})$ with feasible $\boldsymbol{x}$, which is equivalent to

$$\alpha_k \Psi_e''(\log \frac{e^{x_k}}{I_k}) \geq -\phi(\mu_k) e^{x_k} - \sum_{j \in \mathcal{K}} \lambda_j^J \frac{\partial^2}{\partial \bar{x}_k^2} J_j(e^{\boldsymbol{x}}), \qquad \Psi_e''(\log \frac{e^{x_k}}{I_k}) + \Psi_e'(\log \frac{e^{x_k}}{I_k}) \leq 0, \tag{4.79}$$

for feasible $\boldsymbol{x}$, where we defined $y \mapsto \varphi_e''(y) = \frac{\Psi_e''(y)\Psi_e'(y)}{\Psi_e''(y)+\Psi_e'(y)}$ if $\Psi_e''(y) + \Psi_e'(y) \neq 0$ and $\varphi_e''(y) = 0$ otherwise, $y \in \mathbb{R}$.

Proof Let $\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{I},\boldsymbol{\mu},\boldsymbol{\lambda}^I,\boldsymbol{\lambda}^J,\boldsymbol{t}) \in \mathbb{R}^{6K}$ be a stationary point of Lagrangian (4.60) and assume first by contradiction that (4.60) is either a min-max function or a convex-concave function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on some $S(\boldsymbol{x},\boldsymbol{I})$, with $\boldsymbol{x}$ feasible, but $\boldsymbol{z}$ does not correspond to a local solution (4.77). The classical Lagrangian duality results extend straightforwardly to min-max problems (see [85], Section 4) and to generalized Lagrangians (see, e.g. [74], [88], [73]): It is readily seen that $\boldsymbol{z}$ is associated with a local solution (4.77) if and only if

$$\boldsymbol{z} = \arg \min_{\bar{\boldsymbol{x}} \in S(\boldsymbol{x})} \max_{\bar{\boldsymbol{I}} \in S(\boldsymbol{I})} \sup_{(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \inf_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} L(\bar{\boldsymbol{z}}). \tag{4.80}$$

From the fourth expression in (4.71) and Condition 7 iii.) we have that $\nabla^2_{\bar{\boldsymbol{\lambda}}^I} L(\bar{\boldsymbol{z}}) = 0$ whenever $\bar{\boldsymbol{I}} - \bar{\boldsymbol{t}} \leq 0$. Similarly, by the fifth expression in (4.71), the second expression in (4.72) and Condition iii.) we yield $\nabla^2_{(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J)} L(\bar{\boldsymbol{z}}) \preceq 0$, whenever $\bar{\boldsymbol{x}}$ is feasible. In particular, this implies with Appendix A.4.1 that (4.60) is concave-convex in $(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}$, $\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K$ on some $S(\boldsymbol{z})$, and thus, with the assumptions, it is either a min-max-concave-convex function or a convex-concave-concave-convex function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$, $(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}$, $\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K$ on some $S(\boldsymbol{z})$. Thus, the application of Propositions 44 and 45 to the map $(\bar{\boldsymbol{x}},\bar{\boldsymbol{I}},\bar{\boldsymbol{t}}) \mapsto F(\bar{\boldsymbol{x}},\bar{\boldsymbol{I}},\bar{\boldsymbol{t}}) = \sup_{(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \inf_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} L(\bar{\boldsymbol{x}},\bar{\boldsymbol{I}},\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^I,\bar{\boldsymbol{\lambda}}^J,\bar{\boldsymbol{t}})$, $(\bar{\boldsymbol{x}},\bar{\boldsymbol{I}},\bar{\boldsymbol{t}}) \in \mathbb{R}^{3K}$, and the property $\nabla_{(\bar{\boldsymbol{x}},\bar{\boldsymbol{I}})} F(\boldsymbol{x},\boldsymbol{I},\boldsymbol{t}) = 0$ imply (4.80), which contradicts the assumptions. Additionally, the straightforward extension of Proposition 44 to (concatenated) min-max-concave-convex or convex-concave-concave-convex functions shows that we have the saddle point property at $\boldsymbol{z}$, consisting in (4.80) and $\boldsymbol{z} = \max_{(\bar{\boldsymbol{\mu}},\bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \min_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} \min_{\bar{\boldsymbol{x}} \in S(\boldsymbol{x})} \max_{\bar{\boldsymbol{I}} \in S(\boldsymbol{I})} L(\bar{\boldsymbol{z}})$.

By Definition 25, Lagrangian (4.60) is a min-max function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on some (vanishingly small) $S(\boldsymbol{x}, \boldsymbol{I})$, if and only if

$$\nabla^2_{\bar{\boldsymbol{x}}} L(\boldsymbol{z}) - \nabla'^2_{\bar{\boldsymbol{x}},\bar{\boldsymbol{I}}} L(\boldsymbol{z}) (\nabla^2_{\bar{\boldsymbol{I}}} L(\boldsymbol{z}))^{-1} \nabla^2_{\bar{\boldsymbol{x}},\bar{\boldsymbol{I}}} L(\boldsymbol{z}) \succeq 0, \quad \nabla^2_{\bar{\boldsymbol{I}}} L(\boldsymbol{z}) \prec 0, \tag{4.81}$$

where, using assumption (4.58) and $\Psi''_e(y) = \Psi''(e^y)e^{2y} + \Psi'(e^y)e^y$, $y \in \mathbb{R}$, we can write

$$\begin{aligned} (\nabla^2_{\bar{\boldsymbol{I}}} L(\bar{\boldsymbol{z}}))_{kk} &= \frac{\partial^2}{\partial \bar{I}_k^2} L(\bar{\boldsymbol{z}}) = \frac{\alpha_k}{\bar{I}_k^2} (\Psi''_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}) + \Psi'_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k})) \\ (\nabla^2_{\bar{\boldsymbol{x}},\bar{\boldsymbol{I}}} L(\bar{\boldsymbol{z}}))_{kk} &= \frac{\partial^2}{\partial \bar{x}_k \partial \bar{I}_k} L(\bar{\boldsymbol{z}}) = -\alpha_k \Psi''_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}), \end{aligned} \qquad k \in \mathcal{K}, \tag{4.82}$$

and $\frac{\partial^2}{\partial \bar{I}_k \partial \bar{I}_l} L(\bar{\boldsymbol{z}}) = 0$, $\frac{\partial^2}{\partial \bar{x}_k \partial \bar{I}_l} L(\bar{\boldsymbol{z}}) = 0$, $k \neq l$, $k, l \in \mathcal{K}$, for any $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{I}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^I, \bar{\boldsymbol{\lambda}}^J, \bar{\boldsymbol{t}}) \in \mathbb{R}^{6K}$. Thus, with $\boldsymbol{\alpha} > 0$, the second inequality in (4.81) is equivalent to the second expression in (4.78). Together with the first expression in (4.72), this lets us rewrite the first inequality in (4.81) as

$$(\Psi''_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}) + \Psi'_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}))(\phi(\bar{\mu}_k)e^{\bar{x}_k} + \sum_{j \in \mathcal{K}} \bar{\lambda}^J_j \frac{\partial^2}{\partial x_k^2} J_j(e^{\bar{\boldsymbol{x}}})) + \alpha_k \Psi'_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}) \Psi''_e(\log \frac{e^{\bar{x}_k}}{\bar{I}_k}) \leq 0, \quad k \in \mathcal{K},$$

so that the application of φ''_e yields now the first expression in (4.78).

By Appendix A.4.1, Lagrangian (4.60) is a convex-concave function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on a vanishingly small $S(\boldsymbol{x}, \boldsymbol{I})$, if and only if $\nabla^2_{\bar{\boldsymbol{x}}} L(\bar{\boldsymbol{z}}) \succeq 0$, $\nabla^2_{\bar{\boldsymbol{I}}} L(\bar{\boldsymbol{z}}) \preceq 0$. With this, (4.79) is yielded now similarly by applying the first expressions from (4.72) and (4.82), which completes the proof. □

Propositions 20 and 21 make clear that if the iterates generated by (4.66) are attracted by a particular stationary point of (4.60) which satisfies either (4.78) or (4.79), then the iteration solves (4.59) locally. Furthermore, under continuity of second-order characteristics, the convergence to a local solution is quadratic in quotients.

From the proof of Proposition 21 it is evident that a stationary point of (4.60) is a desired point of attraction of iteration (4.66) if it represents a specific saddle point.

Corollary 2 *A stationary point* $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$ *of the Lagrangian (4.60) corresponds to a local problem solution (4.77) if* $\boldsymbol{z}$ *is a saddle point of the type*

$$\boldsymbol{z} = \arg \min_{\bar{\boldsymbol{x}} \in S(\boldsymbol{x})} \max_{\bar{\boldsymbol{I}} \in S(\boldsymbol{I})} \sup_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \inf_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} L(\bar{\boldsymbol{z}}) = \arg \max_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \min_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} \min_{\bar{\boldsymbol{x}} \in S(\boldsymbol{x})} \max_{\bar{\boldsymbol{I}} \in S(\boldsymbol{I})} L(\bar{\boldsymbol{z}}). \tag{4.83}$$

By the proof and theory of convex-concave and min-max functions, it is observed further that the saddle point property (4.83) is equivalent to convex-concavity or min-max property of Lagrangian (4.60) as a function of $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ on some $S(\boldsymbol{x}, \boldsymbol{I})$, and concave-convexity of (4.60) as a function of $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}$, $\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K$ on some $S(\boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J)$.

We verify the convergence of iteration (4.66) in exemplary simulations in Section 4.3.7.

4.3.4 THE UNIQUENESS CASE

The variety of desired and uninteresting points of attraction occurring in general motivates the need for the characterization of functions Ψ and J_k, $k \in \mathcal{K}$, for which any Kuhn-Tucker point of problem (4.59) corresponds to a global problem solution, i.e. (4.77), with $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$. By Proposition 19, this is equivalent to having any stationary point of (4.60) with feasible $\boldsymbol{x} \in \mathbb{R}^K$ being associated with a global problem solution. If such a property is offered by the link QoS function and link performance functions at hand, the proposed power allocation iteration accomplishes the optimization (4.59) *globally* in the following sense: Any point of attraction of iteration (4.66) with feasible $\boldsymbol{x} \in \mathbb{R}^K$ corresponds to a global problem solution (4.77), $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$.

The characterization of a class of performance functions Ψ and interference functions J_k, $k \in \mathcal{K}$, which ensures such feature follows from Proposition 21.

Corollary 3 *Any stationary point* $(\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$ *of Lagrangian (4.60), with* $\boldsymbol{x}$ *feasible, corresponds to a global problem solution (4.77),* $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$*, if Lagrangian (4.60) is a convex-concave function of* $\bar{\boldsymbol{x}} \in \mathbb{R}^K$*,* $\bar{\boldsymbol{I}} \in \mathbb{R}^K$ *for* $\bar{\boldsymbol{\lambda}}^J \geq 0$*,* $\bar{\boldsymbol{I}} - \bar{\boldsymbol{t}} \leq 0$*, and feasible* $\bar{\boldsymbol{x}}$*, which is implied by*

$$\Psi_e''(y) + \Psi_e'(y) \leq 0, \quad \Psi_e''(y) \geq 0, \qquad y \in \mathbb{R}, \tag{4.84}$$

and $\nabla^2 J_k(e^{\bar{\boldsymbol{x}}}) \succeq 0$*,* $\bar{\boldsymbol{x}} \in \mathbb{R}^K$*,* $k \in \mathcal{K}$*. Moreover, the global problem solution (4.77),* $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$*, is unique if the second inequality in (4.84) is strict.*

Proof Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{I}, \boldsymbol{\mu}, \boldsymbol{\lambda}^I, \boldsymbol{\lambda}^J, \boldsymbol{t}) \in \mathbb{R}^{6K}$ be a stationary point of (4.60) with feasible $\boldsymbol{x}$. For the proof of sufficient condition assume first, by contradiction, that $\boldsymbol{z}$ does not correspond to a global solution (4.77), $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$, while (4.60) is convex-concave in $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{I}}$ for $\bar{\boldsymbol{\lambda}}^J \geq 0$. Since $\nabla L(\boldsymbol{z}) = 0$, $\boldsymbol{\alpha} > 0$ and Ψ is decreasing, we get by the third and sixth expression in (4.71) that $\boldsymbol{\lambda}^J \geq 0$. As follows from the proof of Proposition 21, (4.60) is concave-convex in $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}$, $\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K$ whenever $\bar{\boldsymbol{I}} - \bar{\boldsymbol{t}} \leq 0$ and $\bar{\boldsymbol{x}}$ is feasible so that, with the assumptions, (4.60) is convex-concave-concave-convex in $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $\bar{\boldsymbol{I}} \in \mathbb{R}^K$, $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}$, $\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K$ whenever $\bar{\boldsymbol{\lambda}}^J \geq 0$, $\bar{\boldsymbol{I}} - \bar{\boldsymbol{t}} \leq 0$ and $\bar{\boldsymbol{\lambda}}^J \geq 0$. Thus, by Proposition 44 (precisely, by its trivial extension to concatenated convex-concavity), for the stationary point $\boldsymbol{z}$ follows

$$\boldsymbol{z} = \min_{\bar{\boldsymbol{x}} \in \mathbb{R}^K : e^{\bar{\boldsymbol{x}}} - \hat{\boldsymbol{p}} \leq 0} \; \max_{\bar{\boldsymbol{I}} \in \mathbb{R}^K : \bar{\boldsymbol{I}} - \bar{\boldsymbol{t}} \leq 0} \; \sup_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^K \times \mathbb{R}_+^K} \; \inf_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} L(\bar{\boldsymbol{z}}), \tag{4.85}$$

But (4.85) is easily verified to imply (4.77), $S(\boldsymbol{x}, \boldsymbol{I}) = \mathbb{R}^{2K}$ (extension of Lagrangian duality to min-max problem and generalized Lagrangian, see [85], [74], [88], [73]), which contradicts the assumptions.

With Condition 7 ii.), (4.84) and $\nabla J_k(e^{\bar{\boldsymbol{x}}}) \succeq 0$, $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, $k \in \mathcal{K}$, it is evident that (4.79) is satisfied for any $(\bar{\boldsymbol{I}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K} \times \mathbb{R}_+^K$ and feasible $\bar{\boldsymbol{x}}$. With (the proof of) Proposition 21 this further implies convex-concavity of (4.60) in $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{I}}$ for $\bar{\boldsymbol{\lambda}}^J \geq 0$ and feasible $\bar{\boldsymbol{x}}$. By (the proof of) Proposition 21 it is further evident that the first inequality in (4.79) is equivalent to $\nabla^2_{\bar{\boldsymbol{x}}} L(\boldsymbol{z}) \succeq 0$. Thus, when additionally $\Psi_e''(y) > 0$, $y \in \mathbb{R}$, the first inequality in (4.79) is strict and thus (4.60) is strictly convex in $\bar{\boldsymbol{x}} \in \mathbb{R}^K$ whenever $\bar{\boldsymbol{\lambda}}^J \geq 0$ and $\bar{\boldsymbol{x}}$ is feasible. Then, any stationary point $\boldsymbol{z}$ of (4.60) is associated with a unique $\boldsymbol{x} \in \mathbb{R}^K$, which further uniquely defines $I_k = J_k(e^{\boldsymbol{x}})$, $k \in \mathcal{K}$, via $\nabla L(\boldsymbol{z}) = 0$ and Condition 7 (see (4.71)). By Proposition 19, this shows that $(\boldsymbol{x}, \boldsymbol{I}) \in \mathbb{R}^{2K}$ is necessarily the unique solution to (4.59), which completes the proof. □

It can be verified that the approximation of the link capacity $\Psi(\gamma) = -\log(\gamma)$, $\gamma \geq 0$, or the average normalized symbol error rate under Rayleigh fading $\Psi(\gamma) = 1/\gamma$, $\gamma \geq 0$ (see Section 2.2), satisfy together with the linear interference function

$$J_k(\boldsymbol{p}) = (\boldsymbol{V}\boldsymbol{p})_k + \sigma_k^2, \quad \boldsymbol{p} \in \mathcal{P}, \quad k \in \mathcal{K}, \tag{4.86}$$

the conditions in Corollary 3. Thus, such functions guarantee that (4.66) accomplishes the optimization (4.59) globally, according to the above. Moreover, for the link performance function $\Psi(\gamma) = 1/\gamma$, $\gamma \geq 0$, we have $\Psi_e''(y) > 0$, $y \in \mathbb{R}$. By Corollary 3, this implies additionally that the global solution to problem (4.59) is unique and the point of attraction of (4.66) with feasible $\boldsymbol{x}$ is unique up to the component signs.

For any link QoS function satisfying (4.84) and any interference function such that $\bar{\boldsymbol{x}} \mapsto J_k(e^{\bar{\boldsymbol{x}}})$, $\bar{\boldsymbol{x}} \in \mathbb{R}^K$, is convex, any stationary point of Lagrangian (4.60) with feasible $\boldsymbol{x} \in \mathbb{R}^K$ is a saddle point of the type

$$\boldsymbol{z} = \arg \min_{\bar{\boldsymbol{x}} \in \mathbb{R}^K} \max_{\bar{\boldsymbol{I}} \in \mathbb{R}^K} \sup_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \inf_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} L(\bar{\boldsymbol{z}}) = \arg \max_{(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}^J) \in \mathbb{R}^{2K}} \min_{\bar{\boldsymbol{\lambda}}^I \in \mathbb{R}^K} \min_{\bar{\boldsymbol{x}} \in \mathbb{R}^K} \max_{\bar{\boldsymbol{I}} \in \mathbb{R}^K} L(\bar{\boldsymbol{z}}), \tag{4.87}$$

which is unique, up to the component signs, if the second inequality in (4.84) is strict. The global saddle point property (4.87) is an analog of the well-known strong Lagrangian duality property of a canonical minimization problem form and classical Lagrangian [49].

4.3.5 Another problem form

By Corollary 3, Lagrangian (4.60) is convex-concave in $\boldsymbol{x} \in \mathbb{R}^K$, $\boldsymbol{I} \in \mathbb{R}^K$ whenever the link QoS function satisfies (4.84) and function $\boldsymbol{x} \mapsto J_k(e^{\boldsymbol{x}})$, $\boldsymbol{x} \in \mathbb{R}^K$, $k \in \mathcal{K}$, is convex. Thus, with the central property of convex-concave functions (Proposition 44) and Proposition 19 we have the following result.

Corollary 4 *If condition (4.84) is satisfied and $\boldsymbol{x} \mapsto J_k(e^{\boldsymbol{x}})$, $\boldsymbol{x} \in \mathbb{R}^K$, $k \in \mathcal{K}$, is convex, then a solution to problem (4.59) exists, if and only if the solution to the problem*

$$\max_{\boldsymbol{I}} \min_{\boldsymbol{x}} \sum_{k \in \mathcal{K}} \alpha_k \Psi\Big(\frac{e^{x_k}}{I_k}\Big), \quad \textit{subject to} \begin{cases} e^{\boldsymbol{x}} - \hat{\boldsymbol{p}} \leq 0 \\ I_k - J_k(e^{\boldsymbol{x}}) \leq 0, \quad k \in \mathcal{K}, \end{cases} \tag{4.88}$$

exists, and the solutions to (4.59) and (4.88) are equal. Moreover, if the second inequality in (4.84) is strict, the solutions to (4.59) and (4.88) are unique.

The result shows that under the conditions from Corollary 3 we dispose of an alternative expression of the min-max power allocation problem (4.59) in the max-min form (4.88). By Corollaries 3 and 4, it is evident that the alternative form (4.88) is available concurrently with having all stationary points of (4.60) associated with global solutions to (4.59). According to the preceding discussion, an equivalent reformulation of the power control problem (4.59) in the form (4.88) is available under (4.86) and (logarithmic) link capacity approximation or channel-averaged normalized symbol error rate as link QoS function.

4.3.6 Decentralized feedback scheme

In this section we show that algorithm (4.66) can be realized in decentralized manner by means of a modified adjoint network feedback scheme (Algorithm 3), which is, besides quadratic convergence, the second main feature of the approach of variable splitting. We characterize the realization scheme under the assumption of linear receivers, i.e. we assume (4.86).

We utilize the componentwise formulation (4.69) in a slightly more abstract version as

$$\begin{cases} x_k(n+1) = & x_k(n) + \frac{1}{s_k(n)}(\overbrace{(\Psi'_e(\log \frac{e^{x_k(n)}}{I_k(n)}) - \delta_k(n)}^{i.)} + \overbrace{\sum_{j\in\mathcal{K}} V_{jk}\delta_j(n) + \delta_k(n)}^{ii.)} + F^1(x_k(n), \mu_k(n), \alpha_k)) \\ \mu_k(n+1) = & \mu_k(n) + \frac{1}{s_k(n)}(F^2(x_k(n), \mu_k(n), \alpha_k, \hat{p}_k)\overbrace{(\Psi''_e(\log \frac{e^{x_k(n)}}{I_k(n)}) - \Psi'_e(\log \frac{e^{x_k(n)}}{I_k(n)}))}^{i.)} + \\ & F^3(\mu_k(n), \alpha_k)(\underbrace{\Psi''_e(\log \frac{e^{x_k(n)}}{I_k(n)}) - \delta_k(n)}_{i.)} + \underbrace{\sum_{j\in\mathcal{K}} V_{jk}\delta_j(n) + \delta_k(n)}_{ii.)})) \end{cases} \quad k \in \mathcal{K}, \tag{4.89}$$

$$\begin{cases} I_k(n+1) = & \sum_{j\in\mathcal{K}} V_{kj} e^{x_j(n+1)} + \sigma_k^2 \\ \delta_k(n+1) = & -\Psi'_e(\log \frac{e^{x_k(n+1)}}{I_k(n+1)}) \frac{e^{x_k(n+1)}}{I_k(n+1)}, \end{cases} \quad k \in \mathcal{K}, \tag{4.90}$$

where additionally $\phi(\lambda_k^I(n+1)) = \frac{\alpha_k}{e^{x_k(n+1)}}\delta_k(n+1)$, $t_k(n+1) = I_k(n+1)$, $k \in \mathcal{K}$, and

$$s_k(n) = F^4(x_k(n), \mu_k(n), \alpha_k, \hat{p}_k) + \underbrace{\delta_k(n) - \Psi''_e(\log \frac{e^{x_k(n)}}{I_k(n)})}_{i.)} + \underbrace{\sum_{j\in\mathcal{K}} V_{jk}\delta_j(n) + \delta_k(n)}_{ii.)}, \quad k \in \mathcal{K}, \tag{4.91}$$

which is obtained by elementary (re-) grouping, setting $\delta_k(n) = \frac{e^{x_k(n)}}{\alpha_k}\lambda_k^J(n)$, $k \in \mathcal{K}$, and using the property $\frac{\partial}{\partial x_k} J_j(e^{\boldsymbol{x}}) = \frac{\partial^2}{\partial x_k^2} J_j(e^{\boldsymbol{x}})$, $\boldsymbol{x} \in \mathbb{R}^K$, $k, j \in \mathcal{K}$, of function (4.86). The functions F^i, $1 \le i \le 4$, simply group several terms from (4.69), (4.70) and are easily deduced.

The first iterate in (4.90) represents the interference power on a k-th link receiver under the power allocation $e^{\boldsymbol{x}(n+1)}$ and the second one is a function of the resulting SIR on k-th link receiver. Thus, iterates (4.90) can be made available *independently* to any link receiver $k \in \mathcal{K}$ by per-link receiver-side estimation of the interference power and the link SIR. As a consequence, the terms i.) in the iterates (4.89), (4.91) are provided *independently* to any k-th link transmitter by per-link feedback of the link SIR estimated in the preceding iteration. Further, we observe that the term ii.) occurring in (4.89) and (4.91) can be made available *independently* to any k-th link transmitter when an adjoint network transmission is established, as in Algorithm 3, and link transmit powers $\delta_k(n)$, $k \in \mathcal{K}$ are allocated. Finally, it is obvious from (4.89), (4.91) that the values of the functions F^i, $1 \le i \le 4$, are computable at any k-th link transmitter, when the corresponding outdated iterate values $x_k(n)$, $\mu_k(n)$ remain locally stored. This, in summary, ensures the decentralized computation of iteration (4.69) by the following extension of the scheme from Algorithm 3. Analogously to Section 4.1.3, it is justified to assume here that functions Ψ, ϕ are globally known by all transmitters and receivers, any constraint $\hat{p}_k$ is known by the corresponding link transmitter $k \in \mathcal{K}$ and weight $\alpha_k > 0$ is known to the link transmitter and link receiver $k \in \mathcal{K}$.

Algorithm 5

1: *Concurrent transmission with link transmit powers* $e^{x_k(n)}$, $k \in \mathcal{K}$.

2: *Receiver-side estimation of interference power* $I_k(n) = \sum_{j\in\mathcal{K}} V_{kj} e^{x_j(n)} + \sigma_k^2$ *and link SIR* $e^{x_k(n)}/I_k(n)$, *and computation of* $\delta_k(n) = -\Psi'_e(\log \frac{e^{x_k(n)}}{I_k(n)}) \frac{e^{x_k(n)}}{I_k(n)}$ *on each link* $k \in \mathcal{K}$.

3: *Per-link feedback of link SIR* $e^{x_k(n)}/I_k(n)$ *to the corresponding link transmitter on each link* $k \in \mathcal{K}$.
4: *Transmitter-side computation of* $\Psi_e'(\log \frac{e^{x_k(n)}}{I_k(n)})$, $\Psi_e''(\log \frac{e^{x_k(n)}}{I_k(n)})$ *and* $\delta_k(n)$ *on each link* $k \in \mathcal{K}$.
5: *Concurrent transmission of the adjoint network with link transmit powers* $\delta_k(n)$, $k \in \mathcal{K}$.
6: *Transmitter-side (i.e. adjoint network receiver-side) estimation of the received power* $\sum_{j \in \mathcal{K}} V_{jk}\delta_j(n) + \delta_k(n)$ *on each link* $k \in \mathcal{K}$.
7: *Transmitter-side update (4.89) on each link* $k \in \mathcal{K}$ *and* $n \to n+1$ *if termination condition not satisfied.*

As in Algorithm 4, the results from Section 4.1.4 can be straightforwardly extended to apply to the case of noisy iterates in Algorithm 5.

4.3.7 SIMULATION RESULTS

We evaluate the performance of iteration (4.66) and Algorithm 5 by simulations for the case of linear interference function (4.86) and the link capacity approximation $\Psi(\gamma_k) = -\log \gamma_k$, $\gamma_k \geq 0$, and the channel-averaged normalized symbol error rate $\Psi(\gamma_k) = 1/\gamma_k$, $\gamma_k \geq 0$, as link QoS functions. Such settings ensure that any point of attraction of (4.66), with feasible power allocation, corresponds to the problem solution (Section 4.3.4). In Lagrangian (4.60) we take $\phi(\mu) = \mu^2$.

Figure 4.4 shows exemplary convergence of iteration (4.66) in two quite large networks. Such convergence is compared with the convergence of the conventional gradient optimization method applied to problem (4.59) [49]. The step-size of the gradient method is optimized here to achieve the fastest possible descent. As in the case of iteration (4.41), the slight oscillation in the transient phase of convergence of (4.66) is a result of the unconstrained character of the iteration, which allows the iterates $\boldsymbol{x}(n)$, $n \in \mathbb{N}$ be temporarily superior to the actual optimum. As could be expected, the quadratically convergent iteration (4.66) significantly outperforms the (linearly convergent) gradient method.

Figure 4.5 shows the convergence of exemplary iterate sequences obtained by the proposed feedback scheme from Algorithm 5. It can be concluded that the feedback scheme offers good robustness to estimation noise in the simulated realistic case of estimate variance.

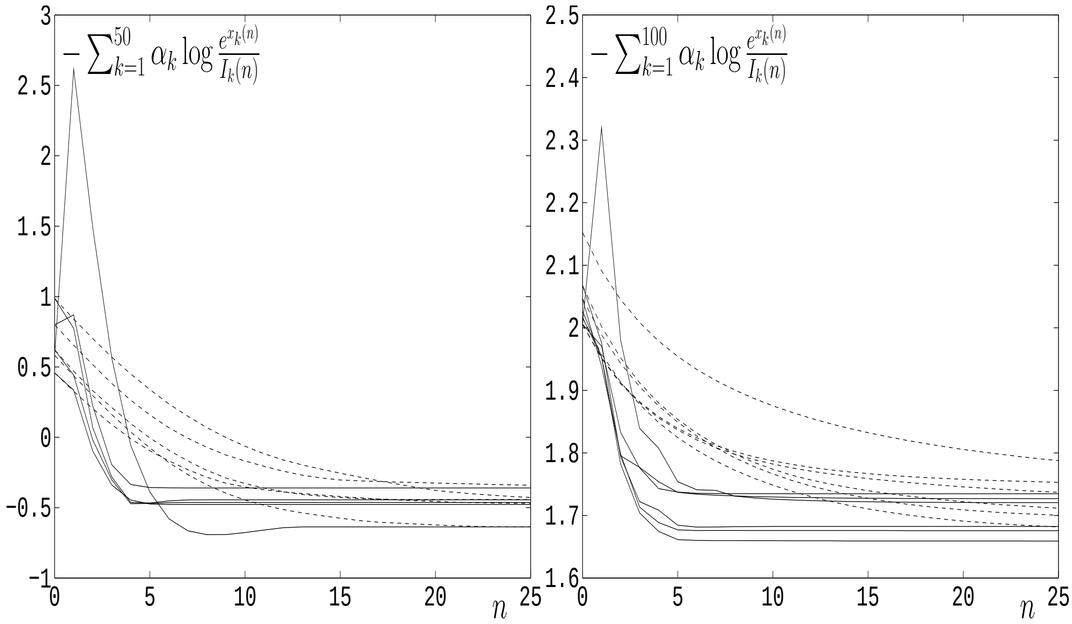

Figure 4.4: Comparison of exemplary convergence of iteration (4.66) (solid lines) with convergence of the conventional gradient optimization method, with constant optimally chosen step-size, applied to problem (4.59) (dashed lines). The settings are $\Psi(\gamma) = -\log(\gamma)$, $\gamma > 0$, $K = 50$ (left) and $K = 100$ (right).

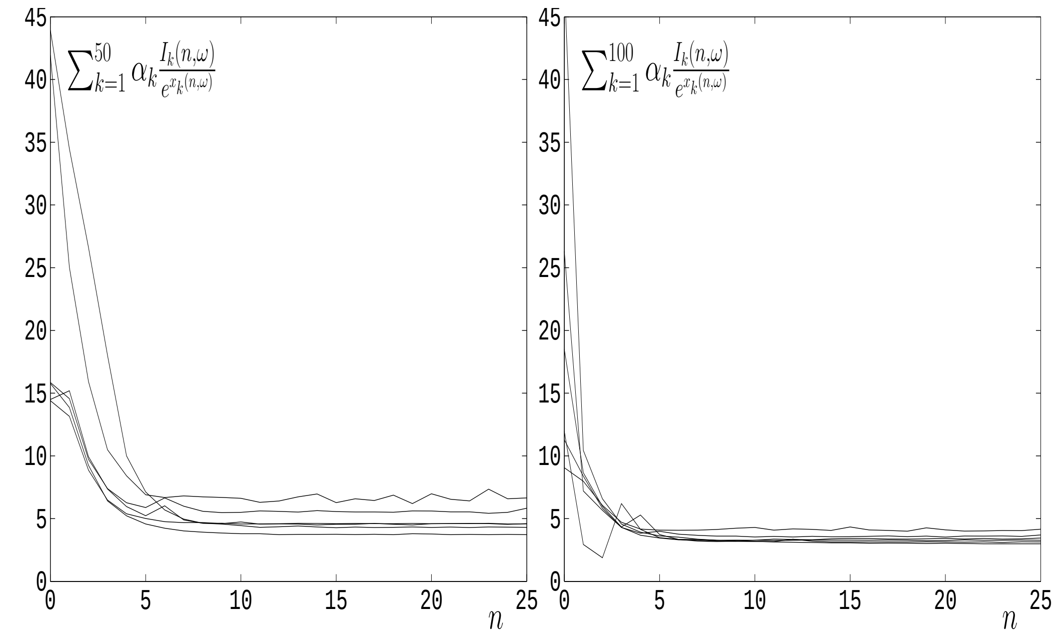

Figure 4.5: Convergence of exemplary iterate sequences generated by Algorithm 5, with no averaging of iterates. The settings are $\Psi(\gamma) = 1/\gamma$, $\gamma > 0$, $K = 50$ (left) and $K = 100$ (right) and the variance of the estimates in steps 2 and 6 is $0.1\sigma_k^2$ for the interference power and received power estimates and $0.05\sigma_k^2$ for the transmit power estimates, $k \in \mathcal{K}$.

5

Particular instance of weighted aggregated performance optimization

The scope of the preceding Chapters 2-4 was on the analysis and algorithmic solutions to the general form of the problem of weighted aggregated performance (2.18), in a general network with interference. In the current chapter we provide an in-depth discussion and propose an algorithmic concept concerning a particular instance of the problem of aggregated performance optimization in a particular network type. The instance of problem (2.18) considered in this chapter is the problem of computation of so-called stability-optimal policy. In broad terms, one can describe a stability-optimal policy as a policy of network operation which ensures stability of all link buffer occupancies for the densest possible traffic arriving at the link buffers. The particular network type considered in this chapter is the multiple access channel with multiple antennas per link [41], [89]. The corresponding model was presented in Section 2.1.1.

In Section 5.1, we first introduce basics and background on queuing networks. Fundamentals of stability considerations in (Markovian) queuing networks follow then at the beginning of Section 5.2. Next, we provide a characterization of the stability-optimal policy of the MIMO multiple access channel, which shows than the corresponding problem of policy computation is in fact an instance of the problem of weighted aggregated performance optimization. The discussed mechanism of the stability-optimal policy, and the intuition behind, is aided by exemplary simulations and extensive interpretation in terms of geometry of the rate region of the multi-antenna multiple access channel. The geometric view of the rate region and specific rate subregions aids the provided results throughout the chapter. It proves to be useful in explaining the mechanisms governing the issues such as stability-optimality of the SIC order or stability-optimality under idle queues, which are addressed in the remainder of Section 5.2.

In Section 5.3 we extend the analysis by providing results based on Kuhn-Tucker conditions of the considered problem. In this way we provide conditions for stability-optimality of link subset transmission and prove a surprising feature of the SIC order, called universal stability-optimality, which consists in a single SIC order becoming superior in terms of stability and capacity issues. Finally, in Section 5.3.5, we propose an approach of splitting of the problem of computation of the stability-optimal policy, which gives rise to an algorithmic solution.

The results of this chapter were presented originally in [90], [91], [92], [93], [27], [94], [95], [96].

Conceptually related to the considered problem are also results from work [97]. This chapter uses the combinatorial geometric notion of a polymatroid, which is explained in Appendix A.5.

5.1 SOME NOTES AND BACKGROUND ON QUEUING NETWORKS

The notion of stability and the corresponding optimal operation is of interest in a *queuing network*, i.e. a network in which an interaction of multiple queues takes place [98], [99]. The state of a queuing network is described by the tuple of its instantaneous queue lengths, referred to as *queue system state*. The policy of handling the reallocation of elements across queues and depletion of queues (so that the queue elements leave the network) in a queuing network is referred usually as the *service policy*.

The best developed theory framework is available for so-called *Markovian* queuing networks, that is, networks with Markov property of either the counting process describing the service of queues or the counting process associated with arrivals at queues (see [98], [99] for details). Better suitable for modeling of real-world queuing networks is the latter case, which is characterizable by the feature that new elements arrive at each queue at time instances independent over time. The corresponding arrival process is referred to as *Poisson arrival process* [99]. Queuing networks with Poisson arrivals allow for the application of powerful Markov chain methods in the analysis. Basic ingredient of such methods are so-called drift conditions and the theory of test functions (or Lyapunov functions), see [100], [101], [102] for the theory and [92], [103] for the application. The considerations throughout the chapter are restricted to the case of Poisson arrivals.

The notion of stability and corresponding optimal operation was originally of interest mainly in automation queuing networks [104]. The establishing of the philosophy of cross-layer design of communications networks caused later a propagation of the view of a communications network as a queuing network [25], [27]. The queue system state corresponds thereby to the tuple of buffer occupancies (queues), measured in [bit], at link transmitters. The service policy is identified with the transmission policy (Section 2.1.1)). Thus, the depletion of a queue and the depletion rate in [bit/s] are identified with the transmission on the link of the queue and the corresponding link data rate, respectively. Furthermore, the arrival process is Poisson when the consecutive bits (or, more generally, bursts) from higher layer processing arrive at any link buffer at independent time instances.

For wired communications networks, crucial insights into the issue of stability and stability-optimal policy were provided in the milestone works [105], [106]. Further stability analysis of wired networks can be found, e.g., in [107], [108] and references therein. One of pioneering general queuing- and information-theoretic approaches to wireless communications can be found in [109]. One of first works dealing with the particular stability-optimal policy in wireless networks was [103], where such policy was considered in a broadcast channel. In [110], [111] the authors considered stability optimality in a (single-antenna) multiple access channel. Particularly challenging remains the issue of stability-optimal policy in a wireless random access network/ channel, or wireless collision network/ channel [112], [113]. In such network type, the general characterization of the stability region, see Definition below, is still open.

5.2 Stability optimality in multi-antenna multiple access channel

At the beginning, note that in the entire chapter we do not distinguish stochastic processes (respectively, random variables) and the corresponding particular process realizations (respectively, variable realizations) in the notation. This is formally loose but introduces no ambiguity.

5.2.1 Stability and related notions in general queuing networks

There are numerous notions of stability of Markovian queuing networks. For instance, we have *nonevanescence* as the weakest established stability principle, *strong stability* as a the strongest widely used notion and *weak stability* as the mostly used principle [112], [113], [108], [107], [104].

Definition 5 *The queuing network with the set of queues $\mathcal{K}$ and its instantaneous queue system states $\boldsymbol{q}(n)$, $n \in \mathbb{N}$, is stable in the weak sense if the limit $\lim_{n\to\infty} Pr(\|\boldsymbol{q}(n)\| > M)$, $M > 0$, is well-defined (exists) and for any $\epsilon > 0$ there exists some $M = M(\epsilon) > 0$ such that*

$$\lim_{n\to\infty} Pr(\|\boldsymbol{q}(n)\| > M) < \epsilon. \tag{5.1}$$

See also [113] for an equivalent alternative formulation. For the definitions of nonevanescence and strong stability we refer to [108], [100].

It is apparent from Definition 5 that weak stability of a queuing network consists in weak convergence of the sequence of probability distributions $1 - Pr(\|\boldsymbol{q}(n)\| > M)$, $n \in \mathbb{N}$, to some limit stationary distribution. One can say equivalently, that the sequence of random variables $\boldsymbol{q}(n)$, $n \in \mathbb{N}$, converges *in distribution* (or *weakly*) to some limit random variable, under which the queue system state remains finite with probability one [82].

An other stability notion, the most intuitive one, is referred to here as *observation-based* and is defined as follows.

Definition 6 *A queuing network with the set of queues $\mathcal{K}$ and its instantaneous queue system states $\boldsymbol{q}(n)$, $n \in \mathbb{N}$, is stable in observation-based sense, if*

$$\lim_{M\to\infty} h_i(M) = 0, \quad i \in \mathcal{K}$$

where

$$h_i(M) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{q_i(\tau)\geq M\}},\ d\tau,$$

with 1_A denoting the indicator function of condition A, where we define $q_i(\tau) = q_i(\max_{n\in\mathbb{N}\,:\,nT\leq\tau} n)$.

Thus, with slight simplification one can say that the system is stable in observation-based sense if the time spent by any queue length above some threshold tends to zero when the threshold increases. One can already recognize from Definitions 5, 6 that when the evolution of the queue system state is ergodic, then it is weakly stable if and only if it is stable in the observation-based sense. Thus, for most real-world cases, Definition 6 provides an *observable* sufficient and necessary characterization of a weakly stable queuing network. In the remainder of this chapter we implicitly assume ergodicity of the queue system state evolution and refer to the notions of weak and observation-based stability simply as to stability.

The notion of stability of a queuing network gives rise to the definition of its stability region.

Definition 7 *The stability region $\mathcal{D}$ of a queuing network is the set of all vectors of arrival rates such that there exists a service policy which achieves stability for any arrival rate vectors from the interior of $\mathcal{D}$.*

Thus, under arrival rate vector from the interior of the stability region, the queuing network can be kept stable by some (existing) service policy. If, however, an arrival rate vector from outside of the interior of $\mathcal{D}$ is given, then there exists no service policy which ensures stability. If the arrival rate vector happens to be included in the boundary of the stability region, then the marginal case of so-called *substability* can occur, see [113], [112].

Definition 8 *The service policy achieving stability for any arrival rate vector from the interior of $\mathcal{D}$ is referred to as stability-optimal.*

Thus, a stability-optimal policy, if existing, is a service policy which ensures stability for the set of arrival rate vectors which is equal to the entire stability region of the queuing network. In other words, the stability-optimal policy is superior in the sense that whenever the queuing network can be kept stable (by some service policy) under given arrival rate vector, then it is stable under the use of the stability-optimal policy.

5.2.2 Stability optimality in the MIMO multiple access channel

According to our cross-layer model, the MIMO multiple access channel represents a special case of a queuing network in which the service policy is identified with the transmission policy. The stability-optimal policy uses the past and instantaneous information on the queue system state (link layer issue) and on the channels (physical layer issue), as figuratively depicted in Fig. 5.1

A general sufficient condition for the stability-optimal policy in Markovian network is well-known and was studied, e.g., for automation networks/ lines in [104], [108], for wired networks in [107], for wireless multi-hop networks in [105], [106], and for wireless single-hop/ cellular networks in [112], [103], [110]. In numerous networks, such condition leads to the characterization of the stability region of the network, as e.g. in [105], [103]. It has to be noted however, that for some network types, as e.g. for the random access network of Aloha-type [112], the stability region is still unknown.

In the case of the MIMO multiple access channel the general stability optimality condition and the resulting stability region characterization take the following particular form.

Proposition 22 *The transmission policy $(\mathcal{Q}(n), \pi(n))$, $n \in \mathbb{N}$, of the multi-antenna multiple access channel with Poisson arrivals is stability-optimal if the sequence of rate vectors $\boldsymbol{R}(n) = \boldsymbol{R}(\mathcal{Q}(n), \pi(n))$, $n \in \mathbb{N}$, satisfies*

$$\boldsymbol{q}'(n)\boldsymbol{R}(n) = \max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}(n))} \boldsymbol{q}'(n)\boldsymbol{R}, \quad n \in \mathbb{N}, \tag{5.2}$$

with $\mathcal{C}(\mathcal{H}(n))$ as the set of achievable (instantaneous) rate vectors in slot $n \in \mathbb{N}$. The stability region of the multi-antenna multiple access channel is equal to its ergodic capacity region, that is, the set of all rate vectors achievable on average.

From Proposition 22 is now evident that the problem of computation of stability-optimal policy in the multi-antenna multiple access channel is an instance of the weighted aggregated performance optimization. According to condition (5.2), the queue system state assumes the role of the weight vector and the rate vector corresponds to the QoS vector. At this point, recall from Section 2.1.1 that under multiple antennas per link the link data rate is regarded as a function of transmission

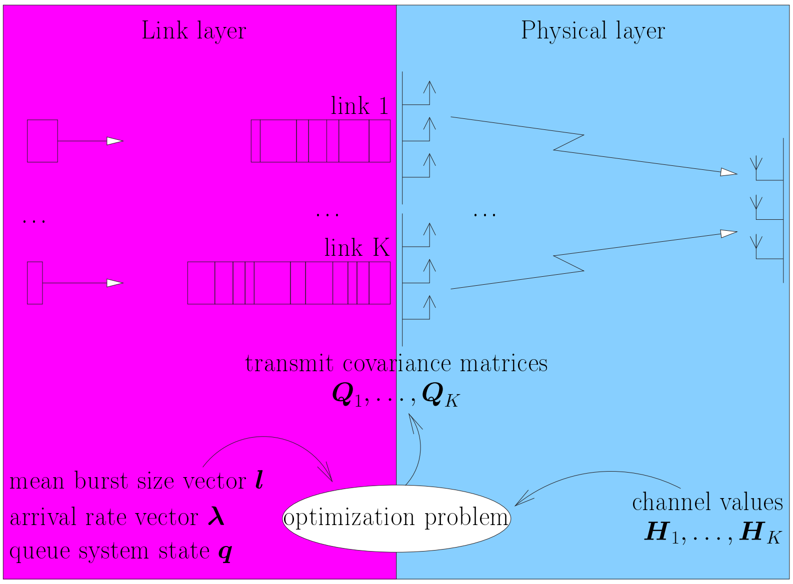

Figure 5.1: The principle of computation of stability-optimal policy in the MIMO multiple access channel.

policy according to (2.12), due to nonexistence of a meaningful SIR notion. This stands in contrast to the single-antenna considerations in Chapters 3-5, where we have a concatenated dependence (2.10) including a general QoS function (in particular also link data rate function) as a function of link SIR.

No explicit features of stability-optimal policy can be seen yet from the condition on the corresponding rate vectors (5.2). Condition (5.2) exhibits the relevance of the set $\mathcal{C}(\mathcal{H}(n))$, $n \in \mathbb{N}$, which we refer to as the *(instantaneous) rate region*. Thus, the structure of the instantaneous rate region is of interest in the context of stability-optimal policy.

The compact stability optimality condition (5.2) arises mainly due to the independence of arrival times (Poisson property) in combination with the assumption of iid (*independently identically distributed*) block fading. Under either of these conditions dropped, the validity of (5.2) is in general lost [98]. The Markovian property of the network, resulting from Poisson arrivals and iid block fading, allows for the application of *drift criteria* (e.g. Foster's Criteria) and *Lyapunov functions* in the stability analysis [100], [98], [108]. Condition (5.2) is a result of application of Foster's Criterion in combination with quadratic Lyapunov function [103], [100]. Since the application of Lyapunov functions other than the quadratic one is thinkable, (5.2) is not a necessary condition for the stability-optimal policy.

Note an important notational difference between the particular aggregated performance optimization (5.2) and the general framework from Chapter 2: In the problem of computation of stability-optimal policy, the link weight (the queue length) is denoted by q_k, $k \in \mathcal{K}$, while this symbol is reserved for the link QoS value in the introduced general framework. We prefer such slightly differing convention in this chapter since it complies with the very most works on queuing networks. Once such difference is noted, no ambiguity should occur in the context of the current chapter.

EVALUATION AND INTERPRETATION OF QUEUE SYSTEM EVOLUTION

Given some slot $n \in \mathbb{N}$, under a policy achieving the maximum in (5.2), the queues with larger lengths are assigned larger rates than the queues of smaller lengths. Thus, the depletion of larger queues in the subsequent slot $n+1$ is faster (in [bit/s]) than the depletion of smaller queues and leads to "more equalized" queues at the end of slot $n+1$. This can be seen as a simple balancing mechanism enforcing permanent drift of excessive queue lengths towards moderate values. With the fact that an instantaneous queue length can be regarded as an indicator of instability potential of the queue (see Definition 6, (5.1)), we get the intuition behind stability optimality of the rate vector sequence (5.2).

In order to expose the queue system behavior under stability-optimal policy and the differences to other transmission policies, we provide a simple simulative comparison in Fig. 5.2. The comparison shows the discrete-time evolution of buffer occupancies $\boldsymbol{q}(n)$, $n \in \mathbb{N}$, that is, the behavior of sample-paths of the corresponding stochastic discrete-time process. We consciously chose an arrival rate vector which is in the interior of the stability region, but lies near the boundary of the stability region. The first row of Fig. 5.2 corresponds to the simple case of so-called *Best-User-Only* policy which consists in single-link transmission of the link with the best channel metric per slot. It is well-known from [18], that in the single-antenna multiple access channel the Best-User-Only policy is optimal in terms of the sum-rate (sum of all link rates). We chose the trace of the channel matrix as the metric of channel quality for the Best-User-Only policy. The second row corresponds to the sum-rate optimal policy, studied e.g. in [114] and [46], and finally the third row corresponds to the stability-optimal policy.

It can be observed that the Best-User-Only policy is overstrained with the arrivals. Fast infinite increase of the queue lengths can not be prevented by the policy and hence the queue system is unstable. The sum-rate optimal policy does better, mainly due to the offered maximal achievable sum-rate of the queue system depletion. However, due to the asymmetry of the arrival rate vector chosen, it also leads to instability of the queue system. In contrast to this, the stability-optimal policy ensures stability of the queue system and the process of queue system evolution approaches a stochastic stationary state (Definition 5). This exposes the essence of the stability-optimal policy. Precisely, as can be seen from the right column in Fig. 5.2, stability is achieved here although the offered average sum-rate is smaller than in the case of instable sum-rate optimal policy.

5.2.3 CAPACITY REGION AND S-RATE REGIONS OF THE MULTI-ANTENNA MULTIPLE ACCESS CHANNEL

It is a fundamental result of information theory that channel capacity can be achieved only asymptotically under an optimal codebook of code-length tending to infinity [41], [115]. For a single slot $n \in \mathbb{N}$, the capacity notion is therefore in general incompatible with our assumption of finiteness of a slot. However, in the particular case of the considered MIMO multiple access channel, the codebook disposes of additional spatial dimensions, so that the the code length can diverge to infinity n_t-times faster than in the single-antenna multiple access channel. Due to such increased code dimensionality, it is justifiable to assume nearly-infinite coding per finite slot for the considered MIMO multiple access channel whenever the number of transmit and receive antennas is "suitably" large. Similar approximation forms the basis of general capacity results for multi-antenna channels e.g. in [21], [116] and references therein.

Under instantaneous channels $\mathcal{H}(n)$ in slot $n \in \mathbb{N}$ and the infinite coding assumption, the (instantaneous) rate region $\mathcal{C}(\mathcal{H}(n))$ of the MIMO multiple access channel can be, for the rest of

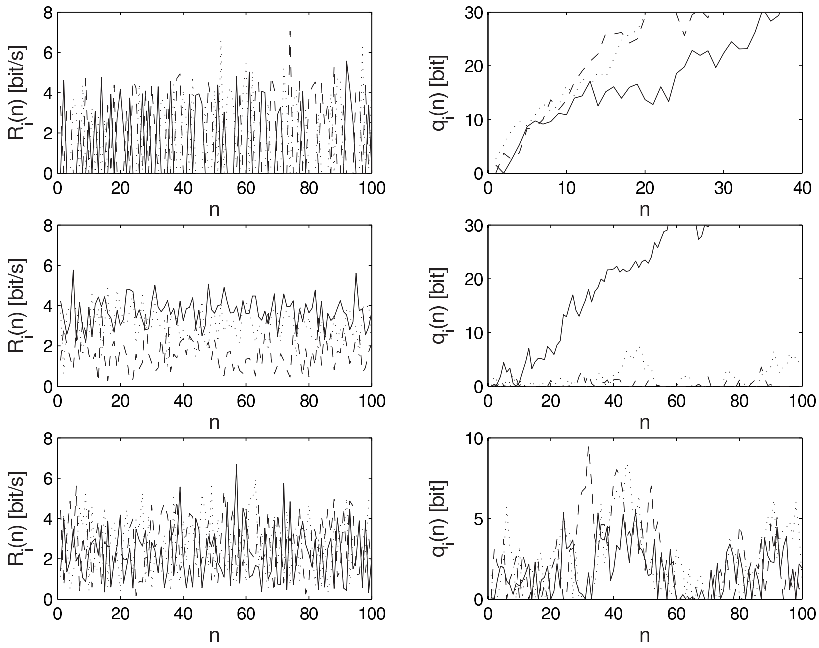

Figure 5.2: Comparison of assigned rates and sample paths of the queue system evolution process under Best-User-Only policy (upper row), sum-rate optimal policy (middle row), and stability-optimal policy (lower row), in the MIMO multiple access channel with $K = 3$ links, $n_t = 2$, $n_r = 2$, individual power. The chosen nonsymmetric bit arrival rate vector lies in the interior of the stability region near its boundary. Note the smaller range of values in the plot corresponding to stability-optimal policy.

this chapter, identified with the corresponding capacity region. The capacity region of a multi-antenna multiple access channel is known to be a union of geometric structures called *polymatroids* [117], [45], [118] (the capacity region of a single-antenna multiple access channel is known to be a polymatroid itself [19]). The definition and fundamental features of polymatroids can be found in Appendix A.5.

The fundamental capacity region

Let us define and fix a set of transmit covariance matrices $\hat{\mathcal{Q}}$ with normalized traces, in the sense

$$tr(\hat{\boldsymbol{Q}}_i) = 1, \quad i \in \mathcal{K}. \tag{5.3}$$

The scaling of $\hat{\mathcal{Q}}$ according to

$$\beta_i \hat{\boldsymbol{Q}}_i, \quad \boldsymbol{\beta} \leq \hat{\boldsymbol{p}}, \quad i \in \mathcal{K}, \tag{5.4}$$

generates a set of transmit covariance matrices included in the power region $\mathcal{P}_{\hat{\boldsymbol{p}}}$ (satisfying individual power constraints $\hat{\boldsymbol{p}}$). The MIMO multiple access channel associated with fixed $\hat{\mathcal{Q}}$ and allowable scaling according to (5.4) corresponds to the MIMO multiple access channel with so-called *scalar* feedback, that is, a feedback which allows for the adjustment/ control of transmit power vector but retains the spatial correlation properties of link signals fixed. The capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ of such MIMO multiple access channel can be expressed as

$$\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}}) = \{\boldsymbol{R} \geq 0 : \sum_{i \in \mathcal{L}} R_i \leq W \log\det(\boldsymbol{I} + \frac{1}{W\sigma^2} \sum_{i \in \mathcal{L}} \hat{p}_i \boldsymbol{H}_i \boldsymbol{Q}_i \boldsymbol{H}_i'), \mathcal{L} \subseteq \mathcal{K}\}. \tag{5.5}$$

From comparison of (5.5) and Definition 27 we deduce the following result.

Lemma 10 *The capacity region $\mathcal{C}(\mathcal{H}, \boldsymbol{p}, \hat{\mathcal{Q}})$ is a polymatroid.*

Proof According to the definition of the polymatroid (Appendix A.5), it has to be shown that the characterization of the region $\mathcal{C}(\mathcal{H}, \boldsymbol{p}, \hat{\mathcal{Q}})$ is of the type

$$\mathcal{C}(\mathcal{H}, \boldsymbol{p}, \hat{\mathcal{Q}}) = \{\boldsymbol{R} \geq 0 : \sum_{i \in \mathcal{L}} R_i \leq R_{\mathcal{H},\boldsymbol{p},\hat{\mathcal{Q}}}(\mathcal{L}), \mathcal{L} \subseteq \mathcal{K}\}, \tag{5.6}$$

where $\mathcal{L} \mapsto R_{\mathcal{H},\boldsymbol{p},\hat{\mathcal{Q}}}(\mathcal{L})$, $\mathcal{L} \subseteq \mathcal{K}$, is a rank function. By comparison of (5.6) and the definition of $\mathcal{C}(\mathcal{H}, \boldsymbol{p}, \hat{\mathcal{Q}})$ in (5.5) follows that this is equivalent to showing that

$$\mathcal{L} \mapsto R_{\mathcal{H},\boldsymbol{p},\hat{\mathcal{Q}}}(\mathcal{L}) = \log\det(\boldsymbol{I} + \frac{1}{\sigma^2}\sum_{i \in \mathcal{L}} p_i \boldsymbol{H}_i \hat{\boldsymbol{Q}}_i \boldsymbol{H}_i'), \quad \mathcal{L} \subseteq \mathcal{K}, \tag{5.7}$$

is a rank function. By the result in [119], any rank function f on the power set of $\mathcal{K}$ can be written in the form

$$f(\mathcal{L}) = h(\sum_{i \in \mathcal{L}} w_i), \quad \mathcal{L} \subseteq \mathcal{K}, \tag{5.8}$$

where $w \mapsto h(w) \geq 0$, $w \geq 0$, is increasing concave and such that $h(0) = 0$, and where $w_i \geq 0$, $i \in \mathcal{L}$. More generally, it is straightforward to show that (5.8) is a formulation of a rank function as well if $w \mapsto h(w) \geq 0$, $w \in \mathbb{S}_+^K$, and $w_i \succeq 0$, $i \in \mathcal{L}$. Now, setting $w_i = p_i \boldsymbol{H}_i \hat{\boldsymbol{Q}}_i \boldsymbol{H}_i'$, $i \in \mathcal{K}$, function (5.7) can be written as $R_{\mathcal{H},\boldsymbol{p},\hat{\mathcal{Q}}}(\mathcal{L}) = h(\sum_{i \in \mathcal{L}} w_i)$, $\mathcal{L} \subseteq \mathcal{K}$, with $h(w) = \log\det(\boldsymbol{I} + w)$, $w \succeq 0$. Since such function h is increasing concave and such that $h(0) = 0$, the proof is completed [47]. □

By (A.12) and some simple linear algebra manipulations, the components of the vertex $\boldsymbol{R}^{\pi_k}$ of the capacity region polymatroid $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ can be written as

$$\begin{aligned} R_i^{\pi_k} &= W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \hat{p}_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \hat{\boldsymbol{Q}}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)}) \\ &- W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i-1} \hat{p}_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \hat{\boldsymbol{Q}}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)}), \quad i \in \mathcal{K}. \end{aligned} \tag{5.9}$$

It can be recognized from (5.9) that $\boldsymbol{R}^{\pi_k}$ represents a rate vector achieved under the use SIC according to the (inverse) SIC order π_k.

We define $K!$ subregions of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, with each one representing the set of rate vectors achievable under the use of some fixed SIC order π_k, $1 \leq k \leq K!$. We refer to such subregions as *S-rate regions* (with S underlining the $s$patial dimension of the SIC operation) and denote them as $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, $1 \leq k \leq K!$. We can express the S-rate-region associated with SIC order π_k as

$$\begin{aligned} \mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}}) = cl\{ \bigcup_{0 \leq \boldsymbol{\beta} \leq \hat{\boldsymbol{p}}} \{\boldsymbol{R} \geq 0 : R_{\pi_k(i)} \leq \\ W \log \frac{\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \beta_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \hat{\boldsymbol{Q}}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)})}{\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i-1} \beta_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \hat{\boldsymbol{Q}}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)})}, i \in \mathcal{K}\}\}, \quad 1 \leq k \leq K!, \end{aligned} \tag{5.10}$$

with cl as the closure operator, which ensures that the uncountable union of closed sets remains closed.

The boundary rate vectors from the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ are achievable by SIC and time sharing among different SIC-based transmission policies [41]. Thus, $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ is a convex hull of all S-rate regions, that is,

$$\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}}) = conv(\bigcup_{1 \le k \le K!} \mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})), \tag{5.11}$$

with convex hull operator $conv$ corresponding to time sharing among different SIC orders. The boundaries of S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ and their arrangement within the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ require some more explanation. It is clear that a vertex rate vector $\boldsymbol{R}^{\pi_k}$, with components (5.9), is included in the boundary of $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, $1 \le k \le K!$. According to (5.9), a vertex rate vector has a distinct position in terms of power consumption, since the use of maximum allowable transmit power of each link is necessary for its achievement, that is, $\boldsymbol{\beta} = \hat{\boldsymbol{p}}$. Furthermore, a vertex rate vector $\boldsymbol{R}^{\pi_k}$ lies at the junction of two boundary parts of $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ with different properties. Precisely, for fixed $\hat{\mathcal{Q}}$ and SIC order π_k, consider first

Condition 9 *We have* $\beta_{\pi_k(1)} = \hat{p}_{\pi_k(1)}$ *and* $\beta_{\pi_k(i)} < \hat{p}_{\pi_k(i)}$, $i \in \mathcal{K}$, $i \ge 2$.

The rate vectors from the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ which satisfy Condition 9 constitute the first of the two boundary parts of the S-rate region. Any such rate vector is included in the boundary of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, since it ensures the maximum achievable rate of the last decoded link $\pi_k(1)$ (recall from the SIC principle that the last decoded link does not suffer any interference).

Now, consider

Condition 10 *We have* $\beta_{\pi_k(K)} = \hat{p}_{\pi_k(K)}$ *and* $\beta_{\pi_k(i)} < \hat{p}_{\pi_k(i)}$, $i \in \mathcal{K}$, $i \le K-1$.

The rate vectors from the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ which satisfy Condition 9 constitute the second mentioned boundary part of the S-rate region. In contrast to the boundary rate vectors satisfying Condition 9, however, any rate vector from $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ which fulfills Condition 10 is included in the interior of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$.

The structure of an exemplary capacity region and S-rate regions of the MIMO multiple access channel with two links from Fig. 5.3 makes the boundary parts characterized by Conditions 9, 10 visible.

Individual Power Constraints

The capacity region of the MIMO multiple access channel under individual power constraints, denoted by $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$, can be expressed by means of the fundamental capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ and fundamental S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, $1 \le k \le K!$ (actually, this was the main motivation for their introduction). Precisely, the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ is a union of all capacity regions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ associated with sets of transmit covariance matrices $\hat{\mathcal{Q}}$ satisfying (5.3). Thus, letting $\mathcal{Q}$ satisfy $\boldsymbol{Q}_i = \hat{p}_i \hat{\boldsymbol{Q}}_i$, $i \in \mathcal{K}$, we can write

$$\begin{aligned} \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}) =& cl\{ \bigcup_{\substack{\hat{\mathcal{Q}} \succeq 0 \\ tr(\hat{\boldsymbol{Q}}_i) \le 1}} \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})\} = \\ & cl\{ \bigcup_{\substack{\mathcal{Q} \succeq 0 \\ tr(\boldsymbol{Q}_i) \le \hat{p}_i}} \{\boldsymbol{R} \ge 0 : \sum_{i \in \mathcal{L}} R_i \le W \log\det(\boldsymbol{I} + \frac{1}{W\sigma^2} \sum_{i \in \mathcal{L}} \boldsymbol{H}_i \boldsymbol{Q}_i \boldsymbol{H}_i'), \mathcal{L} \subseteq \mathcal{K}\}\}. \end{aligned} \tag{5.12}$$

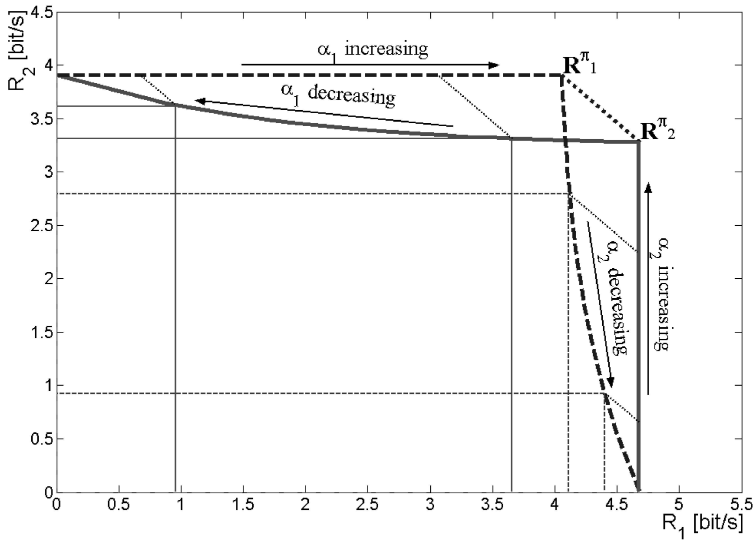

FIGURE 5.3: The structure of the capacity region $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$ of an exemplary MIMO multiple access channel with two links (Rayleigh fading realization, randomly chosen $\hat{\mathcal{Q}}$). The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$, $\pi_1 = 2 \leftarrow 1$, is dashed while the boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$ is solid. The convex hull part of the capacity region boundary is dotted. With thin lines of the corresponding types we plot the capacity regions under smaller power constraints with vertex rate vectors satisfying Conditions 9, 10, respectively.

Analogously to the definition of the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$, we can define the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}})$, $1 \leq k \leq K!$, under individual power constraints. From construction (5.12) one can conclude, that the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}})$, for some fixed $1 \leq k \leq K!$, represents a union of all S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$ corresponding to sets of transmit covariance matrices $\hat{\mathcal{Q}}$ satisfying (5.3). Defining $\delta_i = \frac{\beta_i}{\hat{p}_i}$, $i \in \mathcal{K}$, so that we have $\beta_i \hat{\boldsymbol{Q}}_i = \delta_i \boldsymbol{Q}_i$, $i \in \mathcal{K}$, we can write

$$\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}}) = cl\{ \bigcup_{\substack{\hat{\mathcal{Q}}\succeq 0 \\ tr(\hat{\boldsymbol{Q}}_i)\leq 1}} \mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})\} = cl\{ \bigcup_{\substack{\mathcal{Q}\succeq 0 \\ tr(\boldsymbol{Q}_i)\leq \hat{p}_i}} \bigcup_{0\leq \boldsymbol{\delta} \leq \boldsymbol{1}} \{\boldsymbol{R} \geq 0 : R_{\pi_k(i)} \leq$$
$$W \log \frac{\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i} \delta_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \boldsymbol{Q}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)})}{\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i-1} \delta_{\pi_k(j)} \boldsymbol{H}_{\pi_k(j)} \boldsymbol{Q}_{\pi_k(j)} \boldsymbol{H}'_{\pi_k(j)})}, i \in \mathcal{K}\}\}, \quad 1 \leq k \leq K!. \tag{5.13}$$

Note, that since the fundamental S-rate region is in general a nonconvex set, the same holds for the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}})$, $1 \leq k \leq K!$, as a closed union of fundamental S-rate regions. Since the capacity region $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$ and the S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}})$, $1 \leq k \leq K!$, are in the same relationship as the corresponding fundamental capacity and S-rate regions, the convex hull property

$$\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}}) = conv(\bigcup_{1\leq k\leq K!} \mathcal{S}_{\pi_k}(\mathcal{H},\hat{\boldsymbol{p}})) \tag{5.14}$$

holds also under individual power constraints in the MIMO multiple access channel.

The presented structure of the capacity region of the MIMO multiple access channel with individual power constraints is depicted in Fig. 5.4 for an exemplary case of two links. Similarly,

Fig. 5.5 shows the presented structure of the S-rate regions for the same exemplary MIMO multiple access channel.

Sum Power Constraints

A similar analysis as above can be applied to the capacity region of the MIMO multiple access channel under sum-power constraint. Sum power constrain implies that the links can be allocated arbitrary transmit powers as long as their sum does not exceed some maximal allowed value P. Thus, the capacity region of the multi-antenna multiple access channel under sum-power constraint, denoted as $\mathcal{C}(\mathcal{H}, P)$, is a union of all capacity regions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ corresponding to vectors of individual power constraints $\hat{\boldsymbol{p}}$ satisfying $\hat{\boldsymbol{p}}'\mathbf{1} \leq P$. Precisely,

$$\begin{aligned}\mathcal{C}(\mathcal{H}, P) =& cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})\} = cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \bigcup_{\substack{\hat{\mathcal{Q}}\succeq 0 \\ tr(\hat{\boldsymbol{Q}}_i)\leq 1}} \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})\} = \\ & cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \bigcup_{\substack{\mathcal{Q}\succeq 0 \\ tr(\boldsymbol{Q}_i)\leq \hat{p}_i}} \{\boldsymbol{R} \geq 0 : \sum_{i\in\mathcal{L}} R_i \leq W \log\det(\boldsymbol{I} + \frac{1}{W\sigma^2}\sum_{i\in\mathcal{L}} \boldsymbol{H}_i\boldsymbol{Q}_i\boldsymbol{H}_i'), \mathcal{L} \subseteq \mathcal{K}\}\}.\end{aligned} \tag{5.15}$$

Analogously, the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H}, P)$, $1 \leq k \leq K!$, represents a union of S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}})$, that is,

$$\begin{aligned}&\mathcal{S}_{\pi_k}(\mathcal{H}, P) = cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}})\} = cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \bigcup_{\substack{\hat{\mathcal{Q}}\succeq 0 \\ tr(\hat{\boldsymbol{Q}}_i)\leq 1}} \mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})\} = cl\{\bigcup_{\hat{\boldsymbol{p}}'\mathbf{1}\leq P} \bigcup_{\substack{\mathcal{Q}\succeq 0 \\ tr(\boldsymbol{Q}_i)\leq \hat{p}_i}} \bigcup_{0\leq\boldsymbol{\beta}\leq\mathbf{1}} \\ &\{\boldsymbol{R} \geq 0 : R_{\pi_k(i)} \leq W \log \frac{\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i} \beta_{\pi_k(j)}\boldsymbol{H}_{\pi_k(j)}\boldsymbol{Q}_{\pi_k(j)}\boldsymbol{H}'_{\pi_k(j)})}{\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i-1} \beta_{\pi_k(j)}\boldsymbol{H}_{\pi_k(j)}\boldsymbol{Q}_{\pi_k(j)}\boldsymbol{H}'_{\pi_k(j)})}, i \in \mathcal{K}\}\}, \quad 1 \leq k \leq K!.\end{aligned} \tag{5.16}$$

Analogously to the case of individual power constraints, the S-rate region $\mathcal{S}_{\pi_k}(\mathcal{H}, P)$, $1 \leq k \leq K!$, is a union of (in general) nonconvex sets and thus, is in general nonconvex itself.

Further it is obvious that, analogously to the case of individual power constraints, the convex hull property according to

$$\mathcal{C}(\mathcal{H}, P) = conv(\bigcup_{1\leq k\leq K!} \mathcal{S}_{\pi_k}(\mathcal{H}, P)) \tag{5.17}$$

is satisfied for the multi-antenna multiple access channel with sum-power constraint as well. The presented structure of the capacity region of the MIMO multiple access channel with sum-power constraint is plotted in Fig. 5.6 for an exemplary case of two links. Fig. 5.7 shows the corresponding structure of the S-rate regions for the same exemplary case.

5.2.4 Stability-optimal policy and its computation

The stability optimality condition from Proposition 22 is not sufficient for the characterization of the stability-optimal policy $(\mathcal{Q}(n), \pi(n))$, $n \in \mathbb{N}$, which generates a rate vector sequence satisfying (5.2). The current section addresses the problem of specification of the stability-optimal policy (in a one-slot view).

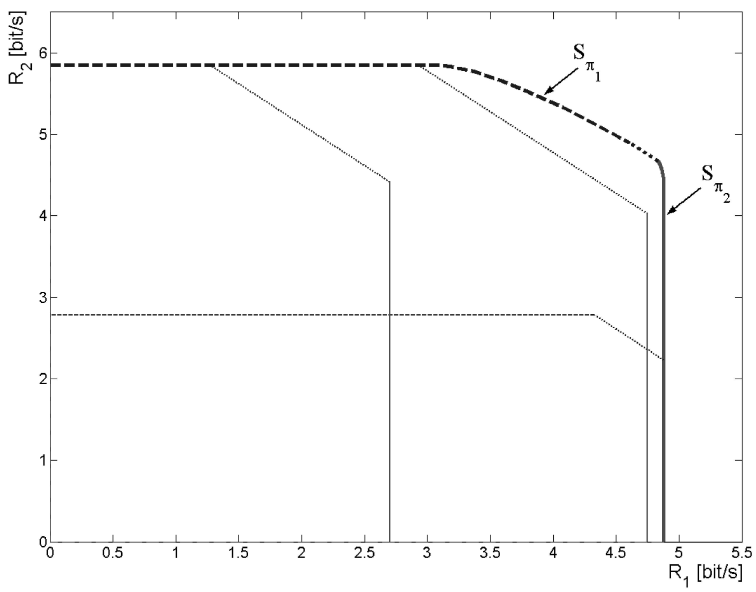

FIGURE 5.4: The structure of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ of an exemplary MIMO multiple access channel with two links and individual power constraints $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2)$ (Rayleigh fading realization) as a union of fundamental subregions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$. The boundary part of the capacity region which is included also in the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, \hat{\boldsymbol{p}})$, $\pi_1 = 2 \leftarrow 1$, is dashed while the boundary part of the capacity region which is included also in the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}})$ is solid. The convex hull part of the capacity region boundary is dotted.

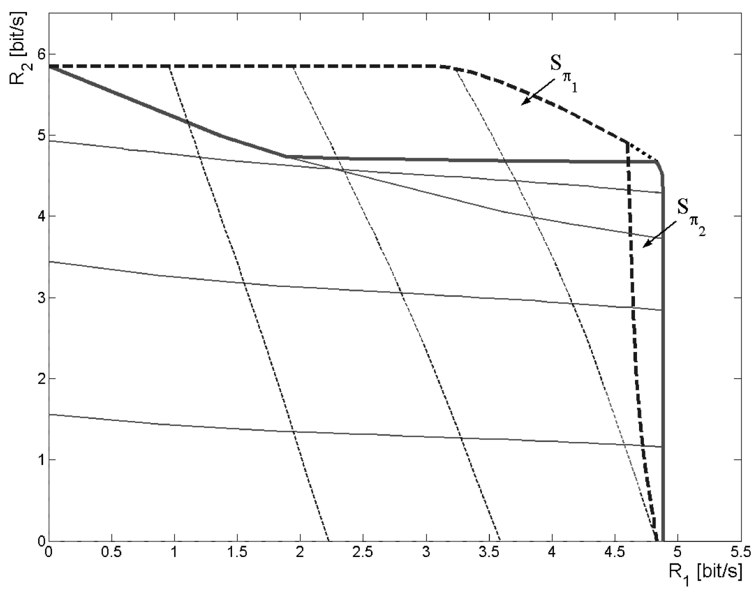

FIGURE 5.5: The structure of the S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H}, \hat{\boldsymbol{p}})$, $1 \leq k \leq 2$, of an exemplary MIMO multiple access channel with two links and individual power constraints $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2)$ (Rayleigh fading realization). The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, \hat{\boldsymbol{p}})$, $\pi_1 = 2 \leftarrow 1$, is dashed and represents a union of S-rate regions $\mathcal{S}_{\pi_1}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ (thin dashed lines). The boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}})$ is solid and represents a union of S-rate regions $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ (thin solid lines).

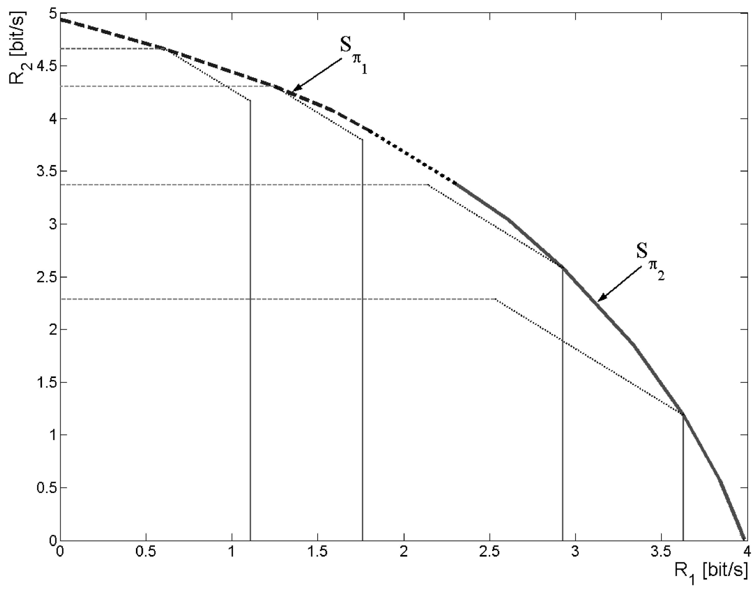

FIGURE 5.6: The structure of the capacity region $\mathcal{C}(\mathcal{H}, P)$ of an exemplary MIMO multiple access channel with two links and sum-power constraint P (Rayleigh fading realization) as a union of fundamental subregions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$. The boundary part of the capacity region which is included also in the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, P)$, $\pi_1 = 2 \leftarrow 1$, is dashed while the boundary part of the capacity region which is included also in the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, P)$ is solid. The convex hull part of the capacity region boundary is dotted.

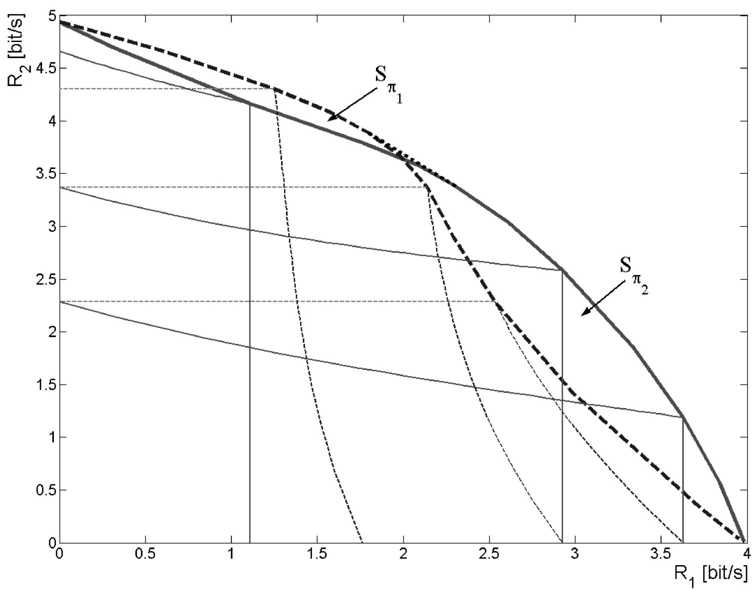

FIGURE 5.7: The structure of the S-rate regions $\mathcal{S}_{\pi_k}(\mathcal{H}, P)$, $1 \leq k \leq 2$, of an exemplary MIMO multiple access channel with two links and sum-power constraint P (Rayleigh fading realization). The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, P)$, $\pi_1 = 2 \leftarrow 1$, is dashed and represents a union of S-rate regions $\mathcal{S}_{\pi_1}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ (thin dashed lines). The boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, P)$ is solid and represents a union of S-rate regions $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ (thin solid lines).

Transformation of the problem

Note, that the characterization of stability optimality from Proposition 22 is independent of the type of power constraints in the multi-antenna multiple access channel. Thus, in related expressions which are constraint type-independent as well, we denote the capacity region and the S-rate regions of the MIMO multiple access channel universally as $\mathcal{C}(\mathcal{H})$ and $\mathcal{S}_\pi(\mathcal{H})$, $\pi \in \Pi_K$, respectively, regardless of the constraint type.

According to Proposition 22, the instantaneous stability-optimal rate vector solves the problem

$$\max_{\boldsymbol{R}\in\mathcal{C}(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R}. \tag{5.18}$$

Since the capacity region $\mathcal{C}(\mathcal{H})$ is a convex set and the objective in (5.18) is linear, the problem (5.18) is convex. Consider fixing the SIC order to some $\pi \in \Pi_K$. Then, with the definition of S-rate regions the problem (5.18) changes to

$$\max_{\boldsymbol{R}\in\mathcal{S}_\pi(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R}, \tag{5.19}$$

and due to $\mathcal{S}_\pi(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H})$ we have obviously

$$\max_{\boldsymbol{R}\in\mathcal{S}_\pi(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R} \leq \max_{\boldsymbol{R}\in\mathcal{C}(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R}. \tag{5.20}$$

Clearly, for stability-suboptimal SIC orders we have strict inequality in (5.20). Furthermore, if the stability-optimal policy does not require time sharing (is *spatial*), then we have equality in (5.20) for a stability-optimal SIC order.

Under fixed SIC order it is now possible to transform the problem (5.19) of optimization of rate vectors to the problem of optimization of (sets of) transmit covariance matrices. Precisely, by (5.13) (respectively (5.16)) we can represent the objective $\boldsymbol{R} \mapsto \boldsymbol{q}'\boldsymbol{R}$, $\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H})$, equivalently as a function $\mathcal{Q} \mapsto f_{\boldsymbol{q},\pi}$, $\mathcal{Q} \in \mathcal{P}$, of the form

$$f_{\boldsymbol{q},\pi}(\mathcal{Q}) = \sum_{i=1}^{K} q_{\pi(i)}(\log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i} \boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)}) - \log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i-1} \boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)})), \tag{5.21}$$

so that we have the equivalence of problem (5.19) and problem

$$\max_{\mathcal{Q}\in\mathcal{P}} f_{\boldsymbol{q},\pi}(\mathcal{Q}) \tag{5.22}$$

in the sense $\max_{\boldsymbol{R}\in\mathcal{S}_\pi(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R} = \max_{\mathcal{Q}\in\mathcal{P}} f_{\boldsymbol{q},\pi}(\mathcal{Q})$. The addends in the objective (5.21) represent concave matrix functions [63], [120], so that the objective itself, as a difference of concave functions, is in general nonconcave. Consequently, the problem (5.22) is in general a nonconvex optimization problem. Note, that the same can be concluded from general nonconvexity of the S-rate region for the problem form (5.19). As already mentioned in Section 2.3, a nonconvex optimization problem is, as a rule, hard to handle in the sense that common locally convergent iterations, e.g. gradient or Newton iteration, in general do not converge to a (global) solution. Further, given satisfied constraint qualification condition, the Kuhn-Tucker conditions are in general merely necessary optimality conditions [48].

Stability-Optimal SIC Order

The question of existence and characterization of the stability-optimal and spatial policy, and thus the stability-optimal SIC order, is answered by the following proposition.

Proposition 23 *Given any queue system state* $\boldsymbol{q} \in \mathbb{R}_+^K$ *in the multi-antenna multiple access channel, we have*

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R} = \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R}, \tag{5.23}$$

for SIC order $\pi \in \Pi_K$ *satisfying*

$$q_{\pi(1)} \geq q_{\pi(2)} \geq \ldots \geq q_{\pi(K)}. \tag{5.24}$$

Proof In Lemma 10 was shown that the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ is a polymatroid. Further, with Lemma 21 follows that, given $\boldsymbol{q} \in \mathbb{R}_+^K$, the vertex rate vector $\boldsymbol{R}^\pi$, with components (5.9) for $\pi = \pi_k \in \Pi_K$, is a solution to the problem (5.18) if the permutation π orders the components of $\boldsymbol{q}$ decreasingly. We know also, that a vertex $\boldsymbol{R}^\pi$ of $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ is also a vertex of the S-rate region $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$. Since further $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$ is included in $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, it is implied that

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})} \boldsymbol{q}'\boldsymbol{R} = \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})} \boldsymbol{q}'\boldsymbol{R} \tag{5.25}$$

for any $\boldsymbol{q} \in \mathbb{R}_+^K$ and $\pi \in \Pi_K$ such that $q_{\pi(1)} \geq \ldots \geq q_{\pi(K)}$.

Consider the case of individual power constraints. By (5.12), the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ is a convex union of fundamental capacity regions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$. Thus, the optimization problem (5.18) over $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ can be split up according to

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})} \boldsymbol{q}'\boldsymbol{R} = \max_{\substack{\mathcal{Q} \succeq 0: \\ tr(\boldsymbol{Q}_i) \leq 1, i \in \mathcal{K}}} \max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}}, \mathcal{Q})} \boldsymbol{q}'\boldsymbol{R}. \tag{5.26}$$

Further, with (5.25) and the feature that the S-rate region $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}})$ is, according to (5.13), a union of S-rate regions $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}}, \hat{\mathcal{Q}})$, we can write

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})} \boldsymbol{q}'\boldsymbol{R} = \max_{\substack{\mathcal{Q} \succeq 0: \\ tr(\boldsymbol{Q}_i) \leq 1, i \in \mathcal{K}}} \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}}, \mathcal{Q})} \boldsymbol{q}'\boldsymbol{R} = \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}})} \boldsymbol{q}'\boldsymbol{R} \tag{5.27}$$

for any $\boldsymbol{q} \in \mathbb{R}_+^K$ and $\pi \in \Pi_K$ such that $q_{\pi(1)} \geq \ldots \geq q_{\pi(K)}$. This completes the proof for the MIMO multiple access channel with individual power constraints.

By (5.15), the capacity region $\mathcal{C}(\mathcal{H}, P)$ is a convex union of capacity regions $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$. Thus, we can similarly split the problem (5.18) over $\mathcal{C}(\mathcal{H}, P)$ as

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, P)} \boldsymbol{q}'\boldsymbol{R} = \max_{\substack{\boldsymbol{p} \geq 0: \\ \sum_{i=1}^K p_i \leq P}} \max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \boldsymbol{p})} \boldsymbol{q}'\boldsymbol{R}. \tag{5.28}$$

With (5.27) and the feature that the S-rate region $\mathcal{S}_\pi(\mathcal{H}, P)$ is, according to (5.16), a union of S-rate regions $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}})$ we can write

$$\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, P)} \boldsymbol{q}'\boldsymbol{R} = \max_{\substack{\boldsymbol{p} \geq 0: \\ \sum_{i=1}^K p_i \leq P}} \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H}, \boldsymbol{p})} \boldsymbol{q}'\boldsymbol{R} = \max_{\boldsymbol{R} \in \mathcal{S}_\pi(\mathcal{H}, P)} \boldsymbol{q}'\boldsymbol{R} \tag{5.29}$$

for any $\boldsymbol{q} \in \mathbb{R}_+^K$ and $\pi \in \Pi_K$ such that $q_{\pi(1)} \geq \ldots \geq q_{\pi(K)}$. This completes the proof. □

In other words, for any instantaneous state of the queue system in the MIMO multiple access channel, there exists a stability-optimal policy which is spatial, whereby the corresponding stability-optimal SIC order orders the queue lengths increasingly (since, according to (5.24) and our model, the inverse SIC order orders them decreasingly).

Proposition 23 is the basis of considerations in the remainder of this chapter. Since it implies the existence of stability-optimal and spatial policy, the use of time sharing within a slot follows to be superfluous in terms of stability optimality. Further, we observe that the stability optimality condition (5.24) for an SIC order is determined solely by the queue system state and does not depend on channel values. Note, that an SIC order remains stability-optimal throughout some slot sequence whenever the queue system evolution does not violate the chain inequality (5.24) within such slot sequence. This is likely to be the case when the queue system state changes "sufficiently" slowly, that is, when the Poisson arrivals are sufficiently sparse and the assigned link rates are sufficiently small.

With the stability optimality condition for the SIC order (5.24) and with problem (5.22), the complete stability-optimal policy (set of transmit covariance matrices plus SIC order) can be determined. A simple regrouping of terms and change in the indices in (5.21) yields

$$\begin{aligned} f_{\boldsymbol{q},\pi}(\mathcal{Q}) = \sum_{i=1}^{K-1}(q_{\pi(i)} - q_{\pi(i+1)})W \log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i}\boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)}) + \\ q_{\pi(K)}W \log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{K}\boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)}) - q_{\pi(1)}W\log\det(\boldsymbol{I}W\sigma^2). \end{aligned} \tag{5.30}$$

The last term in (5.30) can be omitted in optimization, since it does not depend on $\mathcal{Q}$.

When a stability-optimal SIC order π, satisfying (5.24), is incorporated in (5.30), then each concave log det-function is multiplied with a nonnegative scalar. Thus, given (5.24), the objective (5.30) is a concave function and we yeld the following corollary to Proposition 23.

Corollary 5 *Given a stability-optimal SIC order π, satisfying (5.24), the problem (5.22) which determines the set of stability-optimal transmit covariance matrices is convex and takes the form*

$$\begin{aligned} \max_{\mathcal{Q}\in\mathcal{P}}(\sum_{i=1}^{K-1}(q_{\pi(i)} - q_{\pi(i+1)})W \log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i}\boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)}) + \\ q_{\pi(K)}W \log\det(\boldsymbol{I}W\sigma^2 + \sum_{j=1}^{K}\boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)})). \end{aligned} \tag{5.31}$$

According to the discussion in Section 2.3, convexity of the optimization problem (5.31) under optimal SIC order is a significant advantage in terms of its iterative solvability, that is, in terms of algorithmic computation of the stability-optimal policy. Precisely, the set of stability-optimal transmit covariance matrices can be computed by means of a variety of locally convergent iterations, and in particular by means of powerful methods of convex optimization, such as interior point methods [47]. Since the constraint qualification condition can be shown to be satisfied for problem (5.31), the Kuhn-Tucker conditions for (5.31) are necessary and sufficient optimality conditions.

Recall, that problem (5.31) can be one-to-one transformed into the problem form (5.19). Thus, the convex optimization problem translates bijectively to the optimization with nonconvex optimization domain $\mathcal{S}_\pi(\mathcal{H})$. This apparent contradiction is cleared in the next section.

5.2.5 Stability-optimal policy and rates in the geometric view

In this section we use a geometric framework for the treatment of the capacity region and S-rate regions of the multi-antenna multiple access channel. This should aid the understanding of the key

Proposition 23, revised here, and allows us to attack some further questions related with stability optimality and the structure of the rate regions.

First we provide a useful restatement of Proposition 23 in the view of geometry of the capacity region and S-rate regions.

Corollary 6 (Geometric Restatement of Proposition 23) *The boundary of the S-rate region $\mathcal{S}_\pi(\mathcal{H})$, $\pi \in \Pi_K$, coincides with the boundary of the capacity region $\mathcal{C}(\mathcal{H})$ on at least one rate vector which is included in the hyperplane which supports the capacity region $\mathcal{C}(\mathcal{H})$ and has normal vector $\boldsymbol{q} \in \mathbb{R}_+^K$ which satisfies (5.24). Equivalently, when π and $\boldsymbol{q}$ satisfy (5.24), then there exists*

$$\tilde{\boldsymbol{R}} = \arg \max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R} \tag{5.32}$$

such that

$$\tilde{\boldsymbol{R}} \in \delta\mathcal{S}_\pi(\mathcal{H}) \cap \delta\mathcal{C}(\mathcal{H}),$$

with $\delta\mathcal{X}$ denoting the boundary of set $\mathcal{X}$.

A consequence of Corollary 6 is that the boundary part of the S-rate region $\mathcal{S}_\pi(\mathcal{H})$, $\pi \in \Pi_K$, consisting of vectors which are supporting points of a hyperplane with normal vector satisfying (5.24) is convex. This feature follows from convexity of the capacity region since, by Corollary 6, the vectors which are supporting points of a hyperplane with normal vector satisfying (5.24) are common to the boundaries of the S-rate region $\mathcal{S}_\pi(\mathcal{H})$, $\pi \in \Pi_K$, and the capacity region. In other words, the rate vectors which are stability-optimal for a queue system state satisfying (5.24) lie on the convex boundary part of the S-rate region $\mathcal{S}_\pi(\mathcal{H})$, $\pi \in \Pi_K$, which is included in the boundary of the capacity region. The remaining boundary part of any S-rate region is in general nonconvex and included in the interior of the capacity region. Thus, according to the stability optimality condition (5.2), no rate vector included in such boundary part can be stability-optimal.

Relating the provided geometric insights to problems (5.19), (5.22), we conclude that both problem forms are ensured to be convex under stability-optimality of the SIC order according to (5.24). In such case, problem (5.22) takes the form (5.31) with nonnegative weights of each addend and problem (5.19) is its one-to-one transformable version. If however condition (5.24) is not satisfied, both problems (5.19) and (5.22) are in general nonconvex, or, equivalently, negative weights occur in the problem form (5.31).

The geometry described in Corollary 6 is illustrated in Fig. 5.8 for the capacity region of an exemplary multi-antenna multiple access channel with two links and sum-power constraint.

STABILITY-OPTIMAL POLICY AND RATES FOR $N < K$ BUSY QUEUES

We refer to (instantaneously) empty queues as to *idle* queues, in contrast to *busy* queues. The queues of links with sparse traffic (that is, with low arrival rate) and/ or fast depletion rate, e.g. due to good channel conditions, are likely to remain idle throughout several consecutive slots. In any of such slots, queue system state satisfies

$$q_i = 0, \quad i \in \mathcal{L} \subset \mathcal{K}, \qquad q_i > 0, \quad i \notin \mathcal{L}, \tag{5.33}$$

so that the (instantaneous) problem of computing a stability-optimal policy is restricted to a proper subset of links with busy queues $\mathcal{L} \subset \mathcal{K}$. Clearly, there is interest in an efficient utilization of links associated with idle queues, although such links do not influence *directly* the problem of computation of stability-optimal policy (5.31). A particular question of interest is, if, or in which

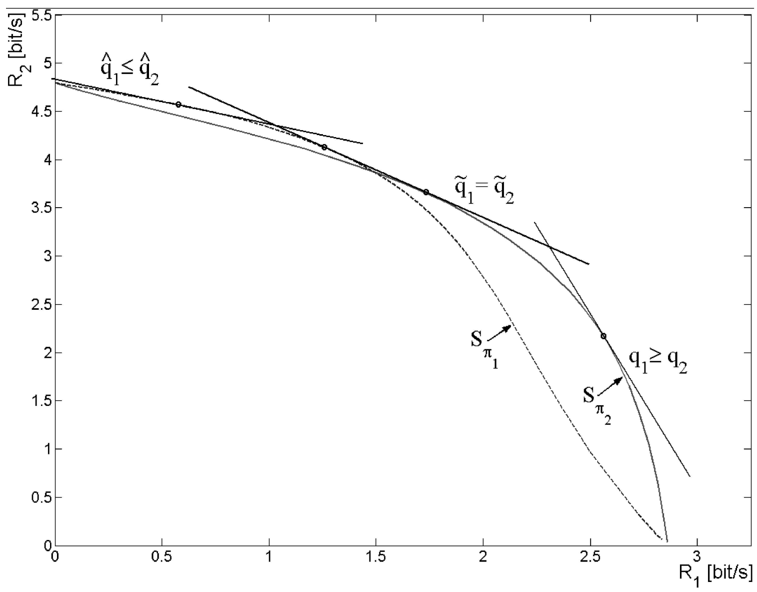

FIGURE 5.8: Illustration of the geometry from Corollary 6 for the capacity region of an exemplary MIMO multiple access channel with two links (Rayleigh fading realization).

case, the transmission of links associated with idle queues influences/ disturbs the stability-optimal transmission policy of links associated with busy queues. If such influence were nonexistent, the links of idle queues could be used for additional transmission of pilots, signaling and control information, etc.

Depending on the type of power constraints, the problem (5.31) exhibits an essentially different behavior in terms of utilization of links of idle queues. First, consider the sum-power constrained multi-antenna multiple access channel and the corresponding result.

Proposition 24 *Let $\mathcal{L} \subset \mathcal{K}$ be the subset of links of idle queues in the sum-power constrained MIMO multiple access channel. For the stability-optimal rate vector $\tilde{\boldsymbol{R}}$ given by (5.32), we have $\tilde{R}_i = 0$, $i \in \mathcal{L}$.*

Proof Since the constraint qualification condition for (5.31) can be easily shown to be satisfied and the problem is convex, the corresponding Kuhn-Tucker conditions are necessary and sufficient optimality conditions. The Kuhn-Tucker condition corresponding to zeroing of the Lagrangian derivative is, given sum-power constraint $\mathcal{P} = \mathcal{P}_P$,

$$\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}\sigma^2 + \sum_{k=1}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)} = \lambda\boldsymbol{I} - \boldsymbol{Z}_{\pi(i)}, \quad 1 \leq i \leq K. \quad (5.34)$$

Thereby, $\lambda \geq 0$ is the Lagrange multiplier associated with the sum-power constraint and $\boldsymbol{Z}_i \succeq 0$, $1 \leq i \leq K$, is the Lagrange multiplier associated with positive semidefiniteness constraint on the corresponding transmit covariance matrix. Clearly, under some idle queues corresponding to link subset $\mathcal{L} \subset \mathcal{K}$, the queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$ takes the form (5.33). Without loss of generality, property (5.33) can be replaced in (5.31) by the assumption

$$q_{\pi(i)} = 0, \quad K - |\mathcal{L}| < i \leq K \quad (5.35)$$

(since regardless of the position of links of idle queues $k \in \mathcal{L}$, the links of busy queues $k \in \mathcal{K} \setminus \mathcal{L}$ have to be reordered analogously to (5.24)). Thus, letting (5.35) in (5.34) yields

$$\sum_{j=i}^{K-|\mathcal{L}|} (q_{\pi(j)} - q_{\pi(j+1)}) \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I}\sigma^2 + \sum_{k=1}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = \lambda \boldsymbol{I} - \boldsymbol{Z}_{\pi(i)}, \quad 1 \le i \le K - |\mathcal{L}| \tag{5.36}$$

and

$$\lambda \boldsymbol{I} = \boldsymbol{Z}_{\pi(i)}, \quad K - |\mathcal{L}| < i \le K. \tag{5.37}$$

Note now that taking $\lambda = 0$ in the optimality condition (5.37) leads to triviality in the optimality condition (5.36). Precisely, in such case (5.36) is satisfied only if both sides are zero (that is, $\boldsymbol{Q}_{\pi(i)} = \boldsymbol{0}$, $1 \le i \le K - |\mathcal{L}|$), since the left-hand side is a positive semidefinite matrix while the right-hand side is a negative semidefinite one. Thus, we must have $\lambda > 0$ which implies with (5.37) that $\boldsymbol{Z}_{\pi(i)}$, $K - |\mathcal{L}| < i \le K$, is a positively scaled unit matrix. Consequently, applying (5.37) to the complementary slackness condition $tr(\boldsymbol{Z}_{\pi(i)} \boldsymbol{Q}_{\pi(i)}) = 0$, $1 \le k \le K$, yields immediately $\boldsymbol{Q}_{\pi(i)} = 0$, $K - |\mathcal{L}| < i \le K$, which completes the proof. □

By Proposition 24, the additional transmission of any signals through links of idle queues is not possible under stability-optimal policy (of links of busy queues) and sum-power constraint. The plausibility proof of Proposition 24 is simple. In the sum-power constrained multi-antenna multiple access channel we have a global power budget to be allocated among single links. The use of any portion of the power budget for additional transmission through links of idle queues automatically reduces the remaining power budget for busy queues and prevents the application of stability-optimal policy to them.

Proposition 24 can be reformulated in terms of geometry of the capacity region. For the reformulation, note that a hyperplane with normal vector $\boldsymbol{q} \in \mathbb{R}^K$ corresponds to an affine subspace $\boldsymbol{c} + span(\mathcal{V}_{\boldsymbol{q}})$, $\boldsymbol{c} \in \mathbb{R}^K$, where $\mathcal{V}_{\boldsymbol{q}}$ is an orthogonal $(K-1)$-system of the form [121]

$$\mathcal{V}_{\boldsymbol{q}} = \{\boldsymbol{v}_i\}_{i=1}^{K-1}, \quad \text{such that} \quad \boldsymbol{v}_i \perp \boldsymbol{v}_j, \quad \boldsymbol{v}_i \perp \boldsymbol{q}, \quad 1 \le i < j \le K-1.$$

Thus, the hyperplane supporting the capacity region at a stability-optimal rate vector takes the form $\arg\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H})} \boldsymbol{q}'\boldsymbol{R} + span(\mathcal{V}_{\boldsymbol{q}})$.

Corollary 7 (Geometric Restatement of Proposition 24) *Any rate vector included in the capacity region $\mathcal{C}(\mathcal{H}, P)$ and in the affine subspace $\arg\max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H},P)} \boldsymbol{q}'\boldsymbol{R} + span(\mathcal{V}_{\boldsymbol{q}})$, with $\boldsymbol{q} \in \mathbb{R}_+^K$ satisfying (5.33), is included in the subspace $span(\{\boldsymbol{e}_i\}_{i \in \mathcal{K} \setminus \mathcal{L}})$ as well. Equivalently, given $\boldsymbol{q} \in \mathbb{R}_+^K$ satisfying (5.33), if*

$$\tilde{\boldsymbol{R}} \in \mathcal{C}(\mathcal{H}, P) \cap \arg \max_{\boldsymbol{R} \in \mathcal{C}(\mathcal{H},P)} \boldsymbol{q}'\boldsymbol{R} + span(\mathcal{V}_{\boldsymbol{q}})$$

then also

$$\tilde{\boldsymbol{R}} \in span(\{\boldsymbol{e}_i\}_{i \in \mathcal{K} \setminus \mathcal{L}}).$$

Note that exemplary illustration of the geometry described in Corollary 7 is already provided in Fig. 5.6 and 5.7.

Consider now the case of MIMO multiple access channel with individual power constraints. In such case, the result paralleling Proposition 24 can be formulated as follows.

Proposition 25 *Let $\mathcal{L} \subset \mathcal{K}$ be the subset of links of idle queues in the MIMO multiple access channel with individual power constraints. Then, the following is true.*

i.) If rate vector $\tilde{\boldsymbol{R}} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ *is stability-optimal in the sense of (5.32), then any rate vector* $\boldsymbol{R} \in \mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ *such that* $R_i = \tilde{R}_i$, $i \in \mathcal{K} \setminus \mathcal{L}$, *is stability-optimal (solves (5.19)) as well.*
ii.) The capacity region available for the link subset $\mathcal{L}$ *under stability-optimal policy for link subset* $\mathcal{K} \setminus \mathcal{L}$ *corresponds to the capacity region of the MIMO multiple access channel (under given channels) with noise covariance matrix*

$$\boldsymbol{N} = \boldsymbol{I} W \sigma^2 + \sum_{i \in \mathcal{K} \setminus \mathcal{L}} \boldsymbol{H}_i \boldsymbol{Q}_i \boldsymbol{H}_i', \tag{5.38}$$

where $\boldsymbol{Q}_i$ *denotes the (stability-optimal) transmit covariance matrix of link* $i \in \mathcal{K} \setminus \mathcal{L}$.

Proof Analogously to the proof of Proposition 24, consider the Kuhn-Tucker condition corresponding to zeroing of the Lagrangian derivative of problem (5.31), given now individual power constraints $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$. Such condition takes now the form

$$\sum_{j=i}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I}\sigma^2 + \sum_{k=1}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = \lambda_{\pi(i)} \boldsymbol{I} - \boldsymbol{Z}_{\pi(i)}, \quad 1 \le i \le K, \tag{5.39}$$

where $\lambda_{\pi(i)} \ge 0$ is the Lagrange multiplier associated with the corresponding link power constraint and $\boldsymbol{Z}_i \succeq 0$, $1 \le i \le K$, is the Lagrange multiplier associated with positive semidefiniteness constraint on the corresponding transmit covariance matrix. Accounting for the set of links of idle queues $k \in \mathcal{L}$ by (5.35), as in the proof of Proposition 24, we yield from (5.39)

$$\sum_{j=i}^{K-|\mathcal{L}|} (q_{\pi(j)} - q_{\pi(j+1)}) \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I}\sigma^2 + \sum_{k=1}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = \lambda_{\pi(i)} \boldsymbol{I} - \boldsymbol{Z}_{\pi(i)}, \quad 1 \le i \le K - |\mathcal{L}|, \tag{5.40}$$

and

$$\lambda_{\pi(i)} \boldsymbol{I} = \boldsymbol{Z}_{\pi(i)}, \quad K - |\mathcal{L}| < i \le K. \tag{5.41}$$

By (5.41) and the complementary slackness conditions

$$\left\{ \begin{array}{l} tr(\boldsymbol{Q}_{\pi(i)} \boldsymbol{Z}_{\pi(i)}) = 0 \\ \lambda_{\pi(i)} (tr(\boldsymbol{Q}_{\pi(i)}) - p_{\pi(i)}) = 0, \end{array} \right. \quad 1 \le i \le K, \tag{5.42}$$

we get a necessary optimality condition as the equation system

$$\left\{ \begin{array}{l} \lambda_{\pi(i)} tr(\boldsymbol{Q}_{\pi(i)}) = 0 \\ \lambda_{\pi(i)} (tr(\boldsymbol{Q}_{\pi(i)}) - p_{\pi(i)}) = 0, \end{array} \right. \quad K - |\mathcal{L}| < i \le K. \tag{5.43}$$

However, by (5.40) can be seen that the variables $tr(\boldsymbol{Q}_{\pi(i)})$, $K - |\mathcal{L}| < i \le K$, occur only in the Kuhn-Tucker condition (5.43) and the Kuhn-Tucker condition representing the individual power constraints

$$0 \le tr(\boldsymbol{Q}_{\pi(i)}) \le \hat{p}_{\pi(i)}, \quad K - |\mathcal{L}| < i \le K. \tag{5.44}$$

Thus, (5.43), (5.44) are necessary and sufficient optimality conditions with respect to variables $tr(\boldsymbol{Q}_{\pi(i)})$, $K - |\mathcal{L}| < i \le K$.

Notice now that (5.43) is solved only under $\lambda_{\pi(i)} = 0$, $K - |\mathcal{L}| < i \le K$. But under such condition (5.43) is solved regardless of $tr(\boldsymbol{Q}_{\pi(i)})$, $K - |\mathcal{L}| < i \le K$. Thus, we conclude that an optimal value of $tr(\boldsymbol{Q}_{\pi(i)})$, $K - |\mathcal{L}| < i \le K$, is an arbitrary value satisfying (5.44), which completes the proof. □

From Proposition 25 follows that in the MIMO multiple access channel with individual power constraints the links of idle queues are allowed to transmit additional signals with arbitrary available rates, i.e. with arbitrary allowable transmit powers. Further, under stability-optimal policy (for links of busy queues), the links of idle queues dispose of a capacity region with noise increased by the interference from links of busy queues. From the proof of Proposition 25 can be concluded that this feature is a consequence of decoding the links of possible idle queues before the links of busy queues.

In broad terms, one can say that the features from Proposition 25 result from the decoupling of link power budgets in the case of individual power constraints. In fact, one kind of decoupling is provided at the (base station) receiver due to SIC, which decouples the link signals decoded later from those decoded earlier in the SIC order. Individual power constraints in the MIMO multiple access channel provide a kind of additional decoupling at the link transmitters.

Analogously to Proposition 25, Proposition 25 can be reformulated in terms of the geometry of the capacity region. The reformulation uses the notion of *exposed subset*, understood as a connected set consisting of points which are supporting points of the same hyperplane (a trivial exposed subset is a simple boundary point) [122].

Corollary 8 (Geometric Restatement of Proposition 25) *If vector* $\boldsymbol{q} \in \mathbb{R}_+^K$ *satisfies (5.33), then the following is true.*
i.) The rate vectors included in the affine subspace $\arg\max_{\boldsymbol{R}\in\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})} \boldsymbol{q}'\boldsymbol{R} + span(\mathcal{V}_{\boldsymbol{q}})$ *and in the capacity region* $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$ *constitute a nontrivial exposed subset (of the capacity region* $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$*). Equivalently,*

$$\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}}) \cap \arg\max_{\boldsymbol{R}\in\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})} \boldsymbol{q}'\boldsymbol{R} + span(\mathcal{V}_{\boldsymbol{q}}) \tag{5.45}$$

is an exposed subset (of the capacity region $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$*).*
ii.) The exposed subset (5.45) of the capacity region $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$ *has the (qualitative) geometric structure of the capacity region* $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}})$ *itself under dimensionality reduced to* $|\mathcal{L}|$.

The geometry described by Corollary 8 is illustrated for the capacity region of an exemplary MIMO multiple access channel with two links in Fig. 5.9.

Some notes on vertex rate vectors

Recall the multi-antenna multiple access channel with individual power constraints $\hat{\boldsymbol{p}}$ and fixed spatial correlation properties of link signals described by $\hat{\mathcal{Q}}$, such that (5.3). In such case, a vertex rate vector $\boldsymbol{R}^\pi$, $\pi \in \Pi_K$, of the corresponding capacity region $\mathcal{C}(\mathcal{H},\hat{\boldsymbol{p}},\hat{\mathcal{Q}})$ is achievable by an extremely easy computable transmission policy. Such policy allocates maximum allowable transmit power to each link and processes the link set by SIC according to the SIC order π, $\pi \in \Pi_K$. Such nice computational properties of a vertex rate vector $\boldsymbol{R}^\pi$, $\pi \in \Pi_K$, imply interest in a rate vector with analogous features in the capacity region of the MIMO multiple access channel with individual power constraints. Such intuitive analogue is the rate vector which is achievable by some given SIC order $\pi \in \Pi_K$ and the set of single-link optimal transmit covariance matrices. Given SIC order $\pi \in \Pi_K$, the single-link optimal transmit covariance matrix of link $\pi(i)$ is hereby known to satisfy

$$\boldsymbol{Q}_{\pi(i)} = \arg\max_{\boldsymbol{Q}\succeq 0 : tr(\boldsymbol{Q})\leq\hat{p}_{\pi(i)}} W \log\det(\boldsymbol{N}_i + \boldsymbol{H}_{\pi(i)}\boldsymbol{Q}\boldsymbol{H}'_{\pi(i)}), \tag{5.46}$$

with noise covariance matrix

$$\boldsymbol{N}_i = \boldsymbol{I}W\sigma^2 + \sum_{j=1}^{i} \boldsymbol{H}_{\pi(j)}\boldsymbol{Q}_{\pi(j)}\boldsymbol{H}'_{\pi(j)}, \quad 1 \leq i \leq K, \tag{5.47}$$

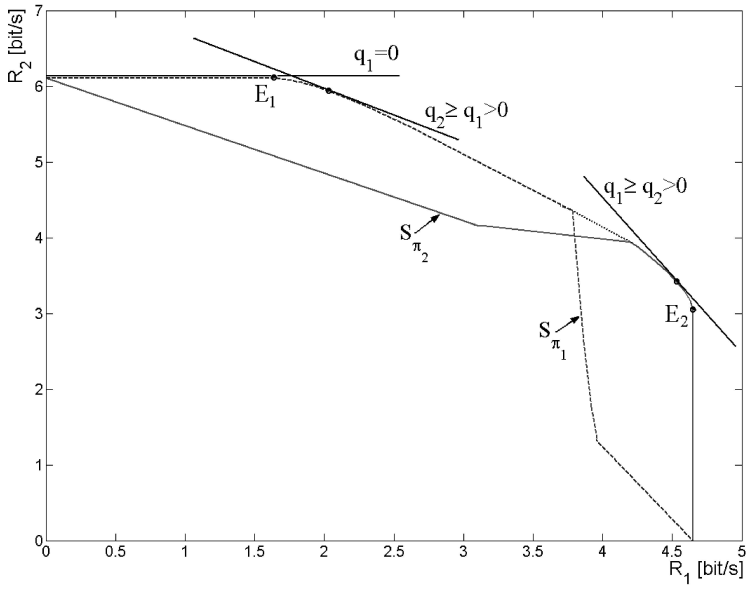

FIGURE 5.9: Illustration of the geometry from Corollary 8 for the capacity region of an exemplary MIMO multiple access channel with two links and individual power constraints (Rayleigh fading realization). For $q_1 = 0$ the exposed subset representing the (trivial) capacity region of link 1 under given transmission of link 2 is a line segment. The two rate vectors achievable under iterative waterfilling with SIC order $\pi_1 = 2 \leftarrow 1$ and $\pi_2 = 1 \leftarrow 2$ are denoted as E_1 and E_2, respectively, and are included in corresponding exposed subsets.

determined by the noise and interference from links decoded later in the SIC order. In other words, the set of transmit covariance matrices (5.46) is computed by so-called *iterative waterfilling*, consisting in the sequence of waterfillings sequentially adapted to the interference resulting under predefined SIC order $\pi \in \Pi_K$ [114].

We can describe the arrangement of rate vectors achievable by iterative waterfilling in terms of geometry of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$. Precisely, let $\mathcal{C}_{\boldsymbol{N}}(\mathcal{H}, \hat{\boldsymbol{p}})$ be the extended notation of the capacity region of the multi-antenna multiple access channel which exhibits that the noise has covariance matrix $\boldsymbol{N} \succeq 0$ (in these terms, we considered up to now the MIMO multiple access channel with capacity region $\mathcal{C}_{\boldsymbol{I}\sigma^2}(\mathcal{H}, \hat{\boldsymbol{p}})$). Then, one easily deduces the following result.

Proposition 26 *The rate vector $\tilde{\boldsymbol{R}}^{\pi}$ achieved by iterative waterfilling under SIC order $\pi \in \Pi_K$ (i.e., by transmit covariance matrices (5.46)) is included in the affine subspaces*

$$\sum_{j=1}^{i} \arg \max_{\boldsymbol{R} \in \mathcal{C}_{\boldsymbol{N}_{j-1}}(\mathcal{H}, \hat{\boldsymbol{p}})} \boldsymbol{e}_j' \boldsymbol{R} + span(\{\boldsymbol{e}_j\}_{j>i}), \quad 1 \leq i \leq K, \tag{5.48}$$

with $\boldsymbol{N}_i$, $1 \leq i \leq K$, defined in (5.47). More precisely, $\tilde{\boldsymbol{R}}^{\pi}$ represents the singleton intersection of the subspaces (5.48).

The case of interest is stability optimality of the rate vector achievable by iterative waterfilling. Equivalently, the interest is in iterative waterfilling as stability-optimal policy. Without formally stating the corresponding result, we only mention here that under certain realistic conditions iterative waterfilling is a stability-optimal policy for certain nonsingleton sets of queue system states (see [95] for details).

The geometric arrangement of the rate vectors achieved under iterative waterfilling is illustrated for the capacity region of an exemplary MIMO multiple access channel with two links in Fig. 5.9.

5.3 Optimization-theoretic analysis of the stability-optimal policy

In the current Section we characterize further features of the stability-optimal policy. In contrast to the basic results from Section 5.2, the results of the current section are based on the Lagrangean approach to optimization problem (5.31). Precisely, the obtained characterizations, e.g. of stability optimality of link subset transmission or stability optimality of the SIC order regardless of the queue system state, rely mainly on the Lagrangean function of (5.31) and the associated Kuhn-Tucker conditions.

5.3.1 Stability-optimal policy under SIC order restriction

The stability-optimal SIC order characterized in Proposition 23 can be in some cases not accessible. This can occur, for instance, when some links have requirements on their position in the SIC order. In particular, the existence of a high-priority link which requires to be decoded last is thinkable. Depending on the queue system state, the computation of the corresponding *(SIC order-) restricted* stability-optimal policy might consist in solving the problem (5.22) (equivalently, (5.31)) under stability-suboptimal SIC order. In particular, the stability optimality condition (5.24) might be not satisfied, so that the problem (5.22) is in general nonconvex. With the results of Section 5.2.5 it is then implied that the nonconvex parts of the boundaries of the S-rate regions become to be of interest in terms of stability optimality.

For the arising nonconvex problem of computation of restricted stability-optimal policy it is at least possible to gain insights in the issue of link power consumption (Proposition 27).

For comparison purposes consider first the problem (5.31) when the stability-optimal SIC order satisfying (5.24) for the given queue system state is accessible. From the resulting nonnegativity of weights in (5.31) and *operator monotony* of the log det-function on positive semidefinite matrices (see [63]) follows that an increase of transmit power $tr(\boldsymbol{Q}_i)$ of any link $i \in \mathcal{K}$ results in an increase of the objective (5.30). This implies that under stability-optimal policy each link is allocated maximum allowed power in the MIMO multiple access channel with individual power constraints, and the entire sum-power is allocated among the links in the sum-power constrained MIMO multiple access channel. The link power consumption under stability-optimal policy restricted to an SIC order violating condition (5.24) has however different features, which can be stated as follows.

Proposition 27 *For the MIMO multiple access channel under some queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$, let a restricted stability-optimal policy $(\mathcal{Q}, \pi)$, such that the set of transmit covariance matrices $\mathcal{Q}$ solves (5.22) but the SIC order π violates (5.24), be given. Then, the following is true.*
i.) Under individual power constraints $\hat{\boldsymbol{p}}$, the link powers satisfy in general $tr(\boldsymbol{Q}_i) \leq \hat{p}_i$, $i \in \mathcal{K}$, and $tr(\boldsymbol{Q}_{\pi(K)}) = \hat{p}_{\pi(K)}$.
ii.) Under sum-power constraint P, the link powers satisfy $\sum_{i \in \mathcal{K}} tr(\boldsymbol{Q}_i) = P$.

Proof Given $\boldsymbol{q} \in \mathbb{R}_+^K$ and some SIC order $\pi \in \Pi_K$ violating (5.24), some of the multipliers

$q_{\pi(i)} - q_{\pi(i+1)}$, $1 \le i \le K-1$, in the objective (5.30) are nonpositive. Thus, define

$$\begin{cases} d_{\pi(i)} = |q_{\pi(i)} - q_{\pi(i+1)}| \\ A = \{i \in \mathcal{K} : d_{\pi(i)} \ge 0\} \quad 1 \le i \le K, \\ B = \{i \in \mathcal{K} : d_{\pi(i)} \le 0\}, \end{cases} \tag{5.49}$$

where we set $q_{\pi(K+1)}=0$. Let now a generalization of the objective (5.30) as a function $(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) \mapsto \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) = f_{\boldsymbol{q},\pi}(\{\alpha_i \hat{\boldsymbol{Q}}_i\}_{i=1}^K)$, $(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) \in \mathbb{R}_+^K \times \mathcal{P}_{\hat{\boldsymbol{p}}}$, $\hat{\boldsymbol{p}} = \mathbf{1}$, be defined. Then, using (5.49), we can rewrite the objective (5.30) according to

$$\begin{aligned} f(\{\alpha_i \hat{\boldsymbol{Q}}_i\}_{i=1}^K) = \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) = & \sum_{i \in A} d_{\pi(i)} W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \alpha_{\pi(j)} \boldsymbol{H}_{\pi(j)} \hat{\boldsymbol{Q}}_{\pi(j)} \boldsymbol{H}'_{\pi(j)}) \\ & - \sum_{i \in B} d_{\pi(i)} W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \alpha_{\pi(j)} \boldsymbol{H}_{\pi(j)} \hat{\boldsymbol{Q}}_{\pi(j)} \boldsymbol{H}'_{\pi(j)}), \end{aligned} \tag{5.50}$$

$(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) \in \mathbb{R}_+^K \times \mathcal{P}_{\hat{\boldsymbol{p}}}$, $\hat{\boldsymbol{p}} = \mathbf{1}$. With matrix differential calculus we can write the partial derivative of (5.50) with respect to a link power as

$$\begin{aligned} \frac{\partial}{\partial \alpha_{\pi(k)}} \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) = & \sum_{\substack{i \in A \\ i \ge k}} d_{\pi(i)} W tr((\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \alpha_{\pi(j)} \boldsymbol{H}_{\pi(j)} \hat{\boldsymbol{Q}}_{\pi(j)} \boldsymbol{H}'_{\pi(j)})^{-1} \boldsymbol{H}_{\pi(k)} \hat{\boldsymbol{Q}}_{\pi(k)} \boldsymbol{H}'_{\pi(k)}) \\ & - \sum_{\substack{i \in B \\ i \ge k}} d_{\pi(i)} W tr((\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \alpha_{\pi(j)} \boldsymbol{H}_{\pi(j)} \hat{\boldsymbol{Q}}_{\pi(j)} \boldsymbol{H}'_{\pi(j)})^{-1} \boldsymbol{H}_{\pi(k)} \hat{\boldsymbol{Q}}_{\pi(k)} \boldsymbol{H}'_{\pi(k)}), \end{aligned} \tag{5.51}$$

$1 \le k \le K$. Since the matrices subject to the trace operators in (5.51) are positive semidefinite, for any $(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) \in \mathbb{R}_+^K \times \mathcal{P}_{\hat{\boldsymbol{p}}}$, $\hat{\boldsymbol{p}} = \mathbf{1}$, we can always find a set of channel values $\mathcal{H}$ and a queue system state $\boldsymbol{q}$ which yield $\partial/(\partial \alpha_{\pi(k)}) \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) < 0$ for some $1 \le k \le K$. Thus, we can find such $\mathcal{H}$ and $\boldsymbol{q}$ in particular for $\boldsymbol{\alpha} = \hat{\boldsymbol{p}}$, which completes the proof of the inequality in part i.).

Note that by definitions (5.49) we have $d_{\pi(K)} > 0$, so that $K \in A$. Thus, $\alpha_{\pi(K)}$ occurs in a single addend in the nonnegative sum in (5.50) and it does not occur in the nonpositive sum in (5.50). This implies by (5.51) that

$$\frac{\partial}{\partial \alpha_{\pi(K)}} \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{\alpha}, \hat{\mathcal{Q}}) > 0, \quad (\boldsymbol{\alpha}, \hat{\mathcal{Q}}) \in \mathbb{R}_+^K \times \mathcal{P}_{\hat{\boldsymbol{p}}}, \quad \hat{\boldsymbol{p}} = \mathbf{1}. \tag{5.52}$$

Taking, in particular, $\{\alpha_i \hat{\boldsymbol{Q}}_i\}_{i=1}^K = \mathcal{Q}$ in (5.52) implies that $tr(\boldsymbol{Q}_{\pi(K)}) = \hat{p}_{\pi(K)}$ under individual power constraints and $\sum_{i=1}^K tr(\boldsymbol{Q}_i) = P$ under sum-power constraint. This completes the proof of part i.) and ii.). □

By Proposition 27, the restricted stability-optimal policy under violated condition (5.24) consumes the entire sum-power in the sum-power constrained multi-antenna multiple access channel. Interestingly, the analogous restricted stability-optimal policy does not assign maximum allowed power to each link under individual power constraints. In the context of Section 5.2.5, the features of power consumption from Proposition 27 are valid under any rate vector included in the nonconvex boundary part of an S-rate region.

The illustration to Proposition 27 i.) is provided in Fig. 5.10. It shows the S-rate regions of an exemplary MIMO multiple access channel with two links and individual power constraints, with the additionally marked path/ manifold of rate vectors achieved under allocation of maximum allowed power to each link.

5.3.2 Kuhn-Tucker conditions of the problem

Since problem (5.31) is convex under given stability-optimal SIC order (5.24) and can be easily shown to satisfy constraint qualification, the associated Kuhn-Tucker conditions are necessary and sufficient optimality conditions [48], [47]. The conventional linear Lagrangean function of the problem (5.31) in the minimization form can be written as

$$\begin{aligned} L(\mathcal{Q}, \mathcal{Z}, \boldsymbol{\lambda}) = &- \sum_{i=1}^{K} (q_{\pi(i)} - q_{\pi(i+1)}) W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \boldsymbol{H}_{\pi(j)} \boldsymbol{Q}_{\pi(j)} \boldsymbol{H}'_{\pi(j)}) \\ &- \sum_{i=1}^{K} tr(\boldsymbol{Z}_{\pi(i)} \boldsymbol{Q}_{\pi(i)}) + \sum_{i=1}^{K} \lambda_{\pi(i)} (tr(\boldsymbol{Q}_{\pi(i)}) - p_{\pi(i)}) \end{aligned} \tag{5.53}$$

in the case of individual power constraints and

$$\begin{aligned} L(\mathcal{Q}, \mathcal{Z}, \lambda) = &- \sum_{i=1}^{K} (q_{\pi(i)} - q_{\pi(i+1)}) W \log\det(\boldsymbol{I} W \sigma^2 + \sum_{j=1}^{i} \boldsymbol{H}_{\pi(j)} \boldsymbol{Q}_{\pi(j)} \boldsymbol{H}'_{\pi(j)}) \\ &- \sum_{i=1}^{K} tr(\boldsymbol{Z}_{\pi(i)} \boldsymbol{Q}_{\pi(i)}) + \lambda (\sum_{i=1}^{K} tr(\boldsymbol{Q}_{\pi(i)}) - P) \end{aligned} \tag{5.54}$$

under sum-power constraint. The hermitian matrix $\boldsymbol{Z}_i$ is the Lagrangean multiplier/ dual variable associated with the constraint of positive semidefiniteness on $\boldsymbol{Q}_i$, $i \in \mathcal{K}$. The dual variables λ_i and λ correspond to the power constraints, the i-th individual power constraint and the sum-power constraint, respectively. The set of Kuhn-Tucker conditions of problem (5.31) under individual power constraints takes the form

$$\begin{cases} -\boldsymbol{Q}_{\pi(i)} \preceq 0 \quad \text{(P1)} \\ tr(\boldsymbol{Q}_{\pi(i)}) - p_{\pi(i)} \leq 0 \quad \text{(P2)} \\ \boldsymbol{Z}_{\pi(i)} \succeq 0 \quad \text{(D1)} \\ \lambda_{\pi(i)} \geq 0 \quad \text{(D2)} \\ tr(\boldsymbol{Q}_{\pi(i)} \boldsymbol{Z}_{\pi(i)}) = 0 \quad \text{(C1)} \\ \lambda_{\pi(i)} (tr(\boldsymbol{Q}_{\pi(i)}) - p_{\pi(i)}) = 0 \quad \text{(C2)} \\ \sum_{j=i}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \sum_{k=1}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = \lambda_{\pi(i)} \boldsymbol{I} - \boldsymbol{Z}_{\pi(i)} \quad \text{(Z)}, \end{cases} \tag{5.55}$$

$i \in \mathcal{K}$. Hereby, the primal constraints are denoted by (P), the dual constraints by (D), the complementary slackness equalities by (C) and the partial derivative of the Lagrangean function set to

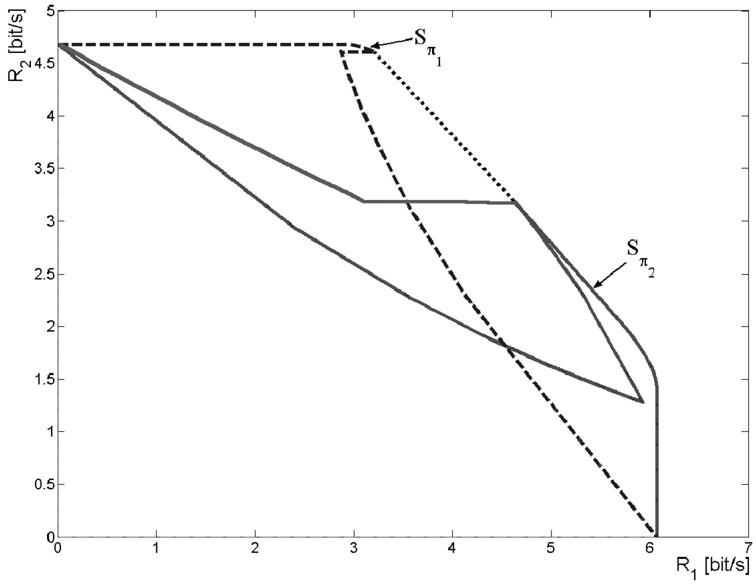

FIGURE 5.10: The S-rate regions in the capacity region of an exemplary MIMO multiple access channel with two links and individual power constraints $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2)$ (Rayleigh fading realization), plotted together with the path of rate pairs achieved under SIC order $\pi_2 = 1 \leftarrow 2$ and maximum link powers $tr(\boldsymbol{Q}_1) = \hat{p}_1$, $tr(\boldsymbol{Q}_2) = \hat{p}_2$. The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, \hat{\boldsymbol{p}})$, $\pi_1 = 2 \leftarrow 1$ is dashed, the boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}})$ is solid. The path lies in the interior of the corresponding S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, \hat{\boldsymbol{p}})$.

zero by (Z). Under sum-power constraint, the set of Kuhn-Tucker conditions changes to

$$\begin{cases} -\boldsymbol{Q}_{\pi(i)} \preceq 0 \quad (\text{P1}) \\ \sum_{i=1}^{K} tr(\boldsymbol{Q}_{\pi(i)}) - P \leq 0 \quad (\text{P2}) \\ \boldsymbol{Z}_{\pi(i)} \succeq 0 \quad (\text{D1}) \\ \lambda \geq 0 \quad (\text{D2}) \\ tr(\boldsymbol{Q}_{\pi(i)} \boldsymbol{Z}_{\pi(i)}) = 0 \quad (\text{C1}) \\ \lambda(\sum_{i=1}^{K} tr(\boldsymbol{Q}_{\pi(i)}) - P) = 0 \quad (\text{C2}) \\ \sum_{j=i}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \sum_{k=1}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = \lambda \boldsymbol{I} - \boldsymbol{Z}_{\pi(i)} \quad (\text{Z}), \end{cases} \tag{5.56}$$

$i \in \mathcal{K}$. Due to problem convexity, strong Lagrangean duality holds for problem (5.31), and additionally the sets of Kuhn-Tucker conditions (5.55) and (5.56) describe saddle points of the Lagrangian (5.53) and (5.54), respectively (recall from Section 4.1.2 that the saddle point property does not hold in general for problems with strong duality which are nonconvex [75], [85]). Clearly, such saddle point, say $(\tilde{\mathcal{Q}}, \tilde{\mathcal{Z}}, \tilde{\boldsymbol{\lambda}})$ in the case of individual power constraints, is such that $\tilde{\mathcal{Q}}$ is the (global) solution to (5.31).

The (Z)-condition in both sets of Kuhn-Tucker conditions is not symmetric with respect to the links $i \in \mathcal{K}$ and depends on link position in the SIC order. In the special case of symmetric queue system state $\boldsymbol{q} = q\boldsymbol{1}$, $q > 0$, the sum in the (Z)-condition reduces to one term for each link $i \in \mathcal{K}$. For such case we know from [114], [46] that the conditions (5.55) and (5.56) characterize the iterative waterfilling solution without and with additional link power adaptation, respectively. As described in Section 5.2.5, under iterative waterfilling the transmit covariance matrix of link $\pi(i)$, $i \in \mathcal{K}$, is a waterfilling solution, optimal in terms of single-link capacity, under the perceived interference from links $\pi(j)$, $j < i$. In this context it is evident that for the first decoded link $\pi(K)$ the sums in the (Z)-conditions in (5.55), (5.56) retain only one term, regardless of the queue system state. This immediately implies the following feature.

Lemma 11 *Let $(\mathcal{Q}, \pi)$ be the stability-optimal policy in the multi-antenna multiple access channel with link set $\mathcal{K}$. Then, the transmit covariance matrix $\boldsymbol{Q}_{\pi(K)}$ of the last decoded link $\pi(K)$ corresponds to the waterfilling solution under the noise covariance matrix determined by the interference from all links $\pi(i)$, $i < K$, and given by (5.47) with $i = K$.*

The intuition behind Lemma 11 is that the first decoded link $\pi(K)$ has to adapt its transmit covariance matrix to all other links $\pi(i)$, $i < K$, since it perceives interference from all of them. Thus, link $\pi(K)$ contributes to the maximization of the weighted sum of rates (5.19) by maximizing its own link capacity, which is actually done by the waterfilling solution.

5.3.3 Stability optimality of N-link regimes

Relying on the Kuhn-Tucker conditions of the problem (5.31), one can formulate useful conditions determining the set of *active* links, that is, links allocated nonzero transmit power by the stability-optimal policy. The evaluation of such conditions before actual solution of the problem (5.31) can reduce the dimension of the optimization problem by the number of links identified as *idle*, that is, not active. Thus, if such conditions can be evaluated with low effort, their evaluation might pay off in significant reduction of computational complexity of the subsequent optimization. In the following we refer to the case when links $i \in \mathcal{N} \subseteq \mathcal{K}$ are active and links $i \in \mathcal{K} \setminus \mathcal{N}$ are idle, where $|\mathcal{N}| = N$, as an N-link regime $\mathcal{N}$.

Individual Power Constraints

For the MIMO multiple access channel with individual power constraints we have the following result.

Proposition 28 *Let $(\mathcal{Q}, \pi)$ be the stability-optimal policy in the MIMO multiple access channel with link set $\mathcal{K}$, queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$ and individual power constraints $\hat{\boldsymbol{p}}$. Then, if link $i \in \mathcal{K}$ is associated with a busy queue $q_i > 0$, then we have $tr(\boldsymbol{Q}_i) = \hat{p}_i$.*

Proof Let $\mathcal{L} \subseteq \mathcal{K}$ be the subset of all links of busy queues $q_i > 0$, $i \in \mathcal{L}$, and assume, by contradiction, that at any solution to (5.31) under individual power constraints only links from a link subset $\mathcal{N} \subset \mathcal{L}$ are allocated corresponding maximum allowed powers $tr(\boldsymbol{Q}_i) = \hat{p}_i$, $i \in \mathcal{N}$ (so that $tr(\boldsymbol{Q}_i) < \hat{p}_i$, $i \in \mathcal{L} \setminus \mathcal{N}$). From the complementary slackness condition (5.55)-(C2) follows then $\lambda_i = 0$, $i \in \mathcal{L} \setminus \mathcal{N}$. Thus, with $q_i = 0$, $i \in \mathcal{K} \setminus \mathcal{L}$, and stability optimality of the SIC order $\pi \in \Pi_K$ in the sense of (5.24), the Kuhn-Tucker condition (5.55)-(Z) implies

$$\sum_{j=i}^{|\mathcal{L}|} (q_{\pi(j)} - q_{\pi(j+1)}) W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \sum_{k=1, k \in \mathcal{L}}^{j} \boldsymbol{H}_{\pi(k)} \boldsymbol{Q}_{\pi(k)} \boldsymbol{H}'_{\pi(k)})^{-1} \boldsymbol{H}_{\pi(i)} = -\boldsymbol{Z}_{\pi(i)}, \quad \pi(i) \in \mathcal{L} \setminus \mathcal{N} \tag{5.57}$$

(see also the corresponding discussion in the proof of Proposition 24, but account for different notation). By $q_{\pi(i)} - q_{\pi(i+1)} \geq 0$, $1 \leq i \leq |\mathcal{L}|$, the left-hand side of (5.57) is positive semidefinite, while the right-hand side is negative semidefinite due to (5.55)-(D1). Thus, (5.57) is satisfied only if both sides are zero matrices, that is, $\boldsymbol{Z}_{\pi(i)} = \boldsymbol{0}$, $\pi(i) \in \mathcal{L} \setminus \mathcal{N}$, and either

$$q_{\pi(j)} = 0, \quad 1 \leq j \leq |\mathcal{L}|, \tag{5.58}$$

or

$$q_{\pi(j)} = q_{\pi(i)}, \quad 1 \leq i, j \leq |\mathcal{L}| + 1.$$

But since $\pi(|\mathcal{L}|+1) \notin \mathcal{L}$ by assumption, we have $q_{\pi(|\mathcal{L}|+1)} = 0$ and thus, (5.58) must hold if (5.57) is satisfied. But this contradicts the definition of $\mathcal{L}$ as the set of links of all busy queues and thus, proves that $tr(\boldsymbol{Q}_i) = \hat{p}_i$, $i \in \mathcal{L}$. □

In other words, under stability-optimal policy in the MIMO multiple access channel with individual power constraints, any busy queue uses the power budget available to its link entirely. Consequently, the dimension of the optimization problem (5.31) is under individual power constraints always equal to the number of busy queues and can not be reduced, irrespective of the queue system state and channel values. For the case of individual power constraints recall in this context from Proposition 25 i.), that any link $i \in \mathcal{K}$ of an idle queue $q_i = 0$ can be allocated arbitrary allowable power without disturbing the stability-optimal policy of links of busy queues.

Exemplary illustration of Proposition 28 in terms of the geometry of the capacity region is already provided in Fig. 5.9.

SUM-POWER CONSTRAINTS

In contrast to the case of individual power constraints, stability optimality of an N-link regime, $N < K$, is possible under sum-power constraint in the MIMO multiple access channel. The following Proposition formulates a necessary and sufficient condition for stability-optimality of an N-link regime.

Proposition 29 *Let $(\mathcal{Q}, \pi)$ be the stability-optimal policy in the MIMO multiple access channel with link set $\mathcal{K}$, queue system state $\boldsymbol{q} \in \mathbb{R}^K_{++}$ and sum-power constraint P. Then, given $\mathcal{N} \subseteq \mathcal{K}$, we have $\boldsymbol{Q}_i \neq \boldsymbol{0}$, $i \in \mathcal{N}$, and $\boldsymbol{Q}_i = \boldsymbol{0}$, $i \in \mathcal{K} \setminus \mathcal{N}$, if and only if*

$$\begin{cases} \lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)}) = \lambda, & \pi(i) \in \mathcal{N} \\ \lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)}) \leq \lambda, & \pi(i) \notin \mathcal{N}, \end{cases} \tag{5.59}$$

with λ_{max} denoting the maximum eigenvalue of a hermitian matrix.

Proof We first prove the necessity. Stability optimality of N-link regime $\mathcal{N}$ implies $\boldsymbol{Q}_{\pi(i)} = 0$, $\pi(i) \notin \mathcal{N}$, which gives with the Kuhn-Tucker condition (5.56)-(Z)

$$\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)} = \lambda\boldsymbol{I} - \boldsymbol{Z}_{\pi(i)}, \quad i \in \mathcal{K} \tag{5.60}$$

(due to convexity of problem (5.31) and satisfied constraint qualification, the corresponding Kuhn-Tucker conditions are necessary and sufficient optimality conditions). Any $\boldsymbol{Q}_{\pi(i)}$, $\pi(i) \in \mathcal{N}$, has some nonzero eigenvector $\boldsymbol{u}^{(l)}_{\pi(i)}$ associated with a positive eigenvalue $\mu^{(l)}_{\pi(i)}$, $1 \leq l \leq M(\pi(i))$, with $M(\pi(i))$ as the rank of $\boldsymbol{Q}_{\pi(i)}$. Thus, the complementary slackness condition (5.56)-(C1) can be rewritten as

$$tr(\boldsymbol{Q}_{\pi(i)}\boldsymbol{Z}_{\pi(i)}) = tr(\sum_{l=1}^{M(\pi(i))}\mu^{(l)}_{\pi(i)}\boldsymbol{u}^{(l)}_{\pi(i)}\boldsymbol{u}'^{(l)}_{\pi(i)}\boldsymbol{Z}_{\pi(i)}) = \sum_{l=1}^{M(\pi(i))}\mu^{(l)}_{\pi(i)}\boldsymbol{u}'^{(l)}_{\pi(i)}\boldsymbol{Z}_{\pi(i)}\boldsymbol{u}^{(l)}_{\pi(i)} = 0, \quad \pi(i) \in \mathcal{N}, \tag{5.61}$$

which further implies

$$\boldsymbol{u}'^{(l)}_{\pi(i)}\boldsymbol{Z}_{\pi(i)}\boldsymbol{u}^{(l)}_{\pi(i)} = 0, \quad 1 \leq l \leq M(\pi(i)), \quad \pi(i) \in \mathcal{N}, \tag{5.62}$$

due to positivity of the eigenvalues in the sum. Without loss of generality, we can normalize $\boldsymbol{u}^{(l)}_{\pi(i)}$ so that $\|\boldsymbol{u}^{(l)}_{\pi(i)}\|_2 = 1$, $1 \leq l \leq M(\pi(i))$, $\pi(i) \in \mathcal{N}$. Now, multiply the equation (5.60) for $\pi(i) \in \mathcal{N}$ on the left-hand side with $\boldsymbol{u}'^{(l)}_{\pi(i)}$ and on the right-hand side with $\boldsymbol{u}^{(l)}_{\pi(i)}$, for some $1 \leq l \leq M(\pi(i))$. Similarly, multiply the equation (5.60) for $\pi(i) \notin \mathcal{N}$ with any vector $\boldsymbol{u} \in \mathbb{R}^K$, $\|\boldsymbol{u}\|_2 = 1$, in the same way. This yields

$$\begin{cases} \boldsymbol{u}'^{(l)}_{\pi(i)}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)})\boldsymbol{u}^{(l)}_{\pi(i)} = \\ \boldsymbol{u}'^{(l)}_{\pi(i)}\lambda\boldsymbol{I}\boldsymbol{u}^{(l)}_{\pi(i)}, \quad \pi(i) \in \mathcal{N} \\ \boldsymbol{u}'(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)})\boldsymbol{u} = \\ \boldsymbol{u}'(\lambda\boldsymbol{I} - \boldsymbol{Z}_{\pi(i)})\boldsymbol{u}, \quad \pi(i) \notin \mathcal{N}. \end{cases} \tag{5.63}$$

Notice, that with the dual Kuhn-Tucker condition (5.56)-(D1) we have [63]

$$\boldsymbol{u}'^{(l)}_{\pi(i)}\lambda\boldsymbol{I}\boldsymbol{u}^{(l)}_{\pi(i)} = \max_{\boldsymbol{u}\in\mathbb{R}^K:\|\boldsymbol{u}\|_2=1}\boldsymbol{u}'(\lambda\boldsymbol{I} - \boldsymbol{Z}_{\pi(i)})\boldsymbol{u}, \quad 1 \leq l \leq M(\pi(i)), \quad \pi(i) \in \mathcal{N}.$$

This implies that the left-hand side of the first equation in (5.63) corresponds to the maximum eigenvalue. Together with taking the maximum of both sides of the second equation over $\boldsymbol{u} \in \mathbb{R}^K$, $\|\boldsymbol{u}\|_2 = 1$, this yields with (5.56)-(D1) finally

$$\begin{cases} \lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)}) = \lambda, \pi(i) \in \mathcal{N} \\ \lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)}) \leq \lambda, \pi(i) \notin \mathcal{N}, \end{cases} \tag{5.64}$$

which completes the proof of necessity.

For the proof of sufficiency, notice first that $\sum_{i\in\mathcal{K}} tr(\boldsymbol{Q}_i) = P$. (The machinery for proving this feature is analogous to the one used in the proof of Proposition 28: The assumption of strict inequality in (5.56)-(P2) implies $\lambda = 0$ due to (5.56)-(C2), which further applied to (5.56)-(Z) yields the contradiction $q_{\pi(j)} = 0$, $j \in \mathcal{K}$.) Thus, the Kuhn-Tucker conditions (5.56)-(P2) and (5.56)-(C2) are obviously satisfied. Now, the equality in (5.59) follows by Weyl's Perturbation Theorem [63] to be equivalent to

$$\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H}'_{\pi(i)}(\boldsymbol{I}W\sigma^2 + \sum_{k=1,\pi(k)\in\mathcal{N}}^{j}\boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)})^{-1}\boldsymbol{H}_{\pi(i)} = \lambda\boldsymbol{I} - \boldsymbol{Z}'_{\pi(i)}, \pi(i) \in \mathcal{N}, \tag{5.65}$$

for some matrices $\boldsymbol{Z}'_{\pi(i)} \succeq 0$, $\pi(i) \in \mathcal{N}$. This corresponds to the Kuhn-Tucker condition (5.56)-(Z) for $\pi(i) \in \mathcal{N}$. Further, the equality in (5.59), which is equivalent to (5.65), can be stated as the first equality in (5.63). Thus, it follows that $\boldsymbol{Z}'_{\pi(i)}$ satisfies the complementary slackness condition (5.56)-(C1) for $\pi(i) \in \mathcal{N}$. Since it is obvious that the remaining Kuhn-Tucker conditions (5.56)-(P1) and (5.56)-(D2) are satisfied as well, we have shown that (5.59) implies the fulfillment of the Kuhn-Tucker condition set (5.56) by the set of transmit covariance matrices $\mathcal{Q}$ which corresponds to an N-link regime $\mathcal{N}$. Since (5.56) are necessary and sufficient optimality conditions for (5.31), the proof is completed. □

According to Proposition 29, a particular condition for stability optimality of a single-link regime $\mathcal{N} = \{j\}$ with, say $j = \pi(n)$, takes the form

$$\begin{cases} \lambda_{max}(q_{\pi(n)} W \boldsymbol{H}'_{\pi(n)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(n)} \boldsymbol{Q}_{\pi(n)} \boldsymbol{H}'_{\pi(n)})^{-1} \boldsymbol{H}_{\pi(n)}) = \lambda \\ \lambda_{max}(q_{\pi(n)} W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(n)} \boldsymbol{Q}_{\pi(n)} \boldsymbol{H}'_{\pi(n)})^{-1} \boldsymbol{H}_{\pi(i)} + \frac{1}{\sigma^2} (q_{\pi(i)} - q_{\pi(n)}) \boldsymbol{H}'_{\pi(i)} \boldsymbol{H}_{\pi(i)}) \leq \lambda, \quad i < n \\ \lambda_{max}(q_{\pi(i)} W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(n)} \boldsymbol{Q}_{\pi(n)} \boldsymbol{H}'_{\pi(n)})^{-1} \boldsymbol{H}_{\pi(i)}) \leq \lambda, \quad n < i \leq K. \end{cases} \tag{5.66}$$

Notice, that the identification of the set of active links $\mathcal{N}$ through condition (5.59) uses the notion of link position in the SIC order. At first glance this may cause some interpretation problems, since, e.g., in the single-link regime $\mathcal{N} = \{\pi(n)\}$ satisfying (5.66) the notion of SIC order is trivial. However, due to stability optimality of the SIC order π, that is, due to satisfied condition (5.24) for the given queue system state, the SIC order is transformable to the order of queue lengths. In these terms, (5.66) is the stability optimality condition for the single-link regime of the link associated with the n-th largest queue.

It is evident that the verification of condition (5.66) for any link $j \in \mathcal{K}$ requires the knowledge of K single-link optimal transmit covariance matrices (waterfilling solutions). Thus, if (5.66) happens to be satisfied for some link $j \in \mathcal{K}$, the subsequent optimization (5.31) becomes superfluous, since the optimal transmit covariance matrix, the waterfilling solution $\boldsymbol{Q}_j$, is already on hand. In other words, the per-link verification of (5.66) allows for the exchange of the K-dimensional multi-link optimization (5.31) with at most K-fold single-link waterfilling and inequality evaluation. Clearly, since condition (5.59) (in particular, (5.66)) can happen to be not satisfied, the additional evaluation of such condition may not pay off in reduced complexity of computation of stability-optimal policy, and even increase the computational effort. To prevent the possible computational effort increase, it is reasonable to evaluate condition (5.59) in a real-world multiple access channel under some other observations which indicate (5.59) "likely to be satisfied". For instance, condition (5.66) is worth evaluating if the largest queue length, say the one of link $j \in \mathcal{K}$, in the queue system state is "sufficiently" larger than any other queue length. In such case, stability optimality of the single-link regime $\mathcal{N} = \{j\}$ appears to be likely.

The illustration to Proposition 29 in terms of the geometry of the capacity region is provided in Fig. 5.11 for an exemplary multi-antenna multiple access channel with two links.

5.3.4 Universal stability optimality of an SIC order

Condition (5.24) was explained in Section 5.2.4 to be a sufficient stability optimality condition for the SIC order. Since it is not a necessary stability optimality condition, there arises the question of stability optimality of SIC orders violating ordering (5.24). A related problem of interest is the existence, and the corresponding existence conditions, of an SIC order which is stability-optimal irrespective of the queue system state. It is intuitive to refer to such SIC order as *universally stability-optimal*. The use from the existence of a universally stability-optimal SIC order $\pi \in \Pi_K$ is immediate. In such case the slot-by-slot reordering of the SIC positions of links according to the queue system evolution is not needed and the order can be kept fixed, equal π. This provides obvious benefits in terms of effort of online computation of the stability-optimal policy. The corresponding effort of determination of transmit covariance matrices from (5.31) remains however the same, since the transmit covariance matrices depend on the (instantaneous) channel values.

Recall from Corollary 6, that any stability-optimal rate vector for a given queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$ is included in the boundary of the sets $\mathcal{S}_\pi(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$, with $\pi \in \Pi_K$ satisfying (5.24), and represents a supporting point of the hyperplane with normal vector $\boldsymbol{q}$. In this context it is evident

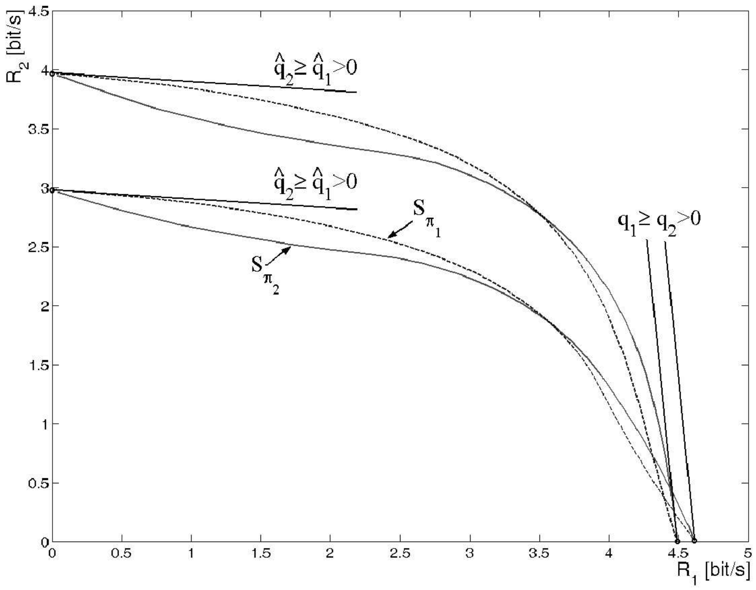

Figure 5.11: Illustration to Proposition 29 in terms of geometry of the capacity region of an exemplary MIMO multiple access channel with two links and sum-power constraint P. The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, P)$, $\pi_1 = 2 \leftarrow 1$, is dashed, the boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, P)$ is solid. The convex hull part of the boundary of the capacity region is dotted. Either of the rate pairs $(R_1^{\max}, 0)$, $(0, R_2^{\max})$ achieved under the two possible single-link regimes is a supporting point of some hyperplane with normal vector $\boldsymbol{q} = (q_1, q_2) > 0$. Thus, either of the single-link regimes can be is stability-optimal for some queue system state $\boldsymbol{q} > 0$.

that universal stability optimality of SIC order π is equivalent to the feature that all boundary rate vectors of $\mathcal{C}(\mathcal{H})$ are also included in (the boundary of) $\mathcal{S}_\pi(\mathcal{H})$ and conversely. Such feature is further equivalent to the relation $\mathcal{C}(\mathcal{H}) = \mathcal{S}_\pi(\mathcal{H})$, that is, to the achievability of all available rate vectors under the use of a single universal SIC order π. From the above argument one can recognize the importance of the issue of universal stability optimality of an SIC order also in terms of capacity considerations in the MIMO multiple access channel [41].

Universal stability optimality subject to power constraints

Consider first the MIMO multiple access channel with individual power constraints. For this case we have the following result.

Proposition 30 *In the MIMO multiple access channel with link set $\mathcal{K}$ and individual power constraints, there exists no set of channel values $\mathcal{H} \in \mathbb{C}^{n_r \times n_t}$, $\boldsymbol{H}_i \neq \boldsymbol{0}$, $i \in \mathcal{K}$, such that an SIC order $\pi \in \Pi_K$ is stability-optimal (in the sense that (5.23) is satisfied) for any queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$.*

Proof Assume, by contradiction, $\pi \in \Pi_K$ is universally stability-optimal in the sense that (5.23) holds for $\boldsymbol{q} \in \mathbb{R}_+^K$. Let now, for instance, $\tilde{\pi} \in \Pi_K$ an SIC order inverse to π, that is, such that

$$\pi(i) = \tilde{\pi}(K - i), \quad i \in \mathcal{K}, \tag{5.67}$$

and consider the set of transmit covariance matrices $\tilde{\mathcal{Q}}$ obtained by iterative waterfilling under SIC order $\tilde{\pi}$ [114]. By Proposition 26 is known that the rate vector, say $\boldsymbol{R}_{\tilde{\pi}}$, achieved by the set of transmit covariance matrices $\tilde{\mathcal{Q}}$ and SIC order $\tilde{\pi}$ is a boundary rate vector of $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ (it represents the intersection of subspaces (5.48)). The inclusion of the rate vector $\boldsymbol{R}_{\tilde{\pi}}$ in the boundary and the construction of $\tilde{\mathcal{Q}}$ imply that $\boldsymbol{R}_{\tilde{\pi}}$ is achieved by no SIC order other than $\tilde{\pi}$, except that we have $\boldsymbol{H}_i = \boldsymbol{0}$, for some $i \in \mathcal{K}$. Let $\boldsymbol{q} \in \mathbb{R}^K_+$ satisfy $q_{\tilde{\pi}(1)} \geq \ldots \geq q_{\tilde{\pi}(N)}$ and $q_{\pi(\tilde{j})} = 0$ for some arbitrary $N < j \leq K$. Then, we conclude by Corollary 8 that $\boldsymbol{R}_{\tilde{\pi}}$ is included in the nontrivial exposed subset of the capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$ which is given by (5.45). Then, by Corollary 8 we have that any rate vector included in exposed subset (5.45) is stability-optimal for $\boldsymbol{q}$, and by the structure of such exposed subset (Proposition 25 ii.)) we have that no rate vector in (5.45) is achievable under SIC order other than $\tilde{\pi}$. Thus, by (5.67), this contradicts stability optimality of SIC order π and completes the proof. □

A simple conclusion from Proposition 30 is that, under individual power constraints, stability optimality of the SIC order is always a queue system state dependent feature. Thus, the notion of universal stability optimality is nonexistent/ obsolete in the MIMO multiple access channel with individual power constraints. Consequently, no effort reduction in the online computation of the stability-optimal policy in such case can be obtained. In terms of geometry of the capacity region, Proposition 30 implies that under no conditions a single S-rate region can become equivalent to the entire capacity region $\mathcal{C}(\mathcal{H}, \hat{\boldsymbol{p}})$, so that the latter one is always a proper union of S-rate regions $\mathcal{S}_\pi(\mathcal{H}, \hat{\boldsymbol{p}})$, $\pi \in \Pi_K$.

In the case of sum-power constrained MIMO multiple access channel, universal stability optimality is not an obsolete feature. The following proposition provides a sufficient condition for the universal stability optimality of the SIC order.

Proposition 31 *Let the MIMO multiple access channel with links set $\mathcal{K}$ be sum-power constrained. Then, an SIC order $\pi \in \Pi_K$ is stability-optimal (in the sense that (5.23) is satisfied) for any queue system state $\boldsymbol{q} \in \mathbb{R}^K_+$, if there exists some queue system state $\tilde{\boldsymbol{q}} \in \mathbb{R}^K_{++}$ satisfying (5.24) such that for any of the link subsets $\{\pi(i)\}_{i=1}^N$, $N \leq K$, the single-link regime of the link with the smallest queue $q_\pi(N)$ is stability-optimal. This condition is equivalent to*

$$\begin{cases} \lambda_{max}(\tilde{q}_{\pi(N)} W \boldsymbol{H}'_{\pi(N)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(N)} \boldsymbol{Q}_{\pi(N)} \boldsymbol{H}'_{\pi(N)})^{-1} \boldsymbol{H}_{\pi(N)}) = \lambda \\ \lambda_{max}(\tilde{q}_{\pi(N)} W \boldsymbol{H}'_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(N)} \boldsymbol{Q}_{\pi(N)} \boldsymbol{H}'_{\pi(N)})^{-1} \boldsymbol{H}_{\pi(i)} + \frac{W}{\sigma^2}(\tilde{q}_{\pi(i)} - \tilde{q}_{\pi(N)}) \boldsymbol{H}'_{\pi(i)} \boldsymbol{H}_{\pi(i)}) \leq \lambda, 1 \leq i < N, \end{cases} \tag{5.68}$$

$1 \leq N \leq K$, with $\boldsymbol{Q}_{\pi(N)}$ as the stability-optimal transmit covariance matrix in the single-link regime $\mathcal{N} = \{\pi(N)\}$ (waterfilling solution).

Proof Given queue system state $\tilde{\boldsymbol{q}} \in \mathbb{R}^K_+$, the equivalence of condition (5.68) and stability optimality of single-link regime $\mathcal{N} = \{\pi(N)\}$ for any link subset $\{\pi(i)\}_{i=1}^N$, $N \leq K$, follows from Proposition 29. Given fixed $N \leq K$, it is apparent from Proposition 29, precisely from (5.59), that condition (5.68) is further equivalent to stability optimality of the single-link regime $\mathcal{N} = \{\pi(N)\}$ for any two-link subset $(\pi(i), \pi(N))$, $i < N$. Let a queue system state $\boldsymbol{q} \in \mathbb{R}^K_+$ violate (5.24) for the given SIC order π. We can express any such queue system state as $\boldsymbol{q} = \tilde{\boldsymbol{q}} + \boldsymbol{\delta}$, for some $\boldsymbol{\delta} = \boldsymbol{\delta}(\boldsymbol{q}) \in \mathbb{R}^K$, where $\tilde{\boldsymbol{q}} \in \mathbb{R}^K_+$ is a fixed queue system state from condition (5.68). Without loss of generality, we are free to down-/ up-scale $\boldsymbol{q}$ or $\tilde{\boldsymbol{q}}$ to yield $q_{\pi(N)} = \tilde{q}_{\pi(N)}$ for any fixed $N \leq K$. Let

$$\mathcal{I}_N = \{\pi(i) : i < N, \delta_{\pi(i)} \leq 0\}, \quad \mathcal{J}_N = \{\pi(i)\}_{i=1}^N \setminus \mathcal{I}_N, \quad N \leq K,$$

and consider first condition (5.68) for $N = K$. Note that $\mathcal{I}_N$ consists of links associated with queues $q_{\pi(i)}$ which are smaller than $\tilde{q}_{\pi(N)}$ (i.e. links $\pi(i)$ such that $\tilde{q}_{\pi(i)} + \delta_{\pi(i)} \leq \tilde{q}_{\pi(N)}$). Clearly, $\mathcal{J}_N$ includes all remaining links from the link subset $\{\pi(i)\}_{i=1}^{N}$. Then, we have $(q_{\pi(i)} - q_{\pi(N)}) \leq (\tilde{q}_{\pi(i)} - \tilde{q}_{\pi(N)})$, $\pi(i) \in \mathcal{I}_N$, so that with (5.68) and Weyl's Inequalities follows [63]

$$\begin{cases} \lambda_{max}(q_{\pi(N)} W \boldsymbol{H'}_{\pi(N)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(N)} \boldsymbol{Q}_{\pi(N)} \boldsymbol{H'}_{\pi(N)})^{-1} \boldsymbol{H}_{\pi(N)}) = \lambda \\ \lambda_{max}(q_{\pi(N)} W \boldsymbol{H'}_{\pi(i)} (\boldsymbol{I} W \sigma^2 + \boldsymbol{H}_{\pi(N)} \boldsymbol{Q}_{\pi(N)} \boldsymbol{H'}_{\pi(N)})^{-1} \boldsymbol{H}_{\pi(i)} + \frac{W}{\sigma^2} (q_{\pi(i)} - q_{\pi(N)}) \boldsymbol{H'}_{\pi(i)} \boldsymbol{H}_{\pi(i)}) \leq \lambda, \pi(i) \in \mathcal{I}_N. \end{cases} \tag{5.69}$$

By Proposition 29, (5.69) means that any link $\pi(i) \in \mathcal{I}_N$ is shut off in the sense $\boldsymbol{Q}_{\pi(i)} = \boldsymbol{0}$ when stability-optimal policy is applied to a two-link subset $(\pi(i), \pi(N))$ under the queue system state $\boldsymbol{q}$. On the other side, considering the application of the stability-optimal policy to a two-link subset $(\pi(i), \pi(N))$ for any link $\pi(i) \in \mathcal{J}_N$, we have then that $\pi(i)$ is either shut off as well or violates the corresponding condition (5.69). Due to $q_{\pi(i)} \geq q_{\pi(N)}$, in the latter case it is implied that the SIC order π satisfies the sufficient stability optimality condition (5.24)) for the two-link subset $(q_{\pi(i)}, q_{\pi(N)})$, $i < N$.

We now proceed in the above manner in evaluating of condition (5.68) under iterative mapping

$$\max_{\pi(i) \in \mathcal{J}_N} i + 1 \mapsto N, \quad 1 \leq N < K. \tag{5.70}$$

Thus, after finishing at smallest possible N from (5.70) we have considered the stability-optimal policy applied to any link pair $(\pi(i), \pi(N))$, $i \leq N$, $N \leq K$, in terms of condition (5.69). By condition (5.69) follows then that SIC order π satisfies the sufficient stability optimality condition (5.24) for any pair of active links among $(\pi(i), \pi(N))$, $i \leq N$, $N \leq K$, under queue system state $\boldsymbol{q}$. But this implies that SIC order π is stability-optimal for the entire set of active links among $1 \leq i \leq K$ under queue system state $\boldsymbol{q}$. Further, since $\boldsymbol{q}$ was chosen arbitrarily, it follows that SIC order π is stability optimal for any queue system state, which completes the proof. □

Proposition 31 implies that in the sum-power constrained MIMO multiple access channel the SIC order may remain (universally) stability-optimal for any queue system state violating the sufficient stability optimality condition (5.24). In terms of geometry, this is equivalent to the feature that the corresponding capacity region $\mathcal{C}(\mathcal{H}, P)$ may become equivalent to a single S-rate-region $\mathcal{S}_\pi(\mathcal{H})$, precisely to the one associated with the universally stability-optimal SIC order $\pi \in \Pi_K$. In this way, the universally stability-optimal SIC order becomes also the only SIC order of interest in terms of capacity considerations in the MIMO multiple access channel.

The evaluation of condition for universal stability optimality (5.68) is of relatively low effort. It requires at most K-fold computation of a single-link optimal transmit covariance matrix, that is, at most K-fold waterfilling.

The illustration of Proposition 31 in terms of geometry of the capacity region is presented in Fig. 5.12 for the capacity region of an exemplary MIMO multiple access channel with two links.

Fig. 5.12 is a nice aid in understanding why some SIC order $\pi \in \Pi_K$ is universally stability-optimal, although it inevitably violates the sufficient stability optimality condition (5.24) for some queue system states.

Irrespective of the universal stability optimality issue, we have from Corollary 6 that any hyperplane with normal vector satisfying (5.24) supports the capacity region $\mathcal{C}(\mathcal{H}, P)$ at some boundary rate vector included in the S-rate region $\mathcal{S}_\pi(\mathcal{H}, P)$ as well. Given now a universally stability-optimal SIC order $\pi \in \Pi_K$, any hyperplane from the complementary class, with normal vector $\boldsymbol{q} \in \mathbb{R}_+^K$ violating (5.24), satisfies either of the following alternatives (see proof of Proposition 31). First, it can support the capacity region at the unique rate vector achieved in the single-link regime $\mathcal{N} = \{\pi(K)\}$ of the link of the smallest queue in $\boldsymbol{q}$. Second, it can support the capacity region in some smaller dimensional orthant $span(\{\boldsymbol{e}_i\}_{i \in \mathcal{N}})$, $|\mathcal{N}| = N < K$, in which the increasing order of queue lengths

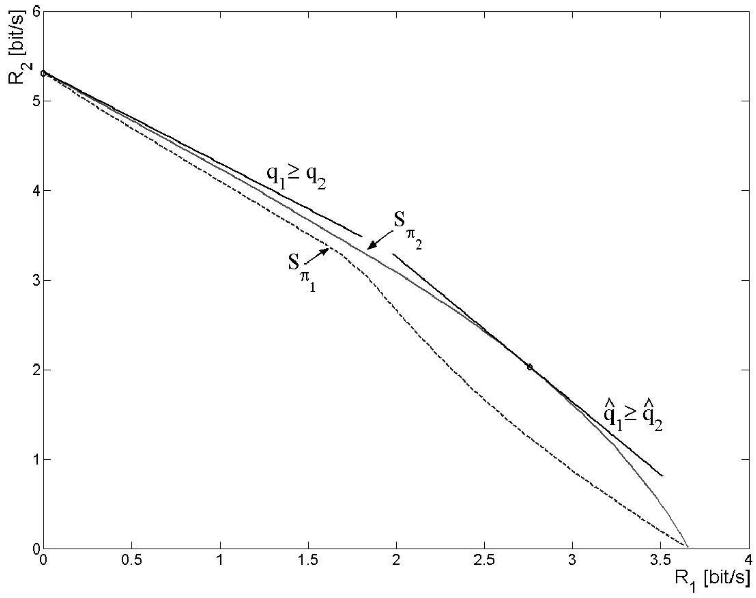

FIGURE 5.12: Illustration to Proposition 31 in terms of geometry of the capacity region of an exemplary multi-antenna multiple access channel with two links and sum-power constraint P. The boundary of the S-rate region $\mathcal{S}_{\pi_1}(\mathcal{H}, P)$, $\pi_1 = 2 \leftarrow 1$, is dashed and the boundary of the S-rate region $\mathcal{S}_{\pi_2}(\mathcal{H}, P)$ is solid. The convex hull part of the capacity region $\mathcal{C}(\mathcal{H}, P)$ is dotted. Any boundary rate vector of the capacity region is a supporting point of some hyperplane with normal vector $\boldsymbol{q} = (q_1, q_2)$ satisfying $q_1 \geq q_2 \geq 0$ (i.e., all such vectors are included in the boundary of $\mathcal{S}_{\pi_2}(\mathcal{H}, P)$), while any hyperplane with normal vector satisfying $0 \leq q_1 < q_2$ supports the capacity region at the rate vector $(0, R_2^{\max})$ corresponding to single-link regime $\mathcal{N} = \{2\}$. This shows that SIC order $\pi_2 = 1 \leftarrow 2$ is (universally) stability-optimal regardless of the queue system state.

q_i, $i \in \mathcal{N}$ (i.e. those associated with active links), coincides with the (corresponding part of) SIC order satisfying the sufficient stability optimality condition (5.24).

The above argument indicates that the existence of a universally stability-optimal SIC order can be seen as a feature of the geometry of the capacity region $\mathcal{C}(\mathcal{H}, P)$.

5.3.5 SPLIT OPTIMIZATION

It is obviously of great interest to dispose of an efficient algorithm which conducts the optimization (5.31) online, slot-by-slot. One of main algorithm design purposes is the optimal trade-off between computational effort and convergence behavior. According to the discussion in Section 2.3, the convexity property of problem (5.31) is hereby of great help in terms of numerical solvability and applicable converging iterations. Nevertheless, in this section we propose a further reformulation of the convex multi-link problem (5.31) in the form of a set of K coupled convex single-link problems. An analogous splitting approach was proposed originally in [114] in the context of the problem of maximization of the sum-rate in the MIMO multiple access channel with individual power constraints. We extend here the splitting idea from [114] to the problem of computation of stability-optimal policy (5.31), and combine it additionally with the algorithmic concept from [46] in order to cover the case of sum-power constrained MIMO multiple access channel. The resulting splitting approach provides certain implementation advantages discussed at the end of the section.

At this point it has to be mentioned, that an efficient alternative algorithmic design solving problem (5.31) was recently presented in [123]. The idea of the algorithm in [123] is completely different from the one presented here.

The splitting approach

Let the SIC order $\pi \in \Pi_K$ satisfy the sufficient stability optimality condition (5.24) for a given queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$. The *single-link problem* of computation of stability-optimal policy, say for link $\pi(i) \in \mathcal{K}$, arises by fixing the transmit covariance matrices $\boldsymbol{Q}_{\pi(j)}$, $j \in \mathcal{K}$, $j \neq i$, in the multi-link problem (5.31). The corresponding single-link objective function $\boldsymbol{Q} \mapsto f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q})$, $\boldsymbol{Q} \succeq 0$, follows straightforwardly from (5.21) or (5.30) for any link $\pi(i) \in \mathcal{K}$ as

$$\begin{aligned} f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q}_{\pi(i)}) = &\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W \log\det(\boldsymbol{N}_{\pi(i)}^{(j)} + \boldsymbol{H}_{\pi(i)}\boldsymbol{Q}_{\pi(i)}\boldsymbol{H}'_{\pi(i)}) \\ &- \sum_{j=1}^{i-1}(q_{\pi(j)} - q_{\pi(j+1)})W \log\det(\boldsymbol{N}_{\pi(i)}^{(j)}), \end{aligned} \tag{5.71}$$

where the matrix $\boldsymbol{N}_{\pi(i)}^{(j)}$ is the corresponding value of a function $\mathcal{Q} \mapsto \boldsymbol{N}_{\pi(i)}^{(j)}(\mathcal{Q})$, $\mathcal{Q} \succeq 0$, defined as

$$\boldsymbol{N}_{\pi(i)}^{(j)}(\mathcal{Q}) = \boldsymbol{I}W\sigma^2 + \sum_{k=1,k\neq i}^{j} \boldsymbol{H}_{\pi(k)}\boldsymbol{Q}_{\pi(k)}\boldsymbol{H}'_{\pi(k)}, \quad j \in \mathcal{K}. \tag{5.72}$$

The arising single-link optimization problem takes the form

$$\max_{\boldsymbol{Q}\in\mathcal{P}_{\pi(i)}(\mathcal{Q})} f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q}), \tag{5.73}$$

with

$$\mathcal{P}_{\pi(i)}(\mathcal{Q}) = \{\boldsymbol{Q} \succeq 0 : tr(\boldsymbol{Q}) \leq \hat{p}_{\pi(i)}\}, \quad i \in \mathcal{K}, \tag{5.74}$$

in the MIMO multiple access channel with individual power constraints and

$$\mathcal{P}_{\pi(i)}(\mathcal{Q}) = \{\boldsymbol{Q} \succeq 0 : tr(\boldsymbol{Q}_{\pi(i)}) \leq P - \sum_{k=1,k\neq i}^{K} tr(\boldsymbol{Q}_{\pi(k)})\}, \quad i \in \mathcal{K}, \tag{5.75}$$

in the case of sum-power constraint. Thus, when the set $\mathcal{Q}$ of transmit covariance matrices is stability-optimal in the sense that it solves the multi-link problem (5.31), then the transmit covariance matrix $\boldsymbol{Q}_{\pi(i)}$ of any link $\pi(i) \in \mathcal{K}$ necessarily solves the single-link problem (5.73). For any link $\pi(i) \in \mathcal{K}$, the second sum in the objective (5.71) can be neglected in the single-link optimization problem (5.73) since, according to (5.72), it is independent of the optimization variable. At this point, note a crucial difference between the solution of the single-link problem (5.73) and the waterfilling solution (5.46).

Since the single-link objective (5.71) is a concave function (see [63]) and both sets (5.74), (5.75) are convex, the single-link optimization problem (5.73) is convex as well, regardless of the type of power constraint and for any link $\pi(i) \in \mathcal{K}$. Since further the corresponding constraint qualification condition is easily shown to be satisfied, the Kuhn-Tucker conditions for problem (5.73) are necessary and sufficient optimality conditions [48].

The single-link problems (5.73), $i \in \mathcal{K}$, remain pairwise coupled. One coupling is through the interference, which is, given link $\pi(i) \in \mathcal{K}$, mirrored in (5.73) by the dependence of any matrix

(5.72) on the transmit covariance matrices of j other links $\pi(k)$, $k \neq i$. In the sum-power constrained MIMO multiple access channel we have an additional kind of coupling through the shared sum-power budget, evident from (5.75).

In the MIMO multiple access channel with individual power constraints, the set of transmit covariance matrices with each matrix solving the single-link problem (5.73) for some link $\pi(i) \in \mathcal{K}$, corresponds to the set of transmit covariance matrices solving the multi-link problem (5.31) as well. This is intuitive, but can be formally proved by showing that duality gaps of all single-link problems equal zero imply zero duality gap of the multi-link problem (such proof goes exactly along the same lines as the proof of the analogous result in [114], so that we omit its statement). In the sum-power constrained multi-antenna multiple access channel, we have the same equivalence between the solutions to problems (5.73), $\pi(i) \in \mathcal{K}$, and (5.31), but provided that the set of the power constraints in (5.75) is chosen optimally.

ITERATIVE SOLUTION

The following Lemma describes a feature of the multi-link objective (5.30) which is of key importance for our algorithm concept. The lemma is a restatement of the corresponding result in [120] (see also [46] for its further use).

Lemma 12 *Let $\tilde{\mathcal{P}} = \{\mathcal{Q} \succeq 0 : tr(\boldsymbol{Q}_i) = 1, i \in \mathcal{K}\}$, let $\boldsymbol{q} \in \mathbb{R}_+^K$ and $\pi \in \Pi_K$ satisfy (5.24), and let a function*

$$(\boldsymbol{p}, \mathcal{Q}) \mapsto \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{p}, \mathcal{Q}) = f_{\boldsymbol{q},\pi}(\{p_i \boldsymbol{Q}_i\}_{i=1}^K), \quad (\boldsymbol{p}, \mathcal{Q}) \in \mathbb{R}_+^K \times \tilde{\mathcal{P}},$$

be defined, with $f_{\boldsymbol{q},\pi}$ given by (5.30). Then, $\tilde{f}_{\boldsymbol{q},\pi}$ is concave for any fixed $\mathcal{Q} \in \tilde{\mathcal{P}}$.

As a straightforward consequence of Lemma 12 we have convexity of the problem

$$\max_{\boldsymbol{p} \in \mathcal{P}_P} \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{p}, \mathcal{Q}), \quad \mathcal{Q} \in \tilde{\mathcal{P}}, \tag{5.76}$$

where set $\mathcal{P}_P$ is understood as the set of power vectors introduced in Section 2.1. Note that problem (5.76) corresponds precisely to optimization/ adjustment of link powers in the MIMO multiple access channel with fixed spatial correlation properties of any link $i \in \mathcal{K}$ (with the capacity region equal to the fundamental capacity region from Section 5.2.3).

The features of the splitting approach of the multi-link problem (5.31) described above and the concavity feature from Lemma 12 give rise to our algorithm concept. The concept consists in sequential K-fold single-link optimization (5.73), similarly to the iterative waterfilling algorithm from [114], and in subsequent link power adjustment in the case of sum-power constraint, analogously to the approach in [46]. Precisely, given a stability-optimal SIC order satisfying (5.24) for a given queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$, the proposed algorithm conducts first the cycle of solutions to single-link problems (5.73), $1 \leq i \leq K$, subject to some value of the power constraint vector. The cycle is repeated iteratively, until the corresponding solutions stabilize up to some predefined accuracy with respect to, e.g., a norm on the difference of consecutive solutions. In the case of sum-power constrained MIMO multiple access channel, the algorithm conducts subsequently the solution to the problem of link power optimization under fixed spatial correlation properties of links $i \in \mathcal{K}$ given by the set of (normalized) transmit covariance matrices obtained from the preceding step. The computed solution is taken as the power constraint vector for the next iteration of the cycle of single-link problem solutions, as described above.

The algorithm terminates when the solutions, both of the cycle of single-link problems and the subsequent link power optimization, stabilize up to some predefined accuracy, e.g., with respect to a norm on the difference of consecutive solutions.

Assuming $\boldsymbol{q} \in \mathbb{R}_+^K$, $\mathcal{Q}^{(1)} \in \mathcal{P}$, $\boldsymbol{p}^{(1)} \in \mathcal{P}_P$ (with $\mathcal{P}_P$ understood as the set of power vectors from Section 2.1), and accuracy constants $\epsilon, \delta > 0$, the algorithm can be stated as follows.

Algorithm 6

1: *set* $(k, l) := (1, 1)$
2: *set* π *from (5.24)*
3: ***for*** $i = 1$ *to* K ***do***
4:     ***for*** $j = i + 1$ *to* K ***do***
5:         *set* $\boldsymbol{N}_{\pi(i)}^{(j),(k)} := \boldsymbol{N}_{\pi(i)}^{(j)}(\mathcal{Q}^{(k)})$ *from (5.72)*
6:     ***end for***
7: ***end for***
8: ***if*** $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$ ***then***
9:     $\boldsymbol{p}^{(1)} := \hat{\boldsymbol{p}}$
10: ***end if***
11: ***repeat*** *{outer loop}*
12:     ***repeat*** *{inner loop}*
13:         ***for*** $i = 1$ *to* K ***do***
14:             *solve* $\max_{\boldsymbol{Q} \succeq 0: tr(\boldsymbol{Q}) \leq p_{\pi(i)}^{(l)}} f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q})$
15:             $\boldsymbol{Q}_{\pi(i)}^{(k)} := \arg\max_{\boldsymbol{Q} \succeq 0: tr(\boldsymbol{Q}) \leq p_{\pi(i)}^{(l)}} f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q})$
16:             ***for*** $j = i + 1$ *to* K ***do***
17:                 *update* $\boldsymbol{N}_{\pi(i)}^{(j),(k)} := \boldsymbol{N}_{\pi(i)}^{(j)}(\mathcal{Q}^{(k)})$ *from (5.72)*
18:             ***end for***
19:         ***end for***
20:         ***for*** $i = 1$ *to* K ***do***
21:             ***for*** $j = i + 1$ *to* K ***do***
22:                 $(\boldsymbol{Q}_{\pi(i)}^{(k+1)}, \boldsymbol{N}_{\pi(i)}^{(j),(k+1)}) := (\boldsymbol{Q}_{\pi(i)}^{(k)}, \boldsymbol{N}_{\pi(i)}^{(j),(k)})$
23:             ***end for***
24:         ***end for***
25:         $k := k + 1$
26:     ***until*** $\sum_{i=1}^K |||\boldsymbol{Q}_{\pi(i)}^{(k)} - \boldsymbol{Q}_{\pi(i)}^{(k-1)}||| \leq \epsilon$
27:     ***if*** $\mathcal{P} = \mathcal{P}_P$ ***then***
28:         ***for*** $i = 1$ *to* K ***do***
29:             $\tilde{\boldsymbol{Q}}_{\pi(i)}^{(k)} := \frac{1}{tr(\boldsymbol{Q}_{\pi(i)}^{(k)})} \boldsymbol{Q}_{\pi(i)}^{(k)}$
30:         ***end for***
31:         *solve* $\max_{\boldsymbol{p} \in \mathcal{P}_P} \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{p}, \tilde{\mathcal{Q}}^{(k)})$
32:         $l := l + 1$
33:         $\boldsymbol{p}^{(l)} := \arg\max_{\boldsymbol{p} \in \mathcal{P}_P} \tilde{f}_{\boldsymbol{q},\pi}(\boldsymbol{p}, \tilde{\mathcal{Q}}^{(k)})$
34:     ***end if***
35: ***until*** $\sum_{i=1}^K \|\boldsymbol{p}^{(l)} - \boldsymbol{p}^{(l-1)}\| \leq \delta$

Due to convexity of the single-link problem (5.73) and the link power optimization problem (5.76), one can apply efficient methods of convex optimization, e.g. interior point methods [47], in the steps 14 and 31 of Algorithm 6. In Algorithm 6 we chose the order of single-link optimization steps in the inner loop complying with the SIC order π obtained in step 2. An arbitrary order in the inner loop is however in general allowable.

Algorithm 6 can be shown to be globally convergent.

Proposition 32 *Given $\epsilon, \delta = 0$, let $\mathcal{Q}^{(k)}$, $k \in \mathbb{N}$, be the sequence of iterates obtained from Algorithm 6. Then, we have*

$$\lim_{k\to\infty} \mathcal{Q}^{(k)} = \tilde{\mathcal{Q}},$$

with $\tilde{\mathcal{Q}} \in \mathcal{P}$ as a set of stability-optimal transmit covariance matrices solving problem (5.31).

Proof Given queue system state $\boldsymbol{q} \in \mathbb{R}_+^K$ and power constraints $\hat{\boldsymbol{p}}$, consider first only the inner loop of Algorithm 6 and note that the SIC order $\pi \in \Pi_K$ is chosen in step 2 to satisfy (5.24). Thus, as already shown in Section 5.3.5, the cycle of single-link optimizations (5.73) in the inner loop is equivalent to optimization of objective $f_{\boldsymbol{q},\pi}$ over all $\mathcal{Q} \in \mathcal{P}_{\hat{\boldsymbol{p}}}$, conducted sequentially with respect to $\boldsymbol{Q}_{\pi(i)}$, $i \in \mathcal{K}$. Since any single-link optimization problem (5.73) in the inner loop is convex, after each single-link optimization conducted in the inner loop an increase in $f_{\boldsymbol{q},\pi}$ is obtained. It is an obvious consequence of the features of the log det function that the set of solutions to any problem (5.73) is compact (in particular, (5.73) has a unique solution) [47]. From convexity of any problem (5.73) follows further, that $|||\nabla f_{\boldsymbol{q},\pi(i)}(\boldsymbol{Q}^{(k)}_{\pi(i)})|||$, $i \in \mathcal{K}$, decrease with the number of conducted cycles $k \in \mathbb{N}$ in the inner loop. By (5.30) and (5.71) it can be seen that this implies the decrease of $|||\nabla f_{\boldsymbol{q},\pi}(\mathcal{Q}^{(k)})|||$ with the number of conducted cycles $k \in \mathbb{N}$ in the inner loop as well. Thus, iterative conduction of the cycle of optimizations in the inner loop must converge to some $\lim_{k\to\infty} \mathcal{Q}^{(k)} = \tilde{\mathcal{Q}} \in \mathcal{P}_{\hat{\boldsymbol{p}}}$. But, with compactness of the solution set to any problem (5.73) in the inner loop, $\tilde{\mathcal{Q}}$ must solve (5.31) for $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$.

The outer loop of Algorithm 6 consists of the cycle of single-link optimizations of the inner loop and, in step 31, the optimization of the objective $\tilde{f}_{\boldsymbol{q},\pi}$ over all $\boldsymbol{p} \in \mathcal{P}_P$ under fixed set $\{\boldsymbol{Q}_i^{(k)}/tr(\tilde{\boldsymbol{Q}}_i^{(k)})\}_{i\ni\mathcal{K}}$ obtained from the preceding k-th conduction of the inner loop. Since the latter problem is convex (by Lemma 12), the same argumentation as to the cycle of optimizations within the inner loop applies to the entire inner loop together with the optimization in step 31. Thus, along the same lines as above, the entire outer loop of Algorithm 6 follows to be convergent to some $\lim_{k\to\infty} \mathcal{Q}^{(k)} = \tilde{\mathcal{Q}} \in \mathcal{P}_P$, which is a solution to (5.31) for $\mathcal{P} = \mathcal{P}_P$. This completes the proof. □

Additionally, we can provide a bound on the distance from the stability-optimal set of transmit covariance matrices which is ensured by Algorithm 6 under termination after one cycle of the inner loop. The distance is measured in terms of the value of the multi-link objective (5.30) and the bound on the distance is obtained by the techniques of bounding of the duality gap presented in [114].

Proposition 33 *Let $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$ (MIMO multiple access channel with individual power constraints) and $\mathcal{Q}^{(1)} = \{\boldsymbol{0}\}_{i=1}^K$ be given in Algorithm 6. Then, we have*

$$0 \leq f_{\boldsymbol{q},\pi}(\mathcal{Q}^{(2)}) - f_{\boldsymbol{q},\pi}(\tilde{\mathcal{Q}}) \leq \sum_{i=2}^{K} q_{\pi(i)} W n_r, \tag{5.77}$$

with $\mathcal{Q}^{(2)}$ as the set of transmit covariance matrices obtained after one cycle of the inner loop of Algorithm 6 and with $\hat{\mathcal{Q}}$ as the set of stability-optimal transmit covariance matrices solving problem (5.31).

Proof The result follows from upper-bounding of the duality gap, that is, the difference between the primal and dual objective [48], [47], after one cycle of the inner loop of Algorithm 6. We shorten the proof at some points which are easily derivable but their derivation is lengthy.

Under strong Lagrangean duality (in particular, under convexity of the problem), the duality gap at the problem solution is zero, see Section 4.1.2 and [47]. In order to yield the dual problem to the single-link problem (5.73) for $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$, we have to rewrite the latter one in equivalent form as

$$\min_{(\mathcal{T}_{\pi(i)},\boldsymbol{Q}_{\pi(i)})\in\mathcal{S}_{\pi(i)}} -\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\log\det(\boldsymbol{T}^{(j)}_{\pi(i)}), \tag{5.78}$$

with $\mathcal{T}_{\pi(i)} = \{\boldsymbol{T}^{(j)}_{\pi(i)}\}_{j=i}^{K}$ and with $\mathcal{S}_{\pi(i)}$ as the set including all pairs $(\mathcal{T}_{\pi(i)}, \boldsymbol{Q}_{\pi(i)})$ satisfying

$$\begin{cases} \boldsymbol{T}^{(j)}_{\pi(i)} - \boldsymbol{N}^{(j)}_{\pi(i)} - \boldsymbol{H}_{\pi(i)}\boldsymbol{Q}_{\pi(i)}\boldsymbol{H}'_{\pi(i)} \preceq 0, \quad i \le j \le K \\ -\boldsymbol{Q}_{\pi(i)} \preceq 0 \\ tr(\boldsymbol{Q}_{\pi(i)}) - \hat{p}_{\pi(i)} \le 0, \end{cases} \tag{5.79}$$

$i \in \mathcal{K}$. The Lagrangian of the problem form (5.78) can be written as

$$\begin{aligned} L_{\pi(i)}(\mathcal{T}_{\pi(i)}, \boldsymbol{Q}_{\pi(i)}, \boldsymbol{Z}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) = &-\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\log\det(\boldsymbol{T}^{(j)}_{\pi(i)}) \\ &+\sum_{j=i}^{K} tr(\boldsymbol{F}^{(j)}_{\pi(i)}(\boldsymbol{T}^{(j)}_{\pi(i)} - \boldsymbol{N}^{(j)}_{\pi(i)} - \boldsymbol{H}_{\pi(i)}\boldsymbol{Q}_{\pi(i)}\boldsymbol{H}'_{\pi(i)})) \\ &- tr(\boldsymbol{Z}_{\pi(i)}\boldsymbol{Q}_{\pi(i)}) + \lambda_{\pi(i)}(tr(\boldsymbol{Q}_{\pi(i)}) - \hat{p}_{\pi(i)}), \end{aligned} \tag{5.80}$$

$i \in \mathcal{K}$, where the set of Lagrange multipliers/ dual variables $\mathcal{F}_{\pi(i)} = \{\boldsymbol{F}^{(j)}_{\pi(i)}\}_{j=i}^{K}$ is associated with the first constraint in (5.79).

The objective function of the dual problem to (5.78) is a function defined as

$$\begin{aligned} (\boldsymbol{Z}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) &\mapsto g_{\boldsymbol{q},\pi(i)}(\boldsymbol{Z}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) \\ &= \min_{\mathcal{T}_{\pi(i)},\boldsymbol{Q}_{\pi(i)}} L_{\pi(i)}(\mathcal{T}_{\pi(i)}, \boldsymbol{Q}_{\pi(i)}, \boldsymbol{Z}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}), \end{aligned}$$

$\mathcal{F}_{\pi(i)} \succeq 0$, $\boldsymbol{Z}_{\pi(i)} \succeq 0$, $\lambda_{\pi(i)} \ge 0$, $i \in \mathcal{K}$, and can be obtained in explicit form by setting the partial derivatives of the Lagrangian (5.80) with respect to $\mathcal{T}_{\pi(i)}$ and $\boldsymbol{Q}_{\pi(i)}$ to zero. This yields precisely

$$\begin{cases} (q_{\pi(j)} - q_{\pi(j+1)})W\left(\boldsymbol{T}^{(j)}_{\pi(i)}\right)^{-1} - \boldsymbol{F}^{(j)}_{\pi(i)} = 0, \quad j \in \mathcal{K} \\ -\sum_{j=i}^{K}\boldsymbol{H}'_{\pi(i)}\boldsymbol{F}^{(j)}_{\pi(i)}\boldsymbol{H}_{\pi(i)} - \boldsymbol{Z}_{\pi(i)} + \lambda_{\pi(i)}\boldsymbol{I} = 0, \end{cases} \tag{5.81}$$

which incorporated in (5.80) gives

$$\begin{aligned} g_{\boldsymbol{q},\pi(i)}(\mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) = &\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\log\det\Big(\frac{1}{(q_{\pi(j)} - q_{\pi(j+1)})W}\boldsymbol{F}^{(j)}_{\pi(i)}\Big) - \\ &\sum_{j=i}^{K} tr(\boldsymbol{F}^{(j)}_{\pi(i)}\boldsymbol{N}^{(j)}_{\pi(i)}) - \lambda_{\pi(i)}\hat{p}_{\pi(i)} + q_{\pi(1)}Wn_r, \end{aligned} \tag{5.82}$$

$i \in \mathcal{K}$. The duality gap of the single-link problem (5.73) corresponds to the function

$$\begin{aligned} (\mathcal{T}_{\pi(i)}, \boldsymbol{Q}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) &\mapsto \gamma_{\pi(i)}(\mathcal{T}_{\pi(i)}, \boldsymbol{Q}_{\pi(i)}, \mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}) \\ &= f_{\boldsymbol{q},\pi(i)}(\mathcal{T}_{\pi(i)}) - g_{\boldsymbol{q},\pi(i)}(\mathcal{F}_{\pi(i)}, \lambda_{\pi(i)}), \quad \boldsymbol{Q}_{\pi(i)} \succeq 0, \quad \mathcal{F}_{\pi(i)} \succeq 0, \quad \lambda_{\pi(i)} \ge 0, \end{aligned}$$

and can be written with (5.78), (5.81) and (5.82) as

$$\gamma_{\pi(i)}(\mathcal{T}_{\pi(i)}, \lambda_{\pi(i)}) = \lambda_{\pi(i)}\hat{p}_{\pi(i)} + \sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})Wtr(\boldsymbol{T}_{\pi(i)}^{(j)\ -1}\boldsymbol{N}_{\pi(i)}^{(j)}) - q_{\pi(i)}Wn_r, \tag{5.83}$$

$i \in \mathcal{K}$. Due to $\boldsymbol{Z}_{\pi(i)} \succeq 0$, $i \in \mathcal{K}$, and the first equality in (5.81), the second equality in (5.81) can be stated as $\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H'}_{\pi(i)}\boldsymbol{T}_{\pi(i)}^{(j)-1}\boldsymbol{H}_{\pi(i)} \preceq \lambda_{\pi(i)}\boldsymbol{I}$, which implies that we have

$$\lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})W\boldsymbol{H'}_{\pi(i)}\boldsymbol{T}_{\pi(i)}^{(j)\ -1}\boldsymbol{H}_{\pi(i)}) = \lambda_{\pi(i)}, \tag{5.84}$$

at the solution of the dual problem $\max_{\mathcal{F}_{\pi(i)}\succeq 0, \lambda_{\pi(i)}\geq 0}\, g_{\boldsymbol{q},\pi(i)}(\mathcal{F}_{\pi(i)}, \lambda_{\pi(i)})$, $i \in \mathcal{K}$, since the dual objective (5.82) is linear and decreasing in $\lambda_{\pi(i)}$. By reformulating the multi-link problem (5.31) for $\mathcal{P} = \mathcal{P}_{\hat{\boldsymbol{p}}}$ similarly to (5.78) and proceeding along the same lines as above, we also yield the expression for the objective $(\mathcal{Z}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{\lambda}}) \mapsto g_{\boldsymbol{q},\pi}(\mathcal{Z}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{\lambda}})$, $\mathcal{Z} \succeq 0$, $\tilde{\mathcal{F}} \succeq 0$, $\tilde{\boldsymbol{\lambda}} \geq 0$, of the dual problem to (5.31) and the corresponding duality gap, which can be written as

$$\gamma(\mathcal{T}, \tilde{\boldsymbol{\lambda}}) = \sum_{i=1}^{K}\tilde{\lambda}_{\pi(i)}\hat{p}_{\pi(i)} + \sum_{j=1}^{K}(q_{\pi(j)} - q_{\pi(j+1)})Wtr(\sigma^2 W\boldsymbol{T}^{(j)^{-1}}) - q_{\pi(1)}Wn_r, \quad \tilde{\boldsymbol{\lambda}} \geq 0. \tag{5.85}$$

Hereby, $\mathcal{T} = \{\boldsymbol{T}^{(j)}\}_{j=1}^{K}$ and $\tilde{\boldsymbol{\lambda}}$ are the Lagrange multipliers/ dual variables of the multi-link problem (5.31) and we have

$$\boldsymbol{T}^{(j)} \preceq \boldsymbol{I}W\sigma^2 + \sum_{l=1}^{j}\boldsymbol{H}_{\pi(l)}\boldsymbol{Q}_{\pi(l)}\boldsymbol{H'}_{\pi(l)}, \quad j \in \mathcal{K}. \tag{5.86}$$

Analogously to (5.82), the objective $g_{\boldsymbol{q},\pi}$ of the dual problem to the multi-link problem (5.31) is linearly decreasing in $\tilde{\boldsymbol{\lambda}}$. Thus, with conditions corresponding to zeroing of the partial derivatives of the Lagrangian (paralleling those in (5.81)), one can show that we have

$$\tilde{\lambda}_{\pi(i)} = \lambda_{max}(\sum_{j=i}^{K}(q_{\pi(j+1)} - q_{\pi(j+1)})W\boldsymbol{H'}_{\pi(i)}\boldsymbol{T}^{(j)^{-1}}\boldsymbol{H}_{\pi(i)}), \tag{5.87}$$

$i \in \mathcal{K}$, at the solution of the dual problem $\max_{\tilde{\mathcal{F}}\succeq 0, \tilde{\boldsymbol{\lambda}}\geq 0}\, g_{\boldsymbol{q},\pi}(\tilde{\mathcal{F}}, \tilde{\boldsymbol{\lambda}})$.

When the order of the single-link problems (5.73) in the inner loop of Algorithm 6 complies with π and when we set $\mathcal{Q}^{(1)} = 0$, the noise covariance matrices $\boldsymbol{N}_{\pi(i)}^{(j),(2)}$ after one cycle of the inner loop can be written as

$$\boldsymbol{N}_{\pi(i)}^{(j),(2)} = \boldsymbol{I}W\sigma^2 + \sum_{l=1, l\neq i}^{\min\{i,j\}}\boldsymbol{H}_{\pi(l)}\boldsymbol{Q}_{\pi(l)}^{(2)}\boldsymbol{H'}_{\pi(l)}, \quad j \in \mathcal{K}, \tag{5.88}$$

$i \in \mathcal{K}$. At the and of one cycle of optimizations in the inner loop we yield $\gamma_{\pi(i)}(\mathcal{T}_{\pi(i)}, \lambda_{\pi(i)}) = 0$, $i \in \mathcal{K}$, since each single-link problem (5.73) is then solved for the arising noise covariance matrices (5.88) and strong Lagrangean duality holds for (5.73). Further, at a solution of version (5.78) of any single-link problem (5.73), $i \in \mathcal{K}$, the first constraint in (5.79) is an equality. This feature implies together with (5.83) and (5.88) that

$$\sum_{j=i}^{K}(q_{\pi(j)} - q_{\pi(j+1)})Wtr((\boldsymbol{I}W\sigma^2 + \sum_{l=1}^{\min\{i,j\}}\boldsymbol{H}_{\pi(l)}\boldsymbol{Q}_{\pi(l)}^{(2)}\boldsymbol{H'}_{\pi(l)})^{-1}(\boldsymbol{I}W\sigma^2 + \sum_{l=1, l\neq i}^{\min\{i,j\}}\boldsymbol{H}_{\pi(l)}\boldsymbol{Q}_{\pi(l)}^{(2)}\boldsymbol{H'}_{\pi(l)})) + \lambda_{\pi(i)}^{(2)}\hat{p}_{\pi(i)} = q_{\pi(i)}Wn_r, \tag{5.89}$$

$i \in \mathcal{K}$, where $\lambda_{\pi(i)}^{(2)}$ denotes the Lagrange multiplier associated with the solution of $\pi(i)$-th single-link problem (5.73) at the end of one cycle of optimizations in the inner loop. By summing up the equalities (5.89) for $i \in \mathcal{K}$ we yield further

$$\sum_{i=1}^{K} q_{\pi(i)} W n_r = \sum_{i=1}^{K} \lambda_{\pi(i)}^{(2)} \hat{p}_{\pi(i)} +$$
$$\sum_{i=1}^{K}\sum_{j=i}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) W tr((\boldsymbol{I} W\sigma^2 + \sum_{l=1}^{\min\{i,j\}} \boldsymbol{H}_{\pi(l)} \boldsymbol{Q}_{\pi(l)}^{(2)} \boldsymbol{H}'_{\pi(l)})^{-1} (\boldsymbol{I} W\sigma^2 + \sum_{l=1, l\neq i}^{\min\{i,j\}} \boldsymbol{H}_{\pi(l)} \boldsymbol{Q}_{\pi(l)}^{(2)} \boldsymbol{H}'_{\pi(l)})). \tag{5.90}$$

We apply now the transmit covariance matrices $\mathcal{Q}^{(2)}$ and noise covariance matrices (5.88) after one cycle of the inner loop to the dual optimality conditions (5.84) and (5.87), to obtain the associated Lagrange multipliers/ dual variables $\lambda_{\pi(i)}^{(2)}$ and $\tilde{\lambda}_{\pi(i)}^{(2)}$, $i \in \mathcal{K}$, in the single-link problem (5.73) and the multi-link problem (5.31), respectively. Then, one can see that for $1 \leq i \leq K-1$ there are additional positive semidefinite terms in the inverses in (5.87) compared to (5.84). Thus, with Weyl's Inequalities we yield then [63]

$$\begin{cases} \tilde{\lambda}_{\pi(i)}^{(2)} \leq \lambda_{\pi(i)}^{(2)}, \quad 1 \leq i \leq K-1 \\ \tilde{\lambda}_{\pi(K)}^{(2)} = \lambda_{\pi(K)}^{(2)}. \end{cases} \tag{5.91}$$

By (5.90) and (5.91) we then obtain after a straightforward but lengthy calculation

$$\sum_{i=1}^{K}\sum_{j=i}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) W tr((\boldsymbol{I} W\sigma^2 + \sum_{l=1}^{\min\{i,j\}} \boldsymbol{H}_{\pi(l)} \boldsymbol{Q}_{\pi(l)}^{(2)} \boldsymbol{H}'_{\pi(l)})^{-1} (\boldsymbol{I} W\sigma^2 + \sum_{l=1, l\neq i}^{\min\{i,j\}} \boldsymbol{H}_{\pi(l)} \boldsymbol{Q}_{\pi(l)}^{(2)} \boldsymbol{H}'_{\pi(l)}))$$
$$\geq \sum_{j=1}^{K} (q_{\pi(j)} - q_{\pi(j+1)}) W tr((\boldsymbol{I} W\sigma^2 + \sum_{l=1}^{j} \boldsymbol{H}_{\pi(l)} \boldsymbol{Q}_{\pi(l)}^{(2)} \boldsymbol{H}'_{\pi(l)})^{-1} \boldsymbol{I} W\sigma^2). \tag{5.92}$$

By (5.85) can be now seen that, given $\mathcal{Q}^{(2)}$, the set of matrices, say $\mathcal{T}^{(2)}$, minimizing the duality gap of the multi-link problem satisfies (5.86) with equality. Thus, by (5.85), (5.90) and (5.92) we have

$$\sum_{i=1}^{K} q_{\pi(i)} W n_r - \sum_{i=1}^{K} \lambda_{\pi(i)}^{(2)} \hat{p}_{\pi(i)} \geq \gamma(\mathcal{T}^{(2)}, \tilde{\boldsymbol{\lambda}}^{(2)}) + q_{\pi(2)} W n_r - \sum_{i=1}^{K} \tilde{\lambda}_{\pi(i)}^{(2)} \hat{p}_{\pi(i)}, \tag{5.93}$$

which further implies with (5.91)

$$\gamma(\mathcal{T}^{(2)}, \tilde{\boldsymbol{\lambda}}^{(2)}) = f_{\boldsymbol{q},\pi}(\mathcal{Q}^{(2)}) - g_{\boldsymbol{q},\pi}(\tilde{\mathcal{F}}^{(2)}, \tilde{\boldsymbol{\lambda}}^{(2)}) \leq \sum_{i=2}^{K} q_{\pi(i)} W n_r, \tag{5.94}$$

where $\tilde{\mathcal{F}}^{(2)}$ are the associated Lagrange multipliers. Since strong Lagrangean duality holds for problem (5.31), we have $f_{\boldsymbol{q},\pi}(\hat{\mathcal{Q}}) = g_{\boldsymbol{q},\pi}(\hat{\mathcal{F}}, \hat{\boldsymbol{\lambda}})$, with $(\hat{\mathcal{Q}}, \hat{\mathcal{F}}, \hat{\boldsymbol{\lambda}})$ as the solution to (5.31) and the associated Lagrange multipliers. This gives finally with duality gap definition and (5.94)

$$\sum_{i=2}^{K} q_{\pi(i)} W n_r \geq f_{\boldsymbol{q},\pi}(\mathcal{Q}^{(2)}) - g_{\boldsymbol{q},\pi}(\hat{\mathcal{F}}, \hat{\boldsymbol{\lambda}}) = f_{\boldsymbol{q},\pi}(\mathcal{Q}^{(2)}) - f_{\boldsymbol{q},\pi}(\hat{\mathcal{Q}}) \geq 0, \tag{5.95}$$

which completes the proof. □

Note, that the bound provided in Proposition 33 is specific for the order of single-link optimization steps in the inner loop complying with stability-optimal SIC order satisfying (5.24).

Algorithm 6 is formally an extension of the iterative waterfilling algorithm from [114] to the problem of maximization of weighted sum of rates (5.19) and the case of sum-power constraint. It is however incorrect to refer to it as extended/ generalized/ weighted iterative waterfilling algorithm, since, except the first decoded link $\pi(K)$, the transmit covariance matrices computed in the inner loop for links $i \in \mathcal{K}$ do not correspond to waterfilling solutions (recall the difference between (5.46) and the solution to (5.73)).

SOME NOTES ON REALIZATION ISSUES

Algorithm 6 is based on the proposed splitting approach and thus lends itself to distribution of the computational work in the MIMO multiple access channel. Precisely, the base station may be exonerated from computational work by assigning the per-link optimization steps 14, 15 to the corresponding link transmitters $1 \leq i \leq K$. This is easily realizable in the sequential manner, if each link transmitter $\pi(i) \in \mathcal{K}$ has up-to-date knowledge on the interference terms $\boldsymbol{N}_{\pi(i)}^{(j),(k)}$, $i \leq j \leq K$, e.g. due to reliable feedback from the base station after each per-link optimization step 14, 15 in iteration $k \in \mathbb{N}$. The detailed discussion on the implementation benefits/ detriments of the partly distributed Algorithm 6, compared to a centralized algorithmic concept solving (5.31) is intricate and shall not be pursued here.

It is evident from the single-link objective (5.71) that under partly distributed computation of the stability-optimal policy, as is e.g. suitable to realize Algorithm 6, some kind of knowledge of the entire instantaneous queue system state is needed at any link transmitter. Similarly, in any centralized algorithmic solution to (5.31), the base station has to be informed on the entire instantaneous queue system state as well. To decrease the effort of feedback which is necessary to provide such knowledge, we can apply some heuristic simplifications. For instance, under centralized algorithmic solution to (5.31), it is thinkable to track the evolution of all queues only on average if the queue evolution process is stationary and ergodic [100], [98]. Precisely, the base station requires the estimated mean number of bit arrivals per slot, say a_i, on each link transmitter $1 \leq i \leq K$ to be fed back once a longer time period. The evolution of each queue estimated/ predicted at the base station for slot $n+1$ is then

$$q_i(n+1) = (q_i(n) + a_i - R_i(n))_+, \quad 1 \leq i \leq K, \quad n \in \mathbb{N},$$

where the rate $R_i(n)$ is known immediately from the transmit strategy $(\mathcal{Q}(n), \pi(n))$ computed for slot $n \in \mathbb{N}$.

A more rough simplification is the assumption of symmetric queue system state throughout. This however reduces the problem of computation of stability-optimal policy (5.31) to the well-studied sum-rate maximization problem [114], [46] (which actually does not require queue system state knowledge at all). As follows from Section 5.2.2 and Fig. 5.2, such rough simplification is likely to deteriorate the performance significantly.

6

Min-max fairness, the fairness gap, and the fairness-performance trade-off

The optimization of aggregated performance according to (2.18) was the concern of Chapters 3-5. In this chapter we deal with achieving min-max fairness (2.30) as a different, but equally important, approach of network performance optimization. In broad terms, the presented results describe a kind of incompatibility, or trade-off, between the approaches of aggregated performance optimization and ensuring fairness in the min-max sense. Further, the interesting relation between the notion of min-max fairness and the notion of so-called max-min fairness is analyzed, which gives rise to the idea of so-called fairness gap.

In Section 6.1 we introduce the concepts of min-max fairness and max-min fairness, standing in a duality-like relation, and prove the related basic features. As a first problem, we deal with the case of concurrent achievement of min-max fairness and optimum aggregated performance, the fairness-performance trade-off, in Sections 6.2, 6.3. By splitting the discussion into the case of so-called entirely interference coupled networks and general networks, we characterize the relations of power vector, weight vector and the performance value under the fairness-performance trade-off. In Section 6.4 we show that the trade-off of min-max fairness and aggregated-performance optimality has the interpretation of a saddle point of the aggregated performance, regarded as a function of power vectors and weight vectors.

As a second separate problem, we focus on the duality-like relation of the concepts of min-max fairness and max-min fairness. Precisely, in Section 6.5 we characterize the class of networks for which the same link performance is achieved under both considered fairness notions. Further, we succeed in describing the corresponding subclass of networks for which there exist power vectors achieving both fairness notions concurrently.

For the statement of the results of this chapter recall the assumption of the SIR function with neglected noise (2.27), the assumption of unconstrained power region (2.28) and the notation of PF eigenvectors and eigenmanifolds of the interference matrix.

The results of this chapter were presented originally in [35], [124], [125]. The chapter makes strong use of Perron-Frobenius Theory and combinatorial theory of nonnegative matrices, with the basic notions explained in Appendices A.1, A.2.

6.1 MIN-MAX FAIRNESS, MAX-MIN FAIRNESS AND THE FAIRNESS-PERFORMANCE TRADE-OFF

In the following Proposition we provide a simple extension of the Collatz-Wielandt min-max formula for the Perron Root [38]. The Collatz-Wielandt formulae are two characterizations, in min-max and max-min form, of the spectral radius of a nonnegative matrix (we refer here to [38], [126] for further details). The proposition characterizes the min-max fair power vector (power allocation or, in short, allocation), in the sense of a solution to the problem of min-max fairness (2.30). Thus, the proposition is fundamental for the remainder of this chapter.

Proposition 34 *For any interference matrix $\boldsymbol{V}$ and any performance function F, we have*

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.1}$$

where $F\left(\frac{(\boldsymbol{V}\boldsymbol{r})_i}{r_i}\right) = F(\rho(\boldsymbol{V}))$, $1 \le i \le K$ whenever $\boldsymbol{r} > 0$.

Since Proposition 34 is essential for the considerations in Section 6.5, we defer the proof of it to this Section.

As explained in Section 2.4, the problem of min-max fairness (2.30), (6.1) is interpretable as improving the worst link performance/ QoS value as much as possible. In analogy, we can think of an in some way complementary goal of degrading the best link QoS value as much as possible. The corresponding fairness notion can be formulated as

$$\sup_{\boldsymbol{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right), \tag{6.2}$$

and can be intuitively referred to as *max-min fairness*, due to the underlying optimization problem form. One is tempted to ask if, or when, the notions of min-max fairness and max-min fairness coincide. This issue is addressed in Section 6.5.

The notion of max-min fairness introduced above must not be confused with the notion of max-min fairness used in the references given in this chapter. As explained in Section 2.4, the notion of max-min fairness in the references corresponds to the notion of min-max fairness in this work.

It may misleadingly appear that any solution to (6.1) is a min-max fair allocation. This is not always the case. Precisely, the following subtlety has to be accounted for. By the definition of the infimum, it follows from (6.1) that for any accuracy $\epsilon > 0$ there exists a power vector $\boldsymbol{p}(\epsilon) > 0$, which is ϵ-near the solution. Precisely,

$$F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\epsilon))_k}{p_k(\epsilon)}\right) \le F(\rho(\boldsymbol{V})) + \epsilon.$$

If the accuracy is increased according to $\epsilon \to 0$, the existence of some link-subset $\mathcal{L} \subset \mathcal{K}$, such that $\boldsymbol{p}(0) = \lim_{\epsilon\to 0} \boldsymbol{p}(\epsilon) = \boldsymbol{r}$ with $r_k = 0$, $k \in \mathcal{L}$, can not be in general excluded. This means that although the link SIR values $\gamma_k(\boldsymbol{r})$, $k \in \mathcal{L}$ are positive at the solution $\boldsymbol{r}$ of (6.1), they in fact represent the limits of ratios with numerator and denominator both approaching zero. In other words, the links $k \in \mathcal{L}$ are practically shut off, while their associated SIR values are formally positive. Consequently, we can not speak of $\gamma_k(\boldsymbol{r})$, $1 \le k \le K$, as of an achieved tuple of SIR values in the network. Thus, any allocation $\boldsymbol{r}$ with zero components can not be regarded as a *valid*, or realizable, allocation in real-world networks. Due to this feature, in [35], [127] such tuple of SIR values (2.27) given a not (strictly) positive power vector, is referred to as *ineffective*.

Clearly, when $\boldsymbol{r} > 0$ exists, then no such difficulty is encountered and $\boldsymbol{r}$ is implied by (6.1) to be valid and min-max fair. Hence, we can summarize as follows.

Observation 1 *The infimum in (6.1) is attained if and only if there exists some right PF eigenvector $\boldsymbol{r} > 0$. In such case, $\boldsymbol{r}$ is a min-max fair allocation. Any right PF eigenvector $\boldsymbol{r}$ which does not satisfy $\boldsymbol{r} > 0$ is not a valid allocation.*

It is important to notice that an allocation $\boldsymbol{r} > 0$ is always valid, regardless how small its elements are. This is a consequence of the multiplicative homogeneity of the SIR function (2.27) (Section 2.4.1). Thus, an arbitrarily small allocation $\boldsymbol{r} > 0$ is equivalent in terms of the SIR to a suitably upscaled allocation $c\boldsymbol{r} > 0$, $c > 0$. Consequently, merely the relations of link powers within an allocation determine the min-max fair performance (6.1).

6.1.1 The fairness-performance trade-off

For particular wired networks, min-max fairness and aggregated performance optimality of bandwidth sharing schemes were shown in [53], [55] and [128] to be incompatible goals. However, such incompatibility is in general strongly topology-dependent. This is an insight from [56], where the corresponding conditions for compatibility/ incompatibility were stated and some examples of min-max fair schemes which achieve also optimal aggregated performance were constructed. A similar kind of incompatibility was observed in [59] in the context of wireless multi-hop ad-hoc networks.

In Sections 6.2-6.4 we address the corresponding interplay of min-max fairness (2.30) and optimality of aggregated performance (2.18) for a wireless (single-hop) network from Chapter 2. For clarity and compactness of notation, we refer to the abstract interplay between min-max fairness and optimality of aggregated performance simply as the *fairness-performance trade-off*. An allocation which is optimal in terms of aggregated performance, that is, a global minimizer of (2.18), is referred to simply as *performance-optimal*. Further, for a given interference matrix $\boldsymbol{V}$, we restrict the analysis of the fairness-performance trade-off to the class of QoS functions $\mathcal{E}(\boldsymbol{V})$ defined as follows.

Definition 9 *Given some interference matrix $\boldsymbol{V}$, we have $F \in \mathcal{E}(\boldsymbol{V})$ if and only if the problem (2.18) is well-defined for any $\boldsymbol{\alpha} \in \mathcal{A}$ and any local minimizer of (2.18) is a global minimizer as well.*

Thus, the class $\mathcal{E}(\boldsymbol{V})$ contains all "well-behaved" performance functions, for which the problem of aggregated performance optimization is globally solvable by locally convergent iterations [49]. Given some $\boldsymbol{V}$, a necessary and sufficient characterization of $\mathcal{E}(\boldsymbol{V})$ is an open problem. However, from Proposition 3, Lemma 1 and the relation of performance functions (2.9), a subclass of performance functions from $\mathcal{E}(\boldsymbol{V})$ can be deduced (note, that under (2.28) the constraint qualification for problem (2.18) is satisfied).

Corollary 9 *For any interference matrix $\boldsymbol{V}$ we have $F \in \mathcal{E}(\boldsymbol{V})$ if*

$$q \mapsto G(q) = \frac{1}{F^{-1}(q)}, \quad q \in \mathbb{R},$$

is log-convex, if and only if $x \mapsto F_e(x) = F(e^{-x})$, $x < \infty$, is convex.

Clearly, examples of QoS functions that correspond to log-convex QoS-SIR dependence G can be obtained by transform (2.9) from the examples given at the end of Section 2.2. These are precisely the following.

- $q = F(\frac{1}{\gamma}) = -\log \frac{\gamma}{1+\gamma}$ (that is, $F(y) = -\log \frac{1}{1+y}$) as the logarithmically (e.g. in dB) expressed effective bandwidth for linear MMSE receivers. In fact, $\gamma = G(q) = \frac{\exp(-q)}{1-\exp(-q)}$ is log-convex.
- $q = F(\frac{1}{\gamma}) = \frac{1}{\gamma^a}$ (that is, $F(y) = y^a$) as the channel-averaged symbol error rate (under receiver diversity $a > 0$ and Rayleigh fading) or as the effective spreading factor in CDMA ($a = 1$). Then, $\gamma = G(q) = \frac{1}{q^{1/a}}$ is log-convex.
- $q = F(\frac{1}{\gamma}) = -\log\gamma$ (that is, $F(y) = \log y$) as the logarithmically (e.g. in dB) expressed SIR, or high-SIR approximation of the link capacity. In fact, $\gamma = G(q) = \exp(-q)$ is log-convex.

6.2 Min-max fair and performance optimal allocation - the uniqueness case

We first concentrate on so-called *entirely interference-coupled* (in short, *entirely coupled*) networks. These are networks with a specific form of coupling of links by interference. The coupling of links is in such case described by an irreducible interference matrix. Let the *interference graph* be defined as a $\boldsymbol{V}$-dependent directed graph on the graph node set $\mathcal{K}$, which has an edge (i,j) whenever $V_{ij} > 0$ [129]. Then, irreducibility of $\boldsymbol{V}$ is equivalent to the property that any pair of graph nodes in the corresponding interference graph is joined by a path [127], [130], [129]. For the interpretation of irreducibility in terms of the canonical form of $\boldsymbol{V}$ see Appendix A.1.

For an entirely coupled network there exists a unique power and weight allocation which combines min-max fairness and aggregated performance optimality. This is shown in the following Proposition.

Proposition 35 *For an irreducible interference matrix $\boldsymbol{V}$, let $F \in \mathcal{E}(\boldsymbol{V})$ and $\boldsymbol{w} = (w_1, \dots, w_K)$, $w_k :=r_k l_k$, $1 \le k \le K$. Then the following is true.*
i.) $\boldsymbol{r}, \boldsymbol{l} > 0$ and $\boldsymbol{r}, \boldsymbol{l}$ are unique up to a scaling constant,
ii.) $\boldsymbol{r} = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right)$ if and only if $\boldsymbol{\alpha} = \boldsymbol{w}$,
iii.) the equality

$$\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) \tag{6.3}$$

is satisfied if and only if $\boldsymbol{\alpha} = \boldsymbol{w}$, with $\boldsymbol{w}$ unique in $\mathcal{A}$.

Proof i.) Follows directly from the properties of nonnegative irreducible matrices [38]. ii.) With $F \in \mathcal{E}(\boldsymbol{V})$ (and satisfied constraint qualification), a power vector solves (2.18) if and only if it satisfies the Kuhn-Tucker conditions for (2.18). From our assumptions on $\mathcal{P}$, the property $\gamma_k(c\boldsymbol{p}) = \gamma_k(\boldsymbol{p})$, $c > 0$, and bijectivity of F follows $\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right) = \min_{\boldsymbol{p}\in\mathbb{R}_+^K} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right)$. Hence, the Kuhn-Tucker conditions for (2.18) correspond simply to the gradient set to zero, which yields

$$\sum_{\substack{j=1\\j\neq k}}^{K} \alpha_j F'\left(\frac{(\boldsymbol{Vp})_j}{p_j}\right) \frac{V_{jk}}{p_j} = \alpha_k F'\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right) \frac{(\boldsymbol{Vp})_k}{p_k^2}, \quad 1 \le k \le K. \tag{6.4}$$

With the definition $\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{p})=(\frac{\alpha_1}{p_1}, \frac{\alpha_2}{p_2}, \dots, \frac{\alpha_K}{p_K})$ we can write (6.4) in an equivalent matrix form

$$(\boldsymbol{F}'(\boldsymbol{p})\boldsymbol{V})'\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{p}) = \boldsymbol{F}'(\boldsymbol{p})\Gamma^{-1}(\boldsymbol{p})\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{p}), \tag{6.5}$$

with the function $\boldsymbol{p} \mapsto \boldsymbol{F}'(\boldsymbol{p})$=diag $\left(F'\left(\frac{(\boldsymbol{Vp})_1}{p_1}\right), \ldots, F'\left(\frac{(\boldsymbol{Vp})_K}{p_K}\right)\right)$, $\boldsymbol{p} \in \mathcal{P}$. By the definition of the right PF eigenvector we can write

$$\frac{r_k}{(\boldsymbol{Vr})_k} = \frac{1}{\rho(\boldsymbol{V})}, \quad 1 \leq k \leq K. \tag{6.6}$$

Hence, by the definition of $\boldsymbol{F}'$ and setting $\boldsymbol{p} = \boldsymbol{r}$ in the optimality condition (6.5), we yield for (6.6)

$$\boldsymbol{V}'\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{r}) = \rho(\boldsymbol{V})\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{r}). \tag{6.7}$$

This implies $\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{r}) = \boldsymbol{l}$ which, by the definition, is equivalent to $\boldsymbol{\alpha} = \boldsymbol{w}$ and completes the proof of the *if* part of ii.). For the *only if* part assume by contradiction that $\boldsymbol{r}$ satisfies the Kuhn-Tucker conditions for some $\boldsymbol{\alpha} \neq \boldsymbol{w}$, $\boldsymbol{\alpha} \in \mathcal{A}$. This means that (6.7) is satisfied for some $\boldsymbol{\beta}(\boldsymbol{\alpha}, \boldsymbol{r}) \neq \boldsymbol{l}$, which is a contradiction and completes the proof of ii.). iii.) From part ii.), the feature $\|\boldsymbol{w}\|_1 = 1$ (due to $\boldsymbol{w} \in \mathcal{A}$) and (6.6), we have

$$\min_{\boldsymbol{p}\in\mathcal{P}} \sum_{k=1}^{K} w_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right) = \sum_{k=1}^{K} w_k F\left(\frac{(\boldsymbol{Vr})_k}{r_k}\right) = \sum_{k=1}^{K} w_k F(\rho(\boldsymbol{V})) = F(\rho(\boldsymbol{V})). \tag{6.8}$$

The uniqueness of $\boldsymbol{w}$ in $\mathcal{A}$ follows directly from its definition and the uniqueness property i.). To show that $\boldsymbol{w}$ is the only vector in $\mathcal{A}$ satisfying (6.8), assume by contradiction that (6.3) is satisfied for some $\boldsymbol{\alpha} \neq \boldsymbol{w}$, $\boldsymbol{\alpha} \in \mathcal{A}$. But by (6.6) and $\|\boldsymbol{\alpha}\|_1 = 1$ we have that $\boldsymbol{r}$ is still a minimizer. This further yields with ii.) that $\boldsymbol{\alpha} = \boldsymbol{w}$, which is a contradiction and completes the proof of iii.). □

The obvious part i.) of Proposition 35 means that for entirely coupled networks the min-max fair allocation exists and is unique (up to a scaling constant, as explained in Section 6.1). Part ii.) says that a min-max fair allocation is performance-optimal for the specific weight vector $\boldsymbol{w}$ corresponding to componentwise product of PF eigenvectors of the interference matrix. Such weighting is unique in the normalized class $\mathcal{A}$ due to the uniqueness of the PF eigenvectors of an irreducible matrix [38]. Moreover, the min-max fair allocation is strictly performance-suboptimal for any other weight vector. Precisely, we have from part ii.)

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vr})_k}{r_k}\right) > \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{Vp})_k}{p_k}\right), \quad \boldsymbol{\alpha} \neq \boldsymbol{w}. \tag{6.9}$$

Up to here, we can summarize as follows.

Observation 2 *Under entire interference coupling in the network, the power and weight allocation* $(\boldsymbol{r}, \boldsymbol{w})$ *combines aggregated performance optimality and min-max fairness and any other power and weight allocation in* $\{\boldsymbol{v} : \|\boldsymbol{v}\|_1 = c\} \times \mathcal{A}$*, for any* $c > 0$*, is either not min-max fair or performance-suboptimal or both.*

From the practical point of view, it has to be noted that the uniqueness of the min-max fair and performance-optimal weight and power allocation in $\{\boldsymbol{v} : \|\boldsymbol{v}\|_1 = c\} \times \mathcal{A}$ is a disadvantage. This is because in order to achieve fairness and aggregated performance optimality at least approximatively, it is necessary that the weights of links be determined by some vector in a sufficiently small neighborhood of a specific unique vector $\boldsymbol{w}$.

If there is a degree of freedom in choosing the weights for the links, and thus the optimization over the weight vectors can be taken into account, Observation 2 becomes interesting from the view of practical power and weight control. It implies that by certain adjustment of link powers and link weights, min-max fairness and optimality in terms of aggregated performance can be achieved concurrently.

6.3 MIN-MAX FAIR AND PERFORMANCE-OPTIMAL ALLOCATION - THE GENERAL CASE

The characterization from Proposition 35 does not hold if the network is not entirely coupled. For such case, even the existence of a min-max fair allocation is not ensured, since $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} > 0$, may not exist (Observation 1) [38]. In a general network, not necessarily entirely coupled, the existence of interference decoupled link pairs is allowed. Equivalently, the corresponding interference graph may include some pair of graph nodes which is not joined by a path [130]. In terms of the representation of $\boldsymbol{V}$ in the canonical form, this means that the network can be partitioned into two or more subnetworks which are entirely coupled in itself and, in general, interfere with each other (see Appendix A.1).

Characterization of the trade-off of min-max fairness and aggregated performance optimality which generalizes Proposition 35 to the case of arbitrary networks is as follows.

Proposition 36 *Let $F \in \mathcal{E}(\boldsymbol{V})$ and $\mathcal{W} = \{\hat{\boldsymbol{w}} = (\hat{w}_1, \ldots, \hat{w}_K) \in \mathcal{A} : \hat{w}_k = \hat{r}_k \hat{l}_k, \hat{\boldsymbol{r}} = (\hat{r}_1, \ldots, \hat{r}_K) \in \mathcal{R}, \hat{\boldsymbol{l}} = (\hat{l}_1, \ldots, \hat{l}_K) \in \mathcal{L}\}$. Then, the following is true.*
i.) For any $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} = \arg\inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$ if and only if $\boldsymbol{\alpha} \in \mathcal{W}$,
ii.) the equality

$$\inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) \tag{6.10}$$

is satisfied if and only if $\boldsymbol{\alpha} \in \mathcal{W}$.

Proof i.) The proof is a straightforward generalization of the proof of Proposition 35 ii.), with $\boldsymbol{r}$ replaced by any $\tilde{\boldsymbol{r}} \in \mathcal{R}$ due to the nonuniqueness of PF eigenvectors for general matrices $\boldsymbol{V} \in \mathbb{R}_+^{K \times K}$. ii.) Construct a matrix $\boldsymbol{V}_\epsilon = \boldsymbol{V} + \epsilon \boldsymbol{1}\boldsymbol{1}'$, $\epsilon > 0$. From the construction follows that $\boldsymbol{V}_\epsilon$ is irreducible for any $\epsilon > 0$ (because it is positive for any $\epsilon > 0$). Further, we have

$$\frac{(\boldsymbol{V}_\epsilon \boldsymbol{p})_k}{p_k} = \frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k} + \epsilon \frac{\|\boldsymbol{p}\|_1}{p_k}, \quad \boldsymbol{p} \in \mathcal{P}, \quad 1 \leq k \leq K. \tag{6.11}$$

From increasingness of F we have also $F((\boldsymbol{V}_\epsilon \boldsymbol{p})_k / p_k) \geq F((\boldsymbol{V}\boldsymbol{p})_k / p_k)$, $1 \leq k \leq K$. Let $\tilde{\boldsymbol{w}}(\epsilon) \in \mathcal{A}$ denote an arbitrary vector parameterized by $\epsilon > 0$. Since $\mathcal{A}$ is compact, there exist sequences $\{\epsilon_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \epsilon_n = 0$ and

$$\lim_{n \to \infty} \|\tilde{\boldsymbol{w}}(\epsilon_n) - \tilde{\boldsymbol{w}}\| = 0 \tag{6.12}$$

for any $\tilde{\boldsymbol{w}} \in \mathcal{A}$. Choose any such sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$. With continuity of the spectral radius as a function of matrix elements, Proposition 35 iii.) and increasingness of F, it follows then

$$\begin{aligned} F(\rho(\boldsymbol{V})) &= \lim_{n \to \infty} F(\rho(\boldsymbol{V}_{\epsilon_n})) = \lim_{n \to \infty} \inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\boldsymbol{V}_{\epsilon_n} \boldsymbol{p})_k}{p_k}\right) \\ &\geq \lim_{n \to \infty} \inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right). \end{aligned} \tag{6.13}$$

On the other side, we can also write

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\boldsymbol{V}_{\epsilon_n}\boldsymbol{p})_k}{p_k}\right) = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \left(\sum_{k=1}^{K} (\tilde{w}_k(\epsilon) - \tilde{w}_k) F\left(\frac{(\boldsymbol{V}_{\epsilon_n}\boldsymbol{p})_k}{p_k}\right) + \sum_{k=1}^{K} \tilde{w}_k \left(F\left(\frac{(\boldsymbol{V}_{\epsilon_n}\boldsymbol{p})_k}{p_k}\right) - F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)\right) + \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)\right). \tag{6.14}$$

The first two sums on the right-hand side of (6.14) can be upper bounded using the Cauchy-Schwarz inequality and the bounds disappear with $n \to \infty$ due to (6.11) and (6.12). Hence, for the limit we get

$$F(\rho(\boldsymbol{V})) = \lim_{n\to\infty} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\boldsymbol{V}_{\epsilon_n}\boldsymbol{p})_k}{p_k}\right) \le \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right). \tag{6.15}$$

Inequalities (6.15) and (6.13) together imply now $F(\rho(\boldsymbol{V})) = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F((\boldsymbol{V}\boldsymbol{p})_k/p_k)$ for $\tilde{\boldsymbol{w}} \in \mathcal{W}$. The if and only if property in ii.) parallels the if and only if property in Proposition 35 iii.). Thus, the proof of the if and only if property is analogous to the corresponding proof in Proposition 35 iii.). □

From Proposition 36 one can conclude that the characterization of the trade-off for entirely coupled networks translates to the general network case except the uniqueness property. Thus, Propositions 35 and 36 can be summarized as follows: Whenever a min-max fair allocation (i.e. a PF eigenvector $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} > 0$) exists, then any such allocation remains performance-optimal for specific weight vectors constituting set $\mathcal{W}$. Moreover, for any weight vector not in $\mathcal{W}$, any min-max fair allocation, if existent, remains strictly performance-suboptimal, that is

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{r})_k}{r_k}\right) > \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \notin \mathcal{W}.$$

In the particular case of entire interference coupling, the sets $\mathcal{W}$ and $\{\boldsymbol{v} : \|\boldsymbol{v}\|_1 = c\} \cap \mathcal{R}$, $c > 0$, become singletons, so that the min-max fair power and weight allocation exists and is unique on $\{\boldsymbol{v} : \|\boldsymbol{v}\|_1 = c\} \cap \mathcal{A}$, $c > 0$.

Hence, together with Observation 1, we can extend Observation 2 as follows.

Observation 3 *Any power and weight allocation* $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}})$ *satisfying* $\tilde{\boldsymbol{r}} \in \mathcal{R} \cap \mathbb{R}^K_{++}$ *and* $\tilde{\boldsymbol{w}} \in \mathcal{W}$ *combines aggregated performance optimality and min-max fairness. Whenever* $\hat{\boldsymbol{r}} \in \mathcal{R}$ *and* $\hat{\boldsymbol{r}} \notin \mathbb{R}^K_{++}$, *then* $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{w}})$ *is not a power and weight allocation. Whenever* $\hat{\boldsymbol{r}} \notin \mathcal{R}$ *or* $\hat{\boldsymbol{w}} \notin \mathcal{W}$, *then the power and weight allocation* $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{w}})$ *is either not min-max fair or performance-suboptimal or both.*

The nonuniqueness of the power and weight allocation $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{w}}) \in \mathcal{R} \cap \mathbb{R}^K_{++} \times \mathcal{W}$ makes Observation 3 practically more relevant than Observation 2. Precisely, in the restricted case of entirely coupled networks, fairness and aggregated performance optimality is approximatively achievable under a power and weight allocation from a neighborhood of $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{w}})$, which is unique in $\{\boldsymbol{v} : \|\boldsymbol{v}\|_1 = c\} \times \mathcal{A}$ (Observation 2). As is implied by Observation 3, in the general case of interference coupling, however, to achieve this goal it suffices to choose a power and weight allocation from a neighborhood of the entire set $\mathcal{R} \cap \mathbb{R}^K_{++} \times \mathcal{W}$. Thus, in the general case it is more likely that some weight vector from the neighborhood of $\mathcal{W}$ is suitable for the link priorities on hand. If this is the case, the choice of a power vector from the neighborhood of the set $\mathcal{R} \cap \mathbb{R}^K_{++}$ allows for the approximative achievement of min-max fairness and aggregated performance optimality concurrently.

6.3.1 Existence of a min-max fair allocation

Recall from Section 6.2 that in entirely coupled networks, a min-max fair allocation exists and is additionally unique. In this section we characterize the class of all networks, including in particular the class of entirely coupled networks, for which a min-max fair allocation is existent. The characterization is in terms of the canonical form of the interference matrix. The result is a straightforward consequence of Theorem 3 in [127], which can be restated for our purposes in the following equivalent form. In the remainder we denote by $\mathcal{I}$ and $\mathcal{M}$ the sets of *isolated* and *maximal* diagonal blocks of an interference matrix (see Appendix A.1 for the definitions of isolation, maximality and other issues related to the canonical form.)

Proposition 37 ([127]) *Let $\{\boldsymbol{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively. Matrix $\boldsymbol{V}$ has a right PF eigenvector $\tilde{\boldsymbol{r}}\in\mathcal{R}$ satisfying $\tilde{\boldsymbol{r}}>0$ if and only if $\mathcal{I}=\mathcal{M}$.*

The isolation property of some diagonal block in $\boldsymbol{V}$ is equivalent to the isolation of the corresponding subnetwork from the interference from other subnetworks (Appendix A.1). Analogously, the nonisolated blocks correspond to subnetworks which include some links which perceive interference from some links in other subnetworks. Since the distinguished subnetworks are entirely interference coupled in itself, we can interpret Proposition 37 as follows.

Observation 4 *A min-max fair allocation exists for any network with interference matrix $\boldsymbol{V}$ such that*
i.) the interference matrix $\boldsymbol{V}^{(n)}$ of each interference-isolated and entirely coupled subnetwork $n\in\mathcal{I}$ satisfies $\rho(\boldsymbol{V}^{(n)})=\rho(\boldsymbol{V})$,
ii.) the interference matrix $\boldsymbol{V}^{(m)}$ of each entirely coupled subnetwork $m\in\mathcal{K}\setminus\mathcal{I}$ perceiving interference from some other entirely coupled subnetwork satisfies $\rho(\boldsymbol{V}^{(m)})<\rho(\boldsymbol{V})$.
For any network violating either i.) or ii.), no min-max fair allocation exists.

It is worth pointing out an interesting relation between the min-max fair allocation for the entire network and for its entirely coupled subnetworks. Denote the left and right eigenvectors of the n-th diagonal block of the (canonical for of) interference matrix $\boldsymbol{V}$ as $\boldsymbol{l}^{(n)}$ and $\boldsymbol{r}^{(n)}$ respectively, and notice that both are unique up to a scaling constant due to irreducibility of each diagonal block. From the eigenvalue equation for the canonical form of $\boldsymbol{V}$ it is then easy to see that the eigenvectors $\boldsymbol{l}^{(n)}$, $\boldsymbol{r}^{(n)}$ of any isolated and maximal diagonal block $\boldsymbol{V}^{(n)}$ (if existent) correspond to the projections of any $\tilde{\boldsymbol{l}}\in\mathcal{L}$ and $\tilde{\boldsymbol{r}}\in\mathcal{R}$, respectively, on the subspace with dimensions restricted to the diagonal block $\boldsymbol{V}^{(n)}$. Precisely, we have

$$\begin{cases}(\tilde{r}_{k_1(n)},\tilde{r}_{k_1(n)+1},\ldots,\tilde{r}_{k_M(n)})=\boldsymbol{r}^{(n)}, & \tilde{\boldsymbol{r}}\in\mathcal{R}\\(\tilde{l}_{k_1(n)},\tilde{l}_{k_1(n)+1},\ldots,\tilde{l}_{k_M(n)})=\boldsymbol{l}^{(n)}, & \tilde{\boldsymbol{l}}\in\mathcal{L},\end{cases}\tag{6.16}$$

whenever the diagonal block of $\boldsymbol{V}^{(n)}$ is isolated and maximal and corresponds to the components $k_1(n),\leq l\leq k_M(n)$, with $1\leq k_1(n),k_M(n)\leq K$ in the matrix $\boldsymbol{V}$. We can interpret this property as follows.

Observation 5 *Let the network satisfy i.) and ii.) in Observation 4. Then, any min-max fair allocation for an entirely coupled and interference-isolated subnetwork corresponds to the restriction of the min-max fair allocation for the entire network to such subnetwork.*

Clearly, the eigenvalue equation implies also that the projection property (6.16), and thus Observation 5, can not hold for nonisolated diagonal blocks of $\boldsymbol{V}$.

6.3.2 Existence of a positive weight allocation

The set $\mathcal{W}$ of performance-optimal and min-max fair weight allocations is in general not guaranteed to include positive weight allocations. In fact, even for networks satisfying i.), ii.) in Observation 4, the existence of $\tilde{\boldsymbol{l}} \in \mathcal{L}$, $\tilde{\boldsymbol{l}} > 0$, is not ensured, so that the construction of $\tilde{\boldsymbol{w}} \in \mathcal{W}$ such that $\tilde{\boldsymbol{w}} > 0$ may be prevented. Therefore, the characterization of the class of networks for which a positive performance-optimal and min-max fair weight allocation exists is here of interest. It is clear from the construction of $\mathcal{W}$ that such class must be included in the class of networks having some $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $v\breve{e}r > 0$, which is characterized in Proposition 37. The corresponding characterization follows straightforwardly from [130] or, equivalently, from Theorems 3 and 4 in [127].

Proposition 38 ([127]) *Let $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the set of maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$. Matrix $\boldsymbol{V}$ has right and left PF eigenvectors $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{l}} \in \mathcal{L}$ satisfying $\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{l}} > 0$ if and only if it is block-irreducible and $\mathcal{M} = \{1, \ldots, N\}$.*

The existence of positive left and right PF eigenvectors following from Proposition 38 makes the construction of a weight vector $\tilde{\boldsymbol{w}} \in \mathcal{W} \cap \mathbb{R}^K_{++}$ possible. Proposition 38 characterizes a subclass of interference matrices from Proposition 37 for which $\mathcal{I} = \mathcal{M} = \{1, \ldots, N\}$, that is, for which no nonisolated diagonal blocks exist. We can interpret Proposition 38 as follows.

Observation 6 *A positive performance-optimal and min-max fair weight allocation exists for any network with interference matrix $\boldsymbol{V}$ such that*
i.) the network consists of a number of entirely coupled and pairwise interference-isolated subnetworks,
ii.) the interference matrix $\boldsymbol{V}^{(n)}$ of each entirely coupled subnetwork satisfies $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$.
For any network violating either i.) or ii.), no positive performance-optimal and min-max fair weight allocation exists.

Obviously, the entirely interference coupled networks are the trivial case of networks satisfying i.), ii.) in Observation 6, as they formally consist of one entirely interference coupled subnetwork.

The role of block irreducibility for aggregated performance optimization

The networks with properties characterized in Observation 6 (that is, with interference matrices characterized in Proposition 38) play a specific role not only in terms of the trade-off between min-max fairness and aggregated performance optimality. Such networks have a specific property of the QoS region, which we describe here in short. As a slight difference to Proposition 38 and Observation 6, the discussion below concerns a weighted interference matrix $\boldsymbol{\Gamma V}$, with $\boldsymbol{\Gamma} = \Gamma(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{P}$, as an SIR matrix (2.4).

From [127] is known that given the SIR function with neglected noise (2.27), the QoS region can be represented as

$$\mathcal{Q} = \{\boldsymbol{q} = (F(\frac{1}{\gamma_1}), \ldots, F(\frac{1}{\gamma_K})) : \rho(\boldsymbol{\Gamma V}) \le 1, \quad \boldsymbol{\Gamma} = \Gamma(\boldsymbol{p}), \quad \boldsymbol{p} \in \mathcal{P}\}. \tag{6.17}$$

From the normal form of the interference matrix we have further

$$\rho(\boldsymbol{\Gamma V}) = \max_{1 \le n \le N} \rho(\boldsymbol{\Gamma}^{(n)} \boldsymbol{V}^{(n)}), \tag{6.18}$$

with $\mathbf{\Gamma}^{(n)} = diag(\gamma_{k_1(n)}, \ldots, \gamma_{k_M(n)})$, with $k_1(n) \leq l \leq k_M(n)$ as the interval of components corresponding to the diagonal block $\boldsymbol{V}^{(n)}$. Consequently, it follows that $\mathcal{Q} = \mathcal{Q}^{(1)} \times \ldots \times \mathcal{Q}^{(N)}$, with

$$\mathcal{Q}^{(n)} = \{\boldsymbol{q}^{(n)} = (F(\frac{1}{\gamma_{k_1(n)}}), \ldots, F(\frac{1}{\gamma_{k_M(n)}})) : \rho(\mathbf{\Gamma}^{(n)}\boldsymbol{V}^{(n)}) \leq c(n)\}, \quad c(n) \leq 1, \quad 1 \leq n \leq N,$$

where the bound on constant $c(n)$, $1 \leq n \leq N$, is due to (6.17) and (6.18). In other words, *the QoS region of the network is the Cartesian product of QoS regions of entirely coupled subnetworks.* With this insight, we can also refine the aggregated performance optimization in the form (2.19), by writing

$$\min_{\boldsymbol{q}\in\mathcal{Q}} \boldsymbol{\alpha}'\boldsymbol{q} = \sum_{n=1}^{N} \min_{\boldsymbol{q}^{(n)}\in\mathcal{Q}^{(n)}} \sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l^{(n)}, \quad \boldsymbol{\alpha} \in \mathcal{A}. \tag{6.19}$$

Let $\boldsymbol{\alpha} > 0$ and notice that the minimum of the partial objective $\boldsymbol{q}^{(n)} \mapsto \sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l^{(n)}$ is achieved on the boundary of the QoS region $\mathcal{Q}^{(n)}$, $1 \leq n \leq N$. Consequently, whenever in n-th subnetwork we have $c(n) < 1$, the corresponding partial objective achieves a value which is strictly suboptimal compared to the case $c(n) = 1$. Consequently, the optimal values of partial objectives $\boldsymbol{q}^{(n)} \mapsto \sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l^{(n)}$, $1 \leq n \leq N$, and hence the minimum in (2.19), is achievable exactly in the case when all weighted subnetwork interference matrices $\mathbf{\Gamma}^{(n)}\boldsymbol{V}^{(n)}$, $1 \leq n \leq N$, correspond to maximal diagonal blocks of $\mathbf{\Gamma}\boldsymbol{V}$, that is,

$$\rho(\mathbf{\Gamma}^{(n)}\boldsymbol{V}^{(n)}) = 1, \quad 1 \leq n \leq N. \tag{6.20}$$

In other words, the "farthest" boundary part of the QoS region $\mathcal{Q}$ is achievable in the aggregated performance optimization exactly when (6.20) is satisfied.

6.4 THE FAIRNESS-PERFORMANCE TRADE-OFF AS A SADDLE POINT

In Section 6.3 we showed that a power and weight allocation of the form $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}})$, $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{w}} \in \mathcal{W}$, combines min-max fairness and aggregated performance optimality. In this section we assume that the link weights are adjustable and study the problems of min-max and max-min form related with the aggregated performance. This approach is followed in order to characterize the role of the power and weight allocation combining fairness and aggregated performance optimality among all power and weight allocations. In this way we characterize the mechanism of the trade-off occurring under combination of fairness and aggregated performance optimality. Precisely, we prove that such trade-off has the interpretation of a saddle point of the aggregated performance function as a function of power and weight allocations.

6.4.1 THE MIN-MAX PROBLEM

Consider first the problem of aggregated performance optimization for a worst-case weight vector. In such case we have the following property.

Lemma 13 *Let $\boldsymbol{V}$ be any interference matrix and let $F \in \mathcal{E}(\boldsymbol{V})$. Then, we have*

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.21}$$

with $\tilde{\boldsymbol{r}} = \arg\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, $\tilde{\boldsymbol{r}} \in \mathcal{R}$. *If* $\boldsymbol{V}$ *is irreducible, then* $\boldsymbol{r} > 0$ *is the unique (up to a scaling constant) vector satisfying* $\boldsymbol{r} = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$.

Proof It is clear that $\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, $\boldsymbol{\alpha} \in \mathcal{A}$. With Proposition 34 it follows further that

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{r}})_k}{\tilde{r}_k}\right) = F(\rho(\boldsymbol{V})), \quad \tilde{\boldsymbol{r}} \in \mathcal{R}. \tag{6.22}$$

By Proposition 35 i.), concerning the case of irreducible $\boldsymbol{V} \in \mathbb{R}_+^{K\times K}$, there is an up to a scaling constant unique vector $\boldsymbol{r} > 0$, and the proof is completed. □

Lemma 13 shows that a right PF eigenvector of $\boldsymbol{V}$ is a power vector which solves the problem of aggregated performance optimization (2.18) for a worst-case vector of weights. Equivalently, the min-max fair allocation $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} > 0$ (which exists whenever the interference matrix $\boldsymbol{V}$ satisfies conditions i.), ii.) in Observation 4), is the performance-optimal allocation, when such a weight vector in $\mathcal{A}$ is chosen which yields the largest value of the aggregated performance. For entirely coupled networks, the lemma shows that given a worst-case weight vector the aggregated performance is optimized under a min-max fair allocation and under no other allocation.

6.4.2 THE MAX-MIN PROBLEM

In what follows we denote the aggregated performance function as a function of powers and weights as

$$(\boldsymbol{p}, \boldsymbol{\alpha}) \mapsto U(\boldsymbol{p}, \boldsymbol{\alpha}) = \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right), \quad (\boldsymbol{p}, \boldsymbol{\alpha}) \in \mathcal{P} \times \mathcal{A}, \tag{6.23}$$

and additionally

$$\boldsymbol{\alpha} \mapsto U_{\boldsymbol{p}}(\boldsymbol{\alpha}) = \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \in \mathcal{A}. \tag{6.24}$$

For the aggregated performance function (6.24) we have first the following insight.

Lemma 14 *Let* $\boldsymbol{V}$ *be any irreducible interference matrix and let* $F \in \mathcal{E}(\boldsymbol{V})$. *Then,* $U_{\boldsymbol{p}}$ *is strictly concave.*

Proof Function $U_{\boldsymbol{p}}$ is concave by definition, due to the properties of the minimum function [47]. Assume now by contradiction that $U_{\boldsymbol{p}}$ is not strictly concave. Then, there exist $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)} \in \mathbb{R}^K$, $\boldsymbol{\alpha}^{(1)} \neq \boldsymbol{\alpha}^{(2)}$, such that

$$U_{\boldsymbol{p}}((1-t)\boldsymbol{\alpha}^{(1)} + t\boldsymbol{\alpha}^{(2)}) = (1-t)U_{\boldsymbol{p}}(\boldsymbol{\alpha}^{(1)}) + tU_{\boldsymbol{p}}(\boldsymbol{\alpha}^{(2)}), \quad \text{for some } t \in (0,1). \tag{6.25}$$

As a first case assume:

i.) If $\boldsymbol{p}^{(1)} = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(1)} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$ and $\boldsymbol{p}^{(2)} = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(2)} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, then $\boldsymbol{p}^{(1)} \neq \boldsymbol{p}^{(2)}$.

Let $\boldsymbol{p}(t) = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}((1-t)\alpha_k^{(1)} + t\alpha_k^{(2)})F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, $t \in (0,1)$. Then,

$$\begin{aligned} U_{\boldsymbol{p}}((1-t)\boldsymbol{\alpha}^{(1)} + t\boldsymbol{\alpha}^{(2)}) &= \sum_{k=1}^{K}((1-t)\alpha_k^{(1)} + t\alpha_k^{(2)})F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(t))_k}{p_k(t)}\right) \\ &= (1-t)\sum_{k=1}^{K}\alpha_k^{(1)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(t))_k}{p_k(t)}\right) + t\sum_{k=1}^{K}\alpha_k^{(2)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(t))_k}{p_k(t)}\right), \quad t \in (0,1). \end{aligned} \tag{6.26}$$

Hence, (6.25) and (6.26) together imply that for some $t \in (0,1)$ we have $\boldsymbol{p}(t) = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k^{(1)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$ and $\boldsymbol{p}(t) = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k^{(2)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$. This contradicts assumption i.) and hence, completes the proof under assumption i.).

Make now the opposite assumption:

ii.) There exists $\tilde{\boldsymbol{p}} \in \mathcal{P}$, such that

$$\tilde{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k^{(1)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \arg\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k^{(2)}F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right).$$

With (2.27) and (2.8) and the assumption $\boldsymbol{\alpha}^{(1)} \neq \boldsymbol{\alpha}^{(2)}$ it follows that assumption ii.) corresponds to the vertex property of the vector $\boldsymbol{q} = (F((\boldsymbol{V}\tilde{\boldsymbol{p}})_1/\tilde{p}_1), \ldots, F((\boldsymbol{V}\tilde{\boldsymbol{p}})_K/\tilde{p}_K)) \in \mathcal{Q}$, which is on the boundary of the QoS region $\mathcal{Q}$. This implies that the Frechet derivative is not defined at $\boldsymbol{q}$ [50]. Clearly, the boundary of $\mathcal{Q}$ can be bijectively mapped, by means of the inverse mapping F, onto the boundary of the manifold $\{\Gamma(\boldsymbol{p}) : \boldsymbol{p} \in \mathcal{P}\}$. With assumptions (2.27) and (2.28), such boundary is known to be equivalent to the manifold $\{\boldsymbol{\Gamma} = diag(\gamma_1, \ldots, \gamma_K) : \rho(\boldsymbol{\Gamma}\boldsymbol{V}) = 1\}$ [127]. Since the spectral radius is a smooth function of matrix elements and F is continuously Frechet differentiable by our assumptions, the boundary of $\mathcal{Q}$ must be Frechet differentiable. This contradicts the existence of a vertex on the boundary of the QoS region. Thus, assumption ii.) is never satisfied and the proof is completed. □

With Lemma 14 we can provide a max-min characterization which is complementary to the min-max characterization from Lemma 13. For clarity, we split the presentation into the one for entirely coupled networks only and its generalization to arbitrary networks.

Proposition 39 *Let $\boldsymbol{V}$ be an irreducible interference matrix and let $F \in \mathcal{E}(\boldsymbol{V})$. Then, we have*

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.27}$$

and $\tilde{\boldsymbol{\alpha}} = \arg\max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$ if and only if $\tilde{\boldsymbol{\alpha}} = \boldsymbol{w}$, with $\boldsymbol{w} = (w_1, \ldots, w_K)$, $w_k = l_k r_k$, $1 \le k \le K$, which is unique in $\mathcal{A}$.

Proof It is clear that $\sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \le \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, $\boldsymbol{p} \in \mathcal{P}$, $\boldsymbol{\alpha} \in \mathcal{A}$. This yields with Proposition 34

$$\min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \le \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.28}$$

where we can write the minimum instead of the infimum since $\boldsymbol{r} > 0$ due to irreducibility of $\boldsymbol{V}$. Inequality (6.28) is further equivalent to

$$U_{\boldsymbol{p}}(\boldsymbol{\alpha}) \leq F(\rho(\boldsymbol{V})), \quad \boldsymbol{\alpha} \in \mathcal{A}. \tag{6.29}$$

By Lemma 14, function $U_{\boldsymbol{p}}$ is strictly concave under irreducibility of $\boldsymbol{V}$, and thus has a unique maximum. Thus, with Proposition 35 iii.) and (6.29) it follows further

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} U_{\boldsymbol{p}}(\boldsymbol{\alpha}) = \max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \min_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{\alpha}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) \tag{6.30}$$

if and only if $\tilde{\boldsymbol{\alpha}} = \boldsymbol{w}$, with vector $\boldsymbol{w} = (w_1, \ldots, w_K)$, $w_k = r_k l_k$, $1 \leq k \leq K$, which is unique in $\mathcal{A}$. This completes the proof. □

The Proposition states that for entirely coupled networks the weight vector $\boldsymbol{w}$, unique in $\mathcal{A}$, is the one for which the optimum value of aggregated performance is worst possible, that is, largest. Moreover, for any other weight vector the optimum aggregated performance is smaller in value, that is, the optimum aggregated performance is better.

Notice that the difference between the min-max and max-min problem in Lemma 13 and Proposition 39, respectively, is subtle in notation but crucial in the sense. The min-max expression (6.21) corresponds to optimum aggregated performance for worst-case weights, while the max-min expression (6.27) represents the maximally degraded optimum aggregated performance among all weight vectors.

The generalization of Proposition 39 to arbitrary networks is as follows.

Proposition 40 *Let $\boldsymbol{V}$ be any interference matrix and let $F \in \mathcal{E}(\boldsymbol{V})$. Then, we have*

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.31}$$

with $\tilde{\boldsymbol{\alpha}} = \arg\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$ *if and only if* $\tilde{\boldsymbol{\alpha}} \in \mathcal{W} = \{\hat{\boldsymbol{w}} = (\hat{w}_1, \ldots, \hat{w}_K) \in \mathcal{A} : \hat{w}_k = \hat{r}_k \hat{l}_k, \hat{\boldsymbol{r}} = (\hat{r}_1, \ldots, \hat{r}_K) \in \mathcal{R}, \hat{\boldsymbol{l}} = (\hat{l}_1, \ldots, \hat{l}_K) \in \mathcal{L}\}$.

Proof As in the proof of Proposition 36, we construct an irreducible (since positive) matrix $\boldsymbol{V}_\epsilon = \boldsymbol{V} + \epsilon\mathbf{1}\mathbf{1}'$, $\epsilon > 0$. Thus, (6.11) is satisfied and implies with increasingness of F that

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \leq \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}_\epsilon\boldsymbol{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \in \mathcal{A}, \quad \boldsymbol{p} \in \mathcal{P}. \tag{6.32}$$

This further implies with Proposition 39

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \leq \max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}_\epsilon\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V}_\epsilon)). \tag{6.33}$$

The left-hand side of (6.33) does not depend on ϵ. Thus, taking the limit of both sides of (6.33) by letting $\epsilon \to 0$ yields with continuity of the spectral radius as a function of matrix elements the inequality

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \leq F(\rho(\boldsymbol{V})). \tag{6.34}$$

From Proposition 36 ii.) we further have $\max_{\boldsymbol{\alpha}\in\mathcal{A}}\inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \geq F(\rho(\boldsymbol{V}))$. Thus, together with (6.34), we must have

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}}\inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\sum_{k=1}^{K}\tilde{\alpha}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) \tag{6.35}$$

for some $\tilde{\boldsymbol{\alpha}} \in \mathcal{A}$. Again, by Proposition 36 ii.) follows that equality (6.35) holds if and only if $\tilde{\boldsymbol{\alpha}} \in \mathcal{W}$, which completes the proof. □

The generalization of Proposition 39 in Proposition 40 is analogous to the generalization of Proposition 35 in Proposition 36. It implies that for arbitrary networks any weight vector from the specific set $\mathcal{W}$, which is a singleton under irreducibility of the interference matrix, makes the corresponding optimum value of aggregated performance the worst among all weight vectors in $\mathcal{A}$. The optimum value of aggregated performance achievable under any weight vector from outside of set $\mathcal{W}$ is superior to the one achieved under any $\tilde{\boldsymbol{\alpha}} \in \mathcal{W}$.

6.4.3 THE SADDLE POINT CONCLUSION

From the min-max max-min characterization of a saddle point (Appendix A.4) it is now easily seen that the min-max and max-min relations from Lemma 13 and Propositions 39, 40 jointly describe a saddle point of the aggregated performance function as a function of weight and power allocations. We can formulate the following Corollary.

Corollary 10 *Let $\boldsymbol{V}$ be any interference matrix and let $F \in \mathcal{E}(\boldsymbol{V})$. Then, any vector pair $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}}) \in \mathcal{S}$, with $\mathcal{S} = \{(\hat{\boldsymbol{r}}, \hat{\boldsymbol{w}}) \in \mathcal{P}\times\mathcal{A} : \hat{\boldsymbol{r}} \in \mathcal{R}, \hat{\boldsymbol{w}} \in \mathcal{W}\}$, and with $\mathcal{W}$ defined as in Propositions 36 and 40, is a saddle point of the aggregated performance function (6.23) and we have*

$$\inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\max_{\boldsymbol{\alpha}\in\mathcal{A}}\sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) = \max_{\boldsymbol{\alpha}\in\mathcal{A}}\inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\sum_{k=1}^{K}\alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right). \tag{6.36}$$

If the interference matrix is irreducible, then $\mathcal{S}$ is a singleton corresponding to the unique saddle point of (6.23).

A consequence of Corollary 10 and Observation 3 is that the set of min-max fair and performance-optimal power and weight allocations corresponds to the subset $\mathcal{S}\cap(\mathbb{R}^K_{++}\times\mathcal{A})$ of the set of saddle points $\mathcal{S}$. Recall that such subset is nonempty if and only if the network satisfies conditions i.), ii.) in Observation 4.

The saddle point property is a compact interpretation of the trade-off between min-max fairness and aggregated performance optimality. It shows that the min-max fair allocation $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} > 0$ (if existent), solves the problem of aggregated performance optimization (2.18) under the penalty that the worst possible weight vector from $\mathcal{A}$ is chosen. For any non-worst-case choice of the weight vector, the optimum value of aggregated performance is better (smaller) than the one achieved by min-max fair allocation. On the other side, any weight vector $\tilde{\boldsymbol{w}} \in \mathcal{W}$ corresponding to the weight allocation achieving min-max fairness and aggregated performance optimality has the property of yielding the worst-case aggregated performance among the performance optima achievable under weight vectors from $\mathcal{W}$. Thus, under any choice $\boldsymbol{\alpha} \notin \mathcal{W}$, the achieved optimal aggregated performance performance is better than under the choice $\tilde{\boldsymbol{w}} \in \mathcal{W}$. These features can be expressed compactly by the chain inequality

$$\begin{aligned}\inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \leq \sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{r}})_k}{\tilde{r}_k}\right) \leq \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \\ \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{r}})_k}{\tilde{r}_k}\right) \leq \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right), \quad (\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}}) \in \mathcal{S},\end{aligned} \tag{6.37}$$

which is equivalent to (6.36).

The chain inequality (6.37) contains the relation $\sum_{k=1}^{K} \alpha_k F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{r}})_k}{\tilde{r}_k}\right) \leq \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{r}})_k}{\tilde{r}_k}\right) \leq \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)$, which is, by the saddle point definition (Appendix A.4) another verification of the saddle point property of any vector pair $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}}) \in \mathcal{S}$.

The saddle point property can be summarized as follows.

Observation 7 *For any power and weight allocation $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{w}}) \in \mathcal{S}$, $\tilde{\boldsymbol{r}} > 0$, which combines aggregated performance optimality and min-max fairness, the following is true.*
i.) The min-max fair allocation $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $\tilde{\boldsymbol{r}} > 0$, yields the optimum value of aggregated performance under the worst-case choice of the weight vector,
i'.) if the network is entirely coupled, then the min-max fair allocation $\tilde{\boldsymbol{r}}$ is the unique (up to a scaling constant) allocation yielding the optimum value of aggregated performance under the worst-case choice of the weight vector,
ii.) under the weight vector $\tilde{\boldsymbol{w}} \in \mathcal{W}$, the worst-case of the optimum value of aggregated performance is yielded,
ii'.) if the network is entirely coupled, the weight vector $\tilde{\boldsymbol{w}}$ is the unique weight vector in $\mathcal{A}$ under which the worst-case of the optimum value of aggregated performance is yielded.

In Fig. 6.1 we illustrate the saddle point property of a min-max fair and performance-optimal power and weight allocation. The visualization is figurative since the power vector and the weight vector are assumed to be one-dimensional.

6.5 The fairness gap

In Sections 6.2-6.4 the focus of our considerations was on the notion of fairness in the min-max sense (2.30), which is interpretable as improving the worst link performance as much as possible. An interesting question in this context is the relation to the problem of max-min fairness (6.2), which can be interpreted as degrading the best link performance as much as possible.

From the interpretation of max-min fairness is apparent, that the applicability of this fairness notion is in general limited. In fact, providing the maximum degradation of the best link QoS is intuitively not a notion of optimality desired in the network (it is rather a notion of a worst-case). However, the situation changes if the notions of min-max fairness and max-min fairness are related by some known deterministic relation. In particular, there is interest in achieving max-min fairness (6.2) if it coincides, in terms of the achieved link performance or even in terms of the optimizers, with the notion of min-max fairness (2.30). In such case, max-min fairness (6.2) can be seen as an alternative characterization of the notion of min-max fairness. This problem of coincidence is addressed in this section.

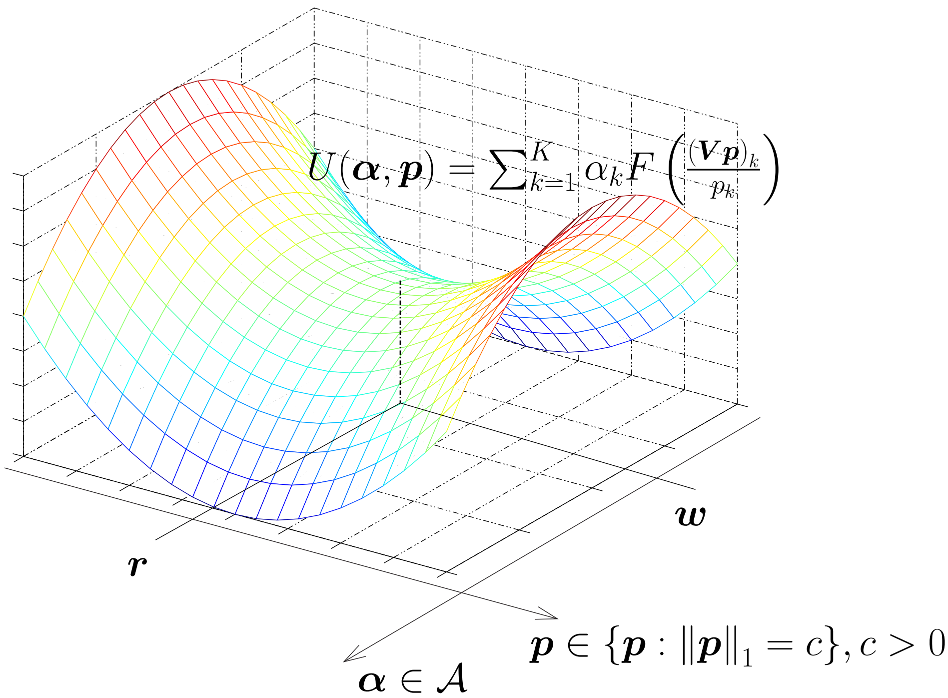

FIGURE 6.1: Figurative visualization of the saddle point property of the min-max fair and performance-optimal power and weight allocation. The two scalar dimensions represent the spaces of power allocations and weight allocations, respectively. The visualized saddle point is unique, as would be the case for entirely coupled networks.

From convex analysis follows that for any network, precisely for any interference matrix $\boldsymbol{V}$, we have

$$\sup_{\boldsymbol{p}\in\mathcal{P}_{++}} \min_{1\leq k\leq K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \leq \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\leq k\leq K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right). \tag{6.38}$$

Inequality (6.38) suggests the following question. *For which class of networks (that is, matrices $\boldsymbol{V}$) we have the property*

Condition 11 *The optimum values in problems (2.30) and (6.2) coincide.*

However, even under equality of optimal values, the optimizers of (2.30) and (6.2) may not coincide. Therefore we are also interested in the answer to the following related question. *For which class of networks we have the property*

Condition 12 *The optimum values in problems (2.30) and (6.2) coincide and there exists a power allocation which solves both problems (2.30) and (6.2).*

By complementarity to Condition 11, a question is also: *For which class of networks we have the property*

Condition 13 *The optimum value in problem (2.30) is larger than the optimum value in problem (6.2).*

Assume for a while that a min-max fair allocation exists, that is, the network satisfies conditions i.), ii.) in Observation 4. Then, the networks satisfying Condition 11 can be regarded as those having no gap between the performance achieved under min-max fairness and max-min fairness, or simply having no (or zero) *fairness gap*. Thus, in networks with zero fairness gap, the maximally degraded best link QoS exactly meets the value of the maximally improved worst link QoS. For networks with no fairness gap which satisfy the stronger Condition 12, we are additionally free to choose between (2.30) and (6.2) as equivalent problem formulations. This provides an alternative in the design of online optimization routines. Depending on hardware constraints, signaling constraints and protocol type, the alternative formulation (6.2) may happen to be favorable in terms of implementation issues.

On the other side, networks satisfying Condition 13 can be interpreted as those with (nonzero) fairness gap. Thus, for networks with fairness gap we know that the maximally degraded best link performance is still superior to the maximally improved worst link performance. In such networks, one can not resort to (6.2) as an equivalent formulation of the problem of min-max fairness for implementation purposes.

6.5.1 Maximal degradation of the best link QoS

Consider for a while the problem of max-min fairness relying on the SIR function including noise (2.1). Due to assumed increasingness of the performance function (2.8), it can be deduced that the best link QoS is maximally degraded under the all-zero power allocation $\boldsymbol{p} = \mathbf{0}$. When neglecting the noise according to (2.27), this is, however, no longer the case. Precisely, let some parameterized allocation $\boldsymbol{p}(\epsilon) \in \mathcal{P}$ converge to $\boldsymbol{p}(0) = \lim_{\epsilon \to 0} \boldsymbol{p}(\epsilon) = \mathbf{0}$. Then, all SIR values converge to finite values, each one representing a ratio of two values approaching zero. This is the same mechanism as the one described in Section 6.1 in the context of validity of allocations. Consequently, we deduce that the optimal value in (6.2) is assumed by a max-min fair allocation which is in general not all-zero. In comparison with the case of the SIR function (2.1), this feature slightly contradicts the intuition. However, from the practical and algorithmic point of view such feature may provide advantages. Precisely, given (2.27) and satisfied Condition 12, the already described degree of freedom occurs: the online optimization algorithms computing the min-max fair allocation can be designed to solve either of the two problems (2.30) or (6.2). Such degree of freedom can not occur if the notion of max-min fairness relies on (2.1).

6.5.2 The cases of zero and nonzero fairness gap

The first step towards the characterization of the network classes having zero and nonzero fairness gap is a simple Lemma.

Lemma 15 *For any interference matrix $\boldsymbol{V}$, we have*

$$\inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \max_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \ge F(\rho(\boldsymbol{V})). \tag{6.39}$$

Proof Construct first a positive matrix $\boldsymbol{V}_\epsilon = \boldsymbol{V} + \epsilon \mathbf{1}\mathbf{1}'$, $\epsilon > 0$ and define a function $(\boldsymbol{X}, \boldsymbol{p}) \mapsto f(\boldsymbol{X}, \boldsymbol{p}) = \max_{1 \le k \le K} \frac{(\boldsymbol{X}\boldsymbol{p})_k}{p_k}$, $\boldsymbol{X} \in \mathbb{R}_+^{K \times K}$, $\boldsymbol{p} \in \mathcal{P}_{++}$. We have obviously $f(\boldsymbol{V}_\epsilon, \boldsymbol{p}) \ge \frac{(\boldsymbol{V}_\epsilon \boldsymbol{p})_k}{p_k}$, $1 \le k \le K$, $\epsilon > 0$, for any $\boldsymbol{p} \in \mathcal{P}_{++}$. Thus, it follows also $f(\boldsymbol{V}_\epsilon, \boldsymbol{p})\boldsymbol{p} \ge \boldsymbol{V}_\epsilon \boldsymbol{p}$, $\epsilon > 0$, $\boldsymbol{p} \in \mathcal{P}_{++}$. Given any $\boldsymbol{p} \in \mathcal{P}$, let $\boldsymbol{l}_\epsilon = \boldsymbol{l}_\epsilon(\boldsymbol{p})$ be the left PF eigenvector of $\boldsymbol{V}_\epsilon$ scaled to satisfy $\boldsymbol{l}'_\epsilon(\boldsymbol{p})\boldsymbol{p} = 1$. Thus, we have

$$f(\boldsymbol{V}_\epsilon, \boldsymbol{p}) = f(\boldsymbol{V}_\epsilon, \boldsymbol{p})\boldsymbol{l}'_\epsilon \boldsymbol{p} \ge \boldsymbol{l}'_\epsilon \boldsymbol{V}_\epsilon \boldsymbol{p} = \rho(\boldsymbol{V}_\epsilon)\boldsymbol{l}'_\epsilon \boldsymbol{p} = \rho(\boldsymbol{V}_\epsilon), \quad \epsilon > 0, \quad \boldsymbol{p} \in \mathcal{P}_{++}. \tag{6.40}$$

Then, by the construction and nondecreasingness of the spectral radius as a function of matrix elements it follows from (6.40) that

$$\max_{1\le k\le K}\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}+\frac{(\epsilon\boldsymbol{1}\boldsymbol{1}'\boldsymbol{p})_k}{p_k}\right)\ge\rho(\boldsymbol{V}),\quad \epsilon>0,\quad \boldsymbol{p}\in\mathcal{P}_{++}. \tag{6.41}$$

Further, notice that $F(\max_{1\le k\le K}x_k)=\max_{1\le k\le K}F(x_k)$, $x_k\ge 0$, $1\le k\le K$, due to increasingness of F. Thus, applying the map F to both sides of (6.41) and taking infimum over $\epsilon>0$ and $\boldsymbol{p}\in\mathcal{P}_{++}$ of both sides of (6.41) yields $\inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\max_{1\le k\le K}F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right)\ge F(\rho(\boldsymbol{V}))$ and completes the proof. □

Lemma 15 specifies a lower bound on the maximally improved worst link QoS. This bound follows to be tight from Proposition 34, stated already in Section 6.1. At this point we can prove Proposition 34 (recall from Section 6.1 that the proposition can be seen as a kind of extension of the Collatz-Wieland characterization to the case of general nonnegative matrices [38]).

Proof (of Proposition 34) Let $\mathcal{S}(n)\subset\mathcal{K}$ denote the set of row (equivalently, column) indices corresponding to the n-th diagonal block $\boldsymbol{V}^{(n)}$, $1\le n\le N$, in the normal form of $\boldsymbol{V}$. Let $\boldsymbol{p}(\lambda)\in\mathbb{R}^K_{++}$ denote any power vector associated with $\lambda>0$, where $\lambda=\lambda(\epsilon)$ is further a function of $\epsilon>0$. The idea of the proof is the construction of a vector $\boldsymbol{p}(\lambda)$, which achieves $\max_{1\le k\le K}F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right)=F(\rho(\boldsymbol{V}))$. Together with Lemma 15 such equality yields the proof.

Define $\boldsymbol{p}(\lambda)=(p_1(\lambda),\ldots,p_K(\lambda))$, $\lambda>0$, precisely as $p_k(\lambda)=\lambda r_k^{(n)}$, $k\in\mathcal{S}(n)$, whenever block $\boldsymbol{V}^{(n)}$ is maximal ($\boldsymbol{r}^{(n)}$ is the right eigenvector of the block $\boldsymbol{V}^{(n)}$ and we have $\boldsymbol{r}^{(n)}>0$, $1\le n\le N$, due to irreducibility of the diagonal blocks). Then, from the construction of the normal form of $\boldsymbol{V}$, we can write for any maximal block $\boldsymbol{V}^{(n)}$

$$\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}=\frac{(\boldsymbol{V}^{(n)}\lambda\boldsymbol{r}^{(n)})_k}{\lambda r_k^{(n)}}+\frac{(\sum_{m=1}^{n-1}\boldsymbol{V}^{(n,m)}\boldsymbol{p}^{(m)}(\lambda))_k}{\lambda r_k^{(n)}},\quad \lambda>0,\quad k\in\mathcal{S}(n), \tag{6.42}$$

where for any $\boldsymbol{p}\in\mathbb{R}^K_{++}$ vector $\boldsymbol{p}^{(n)}$ is defined as a vector in $\mathbb{R}^{|\mathcal{S}(n)|}_{+}$ with components corresponding to p_k, $k\in\mathcal{S}(n)$ (in unchanged order). Define now a map $\boldsymbol{p}\mapsto\boldsymbol{t}^{(n)}(\boldsymbol{p})=\sum_{m=1}^{n-1}\boldsymbol{V}^{(n,m)}\boldsymbol{p}^{(m)}$, $\boldsymbol{p}\in\mathbb{R}^K_{++}$, $1\le n\le N$. Given any $\epsilon>0$, choose $\lambda=\lambda(\epsilon)$ such that $\frac{t_k^{(n)}(\boldsymbol{p}(\lambda))_k}{\lambda r_k^{(n)}}\le\epsilon$, $k\in\mathcal{S}(n)$, is satisfied and transform both sides of (6.42) by F. Then, due to maximality of block $\boldsymbol{V}^{(n)}$ and increasingness of F we yield

$$F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right)=F\left(\rho(\boldsymbol{V})+\frac{t_k^{(n)}(\boldsymbol{p}(\lambda))}{\lambda r_k^{(n)}}\right)\le F(\rho(\boldsymbol{V})+\epsilon),\quad \lambda(\epsilon)>0,\quad \epsilon>0,\quad k\in\mathcal{S}(n). \tag{6.43}$$

Consider now nonmaximal blocks. For any nonmaximal block $\boldsymbol{V}^{(n)}$ define the corresponding components of $\boldsymbol{p}(\lambda)$ as $p_k(\lambda)=\tilde{r}_k$, $\tilde{\boldsymbol{r}}\in\mathcal{R}$, $k\in\mathcal{S}(n)$. Then, from the eigenvalue problem for the normal form of $\boldsymbol{V}$ follows for any nonmaximal block $\boldsymbol{V}^{(n)}$ that

$$\rho(\boldsymbol{V})\boldsymbol{p}^{(n)}(\lambda)=\boldsymbol{V}^{(n)}\boldsymbol{p}^{(n)}(\lambda)+\boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda)),\quad \lambda>0. \tag{6.44}$$

Due to $\rho(\boldsymbol{V}^{(n)})<\rho(\boldsymbol{V})$ this gives $\boldsymbol{p}^{(n)}(\lambda)=(\rho(\boldsymbol{V})\boldsymbol{I}-\boldsymbol{V}^{(n)})^{-1}\boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda))$, $\lambda>0$. Since further $\boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda))\ge 0$, $\lambda>0$, by definition, we have for any nonmaximal block $\boldsymbol{V}^{(n)}$ that $p_k^{(n)}(\lambda)=\tilde{r}_k>0$, $\lambda>0$, $k\in\mathcal{S}(n)$ (see [38]). Thus, from componentwise division of both sides of (6.44) by $p_k(\lambda)$ and transformation by increasing function F we yield

$$F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right)=F\left(\rho(\boldsymbol{V})-\frac{t_k^{(n)}(\boldsymbol{p}(\lambda))}{p_k(\lambda)}\right)\le F(\rho(\boldsymbol{V})),\quad \lambda>0,\quad k\in\mathcal{S}(n), \tag{6.45}$$

for any nonmaximal block $\boldsymbol{V}^{(n)}$. Summarizing now (6.43) and (6.45) we have

$$\max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right) \le F(\rho(\boldsymbol{V})+\epsilon), \quad \lambda(\epsilon)>0, \quad \epsilon>0.$$

Hence, we must have $\lim_{\epsilon\to 0}\max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda(\epsilon)))_k}{p_k(\lambda(\epsilon))}\right) \le F(\rho(\boldsymbol{V}))$, which together with Lemma 15 implies

$$\lim_{\epsilon\to 0}\max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda(\epsilon)))_k}{p_k(\lambda(\epsilon))}\right) = \inf_{\boldsymbol{p}\in\mathcal{P}_{++}}\max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V}))$$

due to assumption (2.28). This proves (6.1). Further, for maximal diagonal blocks $\boldsymbol{V}^{(n)}$ we have by the definition of $\boldsymbol{p}(\lambda)$ and irreducibility of the diagonal blocks in the normal form of $\boldsymbol{V}$ that $p_k(\lambda(0)) = \lim_{\epsilon\to 0} p_k(\lambda(\epsilon)) = \lim_{\epsilon\to 0}\lambda(\epsilon) r_k^{(n)} > 0$, $k\in\mathcal{S}(n)$. Similarly, for nonmaximal diagonal blocks $\boldsymbol{V}^{(n)}$ we have $p_k(0) = \tilde{r}_k$, $k\in\mathcal{S}(n)$. Thus, from the eigenvalue problem for the normal form of $\boldsymbol{V}$ follows $\boldsymbol{p}(\lambda)\in\mathcal{R}$, $\lambda>0$. Consequently, we have also $\boldsymbol{p}(\lambda(0))>0$ and we can then write $F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(0))_i}{p_i(0)}\right) = F(\rho(\boldsymbol{V}))$, $1\le i\le K$, whenever $\tilde{r}_k>0$, $k\in\mathcal{S}(n)$, for a nonmaximal block $\boldsymbol{V}^{(n)}$. The last condition is implied by $\tilde{\boldsymbol{r}}\in\mathcal{R}$, $\tilde{\boldsymbol{r}}>0$ (see Proposition 4), which completes the proof. □

As a consequence of inequality (6.38) and Proposition 34, we have the following result.

Corollary 11 *For any interference matrix $\boldsymbol{V}$, we have*

$$\sup_{\boldsymbol{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \le F(\rho(\boldsymbol{V})). \tag{6.46}$$

To describe the network classes with zero and nonzero fairness gap it remains now to characterize the case in which the bound $F(\rho(\boldsymbol{V}))$ is achieved, respectively not achieved, in (6.46). Strict inequality in (6.46) can be shown to hold for the following network class.

Lemma 16 *Let $\{\boldsymbol{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively. If there exists some $n\in\mathcal{I}$ such that $n\notin\mathcal{M}$, then we have*

$$\sup_{\boldsymbol{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) < F(\rho(\boldsymbol{V})). \tag{6.47}$$

Proof Let $\boldsymbol{V}^{(n)}$ be such that $n\in\mathcal{I}$ and $n\notin\mathcal{M}$ (isolated nonmaximal block). Then, from the construction of the normal form of $\boldsymbol{V}$ follows $\frac{(\boldsymbol{V}^{(n)}\boldsymbol{p})_k}{p_k} = \frac{(\boldsymbol{V}^{(n)}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}$, $\boldsymbol{p}\in\mathbb{R}^K_{++}$, $k\in\mathcal{S}(n)$, where $\mathcal{S}(n)$ and $\boldsymbol{p}^{(n)}$ are defined as in the proof of Proposition 34. Since any diagonal block is, by definition, irreducible, it follows from the Collatz-Wielandt characterization and increasingness of F that [38], [126]

$$F\left(\sup_{\boldsymbol{p}\in\mathbb{R}^K_{++}}\min_{k\in\mathcal{S}(n)}\frac{(\boldsymbol{V}^{(n)}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}\right) = F(\rho(\boldsymbol{V}^{(n)})) < F(\rho(\boldsymbol{V})). \tag{6.48}$$

Obviously, we have $\min_{1\le k\le K}\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k} \le \min_{k\in\mathcal{S}(n)}\frac{(\boldsymbol{V}^{(n)}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}$, $\boldsymbol{p}\in\mathbb{R}_{++}^K$, $1\le n\le N$, so that using five times the increasingness of F we can write

$$\sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K}\min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K} F\left(\min_{1\le k\le K}\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F\left(\sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K}\min_{1\le k\le K}\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \le$$
$$F\left(\sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K}\min_{k\in\mathcal{S}(n)}\frac{(\boldsymbol{V}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}\right) = \sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K} F\left(\min_{k\in\mathcal{S}(n)}\frac{(\boldsymbol{V}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}\right) = \sup_{\boldsymbol{p}\in\mathbb{R}_{++}^K}\min_{k\in\mathcal{S}(n)} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}^{(n)})_k}{p_k^{(n)}}\right). \tag{6.49}$$

The inequalities (6.48) and (6.49) and the assumption (2.28) give together $\sup_{\boldsymbol{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) < F(\rho(\boldsymbol{V}))$, which completes the proof. □

We can interpret the condition in the Lemma 16 as the existence of some entirely coupled subnetwork which is interference-isolated and its interference matrix, say $\boldsymbol{V}^{(n)}$, satisfies $\rho(\boldsymbol{V}^{(n)}) < \rho(\boldsymbol{V})$. By Lemma 16, networks having such property can not pertain to the class satisfying Condition 11 and hence can not pertain to its subclass satisfying Condition 12 as well.

An immediate consequence of Lemma 16 is the following.

Corollary 12 *Let $\{\boldsymbol{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively. If for any increasing performance function F we have*

$$\sup_{\boldsymbol{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})), \tag{6.50}$$

then $\mathcal{I}\subseteq\mathcal{M}$.

By Proposition 34 and inequality (6.38) one can see that Corollary 12 formulates a necessary condition for the inclusion of a network in the class satisfying Condition 11, and hence also in its subclass satisfying Condition 12. This condition is precisely, that the interference matrix, say $\boldsymbol{V}^{(n)}$, of any entirely coupled and interference-isolated n-th subnetwork satisfies $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$. The following lemma shows even more.

Lemma 17 *Let $\{\boldsymbol{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively. If $\mathcal{I}\subseteq\mathcal{M}$, then equality (6.50) is satisfied.*

Proof Let $\mathcal{S}(n)$, $1\le n\le N$, and $\boldsymbol{p}^{(n)}$ (for any $\boldsymbol{p}\in\mathbb{R}_{++}^K$) be defined as in the proof of Proposition 34. Let $\boldsymbol{p}(\lambda)\in\mathbb{R}_{++}^K$ denote any power vector associated with $\lambda>0$, where $\lambda=\lambda(\epsilon)$ is further a function of $\epsilon>0$. As in the proof of Proposition 34, the idea of the proof is the construction of a vector $\boldsymbol{p}(\lambda)$ which achieves equality (6.50).

Consider first $\boldsymbol{V}^{(n)}$ be such that $n\in\mathcal{I}$. Define $\boldsymbol{p}(\lambda)=(p_1(\lambda),\ldots,p_K(\lambda))$, $\lambda>0$, as $p_k(\lambda)=r_k^{(n)}$, $k\in\mathcal{S}(n)$, whenever $n\in\mathcal{I}$ ($\boldsymbol{r}^{(n)}$ is the right eigenvector of the diagonal block $\boldsymbol{V}^{(n)}$ and we have $\boldsymbol{r}^{(n)}>0$, $1\le n\le N$, due to irreducibility of diagonal blocks). From the construction of the normal form of $\boldsymbol{V}$ and increasingness of F follows then

$$F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right) = F\left(\frac{(\boldsymbol{V}^{(n)}\boldsymbol{r}^{(n)})_k}{r_k^{(n)}}\right) = F(\rho(\boldsymbol{V})),\quad \lambda>0,\quad k\in\mathcal{S}(n),\quad n\in\mathcal{I}, \tag{6.51}$$

since we have $n \in \mathcal{M}$ by assumption.

Consider now blocks $\boldsymbol{V}^{(n)}$ such that $n \notin \mathcal{I}$ and additionally $n \in \mathcal{M}$. Define the corresponding components of $\boldsymbol{p}(\lambda)$ as $p_k(\lambda) = r_k^{(n)}$, $k \in \mathcal{S}(n)$, whenever $n \notin \mathcal{I}$, $n \in \mathcal{M}$, and define also a function $\boldsymbol{p} \mapsto \boldsymbol{t}^{(n)}(\boldsymbol{p})$, $\boldsymbol{p} \in \mathbb{R}_{++}^K$, $1 \le n \le N$, as in the proof of Proposition 34. Then, by the construction of the normal form of $\boldsymbol{V}$ and increasingness of F we yield

$$\begin{aligned} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right) &= F\left(\frac{(\boldsymbol{V}^{(n)}\boldsymbol{r}^{(n)})_k}{r_k^{(n)}} + \frac{t_k^{(n)}}{p_k(\lambda)}\right) \\ &= F\left(\rho(\boldsymbol{V}) + \frac{t_k^{(n)}}{p_k(\lambda)}\right) \ge F(\rho(\boldsymbol{V})), \quad \lambda > 0, \quad k \in \mathcal{S}(n), \quad n \notin \mathcal{I}, \quad n \in \mathcal{M}. \end{aligned} \tag{6.52}$$

For the remaining case $n \notin \mathcal{I}$ and $n \notin \mathcal{M}$ define the corresponding components of $\boldsymbol{p}(\lambda)$ as $p_k(\lambda) = \lambda \tilde{r}_k$, $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $k \in \mathcal{S}(n)$. Then, from the eigenvalue problem for the normal form of $\boldsymbol{V}$ follows

$$\rho(\boldsymbol{V})\boldsymbol{p}^{(n)}(\lambda) = \boldsymbol{V}^{(n)}\boldsymbol{p}^{(n)}(\lambda) + \boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda)), \quad \lambda > 0, \quad n \notin \mathcal{I}, \quad n \notin \mathcal{M}. \tag{6.53}$$

This yields further $\boldsymbol{p}^{(n)}(\lambda) = (\rho(\boldsymbol{V})\boldsymbol{I} - \boldsymbol{V}^{(n)})^{-1}\boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda))$, which implies with $n \notin \mathcal{M}$ (that is, $\rho(\boldsymbol{V}^{(n)}) < \rho(\boldsymbol{V})$) and $\boldsymbol{t}^{(n)}(\boldsymbol{p}(\lambda)) \ge 0$, $\lambda > 0$, that $p_k^{(n)}(\lambda) = \lambda \tilde{r}_k > 0$, $\lambda > 0$, $\tilde{\boldsymbol{r}} \in \mathcal{R}$, $k \in \mathcal{S}(n)$ [38]. Given any $\epsilon > 0$, choose now $\lambda(\epsilon)$ such that $\frac{t_k^{(n)}(\boldsymbol{p}(\lambda))_k}{\lambda \tilde{r}_k} \le \epsilon$, $k \in \mathcal{S}(n)$, $n \notin \mathcal{I}$, $n \notin \mathcal{M}$. Then, the componentwise division of both sides of (6.53) by $p_k(\lambda)$ and application of the increasing function F to both sides yields

$$\begin{aligned} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda))_k}{p_k(\lambda)}\right) &= F\left(\rho(\boldsymbol{V}) - \frac{t_k^{(n)}(\boldsymbol{p}(\lambda))}{\lambda \tilde{r}_k}\right) \\ &\ge F(\rho(\boldsymbol{V}) - \epsilon), \quad \lambda(\epsilon) > 0, \quad \epsilon > 0, \quad k \in \mathcal{S}(n), \quad n \notin \mathcal{I}, \quad n \notin \mathcal{M}. \end{aligned} \tag{6.54}$$

Summarizing (6.51), (6.52) and (6.54) we have $\min_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda(\epsilon)))_k}{p_k(\lambda(\epsilon))}\right) \ge F(\rho(\boldsymbol{V}) - \epsilon)$, $\epsilon > 0$. Thus, we must have $\lim_{\epsilon \to 0} \min_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda(\epsilon)))_k}{p_k(\lambda(\epsilon))}\right) \ge F(\rho(\boldsymbol{V}))$, which together with Corollary 11 and assumption (2.28) implies

$$\lim_{\epsilon \to 0} \min_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p}(\lambda(\epsilon)))_k}{p_k(\lambda(\epsilon))}\right) = \sup_{\boldsymbol{p} \in \mathcal{P}_{++}} \min_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V}))$$

This completes the proof. □

As a consequence of Corollary 12 and Lemma 17 we obtain the following necessary and sufficient condition for a network to satisfy Condition 11.

Proposition 41 *Let $\{\boldsymbol{V}^{(n)}\}_{n \in \mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m \in \mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively. Then, the equality*

$$\sup_{\boldsymbol{p} \in \mathcal{P}_{++}} \min_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = F(\rho(\boldsymbol{V})) = \inf_{\boldsymbol{p} \in \mathcal{P}_{++}} \max_{1 \le k \le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \tag{6.55}$$

is satisfied if and only if $\mathcal{I} \subseteq \mathcal{M}$.

Thus, for a network with interference matrix $\boldsymbol{V}$, the condition that any isolated diagonal block in the canonical form of $\boldsymbol{V}$ is maximal is a necessary and sufficient condition for the network to satisfy Condition 11. Automatically we have also that the existence of an isolated diagonal block which is not maximal in the canonical form of the interference matrix is a necessary and sufficient condition for the corresponding network to satisfy Condition 13. Thus, the complete interpretation of Proposition 41 is as follows.

Observation 8 *For a network with interference matrix $\boldsymbol{V}$, the following is true.*
i.) The value of maximally degraded best link performance (performance under max-min fairness) coincides with the value of maximally improved worst link performance (performance under min-max fairness) exactly in the case when the interference matrix $\boldsymbol{V}^{(n)}$ of any entirely coupled and interference-isolated subnetwork $n \in \mathcal{I}$ satisfies $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$.
ii.) The value of maximally degraded best link performance is smaller (that is, better) than the value of maximally improved worst link performance exactly in the case when there exists some entirely coupled and interference-isolated subnetwork with interference matrix $\boldsymbol{V}^{(n)}$ satisfying $\rho(\boldsymbol{V}^{(n)}) < \rho(\boldsymbol{V})$.

6.5.3 The case of common optimizers

It remains to answer the question on the network class satisfying Condition 12. It is precisely the question of description of the subclass of networks with zero fairness gap for which some allocation achieves both min-max fairness in the sense of (2.30) and max-min fairness in the sense of (6.2). The following description of such class is possible with Proposition 34 and Lemma 17.

Proposition 42 *Let $\{\boldsymbol{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\boldsymbol{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\boldsymbol{V}$, respectively.*
i.) There exists some vector $\tilde{\boldsymbol{p}} > 0$ satisfying

$$\tilde{\boldsymbol{p}} = \arg \inf_{\boldsymbol{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) = \arg \sup_{\boldsymbol{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\boldsymbol{p})_k}{p_k}\right) \tag{6.56}$$

if and only if $\mathcal{I} = \mathcal{M}$.
ii.) Moreover, $\tilde{\boldsymbol{p}}$ satisfies (6.56) if and only if $\tilde{\boldsymbol{p}} \in \mathcal{R}\cap\mathcal{P}_{++}$.

Proof We prove the statements i.) and ii.) of the Proposition in the circular manner. For the proof of the if part of i.), assume that some $\tilde{\boldsymbol{p}} \in \mathcal{P}_{++}$ satisfying (6.56) exists. By (6.56) it is implied that $\max_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{p}})_k}{\tilde{p}_k}\right) = \min_{1\le k\le K} F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{p}})_k}{\tilde{p}_k}\right)$, so that by (6.1) (or (6.50)) we have also $F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{p}})_k}{\tilde{p}_k}\right) = F(\rho(\boldsymbol{V}))$, $1 \le k \le K$, and consequently $\tilde{\boldsymbol{p}} \in \mathcal{R}\cap\mathcal{P}_{++}$. By Proposition 41 and (6.56) the existence of such $\tilde{\boldsymbol{p}}$ is already known to imply $\mathcal{I} \subseteq \mathcal{M}$. Thus, assume by contradiction $\mathcal{I} \subset \mathcal{M}$, that is, there exists some nonisolated block $\boldsymbol{V}^{(n)}$ of $\boldsymbol{V}$ such that $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$. For such block follows from the eigenvalue problem for the normal form of $\boldsymbol{V}$ that

$$\rho(\boldsymbol{V})\tilde{\boldsymbol{p}}^{(n)} = \boldsymbol{V}^{(n)}\tilde{\boldsymbol{p}}^{(n)} + \sum_{m=1}^{n-1} \boldsymbol{V}^{(n,m)}\tilde{\boldsymbol{p}}^{(m)}, \quad n \in \mathcal{M}, \quad n \notin \mathcal{I}, \tag{6.57}$$

where $\boldsymbol{p}^{(n)}$ is defined, for any $\boldsymbol{p} \in \mathbb{R}^K_{++}$, as in the proof of Proposition 34. Due to $n \notin \mathcal{I}$, at least one of the matrices $\boldsymbol{V}^{(n,m)}$, $1 \le m \le n-1$ is nonzero. Thus, $\tilde{\boldsymbol{p}} > 0$ for (6.57) that there exists some $k \in \mathcal{S}(n)$ such that $\rho(\boldsymbol{V})\tilde{\boldsymbol{p}}^{(n)}_k > (\boldsymbol{V}^{(n)}\tilde{\boldsymbol{p}}^{(n)})_k$, which implies further $\rho(\boldsymbol{V}^{(n)}) < \rho(\boldsymbol{V})$. This is a contradiction and the if part of i.) is proven. The next step to prove is that $\mathcal{I} = \mathcal{M}$ implies that there exists some $\tilde{\boldsymbol{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$. But this follows already from Proposition 37. The last step to show is the only if part of ii.), precisely, that any $\tilde{\boldsymbol{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$ satisfies (6.56). But with $\tilde{\boldsymbol{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$ we have (as above) $\max_{1 \leq k \leq K} F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{p}})_k}{\tilde{p}_k}\right) = \min_{1 \leq k \leq K} F\left(\frac{(\boldsymbol{V}\tilde{\boldsymbol{p}})_k}{\tilde{p}_k}\right)$, which implies with (6.1) (or (6.50)) that (6.56) holds whenever $\tilde{\boldsymbol{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$. With this, the circle of three if relations is completed and i.), ii.) are proven. □

Thus, the class of networks for which min-max fairness and max-min fairness can be concurrently achieved by some allocation consists of networks for which the isolated diagonal blocks coincide with maximal diagonal blocks in the canonical form of their interference matrices. Consequently, whenever some maximal diagonal block in the canonical form of the interference matrix is not isolated, then there exists no allocation which is both min-max fair and max-min fair for the corresponding network. Similarly, the min-max fair and max-min fair allocation does not exists in the case when some isolated diagonal block is not maximal in the canonical form of the corresponding interference matrix. In both cases, the network satisfies Condition 11, but do not satisfy Condition 12. This can be interpreted as follows.

Observation 9 *For a network with interference matrix $\boldsymbol{V}$, the following is true.*
i.) An allocation which achieves both max-min fairness and min-max fairness exists, when any entirely coupled subnetwork with interference matrix $\boldsymbol{V}^{(n)}$ satisfies $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$ exactly in the case when it is interference-isolated.
ii.) Whenever there exists some not interference-isolated entirely coupled subnetwork with interference matrix $\boldsymbol{V}^{(n)}$ satisfying $\rho(\boldsymbol{V}^{(n)}) = \rho(\boldsymbol{V})$, then an allocation achieving both max-min fairness and min-max fairness does not exist.
iii.) Whenever there exists some interference-isolated entirely coupled subnetwork with interference matrix $\boldsymbol{V}^{(n)}$ satisfying $\rho(\boldsymbol{V}^{(n)}) < \rho(\boldsymbol{V})$, then an allocation achieving both max-min fairness and min-max fairness does not exist as well.

Finally, notice the subtlety that there may exist some vector $\boldsymbol{p} \in \mathcal{P}$ which maximally improves the worst link QoS according to (2.30) and at the same time maximally degrades the best link QoS in the sense of (6.2), without being both a min-max fair and max-min fair allocation. From the discussion in Section 6.1 we know that this is precisely the case when both (2.30) and (6.2) are solved by a non-valid allocation, that is, when $\boldsymbol{p}$ has some zero components. Clearly, the class of networks for which such allocation exists is included in the class satisfying Condition 11, and includes the class satisfying Condition 12 itself.

7 Conclusions and further work

7.1 Summary and conclusions

The goal of this work was the development of an analytic framework and design of specialized algorithms for optimization of wireless single-hop networks with interference. We focused on the optimization of weighted aggregated performance and achieving (relative) min-max fairness as the optimization approaches of greatest interest. The first of the approaches corresponds to the optimization of a weighted sum of QoS/ performance functions, each expressing the perceived performance of the corresponding link (single-hop) or route (multi-hop) [29], [28], [22], [23], [131], [24]. Such approach proved to be suitable under nonstrict service requirements per link/ route or under so-called elastic traffic [34], [16]. The latter approach of achieving min-max fairness corresponds to improving the worst link/ route QoS value as much as possible and proved to be suitable under predefined strict link/ route performance requirements [4], [9], [10], [11], [6], [132], [57], [59].

In our view, the key feature making the developed framework useful is its general applicability. First, it is general in the sense that it is applicable to arbitrary networks with interference as long as the interference topology can be expressed by a nonnegative interference matrix. This is usually the case under interfering links, flat fading channels and linear link receivers [12], [13]. Particular network instances covered by our framework are networks with multiple antennas at link transmitter or link receiver or CDMA networks [89], [7], [40], [39]. The framework is also applicable to networks with a set of separated links sharing the same space/ code/ bandwidth resource, as is the case under single-hop communication within a typical multi-hop ad hoc network [133]. Second, the generality of the framework consists in its applicability to arbitrary monotonous link performance functions, understood as functions of the corresponding link SIR. Due to such feature, the network performance optimization with respect to link capacities, link symbol error rates or link MMSE, which are link performance functions of typical interest, falls automatically into the proposed framework as a special case.

The network model in Chapter 2 exhibits the SIR function and the description of interference topology of the network by a nonnegative interference matrix as the key ingredients of the developed framework. The introductory results in Chapter 2 show that the log-convexity property of the SIR function as a function of the corresponding link performance value plays a crucial role in the optimization of aggregated performance. Such property ensures global solvability of the problem by means of locally convergent methods and, consequently, makes it well-tractable by adaptive online optimization routines in real-world networks. Further, the characterization of the interference topology of the network by a nonnegative interference matrix makes the Perron-Frobenius Theory applicable for our purposes [38], [126].

The application of some basic elements of Perron-Frobenius Theory gives rise to the design of algorithms in Chapter 3. The two algorithms, applicable to networks with individual power constraints and constrained sum-power budget, respectively, rely strongly on the structure of the boundary of the performance region (see also [22]). In our view, they efficiently trade-off computational complexity with convergence rate and thus are an attractive alternative (realizable rather in centralized manner), to algorithms based on conventional iterations, such as gradient iteration [49], [79]. Clearly, under log-convexity of the QoS-SIR dependence both algorithms are globally convergent.

Specific features of the interference matrix and the SIR function are also the key ingredients of the algorithms and feedback schemes proposed in Chapter 4. The basic proposed feedback scheme, the adjoint network feedback scheme, can be regarded as a decentralized interference estimation scheme and makes use of some kind of symmetry, or reversibility, of the SIR function and the interference matrix [24]. It allows for decentralized realization of aggregated performance optimization by means of the conventional projected gradient method [49]. The two specialized algorithms from Chapter 5 are designed to utilize the advantage of ensured decentralized computation provided by the adjoint network feedback scheme. The first of them makes use of a manipulation of the Lagrangean function (see also [71], [73], [74], [72]), while the second utilizes a specific separation/ splitting of the optimization variables corresponding to power and interference at each link. In this way the proposed algorithms obtain advantages in terms of computational complexity and convergence behavior, respectively, compared to conventional iterations. Again, under log-convexity of the QoS-SIR relation, the algorithms from Chapter 4 are globally convergent.

The case of a MIMO network is specific in the sense that a meaningful notion of link SIR function is nonexistent and thus, the link QoS function can not be regarded as a function of link SIR [89]. In Chapter 5 we considered a particular instance of weighted aggregated performance optimization in the MIMO multiple access channel, in the form of the widely-studied problem of computation of the stability-optimal policy [105], [106], [103], [111], [112]. In the context of such problem, particularly interesting is the issue of optimality of the order of SIC. Such question suggests attacking the considered problem within a geometric view of the capacity region of the MIMO multiple access channel and its specific subregions. Such view nicely illustrates the cross-layer interplay between the link layer issues, represented by buffer occupancies, and physical layer issues, in the form of a vector of physical link rates [25]. Relying on the geometric view, we provided, in particular, necessary and sufficient conditions for optimality of the SIC order and link subset transmission. The proposed algorithm computing the stability-optimal policy is based on the approach of splitting of the original multi-link problem into a tuple of coupled single-link problems (see also [123]).

The results concerning the problem of achieving min-max fairness in Chapter 6 make strong use of the Perron-Frobenius Theory, in particular from Collatz-Wielandt formulae, applied to the interference matrix [126]. The resulting characterization of the min-max fair power allocation (and a certain weight vector) as a saddle point of the weighted aggregated performance, understood as a function of link weights and link powers, is an analytic interpretation of the incompatibility of min-max fairness and optimality of aggregated performance [56]. Further, the analysis of combinatorial and spectral properties of the so-called normal form of the interference matrix provides insights in the mechanism of min-max fairness itself. It shows that the maximal improvement of the worst link performance is in general not equivalent to the maximal degradation of the best link performance. However, a class of networks ensuring such equivalence exists and provides advantages in terms of online computation of min-max fair power allocation in real-world networks.

7.2 Some notes on the extension to orthogonal networks

The parallel framework for optimization of networks with orthogonal links (in short, orthogonal networks) suggests itself as an extension of the framework for networks with interference provided in this work. In orthogonal networks, the analogous problems of aggregated performance optimization and achieving min-max fairness become of combined combinatorial and real-valued nature. The combinatorial part corresponds to the assignment of orthogonal (medium) resources, e.g. OFDMA (*Orthogonal Frequency Division Multiple Access*) carriers or TDMA (*Time Division Multiple Access*) slots, among the links [134], [135], [136], [137]. The allocation of link transmit power among the set of resources assigned to each link represents the remaining real-valued part of the problem. Thus, in broad terms, the problem of coupling of link performance functions by interference in networks with interference is in orthogonal networks replaced by the problem of suitable partitioning of the set of orthogonal resources among the links.

The arising dominating intricacy of performance optimization in orthogonal networks is the computational intractability (NP-hardness/ NP-completeness) of the combinatorial problem part in most problems of interest [138]. This is unfortunately the case in aggregated performance optimization under predefined number of assigned resources per link or under the requirement of assigning some (at least one) resource per link. The problem of achieving min-max fairness with respect to link performance aggregated over assigned resources (also under the constraint of predefined number of resources per link) is intractable as well [138].

Due to occurring intractability, there is great interest in efficient polynomial-time heuristics solving the resource assignment problem part in orthogonal networks. Some particular heuristics for the assignment of OFDMA carriers can be found in [134], [135] and references therein. One can also resort to and adapt well-developed heuristic concepts from operations research and economic science, see e.g. [139] and references therein.

When the research focus is on the design of heuristics to NP-hard/ NP-complete problems in orthogonal networks, then the structure of the exact problem solution and the corresponding optimality relations usually can not be recognized or deduced. For this reason, the works [140], [141] address the problem of characterization of the exact solution by novel techniques relying on the concept of so-called blocking and antiblocking systems [142], [143], [144], [145], [146], [147]. We provide there numerous bounds and equality characterizations of the optimum aggregated performance and performance achieved under min-max fairness. The bounds are dependent on certain subnetwork classes, the blocking and antiblocking clutters and polyhedra, so that some key structural features of the optimum resource assignments become recognizable.

The combinatorial nature of the resource assignment part of the performance optimization problems in orthogonal networks suggests its representation as a graph problem. Such view allows for the application of the well-developed theory of graphs and graph-related algorithmic concepts to the resource assignment problem [129]. In particular, the application of the concept of the (Lovasz) Theta-function and the extension of the techniques used in [148], [149], [150] lead to a novel characterization of the link performance achieved under min-max fairness [151]. The link performance under min-max fairness is shown in [151] to be equivalent to the scaled Theta-function value of a certain graph induced by the optimum resource assignment. The scaling constant is a function of the optimum power allocation among all links and resources. Remarkably, the characterization from

[151] applies also to some class of networks with interference when the performance optimization is understood as joint optimization of power allocation and nonorthogonal resources, e.g. in the form of link power allocation and spreading sequence design in CDMA networks. In such case, the graph occurring in the characterization from [151] is induced by the interference topology of the network. Such universality of the characterization from [151] pushes us to a very abstract conclusion that *the interference coupling of links in a network with interference and sharing of resources in an orthogonal network represent the same mechanism in the view of min-max fairness.*

The characterization through the Theta-function from [151] and the application of Szemeredi's Regularity Lemma, a celebrated result in modern graph theory [129], give rise to some scaling laws of the link performance under min-max fairness in large orthogonal networks [152]. Remarkably, the scaling laws from [152] are applicable regardless of the correlation properties of channels across links and resources (and hold also for the class of networks with interference which the Theta-function characterization from [151] applies to).

A
Appendices

A.1 Irreducibility and the normal form of a nonnegative matrix

Definition 10 ([38]) *Matrix* $\boldsymbol{X} \in \mathbb{R}_{+}^{K\times K}$ *is said to be reducible, if there exists a permutation matrix* $\boldsymbol{P} \in \{0,1\}^{K\times K}$ *such that*

$$\boldsymbol{P}'\boldsymbol{X}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{C} \end{pmatrix},$$

with $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *as square matrices. A matrix which is not reducible is said to be irreducible.*

With any nonnegative matrix one can associate a specific directed graph [129].

Definition 11 ([38]) *For* $\boldsymbol{X} = (x_{ij}) \in \mathbb{R}_{+}^{K\times K}$ *let the directed graph* $G(\boldsymbol{X})$ *with vertex set* $\mathcal{K}$ *and edge set* $\mathcal{E}$ *be such that*

$$(i,j) \in \mathcal{E} \quad \textit{if and only if} \quad x_{ij} \neq 0, \quad i,j \in \mathcal{K}.$$

Usually, the following necessary and sufficient characterization of irreducibility in terms of graphs is more useful than Definition 10.

Lemma 18 ([38]) *Matrix* $\boldsymbol{X} \in \mathbb{R}_{+}^{K\times K}$ *is irreducible if and only if any pair of vertices in* $G(\boldsymbol{X})$ *is joined by a directed path.*

Corollary 13 ([130]) *If matrix* $\boldsymbol{X} = (x_{ij}) \in \mathbb{R}_{+}^{K\times K}$ *satisfies*

$$x_{ij} > 0, \quad 1 \leq i,j \leq K, \quad i \neq j, \tag{A.1}$$

then it is irreducible.

It is known that the spectrum and the eigenmanifolds of a matrix are invariant with respect to permutation of its rows and columns [130]. The permutation of rows and columns affects merely the order of dimensions of the eigenmanifold. By such permutation, any nonnegative matrix $\boldsymbol{V}$ can be represented in its *canonical form* (or *normal form*, or *Frobenius normal form*).

Definition 12 ([130]) *For any matrix* $\boldsymbol{X} \in \mathbb{R}_+^{K\times K}$ *there exist some permutation matrix* $\boldsymbol{P} \in \{0,1\}^{K\times K}$*, such that*

$$\boldsymbol{P}'\boldsymbol{X}\boldsymbol{P} = \begin{pmatrix} \boldsymbol{X}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \boldsymbol{X}^{(2,1)} & \boldsymbol{X}^{(2)} & \ddots & \vdots \\ \vdots & \vdots & \cdots & \mathbf{0} \\ \boldsymbol{X}^{(N,1)} & \boldsymbol{X}^{(N,2)} & \cdots & \boldsymbol{X}^{(N)} \end{pmatrix}, \tag{A.2}$$

where $\boldsymbol{X}^{(n)}$*,* $1 \le n \le N$*,* $N \le K$*, are square irreducible matrices and are referred to as diagonal blocks. Matrix (A.2) is referred to as the canonical form of* $\boldsymbol{X}$*.*

For the normal form we have

$$\rho(\boldsymbol{X}) = \max_{1\le m\le N} \rho(\boldsymbol{X}^{(m)}). \tag{A.3}$$

A diagonal block $\boldsymbol{X}^{(n)}$, $1 \le n \le N$, such that $\rho(\boldsymbol{X}^{(n)}) = \rho(\boldsymbol{X})$ is referred to as *maximal*. Further, we refer to a diagonal block $\boldsymbol{X}^{(n)}$, $1 \le n \le N$, as an *isolated* one if $\boldsymbol{X}^{(n,m)} = \mathbf{0}$, $1 \le m < n$. If $\boldsymbol{X}^{(n,m)} = \mathbf{0}$, $1 \le m < n$, $1 \le n \le N$, the matrix is referred to as *block-irreducible*. By Definition 12 we have also that irreducibility is a special case of block-irreducibility with $N = 1$.

A.1.1 Interference interpretation of the canonical form

Let $\boldsymbol{X} = \boldsymbol{V}$ be an interference matrix. Then, the diagonal blocks (by definition irreducible) in the canonical form represent the interference matrices of entirely coupled link subsets, or subnetworks. Clearly, N is then the maximal number of entirely coupled subnetworks into which the network can be partitioned. The nondiagonal block $\boldsymbol{V}^{(n,m)}$, $1 \le m < n \le N$, contains interference coefficients expressing the interference from links in the n-th subnetwork perceived by the links in the m-th subnetwork. The isolation of the diagonal block $\boldsymbol{V}^{(m)}$ means that the m-th subnetwork does not perceive interference from other subnetworks and hence can be referred to as *interference-isolated*. Under block-irreducibility of $\boldsymbol{V}$ the network consists solely of interference-isolated subnetworks.

A.2 Particular results of Perron-Frobenius Theory

Theorem 1 ([38]) *For any* $\boldsymbol{X} \in \mathbb{R}_+^{K\times K}$ *and* $a > 0$*, the solution* $\boldsymbol{y} \ge 0, \boldsymbol{y} \ne \mathbf{0}$ *of the equation*

$$(a\boldsymbol{I} - \boldsymbol{X})\boldsymbol{y} = \boldsymbol{b}$$

exists for any $\boldsymbol{b} \in \mathbb{R}_{++}^K$*, if and only if* $a > \rho(\boldsymbol{X})$*. In such case the solution* $\boldsymbol{p} = (a\boldsymbol{I} - \boldsymbol{X})^{-1}\boldsymbol{b}$ *is unique and such that* $\boldsymbol{p} \in \mathbb{R}_{++}^K$*.*

A.3 Some general notions in optimization theory

A.3.1 Basics of Lagrangean optimization theory

Let an optimization problem

$$\inf_{\boldsymbol{x}} F(\boldsymbol{x}) \quad \text{subject to} \quad \begin{cases} f_k(\boldsymbol{x}) \le 0 & k \in \mathcal{L} \\ g_k(\boldsymbol{x}) = 0 & k \in \mathcal{J}, \end{cases} \tag{A.4}$$

with arbitrary maps $\boldsymbol{x} \mapsto F(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^K$ and $\boldsymbol{x} \mapsto f_k(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X}$, $k \in \mathcal{L}$, be given, where $\mathcal{L} = \{1, \ldots, L\}$, $\mathcal{J} = \{1, \ldots, J\}$.

Definition 13 *Let the infimum in problem (A.4) be bounded. Then, the problem is said to be globally solvable if any local optimizer of (A.4) is also a global optimizer. If there exists some local optimizer of (A.4) which is not a global optimizer, then problem (A.4) is referred to as locally solvable.*

Denoting $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_L)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)$, the Lagrangean function of problem (A.4) can be written as [48], [49]

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = F(\boldsymbol{x}) + \sum_{k \in \mathcal{L}} \lambda_k f_k(\boldsymbol{x}) + \sum_{k \in \mathcal{J}} \mu_k g_k(\boldsymbol{x}), \quad (\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{X} \times \mathbb{R}^L \times \mathbb{R}^J.$$

The variables $\boldsymbol{\lambda} \in \mathbb{R}^L$, $\boldsymbol{\mu} \in \mathbb{R}^J$ are referred to as (vectors of) Lagrange multipliers or (Lagrange) dual variables.

Definition 14 ([48]) *Given problem (A.4) with once differentiable maps F, f_k, $k \in \mathcal{L}$ and g_k, $k \in \mathcal{J}$, the set of inequalities*

$$\begin{aligned} \nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 0 \\ f_k(\boldsymbol{x}) &\leq 0, \quad k \in \mathcal{L} \\ g_k(\boldsymbol{x}) &= 0, \quad k \in \mathcal{J} \\ \lambda_k f_k(\boldsymbol{x}) &= 0, \quad k \in \mathcal{L} \\ \boldsymbol{\lambda} &\geq 0, \end{aligned} \tag{A.5}$$

with $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{X} \times \mathbb{R}^L \times \mathbb{R}^J$, is referred to as Kuhn-Tucker conditions, or Karush-Kuhn-Tucker (KKT) conditions.

For the formulation of optimality conditions for $\boldsymbol{x} \in \mathcal{X}$ relying on the Lagrangean function, it is of crucial importance that the maps f_k, $k \in \mathcal{L}$, and g_k, $k \in \mathcal{J}$, satisfy a constraint qualification (condition) at $\boldsymbol{x}$. For the general problem formulation (A.4) several nonequivalent versions of constraint qualification are known, e.g. Kuhn-Tucker constraint qualification, weak (or modified) Arrow-Hurwicz-Urawa constraint qualification, or the best-known Slater's condition, which is, however, applicable only if f_k, $k \in \mathcal{L}$, are convex and $\{\boldsymbol{x} \in \mathcal{X} : g_k(\boldsymbol{x}) = 0, k \in \mathcal{J}\}$ is a convex set [48].

Definition 15 (Kuhn-Tucker constraint qualification [48]) *Let problem (A.4) with once differentiable maps f_k, $k \in \mathcal{L}$, and g_k, $k \in \mathcal{J}$, be given, and let $\mathcal{L}'(\boldsymbol{x}) = \{k \in \mathcal{L} : f_k(\boldsymbol{x}) = 0\}$, where $\boldsymbol{x} \in \mathcal{X}$ satisfies the constraints in (A.4). Then, the Kuhn-Tucker constraint qualification is said to be satisfied at $\boldsymbol{x}$ if given*

$$\nabla' f_k(\boldsymbol{x}) \boldsymbol{\lambda} \leq 0, \quad \nabla' g_k(\boldsymbol{x}) \boldsymbol{\lambda} = 0 \quad \textit{for some} \quad \boldsymbol{\lambda} \in \mathbb{R}^K,$$

there exists a map $t \mapsto h(t) \in \mathcal{X}$, $t \in [0, 1]$, once differentiable at $t = 0$, such that $h(t)$, $t \in [0, 1]$, satisfies the constraints in (A.4), $h(0) = \boldsymbol{x}$ and $h'(0) = a\boldsymbol{\lambda}$ for some $a > 0$.

There is a well-known sufficient condition for the constraint qualification for differentiable maps. In some works it is even declared as the definition of constraint qualification [85].

Lemma 19 ([49]) *Let $\boldsymbol{x} \in \mathcal{X}$ satisfy the constraints in (A.4), let $\mathcal{L}'(\boldsymbol{x}) = \{k \in \mathcal{L} : f_k(\boldsymbol{x}) = 0\}$, and let f_k, $k \in \mathcal{L}$, and g_k, $k \in \mathcal{J}$, be once differentiable. Then, constraint qualification is satisfied at $\boldsymbol{x}$ if $\nabla g_k(\boldsymbol{x})$, $k \in \mathcal{J}$, $\nabla f_k(\boldsymbol{x})$, $k \in \mathcal{L}'(\boldsymbol{x})$, are pairwise independent.*

Lemma 20 (Kuhn-Tucker necessary and sufficient optimality theorem [48]) *Let problem (A.4) be globally solvable and maps F, f_k, $k \in \mathcal{L}$, and g_k, $k \in \mathcal{J}$, be once differentiable. Then, if $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{X} \times \mathbb{R}^L \times \mathbb{R}^J$ satisfies the Kuhn-Tucker conditions (A.5), then $\boldsymbol{x}$ is a global optimizer of (A.4). If problem (A.4) is locally solvable and constraint qualification is satisfied at a local optimizer $\boldsymbol{x} \in \mathcal{X}$ of (A.4), then $\boldsymbol{x}$ satisfies the Kuhn-Tucker conditions for some $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^L \times \mathbb{R}^J$.*

Slightly simplifying, one says that $\boldsymbol{x} \in \mathcal{X}$ satisfies the Kuhn-Tucker conditions (A.5), if $\boldsymbol{x}$ satisfies (A.5) for some $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^L \times \mathbb{R}^J$.

A further interesting notion related to the Lagrangean function is the following.

Definition 16 *Let $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{X} \times \mathbb{R}^L \times \mathbb{R}^J$ satisfy the Kuhn-Tucker conditions (A.5) and let $\mathcal{L}'(\boldsymbol{x}) = \{k \in \mathcal{L} : f_k(\boldsymbol{x}) = 0\}$. We say that strict complementarity is satisfied at $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ if $\lambda_k > 0$, $k \in \mathcal{L}'(\boldsymbol{x})$.*

Assume that problem (A.4) is perturbed in the sense that some k-th inequality constraint is loosened in the sense $f_k(\boldsymbol{x}) \leq \delta$, $k \in \mathcal{L}$, with $\delta > 0$. The Lagrange price λ_k corresponds to the sensitivity of the optimum value of such perturbed problem as a function of $\delta > 0$ [47]. Thus, strict complementarity at $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{X} \times \mathbb{R}^L \times \mathbb{R}^J$ means that inequality constraints which are tight at $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ are relevant (or nontrivial) in the sense that their loosening provides an improvement in the optimum value.

Besides the Kuhn-Tucker conditions, great interest is in the set of *Second Order Sufficiency Conditions* (SOSC), which are sufficient conditions for the local minimizer property. In this work, we make use of SOSC only for inequality constrained problems.

Definition 17 ([49]) *Let $\mathcal{J} = \varnothing$ and define $\mathcal{L}'(\boldsymbol{x}) = \{k \in \mathcal{L} : f_k(\boldsymbol{x}) = 0\}$, $\boldsymbol{x} \in \mathcal{X}$. Then, the Second Order Sufficiency Conditions (SOSC) are said to be satisfied at a stationary point $(\boldsymbol{x}, \boldsymbol{\lambda}) \in \mathcal{X} \times \mathbb{R}^L$ of the Lagrangian of problem (A.4) if and only if*
i.) $(\boldsymbol{x}, \boldsymbol{\lambda})$ satisfies the Kuhn-Tucker conditions,
ii.) $\boldsymbol{x}'\nabla^2_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda})\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq 0$ satisfying

$$\begin{cases} \nabla' f_k(\boldsymbol{x})\boldsymbol{x} = 0, & k \in \mathcal{L}'(\boldsymbol{x}) \cap \{k \in \mathcal{L} : \lambda_k > 0\} \\ \nabla' f_k(\boldsymbol{x})\boldsymbol{x} \leq 0, & k \in \mathcal{L}'(\boldsymbol{x}) \cap \{k \in \mathcal{L} : \lambda_k = 0\} \end{cases}$$

SOSC are of immense importance in the development and analysis of locally convergent iterations for nonconvex optimization problems. Precisely, they distinguish the local minimizers of the problem from other stationary points of the Lagrangian, as potential points of attraction of the given iteration.

Note that under strict complementarity satisfied at $\boldsymbol{x} \in \mathcal{X}$, $k \in \mathcal{L}'(\boldsymbol{x})$ implies $\lambda_k \neq 0$ so that the last requirement in Definition 17 ii.) is obsolete.

A.3.2 Characterization of numerical convergence

The rate of convergence of an iteration is usually characterized by the kind of convergence of roots or quotients. The root and quotient convergence is measured by the norm-dependent convergence factor and norm-independent convergence order.

Definition 18 ([79]) *Let an iteration $\boldsymbol{x}(n+1) = G(\boldsymbol{x}(n))$, $n \in \mathbb{N}$, be given with $\mathcal{I}$ as the set of all sequences of iterates convergent to a point of attraction $\tilde{\boldsymbol{x}}$. Then, the p-th root convergence factor is defined as*

$$R_p(\mathcal{I}, \tilde{\boldsymbol{x}}) = \sup_{\{\boldsymbol{x}(n)\}_n \in \mathcal{I}} \limsup_{n\to\infty} \|\boldsymbol{x}(n) - \tilde{\boldsymbol{x}}\|^{\frac{p}{n}}, \quad p \geq 1,$$

and the p-th quotient convergence factor takes the form

$$Q_p(\mathcal{I},\tilde{\boldsymbol{x}}) = \sup_{\{\boldsymbol{x}(n)\}_n\in\mathcal{I}} \limsup_{n\to\infty} \frac{\|\boldsymbol{x}(n+1)-\tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}(n)-\tilde{\boldsymbol{x}}\|^p}, \quad p\geq 1, \tag{A.6}$$

and is defined only if $\boldsymbol{x}(n)\neq\tilde{\boldsymbol{x}}$ for all but finitely many $n\in\mathbb{N}$.

Notice that $Q_p = 0$ for $p\in[1,p_0)$, $Q_p = c<\infty$ for $p=p_0$ and $Q_p=\infty$ for $p\in(p_0,\infty)$ (and similarly for R_p).

Definition 19 ([79]) *Let an iteration $\boldsymbol{x}(n+1) = G(\boldsymbol{x}(n))$, $n\in\mathbb{N}$, be given with $\mathcal{I}=\mathcal{I}(\tilde{\boldsymbol{x}})$ as the set of all sequences of iterates convergent to a point of attraction $\tilde{\boldsymbol{x}}$. Then, the root convergence order is defined as*

$$O_R(\mathcal{I},\tilde{\boldsymbol{x}}) = \inf_{p\geq 1: R_p(\mathcal{I},\tilde{\boldsymbol{x}})=\infty} p, \tag{A.7}$$

and the quotient convergence order takes the form

$$O_Q(\mathcal{I},\tilde{\boldsymbol{x}}) = \inf_{p\geq 1: Q_p(\mathcal{I},\tilde{\boldsymbol{x}})=\infty} p.$$

The description of convergence rate in terms of quotients is better-established than in terms of roots, so that we focus on the first one. Based on the definition of quotient convergence order we say that an iteration with an attraction point $\tilde{\boldsymbol{x}}$ exhibits *in any norm*

$$\begin{aligned} \textit{linear quotient convergence if} \quad & O_Q(\mathcal{I},\tilde{\boldsymbol{x}})\geq 1, \\ \textit{quadratic quotient convergence if} \quad & O_Q(\mathcal{I},\tilde{\boldsymbol{x}})\geq 2. \end{aligned}$$

(the notion of cubic quotient convergence exists as well, but is rarely used and hardly occurs in practice). For instance, for gradient-related iterations we usually have linear quotient convergence, while for the Newton iteration quadratic convergence can be achieved [49].

The notion of linear convergence can be refined for the certain metric/ norm considered [79]. Precisely, for an iteration with an attraction point $\tilde{\boldsymbol{x}}$, we have *for the certain norm in* (A.6) [79]

$$\begin{aligned} \textit{superlinear quotient convergence if} \quad & Q_1(\mathcal{I},\tilde{\boldsymbol{x}}) = 0, \\ \textit{sublinear quotient convergence if} \quad & Q_1(\mathcal{I},\tilde{\boldsymbol{x}}) \geq 1,' \end{aligned}$$

while the case $0<Q_1(\mathcal{I},\tilde{\boldsymbol{x}})<1$ is usually unchanged referred to as linear convergence. Note that by the definitions of convergence factors and orders, the quadratic (norm-invariant) convergence implies superlinear convergence in the particular norm considered. In an analogous way we can refine the notion of quadratic convergence to norm-specific superquadratic and subquadratic convergence.

A.4 Some notions of convex analysis

Definition 20 *A set $\mathcal{X}\subseteq\mathbb{R}^K$ is said to be convex if*

$$(1-t)\boldsymbol{x}' + t\boldsymbol{x}'' \in\mathcal{X}, \quad \boldsymbol{x}',\boldsymbol{x}''\in\mathcal{X}.$$

Definition 21 *A function $\boldsymbol{x}\mapsto f(\boldsymbol{x})$, $\boldsymbol{x}\in\mathcal{X}\subseteq\mathbb{R}^K$, is said to be convex if*

$$f(\boldsymbol{x}(t)) \leq (1-t)f(\boldsymbol{x}') + tf(\boldsymbol{x}''), \quad \boldsymbol{x}',\boldsymbol{x}''\in\mathcal{X}, \quad t\in(0,1),$$

with $\boldsymbol{x}(t) = (1-t)\boldsymbol{x}' + t\boldsymbol{x}''$.

It is implied implicitly by Definition (21) that a convex function is defined on a convex set.

Definition 22 *A function* $\boldsymbol{x} \mapsto f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}_+^K$, *is said to be log-convex if* $\log f$ *is a convex function, that is,*

$$f(\boldsymbol{x}(t)) \leq f(\boldsymbol{x}')^{(1-t)} f(\boldsymbol{x}'')^t, \quad \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{X}, \quad t \in (0,1),$$

with $\boldsymbol{x}(t) = (1-t)\boldsymbol{x}' + t\boldsymbol{x}''$.

By the (weighted) geometric arithmetic mean inequality and Definitions 21 and 22 follows that log-convexity implies convexity but not conversely [121].

Definition 23 ([74]) *A vector pair* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathcal{X} \times \mathcal{Y}$ *is referred to as a saddle point of a function* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto F(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$, *if* $F(\tilde{\boldsymbol{x}}, \boldsymbol{y}) \leq F(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \leq F(\boldsymbol{x}, \tilde{\boldsymbol{y}})$, $\boldsymbol{x} \in \mathcal{X}$, $\boldsymbol{y} \in \mathcal{Y}$.

Instead of verifying the pair of inequalities in Definition 23, a saddle point can be identified by means of an equality, referred sometimes to as min-max max-min equality [153].

Proposition 43 ([153]) *A vector pair* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in \mathcal{X} \times \mathcal{Y}$ *is a saddle point of function* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto F(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$, *if and only if*

$$\inf_{\boldsymbol{x} \in \mathcal{X}} \sup_{\boldsymbol{y} \in \mathcal{Y}} F(\boldsymbol{x}, \boldsymbol{y}) = \sup_{\boldsymbol{y} \in \mathcal{Y}} \inf_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}, \boldsymbol{y}),$$

and $\tilde{\boldsymbol{x}} = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \sup_{\boldsymbol{y} \in \mathcal{Y}} F(\boldsymbol{x}, \boldsymbol{y})$, *and* $\tilde{\boldsymbol{y}} = \arg\max_{\boldsymbol{y} \in \mathcal{Y}} \inf_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}, \boldsymbol{y})$.

A.4.1 Min-max functions and convex-concave functions

Convex-concavity property is a straightforward composition of convexity and concavity [88].

Definition 24 ([88]) *We say that a function* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto F(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$, *is convex-concave in* $\boldsymbol{x}, \boldsymbol{y}$ *(equivalently, as a function of* $\boldsymbol{x}, \boldsymbol{y}$*), if* F *is a convex function of* $\boldsymbol{x} \in \mathcal{X}$ *and a concave function of* $\boldsymbol{y} \in \mathcal{Y}$.

Concave-convexity is defined analogously. Strict convex-concavity is an obvious extension of Definition 24. A twice Frechet-differentiable function $(\boldsymbol{x}, \boldsymbol{y}) \mapsto F(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$ is convex-concave in $\boldsymbol{x}, \boldsymbol{y}$ if and only if

$$\nabla^2_{\boldsymbol{x}} F(\boldsymbol{x}, \boldsymbol{y}) \succeq 0, \quad \nabla^2_{\boldsymbol{y}} F(\boldsymbol{x}, \boldsymbol{y}) \preceq 0, \qquad (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}. \tag{A.8}$$

Under strict convex-concavity, the inequalities (A.8) are strict and represent only a sufficient condition [88].

The central property of a convex-concave function is the following.

Proposition 44 ([88]) *If function* $(\boldsymbol{x}, \boldsymbol{y}) \mapsto F(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$, *is convex-concave in* $\boldsymbol{x}$, $\boldsymbol{y}$, *then it has either no stationary points or only saddle-points* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{x}}) \in \mathcal{X} \times \mathcal{Y}$ *of the type [88]*

$$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = \arg\max_{\boldsymbol{x} \in \mathcal{X}} \min_{\boldsymbol{y} \in \mathcal{Y}} F(\boldsymbol{x}, \boldsymbol{y}) = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{y} \in \mathcal{Y}} F(\boldsymbol{x}, \boldsymbol{y}), \tag{A.9}$$

and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *is unique if* F *is strictly convex-concave.*

The concept of a min-max function was introduced in [75] in order to efficiently characterize and classify min-max optimization problems and corresponding iterations.

Definition 25 ([75]) *We say that a function* $(\boldsymbol{x},\boldsymbol{y}) \mapsto F(\boldsymbol{x},\boldsymbol{y})$*,* $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$*, is a min-max function of* $\boldsymbol{x},\boldsymbol{y}$*, if* F *is twice Frechet-differentiable and*

$$\nabla^2_{\boldsymbol{x}} F(\boldsymbol{x},\boldsymbol{y}) - \nabla'^2_{\boldsymbol{x},\boldsymbol{y}} F(\boldsymbol{x},\boldsymbol{y}) (\nabla^2_{\boldsymbol{y}} F(\boldsymbol{x},\boldsymbol{y}))^{-1} \nabla^2_{\boldsymbol{x},\boldsymbol{y}} F(\boldsymbol{x},\boldsymbol{y}) \succeq 0, \quad \nabla^2_{\boldsymbol{y}} F(\boldsymbol{x},\boldsymbol{y}) \prec 0, \quad (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}.$$

A max-min function is defined analogously. The definition of a strict min-max function is a straightforward extension of Definition 25.

A min-max function has the following key property.

Proposition 45 ([75]) *If* $(\boldsymbol{x},\boldsymbol{y}) \mapsto F(\boldsymbol{x},\boldsymbol{y})$*,* $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{X} \times \mathcal{Y}$*, is a min-max function of* $\boldsymbol{x}$*,* $\boldsymbol{y}$*, then it has either no stationary points or only min-max points* $(\bar{\boldsymbol{y}},\bar{\boldsymbol{x}}) \in \mathcal{X} \times \mathcal{Y}$ *such that [75]*

$$(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) = \arg\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{y}\in\mathcal{Y}} F(\boldsymbol{x},\boldsymbol{y}), \tag{A.10}$$

and $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ *is unique if* F *is a strictly min-max function.*

The class of min-max functions generalizes/ contains the class of twice Frechet-differentiable convex-concave functions, for which the second inequality in (A.8) is strict (note that it does *not* generalize the class of twice Frechet-differentiable convex-concave functions which are strictly convex in $\boldsymbol{x} \in \mathcal{X}$ [88]). Consequently, a min-max point (A.10) becomes a saddle point (A.9) if F is also strictly convex-concave.

A.5 Some notes on polymatroids

Definition 26 ([154]) *A set function* $A \mapsto f(A) \in \mathbb{R}_+^{Card(E)}$*,* $A \subseteq E$*, with* $E = \{1,\dots,Card(E)\}$ *is referred to as a rank function if*
i.) $f(\varnothing) = 0$ *(*f *is normalized),*
ii.) if $A \subseteq B \subseteq E$*, then* $f(A) \leq f(B)$ *(*f *is increasing),*
iii.) if $A, B \subseteq E$*, then* $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ *(*f *is submodular).*

The definition of a polymatroid is the following.

Definition 27 ([154]) *A polymatroid is a polytope defined as*

$$\mathcal{P}(f) = \{\boldsymbol{x} \geq 0 : \sum_{i\in A} x_i \leq f(A), \quad A \subseteq E\}, \tag{A.11}$$

with $E = \{1,\dots,Card(E)\}$ *and a rank function* $A \mapsto f(A) \in \mathbb{R}_+^{Card(E)}$*,* $A \subseteq E$*.*

For completeness it has to be noted that a polytope characterized by (A.11) but with reversed inequality is sometimes referred to as a *contra-polymatroid*.

By Definition 27, a polymatroid is a polytope representing an intersection of $2^{Card(E)} - 1$ half-spaces and the nonnegative orthant $\mathbb{R}_+^{Card(E)}$. Every polymatroid has $Card(E)!$ vertices which lie in the interior of the nonnegative orthant and each one of such vertices corresponds to a different permutation of elements in E. Precisely, given a permutation $i \mapsto \pi_k(i)$, $i \in E$, $1 \leq k \leq Card(E)!$, the components of the corresponding vertex $\boldsymbol{x}^{\pi_k} = (x_1^{\pi_k},\dots,x_{Card(E)}^{\pi_k})$ are

$$x_i^{\pi_k} = f(\{\pi_k(j)\}_{j=1}^{i}) - f(\{\pi_k(j)\}_{j=1}^{i-1}), \quad 1 \leq i \leq Card(E). \tag{A.12}$$

We have a following interesting feature of linear programs defined on a polymatroid.

Lemma 21 ([154]) *Given any* $\boldsymbol{c} \in \mathbb{R}_{+}^{Card(E)}$, *the solution to the optimization problem*

$$\max_{\boldsymbol{x} \in \mathcal{P}(f)} \boldsymbol{c}'\boldsymbol{x},$$

where $\mathcal{P}(f)$ *denotes the polymatroid determined by the rank function* f, *is the vertex* $\boldsymbol{x}^{\pi_k}$ *of* $\mathcal{P}(f)$, *where* π_k *orders the elements of* $\boldsymbol{c}$ *decreasingly, that is,*

$$c_{\pi_k(1)} \geq c_{\pi_k(2)} \geq \ldots \geq c_{\pi_k(K)}.$$

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