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ERROR BOUNDS AND EXPONENTIAL IMPROVEMENT FOR HERMITE'S ASYMPTOTIC EXPANSION FOR THE GAMMA FUNCTION

Gergő Nemes

In this paper we reconsider the asymptotic expansion of the Gamma function with shifted argument, which is the generalization of the well-known Stirling series. To our knowledge, no explicit error bounds exist in the literature for this expansion. Therefore, the first aim of this paper is to extend the known error estimates of Stirling's series to this general case. The second aim is to give exponentially-improved asymptotics for this asymptotic series.

1. INTRODUCTION

The Gamma function $\Gamma(z)$ is one of the most important special functions of classical analysis after the so-called elementary functions. It extends the domain of the factorial $n!$ to real and complex arguments except the non-positive integers. Since its creation in 1808, the Gamma function has caught the interest of numerous great mathematicians. A thorough paper on the historical development of the Gamma function is given by Davis [**4**]. For $\mathrm{Re}(z) > 0$, it can be defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t, \tag{1}$$

and then, using the functional equation $\Gamma(z+1) = z\,\Gamma(z)$ and analytic continuation, it can be extended to a meromorphic function on $\mathbb{C}$ with simple poles at the non-positive integers. In (1), the path of integration is the real positive axis and t^{z-1} has its principal value. Many properties of the Gamma function can be found in a number of books on special functions, such as [**6**, pp. 93–134], [**16**], [**17**, pp. 136–147], [**28**, pp. 41–77].

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In this paper we reconsider the known asymptotic expansion of the function $\log\Gamma(z+a)$ ($z \in \mathbb{C} \setminus (-\infty, 0]$, $a \in [0,1]$), the logarithm having its principal value. This expansion takes the form

$$\log\Gamma(z+a) = \left(z+a-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{\nu=2}^{n} \frac{(-1)^\nu B_\nu(a)}{\nu(\nu-1)z^{\nu-1}} + R_n^{(a)}(z). \tag{2}$$

Here $B_\nu(a)$ denotes the ν-th Bernoulli polynomial [**17**, p. 588]. The formula was proved by Barnes [**1**, p. 121] in 1900, who showed that the remainder satisfies $R_n^{(a)}(z) = \mathcal{O}(z^{-n})$ for large z, and gave an explicit formula for it. It is not well known that this expansion was already investigated earlier by Hermite [**9**], who proved another explicit form of $R_{2n+1}^{(a)}(z)$. Therefore, we call the expansion (2) (if extended to $n = \infty$) as Hermite's asymptotic expansion.

Gamma functions with shifted argument occur in many parts of pure and applied mathematics including the theory of hypergeometric series, Mellin–Barnes integrals or the discrete analogue of Laplace's method. Despite this, no exact estimates for $R_n^{(a)}(z)$ are available in the literature, except for the cases $a = 0, 1$ (which are known as Stirling's series). For the latter cases, several error bounds are known. These are given, along with the historical background of the asymptotic expansion (2) and the known results, in Section 2. The first aim of this paper is to extend those error bounds to the general case, when $a \in [0,1]$ is arbitrary. Here we state our results, the proofs are given in Section 3. First, let

$$\ell(\theta) := \begin{cases} 1 & \text{if } |\theta| < \frac{\pi}{4} \\ \csc(2\theta) & \text{if } \frac{\pi}{4} \le |\theta| < \frac{\pi}{2}. \end{cases} \tag{3}$$

Let B_n denote the n-th Bernoulli number [**17**, p. 588].

Theorem 1. *Let a be an arbitrary number in $[0,1]$, and z be a complex number with positive real part. Then we have*

$$\left|R_{2n+1}^{(a)}(z)\right| \le \ell(\theta)\frac{(-1)^{n+1}B_{2n+2}(a)}{(2n+2)(2n+1)|z|^{2n+1}} + \ell(\theta)\frac{2|B_{2n+2}|}{(2n+2)(2n+1)|z|^{2n+1}} + \ell(\theta)\frac{\left|B_{2n+3}(a)\right|}{(2n+3)(2n+2)|z|^{2n+2}} \tag{4}$$

for $n \ge 0$, and

$$\left|R_{2n}^{(a)}(z)\right| \le \ell(\theta)\frac{\left|B_{2n+1}(a)\right|}{2n(2n+1)|z|^{2n}} + \ell(\theta)\frac{(-1)^{n+1}B_{2n+2}(a)}{(2n+2)(2n+1)|z|^{2n+1}} + \ell(\theta)\frac{2|B_{2n+2}|}{(2n+2)(2n+1)|z|^{2n+1}} \tag{5}$$

for $n \geq 1$. *Moreover, if* $z > 0$, *then*

$$R_{2n}^{(a)}(z) = -\theta_4 \frac{B_{2n+1}(a)}{2n(2n+1)z^{2n}} + \theta_1 \frac{B_{2n+2}(a) - B_{2n+2}}{(2n+2)(2n+1)z^{2n+1}} + \theta_2 \frac{B_{2n+2}}{(2n+2)(2n+1)z^{2n+1}} \tag{6}$$

and (26) *hold* (*see Section* 2). *Here* $0 < \theta_4 < 1$ *is a suitable number that depends on* z, a *and* n.

Some restrictions on a lead to the following simpler error estimates.

Theorem 2. *Let* a *be an arbitrary number from the set* $[0,1] \setminus \left(\frac{1}{6}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, \frac{5}{6}\right)$, *and* z *be a complex number with positive real part. Then we have*

$$\left|R_n^{(a)}(z)\right| \leq \ell(\theta)\left(\frac{\left|B_{n+1}(a)\right|}{n(n+1)|z|^n} + \frac{\left|B_{n+2}(a)\right|}{(n+1)(n+2)|z|^{n+1}}\right) \tag{7}$$

for $n \geq 1$. *In addition, if* $z > 0$, *then*

$$R_n^{(a)}(z) = \theta_5 \frac{(-1)^{n+1} B_{n+1}(a)}{n(n+1)z^n} + \theta_6 \frac{(-1)^n B_{n+2}(a)}{(n+1)(n+2)z^{n+1}} \tag{8}$$

for $n \geq 1$ *with* $0 < \theta_i < 1$ $(i = 5, 6)$ *being an appropriate number that depends on* z, a *and* n.

Theorems 1 and 2 are generalizations of Lindelöf's error bound which corresponds to the cases $a = 0, 1$ (see Section 2). Now, for $n \geq 1$, we set

$$M_n(a) := \sup_{t \geq 0} |B_{n+1}(1-a) - B_{n+1}(t - a - \lfloor t - a \rfloor)| < +\infty. \tag{9}$$

An obvious upper bound for the quantity $M_n(a)$ is $2\max_{u \in [0,1]} |B_{n+1}(u)|$. Sharp bounds for the maxima of Bernoulli polynomials in $[0,1]$ are given by Lehmer [**11**].

Theorem 3. *Let* a *be an arbitrary number in* $[0,1]$, *and let* $\theta = \arg z$. *Then we have*

$$\left|R_n^{(a)}(z)\right| \leq \frac{M_n(a)}{n(n+1)|z|^n} \sec^{n+1}\left(\frac{\theta}{2}\right) \quad \textit{if } |\theta| < \pi \textit{ and } n \geq 1. \tag{10}$$

This is a possible extension of a result of Stieltjes we consider in Section 2. Finally, we have the following simpler estimates for the remainder term, that generalizes the error bounds due to Spira (see Section 2).

Theorem 4. *Let* a *be an arbitrary number in* $[0,1]$. *We have*

$$\left|R_{2n+1}^{(a)}(z)\right| \leq \left|\frac{B_{2n+2}}{2n+1}\right| \begin{cases} \dfrac{2}{|z|^{2n+1}} & \textit{if } \mathrm{Re}(z) \geq 0 \\ \dfrac{4}{\left|\Im(z)\right|^{2n+1}} & \textit{if } \mathrm{Re}(z) < 0, \Im(z) \neq 0 \end{cases}$$

and

$$\left|R_{2n}^{(a)}(z)\right| \le \left|\frac{B_{2n}}{2n-1}\right| \begin{cases} \dfrac{1}{|z|^{2n-1}} & \text{if } \mathrm{Re}(z) \ge 0 \\ \dfrac{2}{|\Im(z)|^{2n-1}} & \text{if } \mathrm{Re}(z) < 0, \Im(z) \ne 0 \end{cases}$$

for $n \ge 1$.

Though the asymptotic expansion (2) is valid when $|\arg z| \le \pi - \delta$, $0 < \delta \le \pi$, some applications require more exact information concerning exponentially small subdominant terms. Interest in the exponentially-improved asymptotic expansion for $\Gamma(z+a)$ first arose in an eigenvalue problem related to a physical phenomenon [**3**]. During the investigation of this problem it was necessary to derive an exponentially-improved asymptotics for a certain quotient of Gamma functions, near the negative imaginary axis. The case $a = 0$ was considered by Paris and Wood [**18**], and Berry [**2**]. They showed that infinitely many subdominant exponential terms in the expansion of $\log \Gamma(z)$ appear in the neighbourhood of the rays $\arg z = \pm\frac{\pi}{2}$, and proved a smooth transition property of these. The second aim of this paper is to extend their argument to the general case of $\log \Gamma(z+a)$. The fundamental result, with proof given in Subsection 4.2, is the following representation.

Theorem 5. *Let a be an arbitrary number in* $[0,1]$. *Then we have*

$$R_1^{(a)}(z) = \sum_{\substack{k=-\infty \\ k\ne 0}}^{\infty} \frac{e^{2\pi ika}}{2\pi ik} \sum_{j=1}^{n_k} \frac{(-1)^j (j-1)!}{(2\pi ikz)^j} + \Delta(z,a,n_k) \tag{11}$$

in the sector $|\arg z| \le \pi - \delta < \pi$, $0 < \delta \le \pi$, *where*

$$\Delta(z,a,n_k) := \sum_{\substack{k=-\infty \\ k\ne 0}}^{\infty} \frac{(-1)^{n_k+1} n_k!}{2\pi ik} e^{2\pi ik(z+a)} \Gamma(-n_k, 2\pi ikz) \tag{12}$$

and the indices n_k are arbitrary positive integers. Here $\Gamma(\alpha, z)$ denotes the complementary incomplete Gamma function [**17**, p. 174].

The rest of the paper is organized as follows. In Section 2, we review the known results about the asymptotic series of $\log \Gamma(z+a)$, focusing first on the case when $a = 0$. The proofs of the error bounds are given in Section 3. In Section 4, we describe the exponentially-improved asymptotics of $\log \Gamma(z+a)$.

2. STIRLING'S AND HERMITE'S ASYMPTOTIC EXPANSIONS

In Subsection 2.1, we discuss in detail the known properties of the asymptotic expansion of $\log \Gamma(z)$. The theory of this expansion is well-established. In Subsection 2.2, the asymptotic series of $\log \Gamma(z+a)$ is considered.

2.1. Stirling's series. It is well known that the Gamma function has the expansion

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{\nu=1}^{n} \frac{B_{2\nu}}{2\nu(2\nu-1)z^{2\nu-1}} + R_n(z), \tag{13}$$

where $z \in \mathbb{C} \setminus (-\infty, 0]$ and $n \geq 0$. The series in (13) (if extended to $n = \infty$) is known as Stirling's asymptotic series (see below), although it was first discovered by DE MOIVRE [**8**, pp. 480–484]. As usually happens with asymptotic expansions, Stirling's series is divergent for all $z \in \mathbb{C} \setminus (-\infty, 0]$, necessitating the use of a finite series truncation and a remainder term $R_n(z)$ to make it an exact expression. This remainder has closed form integral representations:

$$R_n(z) = \frac{(-1)^{n+1}}{\pi z^{2n+1}} \int_0^{+\infty} \frac{z^2}{z^2+t^2} t^{2n} \log(1 - e^{-2\pi t}) \mathrm{d}t \tag{14}$$

$$= -\frac{1}{2n+1} \int_0^{+\infty} \frac{B_{2n+1}(t - \lfloor t \rfloor)}{(z+t)^{2n+1}} \mathrm{d}t \tag{15}$$

$$= \frac{1}{2n+2} \int_0^{+\infty} \frac{B_{2n+2} - B_{2n+2}(t - \lfloor t \rfloor)}{(z+t)^{2n+2}} \mathrm{d}t. \tag{16}$$

The representation (14) holds true only for $\mathrm{Re}(z) > 0$. Known bounds are due to LINDELÖF [**12**, p. 99]:

$$|R_n(z)| \leq \ell(\theta) \frac{|B_{2n+2}|}{(2n+2)(2n+1)\,|z|^{2n+1}}, \tag{17}$$

and STIELTJES [**27**, pp. 434–436]:

$$|R_n(z)| \leq \frac{|B_{2n+2}|}{(2n+2)(2n+1)\,|z|^{2n+1}} \sec^{2n+2}\left(\frac{\theta}{2}\right) \quad \text{if } |\theta| < \pi, \tag{18}$$

where $\theta = \arg z$ and $\ell(\theta)$ is given by (3). Also, if $z > 0$ is real then $R_n(z)$ is less than, but has the same sign as, the first neglected term [**28**, p. 65]. It is known that $\csc(2\theta)$, in the sector $\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}$, is the best of n independent bounds for which (17) holds [**21**]. From Stieltjes' estimate (18), we can conclude that Stirling's series is indeed an asymptotic expansion of $\log\Gamma(z)$ as $z \to \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$.

The following simpler bounds are due to SPIRA [**26**]:

$$|R_n(z)| \leq \left|\frac{B_{2n}}{2n-1}\right| \frac{1}{|z|^{2n-1}} \quad \text{for } \mathrm{Re}(z) \geq 0,$$

$$|R_n(z)| \leq 2\left|\frac{B_{2n}}{2n-1}\right| \frac{1}{|\Im(z)|^{2n-1}} \quad \text{for } \mathrm{Re}(z) < 0, \Im(z) \neq 0.$$

Hermite and Sonin [**25**, p. 262] pointed out a surprising expression for $R_n(z)$ when $z > 0$:

$$R_n(z) = \frac{B_{2n+2}}{(2n+2)(2n+1)(z+\rho_n)^{2n+1}}$$

where $0 < \rho_n < \dfrac{1}{\pi\sqrt{2e}}$ is a suitable number that depends on z. For more complex bounds we refer the reader to [**7**, p. 907] and [**22**].

The Binet function $\mu : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is defined by $\mu(z) := R_0(z)$. It is known that $\mu(z)$ is a holomorphic function (see, e.g., [**19**, p. 57]). From (14)–(16), we obtain the following integral expressions:

$$\mu(z) = -\frac{1}{\pi z}\int_0^{+\infty} \frac{z^2}{z^2+t^2} \log\left(1 - e^{-2\pi t}\right) \mathrm{d}t \tag{19}$$

$$= -\int_0^{+\infty} \frac{t - \lfloor t \rfloor - \frac{1}{2}}{z+t} \mathrm{d}t \tag{20}$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{t - \lfloor t \rfloor - (t - \lfloor t \rfloor)^2}{(z+t)^2} \mathrm{d}t. \tag{21}$$

Formula (19) holds true only for $\mathrm{Re}(z) > 0$. Another representations are given by Binet's first and second formula [**29**, pp. 249–251]:

$$\mu(z) = \int_0^{+\infty} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-zt}}{t} \mathrm{d}t \tag{22}$$

and

$$\mu(z) = 2\int_0^{+\infty} \frac{\arctan(t/z)}{e^{2\pi t} - 1} \mathrm{d}t \tag{23}$$

for $\mathrm{Re}(z) > 0$ and the function arctan has its principal value. We remark that (19) follows from (23) using integration by parts.

Kaminski and Paris [**10**, pp. 279–288] derived the following expansion in terms of the complementary incomplete Gamma function:

$$\mu(z) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1}{k} \sum_{j=1}^{n_k - 1} \frac{(-1)^j (2j)!}{(2\pi k z)^{2j+1}} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(2n_k)!}{2\pi i k} e^{2\pi i k z} \Gamma(-2n_k, 2\pi i k z) \tag{24}$$

where $n_k = n_{-k}$ and the n_k's are arbitrary positive integers. An optimal choice of these integers, given by $n_k \sim \pi k |z|$ $(k \geq 1)$, then enabled the smooth transition of certain subdominant exponential terms accross the Stokes lines $\arg z = \pm\frac{\pi}{2}$ (see also [**2**] and [**18**]). For more details we refer the reader to Section 4.

2.2. Hermite's asymptotic expansion. Hermite [**9**] considered in 1895 the function $\log\Gamma(z+a)$, where $z > 0$ and $a \in [0, 1]$. He gave the expansion

$$\log\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{\nu=2}^{2n+1} \frac{(-1)^\nu B_\nu(a)}{\nu(\nu-1)z^{\nu-1}} + R_{2n+1}^{(a)}(z)$$

where $n \geq 0$ and

$$R_{2n+1}^{(a)}(z) = \frac{(-1)^{n+1}}{2\pi z^{2n+1}} \int_0^{+\infty} \frac{z^2}{z^2+t^2} t^{2n} \log\left(1 - 2e^{-2\pi t}\cos(2\pi a) + e^{-4\pi t}\right) \mathrm{d}t + \frac{(-1)^{n+1}}{\pi z^{2n+2}} \int_0^{+\infty} \frac{z^2}{z^2+t^2} t^{2n+1} \arctan\left(\frac{\sin(2\pi a)}{e^{2\pi t} - \cos(2\pi a)}\right) \mathrm{d}t. \tag{25}$$

He also provided an estimate for the remainder as

$$R_{2n+1}^{(a)}(z) = \theta_1 \frac{B_{2n+2}(a) - B_{2n+2}}{(2n+2)(2n+1)z^{2n+1}} + \theta_2 \frac{B_{2n+2}}{(2n+2)(2n+1)z^{2n+1}} - \theta_3 \frac{B_{2n+3}(a)}{(2n+3)(2n+2)z^{2n+2}}, \tag{26}$$

where $0 < \theta_i < 1$ $(i = 1, 2, 3)$ is an appropriate number that depends on z, a and n. The more general formula (2), was given by Barnes [**1**, p. 121] in 1900. For $a \neq 0$, he expressed the remainder $R_n^{(a)}(z)$ as a Mellin–Barnes integral

$$R_n^{(a)}(z) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, a) \mathrm{d}s, \qquad -n < c < -(n-1), \tag{27}$$

where $\zeta(s, a)$ denotes the Hurwitz Zeta function [**17**, p. 607] (see also [**29**, pp. 276–279]). He also showed that $R_n^{(a)}(z) = \mathcal{O}(z^{-n})$ as $z \to \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. This yields what we call Hermite's asymptotic expansion:

$$\log \Gamma(z + a) \sim \left(z + a - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{\nu=2}^{\infty} \frac{(-1)^\nu B_\nu(a)}{\nu(\nu-1)z^{\nu-1}} \tag{28}$$

as $z \to \infty$ in the sector $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$. This is a generalization of Stirling's series. Different proofs are given by Rowe [**20**] and Fields [**13**, pp. 32–33]. We note that, using the notation of the previous subsection, $R_{2n}^{(0)}(z) = R_{2n+1}^{(0)}(z) = R_n(z)$. From the functional equation $\log \Gamma(z + 1) = \log z + \log \Gamma(z)$, it follows that $R_n^{(0)}(z) = R_n^{(1)}(z)$. It is known [**14**, p. 111], [**16**, p. 295] that, for $n \geq 2$,

$$R_n^{(a)}(z) = -\frac{1}{n} \int_0^{+\infty} \frac{B_n(t - a - \lfloor t - a \rfloor)}{(z+t)^n} \mathrm{d}t. \tag{29}$$

This is a generalization of (15). In fact, it also holds true when $n = 1$, since

$$R_1^{(a)}(z) = \frac{B_2(a)}{2z} + R_2^{(a)}(z) = \frac{B_2(a)}{2z} - \frac{1}{2} \int_0^{+\infty} \frac{B_2(t - a - \lfloor t - a \rfloor)}{(z+t)^2} \mathrm{d}t = \frac{1}{2} \int_0^{+\infty} \frac{B_2(a) - B_2(t - a - \lfloor t - a \rfloor)}{(z+t)^2} \mathrm{d}t = -\int_0^{+\infty} \frac{B_1(t - a - \lfloor t - a \rfloor)}{z+t} \mathrm{d}t, \tag{30}$$

where we have used integration by parts in the last step. In general,

$$R_n^{(a)}(z) = \frac{1}{n+1}\int_0^{+\infty} \frac{(-1)^{n+1}B_{n+1}(a) - B_{n+1}(t-a-\lfloor t-a\rfloor)}{(z+t)^{n+1}}\,\mathrm{d}t \qquad (31)$$
$$= \frac{1}{n+1}\int_0^{+\infty} \frac{B_{n+1}(1-a) - B_{n+1}(t-a-\lfloor t-a\rfloor)}{(z+t)^{n+1}}\,\mathrm{d}t$$

which is an extension of (16).

It is convenient to define the function $(z,a) \mapsto \mu_a(z)$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $a \in [0,1]$ by $\mu_a(z) := R_1^{(a)}(z)$. It is clear from the definition that $\mu_a(z)$ is a holomorphic function of z. We call this function as the generalized Binet function, since $\mu(z) = \mu_0(z) = \mu_1(z)$. Using analytic continuation in (25), we find

$$\mu_a(z) = -\frac{1}{2\pi z}\int_0^{+\infty} \frac{z^2}{z^2+t^2}\log(1-2e^{-2\pi t}\cos(2\pi a)+e^{-4\pi t})\,\mathrm{d}t \qquad (32)$$
$$-\frac{1}{\pi z^2}\int_0^{+\infty} \frac{z^2}{z^2+t^2} t\arctan\left(\frac{\sin(2\pi a)}{e^{2\pi t}-\cos(2\pi a)}\right)\mathrm{d}t, \qquad \operatorname{Re}(z) > 0.$$

This generalizes (19). From (30), we have

$$\mu_a(z) = -\int_0^{+\infty} \frac{t-a-\lfloor t-a\rfloor - \frac{1}{2}}{z+t}\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^{+\infty} \frac{a(a-1)+t-a-\lfloor t-a\rfloor - (t-a-\lfloor t-a\rfloor)^2}{(z+t)^2}\,\mathrm{d}t.$$

These are extensions of (20) and (21). The generalization of Binet's first formula (22) is given by Hermite:

$$\mu_a(z) = \int_0^{+\infty}\left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{(1-a)t}}{e^t-1}\right)\frac{e^{-zt}}{t}\,\mathrm{d}t.$$

Although he proved it for $z > 0$, by analytic continuation, it also holds when $\operatorname{Re}(z) > 0$. The generalization of Binet's second formula (23) is apparently not known, however, it follows from (32) using integration by parts:

$$\mu_a(z) = \int_0^{+\infty} \frac{\cos(2\pi a) - e^{-2\pi t}}{\cosh(2\pi t) - \cos(2\pi a)}\arctan\frac{t}{z}\,\mathrm{d}t \qquad (33)$$
$$-\int_0^{+\infty} \frac{\sin(2\pi a)}{\cosh(2\pi t) - \cos(2\pi a)}\frac{1}{2}\log\left(1+\left(\frac{t}{z}\right)^2\right)\mathrm{d}t, \qquad \operatorname{Re}(z) > 0.$$

Here the functions arctan and log have their principal values.

3. PROOFS OF THE ERROR BOUNDS FOR HERMITE'S ASYMPTOTIC EXPANSION

In this section, we prove the generalizations of the error estimates of Lindelöf, Stieltjes and Spira to $R_n^{(a)}(z)$, given in Theorems 1-4.

3.1. Lindelöf-type estimates. For $t>0$ and $a\in[0,1]$ we introduce the functions

$$\psi_a(t) := \log\left(1-2e^{-2\pi t}\cos(2\pi a)+e^{-4\pi t}\right),$$

$$\xi_a(t) := \arctan\left(\frac{\sin(2\pi a)}{e^{2\pi t}-\cos(2\pi a)}\right).$$

By analytic continuation and (25), we have

$$R_{2n+1}^{(a)}(z)=\frac{(-1)^{n+1}}{2\pi z^{2n+1}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n}\psi_a(t)\mathrm{d}t+\frac{(-1)^{n+1}}{\pi z^{2n+2}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n+1}\xi_a(t)\mathrm{d}t \tag{34}$$

for $\mathrm{Re}(z)>0$ and $n\geq 0$. We shall give a similar expression for $R_{2n}^{(a)}(z)$. Using the identity

$$\frac{z^2}{z^2+t^2}=\sum_{\nu=0}^{n-1}(-1)^\nu\frac{1}{z^{2\nu}}t^{2\nu}+(-1)^n\frac{1}{z^{2n}}\frac{z^2}{z^2+t^2}t^{2n}$$

in (32) (with $n=n-1$ in the second integral), we obtain

$$\begin{aligned}\mu_a(z)=&\sum_{\nu=0}^{n-1}\frac{1}{z^{2\nu+1}}\frac{(-1)^{\nu+1}}{2\pi}\int_0^{+\infty}t^{2\nu}\psi_a(t)\mathrm{d}t-\sum_{\nu=0}^{n-2}\frac{1}{z^{2\nu+2}}\frac{(-1)^\nu}{\pi}\int_0^{+\infty}t^{2\nu+1}\xi_a(t)\mathrm{d}t\\&+\frac{(-1)^{n+1}}{2\pi z^{2n+1}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n}\psi_a(t)\mathrm{d}t+\frac{(-1)^n}{\pi z^{2n}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n-1}\xi_a(t)\mathrm{d}t.\end{aligned}$$

From the integral representations

$$\frac{B_{2\nu+2}(a)}{(2\nu+2)(2\nu+1)}=\frac{(-1)^{\nu+1}}{2\pi}\int_0^{+\infty}t^{2\nu}\psi_a(t)\mathrm{d}t \tag{35}$$

and

$$\frac{B_{2\nu+3}(a)}{(2\nu+3)(2\nu+2)}=\frac{(-1)^{\nu}}{\pi}\int_0^{+\infty}t^{2\nu+1}\xi_a(t)\mathrm{d}t \tag{36}$$

we find

$$\mu_a(z)=\sum_{\nu=2}^{2n}\frac{(-1)^\nu B_\nu(a)}{\nu(\nu-1)z^{\nu-1}}+R_{2n}^{(a)}(z),$$

with

$$R_{2n}^{(a)}(z)=\frac{(-1)^{n+1}}{2\pi z^{2n+1}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n}\psi_a(t)\,\mathrm{d}t+\frac{(-1)^n}{\pi z^{2n}}\int_0^{+\infty}\frac{z^2}{z^2+t^2}t^{2n-1}\xi_a(t)\,\mathrm{d}t. \tag{37}$$

This holds true for $\mathrm{Re}(z)>0$ and $n\geq 1$. We introduce the function

$$\rho(\theta) := \max_{u\geq 0}\left|\frac{z^2}{z^2+u^2}\right|,\ \theta=\arg z.$$

Note that $\rho(\theta)$ does not change when the variables z and/or u are multiplied by arbitrary non-zero real numbers. In particular, it is independent of $|z|$. Since

$$\frac{1}{\rho^2(\theta)} = \min_{p\ge 0} \frac{\left(p^2 + |z|^2\cos(2\theta)\right)^2 + |z|^4\sin^2(2\theta)}{|z|^4} = \min_{q\ge 0}\left[\left(q^2+\cos(2\theta)\right)^2 + \sin^2(2\theta)\right],$$

one finds that $\rho(\theta) = \ell(\theta)$, where $\ell(\theta)$ is given by (3). We shall now estimate the remainder $R_n^{(a)}(z)$ for $\mathrm{Re}(z) > 0$.

Proof of Theorem 1. In this case, we allow a to be an arbitrary number in $[0,1]$. A straightforward calculation shows that

$$\psi_a(t) - \psi_0(t) = \log\left(1 + \left(\frac{\sin(\pi a)}{\sinh(\pi t)}\right)^2\right) \ge 0.$$

For fixed a, the function $t \mapsto \xi_a(t)$ does not change sign. It then follows from the representations (34) and (37), and the definition of the function $\rho(\theta)$ $(= \ell(\theta))$ that

$$\begin{aligned}\left|R_{2n+1}^{(a)}(z)\right| \le{}& \frac{\ell(\theta)}{|z|^{2n+1}}\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\left(\psi_a(t) - \psi_0(t)\right)\mathrm{d}t \\ &+ \frac{\ell(\theta)}{|z|^{2n+2}}\left|\frac{1}{\pi}\int_0^{+\infty} t^{2n+1}\xi_a(t)\mathrm{d}t\right| + \frac{\ell(\theta)}{|z|^{2n+1}}\left|\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\psi_0(t)\,\mathrm{d}t\right|,\\ \left|R_{2n}^{(a)}(z)\right| \le{}& \frac{\ell(\theta)}{|z|^{2n+1}}\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\left(\psi_a(t) - \psi_0(t)\right)\mathrm{d}t \\ &+ \frac{\ell(\theta)}{|z|^{2n}}\left|\frac{1}{\pi}\int_0^{+\infty} t^{2n-1}\xi_a(t)\mathrm{d}t\right| + \frac{\ell(\theta)}{|z|^{2n+1}}\left|\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\psi_0(t)\mathrm{d}t\right|.\end{aligned}$$

Using the integral representations (35) and (36), we deduce the estimates (4) and (5). To prove the bound (6), note that when $z > 0$ is real, we have $0 < \frac{z^2}{z^2+t^2} < 1$ for all $t > 0$.

Proof of Theorem 2. In this case, we restrict a to the set $[0,1] \setminus \left(\frac{1}{6},\frac{1}{4}\right) \cup \left(\frac{3}{4},\frac{5}{6}\right)$. The functions $t \mapsto \psi_a(t)$ and $t \mapsto \xi_a(t)$ do not change sign for these values of a. Consequently, from the representations (34) and (37), and the definition of the function $\rho(\theta)$ $(= \ell(\theta))$, we obtain

$$\begin{aligned}\left|R_{2n+1}^{(a)}(z)\right| &\le \frac{\ell(\theta)}{|z|^{2n+1}}\left|\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\psi_a(t)\mathrm{d}t\right| + \frac{\ell(\theta)}{|z|^{2n+2}}\left|\frac{1}{\pi}\int_0^{+\infty} t^{2n+1}\xi_a(t)\mathrm{d}t\right|,\\ \left|R_{2n}^{(a)}(z)\right| &\le \frac{\ell(\theta)}{|z|^{2n+1}}\left|\frac{1}{2\pi}\int_0^{+\infty} t^{2n}\psi_a(t)\mathrm{d}t\right| + \frac{\ell(\theta)}{|z|^{2n}}\left|\frac{1}{\pi}\int_0^{+\infty} t^{2n-1}\xi_a(t)\mathrm{d}t\right|.\end{aligned}$$

Using the integral representations (35) and (36), we deduce the estimate (7). To prove the bound (8), note that when $z > 0$ is real, we have $0 < \frac{z^2}{z^2+t^2} < 1$ for all $t > 0$.

3.2. Stieltjes-type estimates. The proof of Stieltjes' error bound (18) is based on the representation (16) and the fact that $B_{2n+2} - B_{2n+2}(t - \lfloor t \rfloor)$ does not change sign. A generalization of Stieltjes' estimate for $R_n^{(a)}(z)$ may come from (31). However, in general, $B_{n+1}(1-a) - B_{n+1}(t - a - \lfloor t - a \rfloor)$ might change sign for fixed $a \in (0,1)$. Indeed, it changes at least when $a \neq \frac{1}{2}$, as can be seen from the representations (35) and (36). Nevertheless, we can proceed as follows. Set $\theta = \arg z$, then

$$|z+t|^2 = |z|^2 + 2|z|t\cos\theta + t^2 = (|z|+t)^2 - 4|z|t\sin^2\left(\frac{\theta}{2}\right) \geq (|z|+t)^2\cos^2\left(\frac{\theta}{2}\right).$$

Applying this and the definition (9) in (31) yields

$$\begin{aligned} \left|R_n^{(a)}(z)\right| &\leq \frac{1}{n+1}\int_0^{+\infty}\frac{|B_{n+1}(1-a) - B_{n+1}(t-a-\lfloor t-a \rfloor)|}{|z+t|^{n+1}}\mathrm{d}t \\ &\leq \frac{M_n(a)}{n+1}\sec^{n+1}\left(\frac{\theta}{2}\right)\int_0^{+\infty}\frac{1}{(|z|+t)^{n+1}}\mathrm{d}t. \end{aligned}$$

Evaluating this integral then leads us to the error bound (10) and the proof of Theorem 3.

3.3. Spira-type estimates. It is well known that $\max_{u\in[0,1]}|B_{2n}(u)| \leq |B_{2n}|$. From the integral expressions (29) and (31), we obtain

$$\left|R_{2n+1}^{(a)}(z)\right| \leq \frac{1}{2n+2}2|B_{2n+2}|\int_0^{+\infty}\frac{\mathrm{d}t}{|z+t|^{2n+2}}$$

and

$$\left|R_{2n}^{(a)}(z)\right| \leq \frac{1}{2n}|B_{2n}|\int_0^{+\infty}\frac{\mathrm{d}t}{|z+t|^{2n}}.$$

Spira showed that

$$\int_0^{+\infty}\frac{\mathrm{d}t}{|z+t|^{2n}} \leq \left|\frac{2n}{2n-1}\right|\frac{1}{|z|^{2n-1}} \quad \text{for } \operatorname{Re}(z) \geq 0,$$

$$\int_0^{+\infty}\frac{\mathrm{d}t}{|z+t|^{2n}} \leq 2\left|\frac{2n}{2n-1}\right|\frac{1}{|\Im(z)|^{2n-1}} \quad \text{for } \operatorname{Re}(z) < 0, \Im(z) \neq 0.$$

Applying these in the estimates above, we deduce Theorem 4.

4. EXPONENTIALLY-IMPROVED ASYMPTOTICS

Although the asymptotic series (28) is valid when $|\arg z| \leq \pi - \delta$, $0 < \delta \leq \pi$, some applications require more precise information concerning the subdominant terms, usually exponentially small. Interest in the exponentially-improved asymptotic expansion for $\Gamma(z+a)$ first arose in connection with a simple model describing the leakage of energy from weakly bent optical fibre waveguides [**3**]. In the course of this it was necessary to derive exponentially-improved asymptotics for the quotient

$$\frac{\Gamma\left(z+\frac{1}{4}\right)}{\Gamma\left(z+\frac{3}{4}\right)} \tag{38}$$

as $z \to \infty$, near the negative imaginary axis. The case $a = 0$ was considered by Paris and Wood [**18**], who showed that infinitely many subdominant exponential terms in the expansion of $\log\Gamma(z)$ appear in the neighbourhood of the rays $\arg z = \pm\frac{\pi}{2}$. They also smoothed the first (leading) subdominant exponential. The smoothing of the higher subdominant exponentials was carried out by Berry [**2**]. An alternative proof of Berry's result is given by Kaminski and Paris [**10**, pp. 279–288]. Our goal is to extend their argument to the general case of $\log\Gamma(z+a)$. First, we shall find the Stokes multipliers in the sectors $\frac{\pi}{2} < |\arg z| < \pi$, $|\arg z| < \frac{\pi}{2}$.

4.1. The discontinuous treatment of the Stokes multipliers. The asymptotic expansion of $\log\Gamma(z+a)$ in the sector $|\arg z| < \frac{\pi}{2}$ is given by Hermite's expansion (28). We shall obtain an expansion in the sector $|\arg z| > \frac{\pi}{2}$. We will use the reflection formula

$$\Gamma(z+a) = \frac{\pi}{\sin(\pi(z+a))\Gamma(-z+1-a)}. \tag{39}$$

First, we consider the case $\frac{\pi}{2} < \arg z < \pi$ and put $-z = ze^{-\pi i}$ so that $-\frac{\pi}{2} < \arg(-z) < 0$. The asymptotic expansion of $\log\Gamma(-z+1-a)$ can be obtained from (28). Taking logarithms in (39) yields

$$\log\Gamma(z+a) \sim \log\pi - \log\sin(\pi(z+a)) - \left(-z-a+\frac{1}{2}\right)\log(ze^{-\pi i}) - z$$
$$- \frac{1}{2}\log(2\pi) - \sum_{\nu=2}^{\infty}\frac{(-1)^\nu B_\nu(1-a)}{\nu(\nu-1)(-z)^{\nu-1}}.$$

After a simple rearrangement, the result is

$$\log\Gamma(z+a) \sim \left(z+a-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) - \log\left(1-e^{2\pi i(z+a)}\right)$$
$$+ \sum_{\nu=2}^{\infty}\frac{(-1)^\nu B_\nu(a)}{\nu(\nu-1)z^{\nu-1}}, \tag{40}$$

as $z \to \infty$ in the sector $\frac{\pi}{2} < \arg z < \pi$. A similar argument shows that in the conjugate sector $-\pi < \arg z < -\frac{\pi}{2}$ an analogous expansion holds with $e^{2\pi i(z+a)}$ replaced by $e^{-2\pi i(z+a)}$.

The case $|\arg z| = \dfrac{\pi}{2}$ must be considered separately. On the positive imaginary axis we put $z = iy$, $y > 0$. Using the Fourier series of $t \mapsto B_1\left(t - a - \lfloor t - a\rfloor\right)$ (see, e.g., [**17**, p. 592]) in (30), we find

$$\mu_a(iy) = \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{\sin(2\pi k t)}{iy + t}\,\mathrm{d}t - \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{\cos(2\pi k t)}{iy + t}\,\mathrm{d}t.$$

Separation of the real and imaginary parts yields

$$\begin{aligned}\mu_a(iy) = {}& \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{t\sin(2\pi k t)}{y^2 + t^2}\,\mathrm{d}t + iy \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{\cos(2\pi k t)}{y^2 + t^2}\,\mathrm{d}t \\ & - iy \sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{\sin(2\pi k t)}{y^2 + t^2}\,\mathrm{d}t - \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{k} \frac{1}{\pi} \int_0^{+\infty} \frac{t\cos(2\pi k t)}{y^2 + t^2}\,\mathrm{d}t.\end{aligned}$$

The two integrals in the first line can be evaluated using Fourier transforms, and we find

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\cos(2\pi k a)}{k} \frac{e^{-2\pi k y}}{2} + i \sum_{k=1}^{\infty} \frac{\sin(2\pi k a)}{k} \frac{e^{-2\pi k y}}{2} &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{2\pi i k(iy+a)}}{k} \\ &= -\frac{1}{2} \log\left(1 - e^{2\pi i(iy+a)}\right).\end{aligned}$$

The two integrals in the second line can be written in terms of the Exponential integral [**5**, eq. 3.723/1. and 3.723/5.], hence we arrive at

$$\mu_a(iy) = -\frac{1}{2} \log\left(1 - e^{2\pi i(iy+a)}\right) - i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{e^{2\pi i k a}}{2\pi k}\, e^{-2\pi k y} \mathrm{Ei}\,(2\pi k y). \tag{41}$$

From Watson's lemma [**16**, p. 71], we obtain that

$$e^{\mp 2\pi k y} \mathrm{Ei}(\pm 2\pi k y) = \pm \int_0^{+\infty} \frac{e^{-2\pi k y t}}{1 \mp t}\,\mathrm{d}t \sim \sum_{\nu=1}^{\infty} \frac{(\nu-1)!}{(\pm 2\pi k y)^{\nu}},\ k > 0,$$

as $y \to +\infty$. Substituting this into (41), changing the order of summation and using the Fourier series of the Bernoulli polynomials [**17**, p. 592], we find

$$\mu_a(iy) \sim -\frac{1}{2} \log\left(1 - e^{2\pi i(iy+a)}\right) + \sum_{\nu=2}^{\infty} \frac{(-1)^{\nu} B_{\nu}(a)}{\nu(\nu-1)(iy)^{\nu-1}},$$

as $y \to +\infty$. Thus, by definition

$$\text{(42)}\quad \log\Gamma(iy+a) \sim \left(iy+a-\frac{1}{2}\right)\log(iy) - iy + \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\bigl(1-e^{2\pi i(iy+a)}\bigr) + \sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}B_{\nu}(a)}{\nu(\nu-1)(iy)^{\nu-1}}$$
$$\sim \left(z+a-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) - \frac{1}{2}\log\left(1-e^{2\pi i(z+a)}\right) + \sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}B_{\nu}(a)}{\nu(\nu-1)z^{\nu-1}}.$$

A similar result holds in the negative imaginary axis with $e^{2\pi i(z+a)}$ replaced by $e^{-2\pi i(z+a)}$. Using the Taylor expansion of the logarithm, we can unify the series (39), (40), (42) and their analogues in the conjugate sectors as follows:

$$\text{(43)}\qquad \log\Gamma(z+a) \sim \left(z+a-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} S_k^{(a)}(\theta)e^{\pm 2\pi ikz} + \sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}B_{\nu}(a)}{\nu(\nu-1)z^{\nu-1}},$$

where

$$\text{(44)}\qquad S_k^{(a)}(\theta) := \begin{cases} \dfrac{e^{\pm 2\pi ika}}{k} & \text{if } \dfrac{\pi}{2} < |\theta| < \pi \\ \dfrac{e^{\pm 2\pi ika}}{2k} & \text{if } \theta = \pm\dfrac{\pi}{2} \\ 0 & \text{if } |\theta| < \dfrac{\pi}{2} \end{cases}$$

and $\theta = \arg z$. The upper or lower sign is taken in (43) and (44) according as z is in the upper or lower half-plane. It seems that there is a discontinuous change in the coefficients of the exponential terms when $\arg z$ changes continuously across $\arg z = \pm\frac{\pi}{2}$. We have encountered a Stokes phenomenon with Stokes lines $\arg z = \pm\frac{\pi}{2}$. The function $S_k^{(a)}(\theta)$ is called the Stokes multiplier of the subdominant exponential $e^{\pm 2\pi ikz}$. In the following subsection we will prove Theorem 5, and use it to show how the discontinuous behaviour across the Stokes lines can be smoothed by an error function.

We remark that the appearance of the exponentially small terms in Hermite's asymptotic series was first investigated by Slavić [**23**]. Unfortunately, his result and the slightly improved version in [**24**] are incorrect.

Finally, we can deduce from (43) that close to the negative imaginary axis, the ratio (38) has the asymptotic expansion

$$\frac{\Gamma\left(z+\frac{1}{4}\right)}{\Gamma\left(z+\frac{3}{4}\right)} \sim \frac{1}{\sqrt{z}}\sqrt{\frac{1-ie^{-2\pi iz}}{1+ie^{-2\pi iz}}}\left(1-\frac{1}{64z^2}+\frac{21}{8192z^4}-\cdots\right)$$
$$\sim \frac{1}{\sqrt{z}}\left(1-ie^{-2\pi iz}-\frac{e^{-4\pi iz}}{2}+\cdots\right)\left(1-\frac{1}{64z^2}+\frac{21}{8192z^4}-\cdots\right)$$

as $z \to \infty$.

4.2. Smoothing the Stokes discontinuities. We begin with the proof of Theorem 5. Substituting $n = 1$ and $s = -t$ in (27) shows that the generalized Binet function $\mu_a(z)$ has the Mellin–Barnes integral representation given by

$$\mu_a(z) = -\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\pi z^{-t}}{t\sin(\pi t)}\zeta(-t,a)\,\mathrm{d}t$$

valid when $|\arg z| \le \pi - \delta < \pi$, $0 < \delta \le \pi$ and $a \in (0,1]$.

We now proceed to manipulate the above integral for $\mu_a(z)$ to extract the sequence of exponentially small terms. First, we apply the Hurwitz formula [**6**, p. 115]

$$\zeta(-t,a) = \frac{\Gamma(t+1)}{(2\pi i)^{t+1}}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{e^{2\pi ika}}{k^{t+1}},\ \mathrm{Re}(t) > 0,\ a \in (0,1],$$

to obtain the result

$$\mu_a(z) = -\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{e^{2\pi ika}}{2\pi ik}\Lambda_k(z), \tag{45}$$

where

$$\Lambda_k(z) := \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\pi\Gamma(t)}{\sin(\pi t)}\frac{1}{(2\pi ikz)^t}\,\mathrm{d}t.$$

The reversal of the order of summation and integration is justified when $|\arg z| \le \pi - \delta < \pi$, $0 < \delta \le \pi$ by absolute convergence. We now move the path of integration to the right over the simple poles of the integrand at $t = 1, 2, \ldots, n_k$, where n_k is an arbitrary positive integer. Use of the residue theorem then shows that

$$\Lambda_k(z) = \sum_{j=1}^{n_k}\frac{(-1)^{j-1}(j-1)!}{(2\pi ikz)^j} + \Delta_k(z,n_k),$$

provided that $|\arg z| \le \pi - \delta < \pi$. The remainder term is given by

$$\Delta_k(z,n_k) := \frac{1}{2\pi i}\int_{n_k+\frac{1}{2}-i\infty}^{n_k+\frac{1}{2}+i\infty}\frac{\pi\Gamma(t)}{\sin(\pi t)}\frac{1}{(2\pi ikz)^t}\,\mathrm{d}t$$
$$= (-1)^{n_k} n_k! e^{2\pi ikz}\Gamma(-n_k, 2\pi ikz)$$

in the sector $|\arg z| \le \pi - \delta < \pi$, $0 < \delta \le \pi$. Plugging all these results into (45), we arrive at the representation given in Theorem 5. The absolute convergence of the series (12) follows from the large argument asymptotics of the complementary incomplete Gamma function. The validity of formula (11) for $a = 0$ follows from the fact that $\mu_0(z) = \mu_1(z)$.

If, for example, we choose $n_k = n - 1$ $(n \ge 2)$ for all k, the double sum in (11) becomes

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k a}}{2\pi i k} \sum_{j=1}^{n-1} \frac{(-1)^j (j-1)!}{(2\pi i k z)^j} = \sum_{k=1}^{n-1} \frac{(-1)^k \zeta(-k,a)}{k z^k} = \sum_{k=2}^{n} \frac{(-1)^k B_k(a)}{k(k-1)z^{k-1}},$$

whence, by (12) and the definition of $R_n^{(a)}(z)$,

$$R_n^{(a)}(z) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^n (n-1)!}{2\pi i k} e^{2\pi i k (z+a)} \Gamma(1-n, 2\pi i k z)$$

for $|\arg z| \le \pi - \delta < \pi$, $0 < \delta \le \pi$ and $n \ge 2$. This is the generalization of the result obtained by Paris and Wood [**18**]. The problem with this formula is that the optimal truncation $n = N \sim 2\pi|z|$ enables only the smooth transition of the leading subdominant exponential with $k = 1$, but it is not appropriate to describe the transition of the smaller exponentials corresponding to $k \ge 2$.

To overcome this problem, we choose the optimal truncation $n_k = n_{-k} = N_k \sim 2\pi k|z|$ $(k \ge 1)$ for each finite series in (11). It is expedient to write the remainder (12) in the form

$$\Delta(z,a,N_k) = \sum_{k=1}^{\infty} \frac{1}{k}\left(e^{2\pi i k(z+a)} \widehat{T}_{N_k+1}(2\pi i k z) - e^{-2\pi i k(z+a)} \widehat{T}_{N_k+1}(-2\pi i k z)\right), \tag{46}$$

where $\widehat{T}_\nu(w)$ is the so-called Terminant function, defined by

$$\widehat{T}_\nu(w) := \frac{\Gamma(\nu)}{2\pi i} e^{\pi i \nu} \Gamma(1-\nu, w).$$

When $\nu \sim |w|$, the Terminant function has the leading order behaviour as $w \to \infty$ [**10**, p. 263–264], [**15**, p. 1473]:

$$\widehat{T}_\nu(w) \sim \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(c(\varphi)\sqrt{\frac{1}{2}|w|}\right), \ -\pi + \delta \le \arg w \le 3\pi - \delta, \ 0 < \delta \le 2\pi; \tag{47}$$

$$\widehat{T}_\nu(w) \sim -\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(-\overline{c(-\varphi)}\sqrt{\frac{1}{2}|w|}\right), \ -3\pi + \delta \le \arg w \le \pi - \delta, \ 0 < \delta \le 2\pi. \tag{48}$$

Here $\varphi = \arg w$ and erf denotes the Error function [**17**, p. 160]. The quantity $c(\varphi)$ is defined implicitly by the equation

$$\frac{1}{2}c^2(\varphi) = 1 + i(\varphi - \pi) - e^{i(\varphi - \pi)},$$

and corresponds to the branch of $c(\varphi)$ which has the following expansion in the neighbourhood of $\varphi = \pi$:

$$c(\varphi) = (\varphi - \pi) + \frac{i}{6}(\varphi - \pi)^2 - \frac{1}{36}(\varphi - \pi)^3 - \frac{i}{270}(\varphi - \pi)^4 + \cdots . \tag{49}$$

In the upper half-plane, the dominant contribution to $\Delta(z, a, N_k)$ is controlled by the terms involving $\widehat{T}_{N_k+1}(2\pi ikz)$. From (47), the Stokes multiplier of each subdominant exponential $e^{2\pi ikz}$ in (46) then has the leading form given by

$$\frac{e^{2\pi ika}}{k}\widehat{T}_{N_k+1}(2\pi ikz) \sim \left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(c\left(\theta + \frac{\pi}{2}\right)\sqrt{\pi k|z|}\right)\right)\frac{e^{2\pi ika}}{k},$$

where $\theta = \arg z$. Similarly, in the lower half-plane, the dominant contribution arises from the terms involving $\widehat{T}_{N_k+1}(-2\pi ikz)$. From (48), the leading order behaviour of the Stokes multiplier of each subdominant exponential $e^{-2\pi ikz}$ takes the form

$$-\frac{e^{-2\pi ika}}{k}\widehat{T}_{N_k+1}(-2\pi ikz) \sim \left(\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(-\overline{c\left(-\theta + \frac{\pi}{2}\right)\sqrt{\pi k|z|}}\right)\right)\frac{e^{-2\pi ika}}{k}.$$

By (49), the approximate functional form of the Stokes multipliers near $\arg z = \pm\frac{\pi}{2}$ is then found to be

$$\left(\frac{1}{2} \pm \frac{1}{2}\mathrm{erf}\left(\left(\theta \mp \frac{\pi}{2}\right)\sqrt{\pi k|z|}\right)\right)\frac{e^{\pm 2\pi ika}}{k}.$$

For example, in the upper half-plane, the Stokes multipliers change from approximately 0 when $\theta < \frac{\pi}{2}$ to approximately $k^{-1}e^{2\pi ika}$ when $\theta > \frac{\pi}{2}$. On the Stokes line $\arg z = \frac{\pi}{2}$ they have the value $(2k)^{-1}e^{2\pi ika}$. Hence, we have proved that the transition of the Stokes multipliers across the Stokes lines $\arg z = \pm\frac{\pi}{2}$ is smooth.

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