Dynamic Portfolio Optimization with Transaction Costs: Heuristics and Dual Bounds

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First version: August 10, 2010
This version: February 3, 2011

Abstract

We consider the problem of dynamic portfolio optimization in a discrete-time, finite-horizon setting. Our general model considers risk aversion, portfolio constraints (e.g., no short positions), return predictability, and transaction costs. This problem is naturally formulated as a stochastic dynamic program. Unfortunately, with non-zero transaction costs, the dimension of the state space is at least as large as the number of assets and the problem is very difficult to solve with more than one or two assets.

In this paper, we consider several easy-to-compute heuristic trading strategies that are based on optimizing simpler models. We complement these heuristics with upper bounds on the performance with an optimal trading strategy. These bounds are based on the dual approach developed in Brown, Smith and Sun (2009). In this context, these bounds are given by considering an investor who has access to perfect information about future returns but is penalized for using this advance information. These heuristic strategies and bounds can be evaluated using Monte Carlo simulation.

We evaluate these heuristics and bounds in numerical experiments with a risk-free asset and three or ten risky assets. In many cases, the performance of the heuristic strategy is very close to the upper bound, indicating that the heuristic strategies are very nearly optimal.

Subject Classifications: Dynamic Programming, Portfolio Optimization

*We thank the anonymous Associate Editor and referees for their helpful comments.

1. Introduction

Dynamic portfolio theory – dating from the seminal work of Mossin (1968), Samuelson (1969) and Merton (1969, 1971) – provides a rigorous framework for determining optimal investment strategies in idealized environments that assume there are no transaction costs. The solutions to these models rely heavily on the absence of transaction costs. For example, in some models, the optimal solutions recommend holding constant fractions of the investor's wealth in different assets. To implement such a strategy, an investor must continually buy and sell assets in order to maintain the target fractions as asset prices fluctuate. In practice, transactions are costly and continual rebalancing can be quite expensive. If expected returns vary over time, the situation may be even worse as the investor continually trades to adjust to a moving target.

Constantinides (1979) studied a general discrete-time model of portfolio optimization with transaction costs and obtained the strongest structural results in the setting with power utility, proportional transaction costs, and two assets. Specifically, he showed that there is a two-dimensional convex cone of asset positions where it is optimal to not trade and that, when the asset position is outside of this cone, it is optimal to trade to bring the asset position to the boundary of the cone. Davis and Norman (1990) and Shreve and Soner (1994) analyzed continuous-time versions of this model with one risky and one risk-free asset and established analogous results. Muthuraman (2006) and others have developed numerical methods for the case with a single risky asset.

Of course, practitioners must actually contend with multiple risky assets and, unfortunately, this problem is much more difficult to solve. The portfolio optimization problem is naturally formulated as a stochastic dynamic program. With no transaction costs, the optimal investments typically depend on the investor's wealth but not the investor's asset positions. However, with transaction costs, the optimal investments depend on the investor's initial asset positions and the dimension of the state space is at least as large as the number of assets considered. The resulting dynamic program suffers from the "curse of dimensionality" and is very difficult to solve with more than one or two risky assets.[1]

In this paper, we consider the problem of dynamic portfolio optimization in a discrete-time, finite-horizon setting. Our general model considers risk aversion (we maximize the expected utility of terminal wealth), portfolio constraints (e.g., no short positions), the possibility of predictable returns, and convex transaction costs. We introduce several easy-to-compute heuristic trading strategies that are based on solving simpler optimization problems. The first heuristic is a "cost-blind" strategy that follows the trading strategy given by ignoring transaction costs; this is considered primarily as a benchmark for evaluating the performance of the other heuristics and bounds. The second heuristic is a "one-step" strategy that can be viewed

[1] The special case with a quadratic utility and quadratic transaction costs and no portfolio constraints can be formulated as a linear quadratic control problem that is straightforward to solve; see Gârleanu and Pedersen (2009).

as approximating the dynamic programming recursion by taking the continuation value to be the value function for the model that ignores transaction costs; transaction costs are considered in the current period only. Finally, we consider a "rolling buy-and-hold" strategy where, in each period, we solve an optimization problem with transaction costs with the simplifying assumption that there will be no further opportunities to trade over a fixed horizon; the continuation value at the end of the horizon is again taken to be the value function for the model that ignores transaction costs.

We complement these heuristics with upper bounds on the performance with an optimal trading strategy. These bounds are based on the dual approach developed in Brown, Smith and Sun (2009). In this context, the bounds are given by considering an investor who has access to perfect information about future returns but is penalized for using this advance information. The dual approach of Brown, Smith and Sun (2009) generalizes the dual approach developed for option pricing problems by Rogers (2002), Haugh and Kogan (2004), and Andersen and Broadie (2004) (see also related earlier work by Davis and Karatzas (1994)) to consider general stochastic dynamic programs; this generalization is essential for the application to portfolio optimization problems. In this paper, we generate penalties using the approach for "good penalties" suggested in Brown, Smith and Sun (2009) and develop a new gradient-based approach for generating penalties that exploits the convex structure of the underlying stochastic dynamic program. These heuristic strategies and bounds can be simultaneously evaluated using Monte Carlo simulation.

We evaluate these heuristics and bounds in numerical experiments with a risk-free asset and three risky assets (with predictable returns) or with ten risky assets (without predictability). The results are promising: the run times are reasonable and, in many cases, the performance of the heuristic strategy is very close to the upper bound, indicating that the heuristic strategies are very nearly optimal.

Brandt (2009) provides a recent survey of research in portfolio optimization and touches briefly on issues related to transaction costs. Closer to this paper, Akian, Menaldi and Sulem (1996), Leland (2000), Muthuraman and Kumar (2006) and Lynch and Tan (2009) consider portfolio choice problems with multiple risky assets and transaction costs in various settings. These papers develop analytic frameworks for the case with many assets, but focus on numerical examples with two risky assets; the numerical methods employed are based on grid approximations of the state space and do not scale well for problems with more risky assets. Muthuraman and Zha (2008) describe a numerical procedure that scales better: their procedure assumes a particular form of trading strategy, estimates the value function given this trading strategy using simulation, and then updates the trading strategy. The computational effort required by this scheme scales polynomially in the number of assets. Their example with seven risky assets requires approximately 62 hours to compute a trading strategy; there is no guarantee that the resulting strategy is optimal or any indication of how much better one might do with an optimal strategy. Chryssikou (1998) studies portfolio optimization with quadratic transaction costs and no constraints using a pair of heuristic strategies, one of which is similar to our rolling buy-and-hold strategy.

We view our contributions to be (i) the study of some easy-to-compute heuristic trading strategies that may be useful in practice and (ii) the development of a dual bounding approach that can be used to evaluate the quality of these and other heuristics. Both the heuristics and dual bounds are fairly flexible and can be adapted to problems with different forms of utility functions, different forms of transaction costs (provided they are convex), different forms of portfolio constraints and different models of returns.

There is also a recent literature that uses dual methods to evaluate portfolio strategies with portfolio constraints. For example, Haugh, Kogan, and Wang (2006) and Haugh and Jain (2010), following Cvitanić and Karatzas (1992) and others, use Lagrange multiplier methods to "dualize" portfolio constraints in continuous time portfolio allocation models; see Rogers (2003) for a review and synthesis of related theory. In contrast, we treat portfolio constraints directly and "dualize" the nonanticipativity constraints that require the investor to use only the information available at the time a decision is made; our penalties can be viewed as Lagrange multipliers associated with these nonanticipativity constraints. This interpretation is discussed in more detail in Brown, Smith and Sun (2009).

In the next section of the paper, we describe our basic model and note some structural properties of the model. In §3, we describe our heuristics. In §4, we describe our approach to dual bounds in general and discuss the specific bounds we consider in our numerical experiments. In §5, we describe our numerical experiments and results. The appendix contains proofs the propositions in the paper as well as some detailed assumptions and results for the numerical experiments.

2. The Portfolio Optimization Model

Time is discrete and indexed as $t = 0, \ldots, T$, with $t = 0$ being the current period and T being the terminal period. There are n risky assets and a risk-free asset (cash). The risk-free rate r_f is assumed to be known and constant over time. The returns of the risky assets are stochastic and denoted by $\boldsymbol{r}_t = (r_{t,1}, \ldots, r_{t,n})$ where $r_{t,i} \geq 0$ is the (gross) return on asset i from period t-1 to period t.

The monetary values of the risky asset holdings at the beginning of period t are described by the vector $\boldsymbol{x}_t = (x_{t,1}, \ldots, x_{t,n})$; the cash position in period t is denoted c_t. We let the trade vector $\boldsymbol{a}_t = (a_{t,1}, \ldots, a_{t,n})$ denote the amounts of risky assets bought (if $a_{t,n} > 0$) or sold (if $a_{t,n} < 0$) in period t. The transaction costs associated with trade vector $\boldsymbol{a}_t$ are given by $\kappa(\boldsymbol{a}_t)$. In our general analysis and approach, we will assume that $\kappa(\boldsymbol{a}_t)$ is a nonnegative and convex function of the trades $\boldsymbol{a}_t$ with $\kappa(\boldsymbol{0}) = 0$. In our numerical experiments, we will focus on the special case of proportional transaction costs with

$$\kappa(\boldsymbol{a}_t) = \sum_{i=1}^{n} \left(\delta_i^+ a_{t,i}^+ - \delta_i^- a_{t,i}^- \right) \ , \tag{1}$$

where $a_{t,i}^+ = \max(a_{t,i}, 0)$ and $a_{t,i}^- = \min(a_{t,i}, 0)$ denote the positive and negative components of the trades and $\delta_i^+, \delta_i^- \geq 0$ are the proportional costs for buying and selling (respectively) asset i. Alternatively, we could use a quadratic function for transaction costs to capture a "linear price impact," where trades lead to temporary linear changes in prices. Many other forms are also possible.

Taking transaction costs into account, the asset holdings and cash position evolve according to:

$$\boldsymbol{x}_{t+1} = \boldsymbol{r}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t), \tag{2}$$

$$c_{t+1} = r_f(c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)). \tag{3}$$

Here $\cdot$ denotes the componentwise product of two vectors (so $x_{t+1,i} = r_{t+1,i}(x_{t,i} + a_{t,i})$) and $\mathbf{1}$ is an n-vector of ones. The investor's wealth w_t in period t is the sum of the total dollar holdings across the risky assets and cash, i.e.,

$$w_t = \mathbf{1}'\boldsymbol{x}_t + c_t. \tag{4}$$

The investor's goal is to maximize the expected utility of terminal wealth, $\mathbb{E}\left[U(w_T)\right]$, where U is a non-decreasing and concave utility function. Note that in this formulation, we define wealth in terms of the market value of the portfolio. We could have instead defined wealth in terms of the liquidation value of the portfolio, including the transaction costs associated with liquidation. In this case, we would take $w_t = \mathbf{1}'\boldsymbol{x}_t - \kappa(-\boldsymbol{x}_t) + c_t$. Our general approach works in either case, though the numerical results would obviously be somewhat different.

We will assume that the investor's trades $\boldsymbol{a}_t$ in period t are restricted to a closed, convex set $\mathbb{A}_t(\boldsymbol{x}_t, c_t)$. In our numerical experiments, we will focus on the case where the investor is not allowed to have short positions in risky assets or cash, so given an asset position $(\boldsymbol{x}_t, c_t)$, the allowed trades are

$$\mathbb{A}_t(\boldsymbol{x}_t, c_t) = \left\{ \boldsymbol{a}_t \in \mathbb{R}^n \ : \ \boldsymbol{x}_t + \boldsymbol{a}_t \geq 0, \ c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t) \geq 0 \right\}. \tag{5}$$

We will also consider numerical results for the case where short positions are allowed, but there is a margin requirement that limits the total (long or short) position in risky assets to be no more than ℓ times the investor's post-trade wealth. In this case, the set of allowed trades is also convex and can be written:

$$\mathbb{A}_t(\boldsymbol{x}_t, c_t) = \left\{ \boldsymbol{a}_t \in \mathbb{R}^n \;\; : \;\; \sum_{i=1}^{n} |x_{t,i} + a_{t,i}| \;\leq\; \ell \left(\mathbf{1}'\boldsymbol{x}_t + c_t - \kappa(a_t) \right) \right\}. \tag{6}$$

In general, we consider sets of allowed trades $\mathbb{A}_t(\boldsymbol{x}_t, c_t)$ defined in terms of a set $\mathbb{H}_t$ of allowed final positions (or holdings): $\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t)$ if and only if $(\boldsymbol{x}_t + \boldsymbol{a}_t, c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)) \in \mathbb{H}_t$. We assume that the allowed set of final positions $\mathbb{H}_t$ is closed, convex and nondecreasing in c_t (if $(\boldsymbol{x}_t, c_t^1) \in \mathbb{H}_t$ and $c_t^1 \leq c_t^2$ then $(\boldsymbol{x}_t, c_t^2) \in \mathbb{H}_t$). This implies that $\mathbb{A}_t(\boldsymbol{x}_t, c_t)$ is convex for each $(\boldsymbol{x}_t, c_t)$.

We will allow the possibility that returns exhibit some degree of predictability. To model this, we let $\boldsymbol{z}_t$ denote a vector of observable market state variables that provides information about the returns $\boldsymbol{r}_{t+1}$ of the risky assets. We will assume that $\boldsymbol{z}_t$ follows a Markov process. The returns $\boldsymbol{r}_{t+1}$ may depend on $\boldsymbol{z}_t$ but, given $\boldsymbol{z}_t$, the returns are assumed to be conditionally independent of prior returns and earlier values of the market state variable.

This portfolio optimization problem can be formulated as a stochastic dynamic program with state variables consisting of the current positions in risky assets and cash $(\boldsymbol{x}_t, c_t)$ and the market state variable $(\boldsymbol{z}_t)$. We take the terminal value function to be the utility of terminal wealth, $V_T(\boldsymbol{x}_T, c_T, \boldsymbol{z}_T) = U(\mathbf{1}'\boldsymbol{x}_T + c_T)$, and earlier value functions V_t are given recursively as

$$V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t) \;=\; \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t, \boldsymbol{z}_t) \tag{7}$$

$$W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t, \boldsymbol{z}_t) \;=\; \mathbb{E}\left[V_{t+1}(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t),\; r_f(c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\; \tilde{\boldsymbol{z}}_{t+1}) \;|\; \boldsymbol{z}_t \right]. \tag{8}$$

In (8), expectations are taken over the random asset returns $\tilde{\boldsymbol{r}}_{t+1}$ and the next-period market state $\tilde{\boldsymbol{z}}_{t+1}$. We will assume that these expectations (and other expectations in the paper) are well defined and that maxima in (7) are attained by some set of trades.

The following proposition states some key properties of this portfolio optimization model.

Proposition 2.1. ***Properties of the portfolio optimization model.***

1. *For any market state $\boldsymbol{z}_t$, $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$ is nondecreasing in cash c_t and jointly concave in the asset position $(\boldsymbol{x}_t, c_t)$.*

2. *For any market state $\boldsymbol{z}_t$, $W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$ is jointly concave in the trades $\boldsymbol{a}_t$ and asset position $(\boldsymbol{x}_t, c_t)$.*

Thus, for any given market state $\boldsymbol{z}_t$ and asset and cash position $(\boldsymbol{x}_t, c_t)$, the optimization problem (7) is convex: we are maximizing a concave function over a convex set. Unfortunately the dimension of the state space makes the portfolio optimization problem very difficult to solve, even with just a few risky assets.

For example, suppose the market state variable $\boldsymbol{z}_t$ is one-dimensional. If we approximated the state space using a grid with 20 points for this market state variable and 100 points for each of the n+1 asset positions, the state space would consist of $20 \times 100^{n+1}$ states. To determine the value function $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$ on this grid, we would have to solve the optimization problem (7) for each of these $20 \times 100^{n+1}$ states in each period. In our numerical examples with $n = 3$ risky assets and predictability, the state space would include $20 \times 100^4 = 2$ billion elements. With $n = 10$ risky assets and no predictability, the state space would include $100^{11} = 10^{22}$ elements. Moreover, each of these optimization problems involves expectations (8) over the (n+1)-dimensional space of $(\boldsymbol{r}_{t+1}, \boldsymbol{z}_{t+1})$ outcomes and we would have to somehow interpolate between grid points when solving for the optimal trades.

If there are no transaction costs ($\kappa = 0$), the portfolio optimization problem can be greatly simplified by taking the dynamic programming state variables to be the current wealth (w_t) and market state variable ($\boldsymbol{z}_t$); we no longer need to consider the specific asset positions $(\boldsymbol{x}_t, c_t)$. In this simpler dynamic program, the decision variables are the post-trade positions in risky assets $\hat{\boldsymbol{x}}_t = \boldsymbol{x}_t + \boldsymbol{a}_t$. Let $\mathbb{X}_t(w_t)$ denote the set of possible post-trade positions in risky assets given initial wealth w_t; that is $\mathbb{X}_t(w_t) = \{\hat{\boldsymbol{x}}_t : (\hat{\boldsymbol{x}}_t, w_t - \mathbf{1}'\hat{\boldsymbol{x}}_t) \in \mathbb{H}_t\}$. For example, with no transaction costs, the case described by (5) where the investor is not allowed to have short positions corresponds to a feasible set of post-trade asset positions of the form

$$\mathbb{X}_t(w_t) = \{\hat{\boldsymbol{x}}_t \in \mathbb{R}^n \ : \ \hat{\boldsymbol{x}}_t \geq 0, \ \mathbf{1}'\hat{\boldsymbol{x}}_t \leq w_t\}. \tag{9}$$

We can then write the recursion for this "frictionless model" as follows: The terminal value function is $V_T^f(w_T, \boldsymbol{z}_T) = U(w_T)$ and earlier value functions are

$$V_t^f(w_t, \boldsymbol{z}_t) = \max_{\hat{\boldsymbol{x}}_t \in \mathbb{X}_t(w_t)} W_t^f(\hat{\boldsymbol{x}}_t, w_t, \boldsymbol{z}_t) \ , \tag{10}$$

$$W_t^f(\hat{\boldsymbol{x}}_t, w_t, \boldsymbol{z}_t) = \mathbb{E}\left[V_{t+1}^f\big(\tilde{\boldsymbol{r}}_{t+1}'\hat{\boldsymbol{x}}_t + r_f(w_t - \mathbf{1}'\hat{\boldsymbol{x}}_t), \ \tilde{\boldsymbol{z}}_{t+1}\big) \ \mid \boldsymbol{z}_t\right] . \tag{11}$$

This frictionless model also has a convex structure and its results can be related to those of the more complicated model with transaction costs.

Proposition 2.2. ***Properties of the frictionless portfolio optimization model.***

1. *For any market state $\boldsymbol{z}_t$, $V_t^f(w_t, \boldsymbol{z}_t)$ is nondecreasing and concave in wealth w_t.*

2. *For any market state $\boldsymbol{z}_t$, $W_t^f(\hat{\boldsymbol{x}}_t, w_t, \boldsymbol{z}_t)$ is jointly concave in the post-trade asset positions $\hat{\boldsymbol{x}}_t$ and wealth w_t.*

3. *For any market state $\boldsymbol{z}_t$ and asset position $(\boldsymbol{x}_t, c_t)$, $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t) \leq V_t^f(\mathbf{1}'\boldsymbol{x}_t + c_t, \boldsymbol{z}_t)$.*

Thus, to solve the frictionless model, we need to solve a convex optimization problem for each market state $\boldsymbol{z}_t$ and wealth w_t. For example, if the market state variable $\boldsymbol{z}_t$ is one-dimensional, we could solve this dynamic program on a two-dimensional grid involving $\boldsymbol{z}_t$ and w_t. The expectations over $(\tilde{\boldsymbol{r}}_{t+1}, \tilde{\boldsymbol{z}}_{t+1})$ in (11) will still be high-dimensional if we have many assets, but can be evaluated using various methods. In our numerical experiments, we will approximate these expectations using discrete approximations of the underlying distributions; see §5.1 below.

If the investor has a power utility function, the frictionless model simplifies further. Specifically, suppose

$$U(w_T) = \frac{1}{1-\gamma}\, w_T^{1-\gamma}, \tag{12}$$

where $\gamma > 0$ is the coefficient of relative risk aversion; in the case where $\gamma = 1$, $U(w_T) = \ln(w_T)$. We can then write the value function as

$$V_t^f(w_t, \boldsymbol{z}_t) = \frac{1}{1-\gamma}\, w_t^{1-\gamma}\, \phi_t(\boldsymbol{z}_t)\ , \tag{13}$$

where $\phi_t(\boldsymbol{z}_t)$ is defined recursively with $\phi_T(\boldsymbol{z}_T) = 1$ and

$$\frac{1}{1-\gamma}\phi_t(\boldsymbol{z}_t) = \max_{\hat{\boldsymbol{\theta}}_t \in \mathbb{X}_t(1)} \mathbb{E}\left[\frac{1}{1-\gamma}(\tilde{\boldsymbol{r}}_{t+1}' \hat{\boldsymbol{\theta}}_t + r_f(1 - \mathbf{1}'\hat{\boldsymbol{\theta}}_t))^{1-\gamma}\ \phi_{t+1}(\tilde{\boldsymbol{z}}_{t+1}) \mid \boldsymbol{z}_t\right]. \tag{14}$$

Here $\hat{\boldsymbol{\theta}}_t = (\hat{\theta}_1, \ldots, \hat{\theta}_n)$ are the post-trade fractions of wealth w_t invested in the risky assets. In this case, the dimension of the state space is equal to the dimension of the market state variable $\boldsymbol{z}_t$.

Note that if the next-period market state $\tilde{\boldsymbol{z}}_{t+1}$ and the returns $\tilde{\boldsymbol{r}}_{t+1}$ are independent given $\boldsymbol{z}_t$, then equation (14) factors and it is optimal to pursue a myopic trading strategy that maximizes the expected utility of next period's wealth. However, if the next-period market state $\tilde{\boldsymbol{z}}_{t+1}$ and the returns $\tilde{\boldsymbol{r}}_{t+1}$ are not independent and the investor has a relative risk aversion coefficient $\gamma > 1$ (as is considered to be typical), the optimal trading strategies will include some degree of hedging against unfavorable changes in the market state variable: compared to the myopic strategies, these strategies tend to have higher next-period wealth in scenarios with poor future prospects (i.e., with high values of $\phi_{t+1}(\tilde{\boldsymbol{z}}_{t+1})$) and lower next-period wealth in scenarios with better future prospects (i.e., with low values of $\phi_{t+1}(\tilde{\boldsymbol{z}}_{t+1})$).

3. Some Heuristic Trading Strategies

Given that the portfolio optimization model with transaction costs is difficult to solve, it is natural to consider heuristic trading strategies based on solutions to approximate models that are simpler to solve. We will consider several such heuristic strategies.

3.1. Cost-Blind Strategies

First, we consider *cost-blind strategies* that ignore transaction costs and simply follow the optimal trading strategy recommended by the frictionless model. To simulate such a strategy, we need to first solve the dynamic program for the frictionless model (10) to find the optimal post-trade asset positions $\hat{\boldsymbol{x}}_t$ or fractions $\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{x}}_t / w_t$ for each period t, market state $\boldsymbol{z}_t$ and wealth level w_t; we do this once and store the results. In the body of the simulation, in each period, we choose trades $\boldsymbol{a}_t$ to move to the investor to the recommended fractions $\hat{\boldsymbol{\theta}}_t$ for the current market state $\boldsymbol{z}_t$ and wealth level w_t. Given this trade, we generate random returns $\boldsymbol{r}_{t+1}$ for the risky assets and calculate next-period asset positions $(\boldsymbol{x}_{t+1}, c_{t+1})$ using equations (2) and (3), deducting transaction costs from the cash position. We then generate the next-period market state $\boldsymbol{z}_{t+1}$ and continue the simulation process for the next period.

Though this cost-blind strategy may perform reasonably well when the transaction costs are small or when it is optimal to put all of the investor's wealth in a single asset, with larger transaction costs and more balanced investments we would expect this strategy to trade too much in pursuit of marginal improvements in asset positions that do not exceed the cost of executing the trade.

3.2. One-Step Strategies

Second, we consider a heuristic strategy where the investor uses the value function from the frictionless model as an approximate continuation value, but includes transaction costs in the current period. In this case, in each period, the investor chooses trades $\boldsymbol{a}_t$ that solve:

$$\max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t)} \mathbb{E}\left[\ V_{t+1}^f\big(\tilde{\boldsymbol{r}}_{t+1}'(\boldsymbol{x}_t + \boldsymbol{a}_t) + r_f(c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+1}\big)\ \mid \boldsymbol{z}_t\right]. \tag{15}$$

To simulate such a *one-step strategy*, we first solve the dynamic program for the frictionless model (10) to determine the value function $V_t^f(w_t, \boldsymbol{z}_t)$; we do this in advance of the simulation and store the value function. In the body of the simulation, for each period in each trial, we solve the optimization problem (15) to find the recommended trade $\boldsymbol{a}_t$ for the then-prevailing $(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$ scenario. We then generate random returns $\boldsymbol{r}_{t+1}$ for the risky assets, calculate next-period asset positions $(\boldsymbol{x}_{t+1}, c_{t+1})$ using equations (2) and (3), generate a new market state $\boldsymbol{z}_{t+1}$ and continue to the next period. Note that the objective function in (15) is concave in the trades $\boldsymbol{a}_t$ (this follows from the assumption that the transaction cost function $\kappa(\boldsymbol{a}_t)$ is convex and the fact that $V_t^f(w_t, \boldsymbol{z}_t)$ is nondecreasing and concave in wealth w_t; see Proposition 2.2), so (15) is a convex optimization problem. The optimization problem is, however, complicated by the presence of the high-dimensional expectation in (15): these expectations must be evaluated to calculate the objective function for any candidate trade $\boldsymbol{a}_t$.

While the cost-blind strategies are likely to trade too much because they neglect the costs of trading, we would expect these one-step strategies to trade too little because they underestimate the benefits of moving towards the optimal position without transaction costs. Or, put another way, the frictionless model underestimates the cost of being out of the optimal position because it assumes the investor can costlessly adjust the position in the next period. If the optimal asset positions change slowly over time, moving towards the optimal position in one period may provide benefits in future periods as well as the current one.

Though we would have to solve the full portfolio optimization problem (7) to exactly capture the long-term impacts of adjusting portfolio positions, we can perhaps approximate this effect by reducing the transaction costs $\kappa(\boldsymbol{a}_t)$ appearing in the objective function in (15). There are a variety of ways we might modify these costs and we can experiment to find a good modification. In our numerical experiments, we consider monthly trades and focus on the case where we adjust the transaction costs by dividing $\kappa(\boldsymbol{a}_t)$ by dividing by a time-dependent constant. Specifically, we focus on the case where we divide by the smaller of 6 or the number of periods remaining $(T-t)$. The intuitive interpretation of this adjustment is that the benefit of adjusting the asset positions lasts approximately 6 months. We will call these trading strategies *modified one-step strategies* and consider alternative divisors in §5.4.

3.3. Rolling Buy-and-Hold Strategies

Finally, we consider a heuristic trading strategy where, in each period, the investor chooses trades to maximize the expected utility of wealth at some horizon h periods into the future, taking transaction costs into account, but assuming that there will be no opportunities to adjust the portfolio over this time horizon. As in the one-step strategy, the continuation value at the horizon is approximated by the value function for the frictionless model. That is, in period t, the investor chooses trades $\boldsymbol{a}_t$ that solve:

$$\max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t)} \mathbb{E}\left[V^f_{t+h}\big((\tilde{\boldsymbol{r}}_{t+h} \cdot \ldots \cdot \tilde{\boldsymbol{r}}_{t+1})'(\boldsymbol{x}_t + \boldsymbol{a}_t) + r_f^h(c_t - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+h}\big) \ \mid \boldsymbol{z}_t\right], \tag{16}$$

with the understanding that we use the terminal utility $U(w_T)$ in place of $V^f_{t+h}(w_{T+h})$ whenever $t+h > T$. Though this objective function assumes that there are no future opportunities to adjust the portfolio over this time horizon, when simulating or executing this strategy, the investor solves the same problem in the next period. As with the one-step strategies, the objective function in (16) is concave in $\boldsymbol{a}_t$ and, when simulating with this strategy, we must solve the convex optimization problem (16) once for each period of each simulated trial.

As with the modification of the one-step strategies, some degree of experimentation may be required to identify a good horizon h for a particular problem. In our numerical experiments with monthly trading, we will focus on the case where the horizon h is 6 months, but will consider alternatives in §5.4. We will refer to these heuristic trading strategies as the *rolling buy-and-hold strategies.* Chryssikou (1998) studies a similar heuristic but with the horizon fixed at the terminal period T, i.e., with terminal utility $U(w_T)$ in place of $V^f_{t+h}(w_{T+h})$ in (16).

4. Dual Bounds

We can evaluate the heuristic strategies of §3 using simulation and we can rerun these simulations with variations of these strategies (e.g., adjusting the modification of transaction costs for the one-step strategies or the horizon for the rolling buy-and-hold strategies) in an attempt to improve their performance. When doing these experiments, it would be helpful to know how much better we could possibly do. The frictionless model provides an upper bound on performance (see Proposition 2.2), but when transaction costs are substantial, this "no transaction cost bound" may be rather weak.

In this section, we will derive upper bounds on performance using the dual approach developed in Brown, Smith and Sun (2009). This dual approach consists of two elements: (i) we relax the "nonanticipativity" constraints that require the trading decisions to depend only on the information available at the time the decision is made and (ii) we impose penalties that punish violations of these nonanticipativity constraints. We first describe this dual approach in general and then describe the penalties we will use in our numerical experiments. As we will see, these dual bounds are typically tighter than the bound given by the model that ignores transaction costs.

4.1. The Dual Approach

In our discussion of the dual bounds, it helps to introduce notation to describe the full sequences of market states $\boldsymbol{z} = (\boldsymbol{z}_0, \ldots, \boldsymbol{z}_T)$, returns $\boldsymbol{r} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T)$ and trades $\boldsymbol{a} = (\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{T-1})$. Using this notation, we write the pre-trade asset and cash positions as $\boldsymbol{x}_t(\boldsymbol{a}, \boldsymbol{r})$ and $c_t(\boldsymbol{a})$ and wealth as $w_t(\boldsymbol{a}, \boldsymbol{r})$; these are calculated according to equations (2)-(4) and are given by

$$\begin{aligned}
\boldsymbol{x}_t(\boldsymbol{a}, \boldsymbol{r}) &= \sum_{\tau=0}^{t-1} (\boldsymbol{r}_t \cdot \ldots \cdot \boldsymbol{r}_{\tau+1}) \cdot \boldsymbol{a}_\tau \;+\; (\boldsymbol{r}_t \cdot \ldots \cdot \boldsymbol{r}_1) \cdot \boldsymbol{x}_0 \\
c_t(\boldsymbol{a}) &= \sum_{\tau=0}^{t-1} r_f^{t-\tau} \left(-\mathbf{1}'\boldsymbol{a}_\tau - \kappa(\boldsymbol{a}_\tau)\right) \;+\; r_f^t c_0
\end{aligned}$$

and $w_t(\boldsymbol{a}, \boldsymbol{r}) = \mathbf{1}'\boldsymbol{x}_t(\boldsymbol{a}, \boldsymbol{r}) + c_t(\boldsymbol{a})$. Similarly, we write the set of feasible trade sequences $\boldsymbol{a}$ as $\mathbb{A}(\boldsymbol{r})$. Note that, for any given return sequence $\boldsymbol{r}$, the position in risky assets $\boldsymbol{x}_t(\boldsymbol{a}, \boldsymbol{r})$ is linear in the trade sequence $\boldsymbol{a}$ and, with convex transaction costs, the cash position $c_t(\boldsymbol{a})$ and wealth $w_t(\boldsymbol{a}, \boldsymbol{r})$ are concave in the trade sequence $\boldsymbol{a}$.

A trading strategy can be viewed as a function $\alpha(\boldsymbol{r}, \boldsymbol{z})$ that maps from sequences of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$ to a trade sequence $\boldsymbol{a}$. A trading strategy α is feasible if (i) $\alpha(\boldsymbol{r}, \boldsymbol{z})$ is in $\mathbb{A}(\boldsymbol{r})$ for each $(\boldsymbol{r}, \boldsymbol{z})$, and (ii) α is nonanticipative in that the trade $\boldsymbol{a}_t$ selected in period t depends only on what is known in period t; that is, $\boldsymbol{a}_t$ depends on the market states $(\boldsymbol{z}_0, \ldots, \boldsymbol{z}_t)$ and the asset returns $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_t)$, but not the future market states or returns. We let $\mathcal{A}$ denote this set of feasible strategies. In this notation, we can rewrite the portfolio optimization problem (7) compactly as

$$\max_{\alpha \in \mathcal{A}} \mathbb{E}\left[U(w_T(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right]. \tag{17}$$

Here the expectations are taken over sequences of returns $\tilde{\boldsymbol{r}}$ and market states $\tilde{\boldsymbol{z}}$, with trading strategy α selecting trades in each $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})$ scenario.

In deriving dual upper bounds for this problem, we will focus on a "perfect information relaxation" that assumes the investor knows all market states $\boldsymbol{z}$ and all asset returns $\boldsymbol{r}$ before making any trading decisions. The penalties $\pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ depend on the sequence of trades $\boldsymbol{a}$, returns $\boldsymbol{r}$, and market states $\boldsymbol{z}$ in a given scenario; we say a penalty π is *dual feasible* if $\mathbb{E}\left[\pi(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] \leq 0$ for any feasible trading strategy α. We can then state the duality result as follows.

Proposition 4.1. ***Dual bound.*** *For any feasible trading strategy α and any dual feasible penalty π,*

$$\mathbb{E}\left[U(w_T(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right] \;\leq\; \mathbb{E}\left[\max_{\boldsymbol{a} \in \mathbb{A}(\tilde{\boldsymbol{r}})} \left\{U(w_T(\boldsymbol{a}, \tilde{\boldsymbol{r}})) - \pi(\boldsymbol{a}, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right\}\right]. \tag{18}$$

The problem on the right of (18) is perhaps easiest to understand by considering how we estimate this expression using simulation. In each trial of the simulation, we generate a sequence of market states $\boldsymbol{z}$ and asset returns $\boldsymbol{r}$, drawing samples according to their joint stochastic process. We then solve a deterministic "inner problem" of the form:

$$\max_{\boldsymbol{a} \in \mathbb{A}(\boldsymbol{r})} \left\{U(w_T(\boldsymbol{a}, \boldsymbol{r})) - \pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})\right\} \tag{19}$$

to find the sequence of trades $\boldsymbol{a}$ in $\mathbb{A}(\boldsymbol{r})$ that maximizes the penalized objective, $U(w_T(\boldsymbol{a}, \boldsymbol{r})) - \pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$, assuming perfect foresight, i.e., assuming that the full sequences of market states $\boldsymbol{z}$ and asset returns $\boldsymbol{r}$ are known. We obtain an estimate of the dual bound (18) by averaging the optimal values from these inner problems across the trials of the simulation. Note that since wealth $w_T(\boldsymbol{a}, \boldsymbol{r})$ is concave in the trade sequence $\boldsymbol{a}$ and the utility function is increasing and concave in wealth, the utility of final wealth is concave in $\boldsymbol{a}$ for a given return sequence $\boldsymbol{r}$. If we consider penalties $\pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ that are convex in the trade sequence $\boldsymbol{a}$ for given sequences of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$, the inner problem (19) will be a deterministic convex optimization problem in $\boldsymbol{a}$ and will not be difficult to solve.

It is not hard to see that inequality (18) holds with any dual feasible penalty, for any feasible strategy α. To see this, note:

$$\begin{aligned}
\mathbb{E}\left[U(w_T(\alpha(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}}))\right] &\leq \mathbb{E}\left[U(w_T(\alpha(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}})) - \pi(\alpha(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}})\right] \\
&\leq \mathbb{E}\left[\max_{\boldsymbol{a}\in\mathbb{A}(\tilde{\boldsymbol{r}})}\left\{U(w_T(\boldsymbol{a},\tilde{\boldsymbol{r}})) - \pi(\boldsymbol{a},\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}})\right\}\right].
\end{aligned} \tag{20}$$

The first inequality follows from the assumption that π is dual feasible; since α is assumed to be a feasible trading strategy, this implies $\mathbb{E}\left[\pi(\alpha(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}})\right] \leq 0$. The second inequality follows from the fact that the value with perfect foresight must meet or exceed the value of any feasible trading strategy α: The investor with perfect foresight could choose the sequence of trades that would be chosen by α in each scenario and obtain the same value. However, the investor with perfect foresight can usually do better by choosing a different sequence of trades that maximizes the penalized objective in the given scenario.

The bound (18) is a special case of the weak duality result from Brown, Smith and Sun (2009). Brown, Smith and Sun (2009) also show that strong duality holds in this framework in that there exists an optimal penalty π^* such that the inequality in (18) will hold with equality with this π^* and an optimal strategy α^*; "complementary slackness" also holds in that an optimal penalty π^* will lead to trades $\boldsymbol{a}$ in the dual problem that match those of an optimal strategy.

Note that the penalty $\pi = 0$ is trivially dual feasible. In this case, the inner problem (19) amounts to finding an optimal trading strategy given perfect knowledge of all future returns and the dual bound (18) is the expected utility with perfect information. This inner problem is straightforward to solve, but the bound is typically quite weak. To obtain tighter bounds, we need to choose a penalty that reduces the benefit provided by having advance knowledge of future market states and returns. In addition, to ensure reasonable computational times, the penalties should be easy to compute and lead to an inner problem (19) that is easy to solve. We will consider two types of penalties. First we will consider penalties that are constructed following the prescription for "good penalties" from Brown, Smith and Sun (2009). Second, we will consider a new type of penalty that exploits the convex structure of the primal problem.

4.2. Penalties Based on Approximate Value Functions

Brown, Smith and Sun (2009) suggest constructing penalties by choosing a sequence of generating functions $(g_0, \ldots, g_T)$ that approximate the continuation value functions for the dynamic programming model. In this context, the generating functions $g_t(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})$ may depend on the full sequences of returns $\boldsymbol{r}$ and market states

$\boldsymbol{z}$, but depend only on trades up to period t, $(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_t)$. The penalty is then taken to be

$$\pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) = \sum_{t=0}^{T} \left(g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) - \mathbb{E}\left[g_t(\boldsymbol{a}, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}) \mid \boldsymbol{r}_1, \ldots, \boldsymbol{r}_t, \boldsymbol{z}_0, \ldots, \boldsymbol{z}_t \right] \right). \tag{21}$$

Brown, Smith and Sun (2009) show that penalties constructed this way are dual feasible. Moreover, if we take the generating functions $g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ to be the optimal continuation values $V_{t+1}(\boldsymbol{x}_{t+1}(\boldsymbol{a}, \boldsymbol{r}), c_{t+1}(\boldsymbol{a}), \boldsymbol{z}_{t+1})$ for the original portfolio optimization program (7), the resulting "ideal penalty" is optimal: it provides a dual bound equal to the optimal value for the primal and the optimal trades in the dual problem are feasible and optimal for the primal problem.

We can approximate this ideal penalty using approximations of the continuation values as generating functions. For example, consider the continuation values for the one-step strategies; we could approximate the continuation value using the continuation value from the frictionless model (10) by taking the generating function $g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ to be $V^f_{t+1}(w_{t+1}(\boldsymbol{a}, \boldsymbol{r}), \boldsymbol{z}_{t+1})$. Though this frictionless value function is reasonably easy to compute, it leads to an inner problem that is not easy to solve: $V^f_{t+1}(w_{t+1}(\boldsymbol{a}, \boldsymbol{r}), \boldsymbol{z}_{t+1})$ is a concave function of the trades $\boldsymbol{a}$ but when used to generate a penalty π using (21), V^f_{t+1} enters into the objective for the inner problem with both positive and negative signs, leading to an objective function that is neither convex nor concave and an inner problem that is generally not easy to solve.

To overcome this difficulty, we will take $g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ to be a linear approximation of $V^f_{t+1}(w_{t+1}(\boldsymbol{a}, \boldsymbol{r}), \boldsymbol{z}_{t+1})$ based on a first-order Taylor series expansion in the trades $\boldsymbol{a}$ around the trades $\boldsymbol{a}^*$ given by some fixed trading strategy. For example, in the case with proportional transaction costs given by equation (1), the linear approximation of V^f_{t+1} yields a generating function of the form

$$\begin{aligned} g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) \quad &= \quad V^f_{t+1}(w_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}),\ \boldsymbol{z}_{t+1}) \\ &\quad + V^{f\prime}_{t+1}(w_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}),\ \boldsymbol{z}_{t+1}) \sum_{\tau=1}^{t} \sum_{i=1}^{n} \left(\frac{\partial w_{t+1}}{\partial a^+_{\tau,i}} \left(a^+_{\tau,i} - a^{*+}_{\tau,i} \right) + \frac{\partial w_{t+1}}{\partial a^-_{\tau,i}} \left(a^-_{\tau,i} - a^{*-}_{\tau,i} \right) \right), \end{aligned} \tag{22}$$

where $V^{f\prime}_{t+1}$ denotes the derivative of V^f_{t+1} with respect to wealth and $a^+_{t,i}, a^-_{t,i}, a^{*+}_{t,i}$, and $a^{*+}_{t,i}$ denote the positive and negative components of $\boldsymbol{a}$ and $\boldsymbol{a}^*$. With a power utility function, V^f_{t+1} can be calculated analytically as $V^{f\prime}_t(w_t, \boldsymbol{z}_t) = w_t^{-\gamma} \phi_t(\boldsymbol{z}_t)$, where $\phi_t(\boldsymbol{z}_t)$ is determined when solving the frictionless model (14); in other cases,

V_{t+1}^f may have to be estimated numerically. The partial derivatives in (22) are given by:

$$
\begin{aligned}
\frac{\partial w_{t+1}}{\partial a_{\tau,i}^+} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \;-\; r_f^{t+1-\tau}\left(1+\delta_i^+\right) \\
\frac{\partial w_{t+1}}{\partial a_{\tau,i}^-} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \;-\; r_f^{t+1-\tau}\left(1-\delta_i^-\right).
\end{aligned}
$$

With a generating function of the form of (22), using (21) we obtain a dual feasible penalty π that is linear in the trades $\boldsymbol{a}$ (or, more precisely, linear in the positive and negative components of $\boldsymbol{a}$), for any sequence of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$. The objective for the inner problem (19) is then concave in $\boldsymbol{a}$ and the resulting inner problem is a deterministic convex optimization problem that is not difficult to solve. Note that there are high-dimensional expectations (over returns $\boldsymbol{r}_{t+1}$ and the market state $\boldsymbol{z}_{t+1}$) in the definition of the penalty (21); however, these expectations only affect the weights associated with the trades in this linear penalty and the weights need only be calculated once when solving the inner problem in a simulated scenario. In our numerical experiments, we will consider bounds generated by penalties of this form, taking $\boldsymbol{a}^*$ to be the trades suggested by the modification of the one-step heuristic strategy. We will call these bounds the *modified one-step bounds.*

We can construct a similar penalty using a generating function based on the analogue of the continuation value used to determine the rolling buy-and-hold strategy,

$$
\mathbb{E}\left[\; V_{t+h}^f\Big((\tilde{\boldsymbol{r}}_{t+h}\cdot\ldots\cdot\tilde{\boldsymbol{r}}_{t+2})'\boldsymbol{x}_{t+1}(\boldsymbol{a},\boldsymbol{r}) + r_f^{h-1}c_{t+1}(\boldsymbol{a}),\ \tilde{\boldsymbol{z}}_{t+h}\Big) \;\mid \boldsymbol{z}_{t+1}\right]. \tag{23}
$$

Note that this is a function of the period-t+1 market-state $\boldsymbol{z}_{t+1}$ (by conditioning) and returns $\boldsymbol{r}_{t+1}$ (through $\boldsymbol{x}_{t+1}$), but does not depend on later market states or returns as these are integrated out in the expectations. Here too we will consider a generating function based on a first-order Taylor series expansion of (23) in the trades $\boldsymbol{a}$ around the trades $\boldsymbol{a}^*$ given by some heuristic strategy; the details are provided in the appendix. As with the modified one-step penalty, this leads to a dual feasible penalty π that is linear in the positive and negative components of $\boldsymbol{a}$ for any sequence of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$. In our numerical experiments, we will consider bounds generated by penalties of this form, taking $\boldsymbol{a}^*$ to be the trades suggested by a rolling buy-and-hold strategy. We will call these the *rolling buy-and-hold dual bounds.*

4.3. Gradient-Based Penalties

We will also consider gradient-based penalties that exploit the convex structure of the primal optimization problem. To understand the motivation for this approach, assume (for the sake of this motivating discussion) that the utility of terminal wealth $U(w_T(\boldsymbol{a},\boldsymbol{r}))$ is differentiable in the sequence of trades $\boldsymbol{a}$ and that an optimal trading strategy α^* for the portfolio optimization problem with transaction costs is known. Then suppose we take the penalty $\pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z})$ to be

$$\pi(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) = \nabla_{\boldsymbol{a}} U(w_T(\alpha^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))' \, (\boldsymbol{a} - \alpha^*(\boldsymbol{r}, \boldsymbol{z})) \tag{24}$$

where $\nabla_{\boldsymbol{a}} U(w_T(\boldsymbol{a}, \boldsymbol{r}))$ is the gradient of terminal utility with respect to the trade sequence $\boldsymbol{a}$, for a given sequence of returns $\boldsymbol{r}$. Note that this penalty is linear in the trade sequence $\boldsymbol{a}$.

We can view the primal problem (17) as a convex optimization problem with the decision variables being the trading strategy α; this will be formalized in the proof of the proposition below. The first-order conditions for this optimization problem can be shown to imply that

$$\mathbb{E}\left[\pi(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[\nabla_{\boldsymbol{a}} U(w_T(\alpha^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))' \, (\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}) - \alpha^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}))\right] \leq 0 \tag{25}$$

for any feasible strategy α. This means the penalty (24) is dual feasible.

Now consider the deterministic inner problem (19) given by this penalty:

$$\max_{\boldsymbol{a} \in \mathbb{A}(\boldsymbol{r})} \left\{ U(w_T(\boldsymbol{a}, \boldsymbol{r})) - \nabla_{\boldsymbol{a}} U(w_T(\alpha^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))' \, (\boldsymbol{a} - \alpha^*(\boldsymbol{r}, \boldsymbol{z})) \right\}. \tag{26}$$

Since the penalty is linear in the trade sequence $\boldsymbol{a}$, the inner problem (26) is a convex optimization problem and its first-order conditions are necessary and sufficient for an optimal solution. The gradient of the objective function in (26) with respect to the trade sequence $\boldsymbol{a}$ is

$$\nabla_{\boldsymbol{a}} U(w_T(\boldsymbol{a}, \boldsymbol{r})) - \nabla_{\boldsymbol{a}} U(w_T(\alpha^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r})) \ . \tag{27}$$

Now note that if we take the trade sequence to be that selected by the optimal strategy, i.e., $\boldsymbol{a} = \alpha^*(\boldsymbol{r}, \boldsymbol{z})$, the gradient (27) is equal to zero. Since this $\boldsymbol{a}$ is in $\mathbb{A}(\boldsymbol{r})$ and it sets the gradient equal to zero, this $\boldsymbol{a}$ must be an optimal solution for the inner problem (26). Moreover, with $\boldsymbol{a} = \alpha^*(\boldsymbol{r}, \boldsymbol{z})$, the penalty (24) is zero and the objective for the inner problem (26) reduces to $U(w_T(\alpha^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))$ and the dual bound is $\mathbb{E}\left[U(w_T(\alpha^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right]$. Thus, the penalty (24) is optimal: it yields a dual trading strategy that is optimal for the primal problem and a dual bound equal to the optimal value for the primal.

Of course, in practice we do not know the optimal strategy α^* for the portfolio optimization problem and cannot use the penalty (24). We can, however, approximate the original problem and use similar penalties based on the optimal solution to this approximate problem. For example, we can approximate the original problem by considering the frictionless model (10). If we take $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$ to be the terminal wealth without transaction costs and $\hat{\mathcal{A}}$ to be the set of feasible trading strategies without transaction costs (and let $\hat{\mathbb{A}}(\boldsymbol{r})$ be the set of feasible trades in the approximate model given returns $\boldsymbol{r}$), we can then write the approximate optimization problem based on the frictionless model as

$$\max_{\alpha \in \hat{\mathcal{A}}} \mathbb{E}\left[U(\hat{w}_T(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right]. \tag{28}$$

Let $\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})$ be an optimal trading strategy for this approximating frictionless model and consider a gradient-based penalty of the form of (24), but with $\hat{w}_T$ in place of w_T and $\hat{\alpha}^*$ in place of α^*, i.e.,

$$\hat{\pi}(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) = \nabla_{\boldsymbol{a}} U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))' \; (\boldsymbol{a} - \hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})) \; . \tag{29}$$

The argument leading to (25) requires the strategy to be optimal for the chosen wealth function (i.e., for (28)), but it does not require the strategy to be optimal for the true wealth function with transaction costs. If the set of feasible strategies for the approximate model $\hat{\mathcal{A}}$ includes those for the real model $\mathcal{A}$ (i.e., $\mathbb{A}(\boldsymbol{r}) \subseteq \hat{\mathbb{A}}(\boldsymbol{r})$ for all $\boldsymbol{r}$), then $\mathbb{E}\left[\hat{\pi}(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right]$ will hold for all α in $\hat{\mathcal{A}}$ and this approximate penalty $\hat{\pi}$ will be dual feasible for the original problem. However, the inner problem with this approximate penalty,

$$\max_{\boldsymbol{a} \in \mathbb{A}(\boldsymbol{r})} \left\{ U(w_T(\boldsymbol{a}, \boldsymbol{r})) - \nabla_{\boldsymbol{a}} U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))'(\boldsymbol{a} - \hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})) \right\}, \tag{30}$$

will generally not be optimized by taking the trade sequence to be $\boldsymbol{a} = \hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})$. Nevertheless, because the penalty is dual feasible, the dual problem with this approximate penalty will provide a valid upper bound on the performance of any feasible trading strategy.

We can use this gradient-based approach with a variety of approximations of the wealth function as long as the approximate wealth function is concave and we can identify an optimal strategy for the approximate problem. The following proposition formalizes this gradient-based approach to penalties.

Proposition 4.2. ***Gradient-based penalties***. *Let $\hat{\alpha}^*$ be an optimal trading strategy for the portfolio choice problem (28) with modified terminal wealth $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$, assumed concave in $\boldsymbol{a}$, and modified allowable trades $\hat{\mathbb{A}}(\boldsymbol{r})$, assumed convex and satisfying $\mathbb{A}(\boldsymbol{r}) \subseteq \hat{\mathbb{A}}(\boldsymbol{r})$ for each return sequence $\boldsymbol{r}$. Consider the penalty $\hat{\pi}$ given by equation (29):*

1. *$\hat{\pi}$ is dual feasible.*

2. *If $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r}) = w_T(\boldsymbol{a}, \boldsymbol{r})$ and $\mathbb{A}(\boldsymbol{r}) = \hat{\mathbb{A}}(\boldsymbol{r})$ for each return sequence $\boldsymbol{r}$, then the dual bound (18) holds with equality with penalty $\hat{\pi}$.*

3. If $w_T(\boldsymbol{a},\boldsymbol{r}) \leq \hat{w}_T(\boldsymbol{a},\boldsymbol{r})$, then,

$$\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})} \Big\{ U(w_T(\boldsymbol{a},\boldsymbol{r})) - \hat{\pi}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z}) \Big\} \quad \leq \quad U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r})). \tag{31}$$

The first two parts of the proposition formalize the results discussed earlier. We will discuss the last part of the proposition in a moment. Note that our definition of gradient-based penalties $\hat{\pi}$ in equation (29) implicitly assumes that U and $\hat{w}_T$ are differentiable so the necessary gradients exist and $\hat{\pi}$ corresponds to a directional derivative. If the gradient does not exist, we can define $\hat{\pi}$ in terms of the directional derivative instead; this directional derivative will exist whenever $U \circ \hat{w}_T$ is concave. The use of the directional derivatives is discussed in more detail in the proof of the result in the appendix.

In our numerical experiments, we will consider two examples of gradient-based penalties. The first is based on the frictionless model, as discussed above. In this case, the approximate terminal wealth $\hat{w}_T(\boldsymbol{a},\boldsymbol{r})$ is given by taking the transaction costs to be zero and the allowable trade sequences $\hat{\mathbb{A}}(\boldsymbol{r})$ are the same as in the original model but without the transaction costs: $\mathbb{A}(\boldsymbol{r}) \subseteq \hat{\mathbb{A}}(\boldsymbol{r})$ then follows from our assumption that the set of feasible asset positions $\mathbb{H}_t$ is nondecreasing in cash. In this frictionless model, the gradient of the utility of terminal wealth is $\nabla_{\boldsymbol{a}} U(\hat{w}_T(\boldsymbol{a},\boldsymbol{r})) = U'(\hat{w}_T(\boldsymbol{a},\boldsymbol{r}))\nabla_{\boldsymbol{a}}\hat{w}_T(\boldsymbol{a},\boldsymbol{r})$, where U' is the derivative of the utility function and $\nabla_{\boldsymbol{a}}\hat{w}_T(\boldsymbol{a},\boldsymbol{r})$ is a $nT \times 1$ vector with entries corresponding to trade $\boldsymbol{a}_t$ given by

$$\nabla_{\boldsymbol{a}_t}\hat{w}_T(\boldsymbol{a},\boldsymbol{r}) \quad = \quad (\boldsymbol{r}_T \cdot \ldots \cdot \boldsymbol{r}_{t+1}) - r_f^{T-t}\mathbf{1} \ .$$

We will call the resulting penalty the *frictionless gradient-based penalty.*

Note that with this approximation, the wealth with transaction costs $w_T(\boldsymbol{a},\boldsymbol{r})$ are less than or equal to the wealth without transaction costs $\hat{w}_T(\boldsymbol{r},\boldsymbol{z})$ (for all $\boldsymbol{r}$ and $\boldsymbol{z}$), so the last part of Proposition 4.2 applies: the optimal values for the inner problems with this penalty (on the left side of (31)) will be less than or equal to the utility of final wealth with no transaction costs (on the right side of (31)) for every $\boldsymbol{r}$ and $\boldsymbol{z}$. This implies that the dual bounds using this frictionless gradient penalty must be at least as tight as the no transaction cost bound given by the value function for the frictionless model.

Although the frictionless gradient penalty leads to tighter bounds than the frictionless model, we can perhaps do better if we somehow incorporate the effects of transaction costs in the approximate model. The key for the gradient penalty approach is to do this in a way that still allows us to find the optimal solution for the approximate model. One way to do this is to consider a variation of the original model where the transaction costs depend on the post-trade asset positions rather than the trades. In this case, the transaction costs are of the form $\hat{\kappa}(\hat{\boldsymbol{x}}_t)$, where $\hat{\boldsymbol{x}}_t = \boldsymbol{x}_t + \boldsymbol{a}_t$ and the cash position evolves according to

$c_{t+1} = r_f(c_t - \mathbf{1}'\boldsymbol{a}_t - \hat{\kappa}(\boldsymbol{x}_t + \boldsymbol{a}_t))$, rather than equation (3). In such a case, we can represent the portfolio problem as a dynamic program like that of the frictionless model (10) with wealth w_t and the market state $\boldsymbol{z}_t$ as state variables and post-trade asset positions $\hat{\boldsymbol{x}}_t$ as decision variables, without considering the specifics of the asset positions $(\boldsymbol{x}_t, c_t)$.

In our numerical experiments, we will consider a *modified gradient-based penalty* where we take $\hat{\kappa}(\hat{\boldsymbol{x}}_t)$ to be proportional to the post-trade asset positions $\hat{\boldsymbol{x}}_t$. Specifically, we will take the proportional fee for the asset positions to be equal to the proportional fee for trades divided by the number of periods (T) in the model. More generally, we could consider $\hat{\kappa}(\hat{\boldsymbol{x}}_t)$ that assume transaction costs are proportional to the difference between the post-trade position $\hat{\boldsymbol{x}}_t$ and some reference position. For example, we might take this reference position to be the asset allocation recommended by the frictionless model for the same market state or, alternatively, the initial (period 0) asset position. Our modified gradient bound can be viewed as an example of this general form where the reference position is taken to be a zero position (i.e., with zero investment in each risky asset), which is the assumed initial position in the experiments. Of course, there are a number of possible variations on these ideas and we could experiment to perhaps find better bounds.

5. Numerical Experiments

In this section, we describe the experiments that we use to test the proposed trading strategies and dual bounds. We first describe the details of the models considered and then discuss the run times and numerical results. We then consider variations on the heuristics and bounds as well as the constraints.

5.1. Model Details

We will test the heuristic strategies and dual bounds by evaluating these heuristics and bounds in a series of simulations with varying parameter values. In all cases, we begin by solving the dynamic program for the frictionless model (equations (10) and (11)) for the given parameter values; we also solve the analogous dynamic program used to determine the modified gradient penalty of §4.3. We then repeatedly generate random sequences of market states and returns. For each sequence of market states and returns, we "run" the heuristic strategies of §3, determining the sequence of trades selected by the heuristic and the corresponding terminal wealth and utility. We also solve the inner problem for each of the dual bounds in this same scenario. We repeat this simulation process for a given number of trials.

In our experiments, we assume monthly time steps, proportional transaction costs $\kappa(\boldsymbol{a}_t) = \delta \sum_{i=1}^{n} |a_{t,i}|$, and power utilities. We will consider a variety of parameters:

- time horizons T of 6, 12, 24, and 48 months;
- transaction cost rates δ of 0.5%, 1.0% and 2.0%;

- relative risk aversion coefficients γ of 1.5, 3.0 or 8.0, reflecting low, medium and high degrees of risk aversion.

In the next three subsections, we will focus on the case with constraint (5) ruling out short positions; we consider limited leverage constraints of the form of (6) in §5.5. In all cases, we assume the investor starts with all wealth invested in cash and normalize wealth to one, i.e., we assume $\boldsymbol{x}_0 = \boldsymbol{0}$ and $c_0 = 1$. We will consider 1000 trials in each simulation.

We consider two different models of returns. The first highlights the role of predictability and the second considers a larger number of risky assets.

Model with Three Risky Assets and Predictability. We first consider a model with three risky assets and one market state variable, based on Lynch (2001); Lynch studied the impact of predictability on portfolio choices, without considering transaction costs. Specifically, letting $\boldsymbol{\rho}_t = \ln \boldsymbol{r}_t$, the model assumes returns and market states evolve according to

$$\begin{bmatrix} \boldsymbol{\rho}_{t+1} \\ \boldsymbol{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_r + \boldsymbol{b}_r \boldsymbol{z}_t \\ \boldsymbol{a}_z + \boldsymbol{b}_z \boldsymbol{z}_t \end{bmatrix} + \begin{bmatrix} \boldsymbol{e}_{t+1} \\ \boldsymbol{v}_{t+1} \end{bmatrix} , \tag{32}$$

where the stochastic increments $(\boldsymbol{e}_{t+1}\ \boldsymbol{v}_{t+1})$ are multivariate normal with mean zero and covariance $\boldsymbol{\Sigma}_{ev}$. The three risky assets correspond to value-weighted equity portfolios sorted by the size of the underlying firms (i.e, small-, medium- and large-cap stocks) and the market state variable is a normalized index reflecting the term spread (specifically, the difference between 20-year and one-month treasury bonds). Lynch estimates this model using data from 1927 to 1996. The returns are inflation-adjusted and the risk-free rate r_f is 1.00042. The other numerical assumptions are discussed in the appendix.

In this model, the market state variable has a significant impact on expected returns. With no transaction costs and medium risk aversion ($\gamma = 3.0$), we find that with high values of $\boldsymbol{z}_t$ (i.e., with a large term spread), the investor should invest heavily in a mix of small- and medium-sized stocks and hold no cash. With $\boldsymbol{z}_t = 0$, the investor should invest in a mix of all three assets, while holding substantial reserves in cash. With negative values for $\boldsymbol{z}_t$, the investor should invest most of his wealth in cash. In our numerical experiments, we will assume the initial market state is neutral (i.e., $\boldsymbol{z}_0 = 0$).

We follow Lynch (2001) and use discrete approximations of the uncertainties to calculate expectations. We approximate the market state variable using a grid with 19 points. The idiosyncratic returns ($\boldsymbol{e}_{t+1}$ in equation (32)) are approximated using a Gaussian quadrature approach with three points per asset. This Gaussian quadrature approximation exactly matches the mean and covariance structure for log returns $\boldsymbol{\rho}_t$ and matches higher-order moments (3rd-5th) of this joint distribution as well. (See, e.g., Judd 1998, for an introduction to Gaussian quadrature methods.) Taken together, the joint distribution for returns and the market-state variable is approximated using a four-dimensional grid with a total of $3^3 \times 19 = 513$ elements. This discrete approximation scheme is used to calculate the expectations required to solve the dynamic programming model for the frictionless model, to evaluate the expectations in the optimization problems for the heuristic trading strategies (in §3), and to evaluate the expectations appearing in the penalties based on approximate value functions (in §4.2). For consistency, we also use this discrete approximation in the simulations, i.e., we generate sample returns and market states from this grid according to the probabilities of the discrete approximation.

Model with Ten Risky Assets and No Predictability. We also consider examples with ten risky assets and no predictability. In this case, we assume that the asset returns follow a discrete-time multivariate geometric Brownian motion process, a special case of (32) with $\boldsymbol{\rho}_t = \ln \boldsymbol{r}_t$ evolving according to $\boldsymbol{\rho}_{t+1} = \boldsymbol{a}_r + \boldsymbol{e}_{t+1}$ where the stochastic increments $\boldsymbol{e}_{t+1}$ are multivariate normal with mean zero and covariance $\boldsymbol{\Sigma}_e$. In this example, the ten risky assets correspond to five equity indices (S&P 500, Russell 2000 Value, MSCI World Gross, Russell 1000 Value Index, Russell MidCap Index), three bond indices (Lehman Brothers' US government and corporate bond indices and Lehman Brothers' Fixed Rate Mortgage Backed Securities Index), a real estate index trust (NAREIT), and a composite index of 1-5 Year US Treasuries. The parameters were estimated using monthly return data from 1981-2006 and are provided in the appendix. The risk-free rate $r_f = 1.0048$ is the average return on three-month US treasuries, estimated from the same data set. These returns are not inflation-adjusted.

We also use discrete approximations of the uncertainties to calculate expectations in this model. The idiosyncratic returns ($\boldsymbol{e}_{t+1}$) are approximated using a multidimensional quadrature (or cubature) formula in Stroud (1971, p. 317) that includes $2^n + 2n$ points and exactly matches the first five moments of the return distribution. With 10 assets, the return distribution is approximated using a ten-dimensional grid with a total of 1044 elements. There are a variety of different approaches we could use to calculate expectations in these models and there is a tradeoff between the accuracy of the approximation and the amount of work involved. Stroud (1971) provides a comprehensive review of multidimensional quadrature formulas; one such formula matches three moments of the underlying distribution and involves only $2n$ points. Alternatively, we might consider evaluating these expectations using Monte Carlo or Quasi Monte Carlo methods; see, e.g., Judd (1998) or Glasserman (2004) for discussions of these approaches.

5.2. Run Times

Table 1 provides the run times for evaluating the heuristics and dual bounds in a simulation with 1000 trials for the two different return models and four different time horizons (T). In all cases, we consider the case

Table 1: Run Times (seconds) Required to Evaluate Heuristic Strategies and Dual Bounds

	Simple DP Models		Heuristic Strategies				Dual Bounds				
Horizon (T)	No Trans. Cost	Modified Trans. Cost Model	Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient
				Three-Asset Model with Predictability							
6	5.1	5.3	5.7	293.1	291.1	285.2	9.1	12.0	13.6	9.0	10.8
12	10.1	10.4	10.2	622.3	591.9	583.9	12.2	17.0	21.3	12.0	14.7
24	20.0	20.8	20.0	1214.8	1180.9	1215.4	21.6	33.0	41.8	22.2	29.9
48	39.9	41.8	40.9	2482.6	2363.9	2558.7	72.2	80.0	105.1	62.4	89.8
				Ten-Asset Model without Predictability							
6	0.8	0.8	9.5	743.0	839.0	870.2	10.3	14.8	20.6	12.4	13.1
12	1.5	1.5	16.9	1485.5	1643.7	1784.9	20.4	25.5	34.4	21.3	22.2
24	2.8	3.1	34.0	3020.5	3243.0	3354.9	55.4	60.3	77.8	57.7	62.9
48	5.6	6.3	66.6	6150.6	6742.6	7238.9	269.0	221.4	250.1	253.4	291.4

with risk aversion coefficient $\gamma = 3$ and the transaction cost rate $\delta = 0.01$; changes in these two parameters do not appreciably affect the run times. These computations were run on a Dell personal computer with a 2.55GHz Intel Core 2 Quad CPU processor and 3.25GB of RAM, running Windows XP. The calculations were done using Matlab with a single processor; the run times were estimated using Matlab's Profiler utility. In our calculations, we used the general purpose MOSEK optimization toolbox for Matlab to solve the convex optimization problems. We could almost certainly improve the run times by developing more specialized code for the particular forms of optimization problems that we consider.

The first two columns in Table 1 report the time required to solve the dynamic program for the frictionless model. We also show the time required to solve the dynamic program with modified transaction costs that is used to calculate the modified gradient bound. These models must be solved once, before running the simulation. As expected, the run times for these two models are quite similar and grow linearly with the number of periods in the model. The models without predictability take less time to solve: though they have more assets (10 rather 3), they do not involve a market state variable and there is only one scenario to evaluate in each period, as opposed to the 19 market states considered in the model with predictability.

Most of the time in the simulation is spent evaluating the heuristic strategies. The cost-blind heuristic is quite easy to evaluate, as we simply move to the post-trade asset allocations recommended by the frictionless model. The other heuristics require solving a convex optimization problem in each period to determine the trades that optimize the heuristic's objective in that scenario. The run times thus grow linearly with the number of periods and the number of trials. The complexity of each of these convex optimization problems grows more than linearly in the number of assets (in theory no worse than polynomially), but this depends on the details of the optimization methods used.

The dual bounds take less time to calculate. Here we solve one deterministic inner problem for each trial; the number of decision variables is the number of assets n times the number of periods T or $2nT$ when we decompose the trades into their positive and negative components. The run times grow linearly in the number of trials and the complexity of the convex optimization problem grows more than linearly in the number of decision variables involved (nT or $2nT$); this polynomial growth is evident in the run times in Table 1 for the dual problems with increasing horizon T. The run times required to evaluate the heuristic strategies are longer than the run times for the dual problems because the optimization problems for the heuristic strategies involve high-dimensional expectations (over returns and market states) to calculate objective function values for each setting of the decision variables.

Finally, remember that the run times in Table 1 are the times required to evaluate the quality of the heuristic strategies and dual bounds. In practice, if we want to use the modified one-step or rolling buy-and-hold heuristics to recommend a trade, we need only solve the corresponding optimization problems once for the current state and period. Dividing the run times in Table 1 by the 1000 trials in the simulation and the number of periods considered (T), we see that trades recommended by these heuristic strategies can be determined in a fraction of a second on a desktop PC.

5.3. Results

In each simulation, for each heuristic and dual bound, we calculate

- the average utility of final wealth or value for the dual bound,
- the mean "turnover," defined as the average volume of trade in each period, $\frac{1}{T}\sum_{t=0}^{T-1}\sum_{i=1}^{n}|a_{t,i}|$

To simplify the interpretation of the results, we will convert the utilities and bounds to annualized certainty equivalent returns. Given a time horizon of T months and a mean utility calculated in a simulation of $\hat{\mu}$, the annualized certainty equivalent return is defined as the constant annual return $\hat{r}$ that yields utility $\hat{\mu}$, i.e., the $\hat{r}$ that solves:

$$\hat{\mu} = U\Big(w_0 \ \hat{r}^{T/12}\Big) \tag{33}$$

where w_0 is the initial wealth. We estimate mean standard errors for these certainty equivalent returns and the duality gaps (the differences between upper and lower bounds on optimal returns) using the "delta method" (see, e.g., Casella and Berger 2002, p. 240) based on a first-order Taylor series expansion of the certainty equivalent formula (i.e., the inverse of equation (33)).

To reduce the variance in our estimates of the expected utilities, we use a simple control variate technique (see, e.g., Glasserman 2004, p. 185) using the utilities for the frictionless model as a control variate.

Specifically, for a given strategy α, we estimate its expected utility as

$$\hat{\mu} \quad = \quad \frac{1}{S}\sum_{s=1}^{S}\left\{U(w_T(\alpha(\boldsymbol{r}_s,\boldsymbol{z}_s),\boldsymbol{r}_s)) + \beta\left(V_0^f(w_0,\boldsymbol{z}_0) - U(w_T^f(\alpha^f(\boldsymbol{r}_s,\boldsymbol{z}_s),\boldsymbol{r}_s))\right)\right\} \tag{34}$$

where S is the number of trials, $\boldsymbol{r}_s$ and $\boldsymbol{z}_s$ are the sequences of returns and market states in trial s, $\alpha(\boldsymbol{r}_s,\boldsymbol{z}_s)$ and $\alpha^f(\boldsymbol{r}_s,\boldsymbol{z}_s)$ are the trades for the chosen strategy and frictionless strategy in trial s, and w_T and w_T^f are the terminal wealths with and without transaction costs. $V_0^f(w_0,\boldsymbol{z}_0)$ is the expected utility for the frictionless model in the initial state; this is computed before we begin the simulation. The term inside the parentheses in (34) has zero mean, so adjusting the estimate of expected utility by adding this term does not bias the estimate. The regression coefficient β in (34) is given as $(\sigma_y/\sigma_x)\rho_{xy}$ where σ_x is the standard deviation of $U(w_T^f(\alpha^f(\boldsymbol{r}_s,\boldsymbol{z}_s),\boldsymbol{r}_s))$, σ_y is the standard deviation of $U(w_T(\alpha(\boldsymbol{r}_s,\boldsymbol{z}_s),\boldsymbol{r}_s))$ and ρ_{xy} is the correlation between these quantities. The estimates for the gradient-based dual bounds of §4.3 are similarly adjusted using control variates.

Table 2 shows the simulation results for the three-asset model with predictability, for a time horizon (T) of 12 months; results for the other time horizons are shown in Table A3 in the appendix. Figure 1 summarizes the results for all time horizons, transaction costs, and risk aversion levels, showing the certainty equivalent returns for the best heuristic policy (the bottom end of the error bars) and the best dual bound (the upper end of the error bar); the length of the error bar thus represents the duality gap for a particular set of parameters. In these results, the annualized certainty returns are stated in percentage terms. For example, in the first row of Table 2, we see that in the case with risk aversion coefficient $\gamma = 1.5$ and transaction cost rate $\delta = 0.5\%$, the modified one-step heuristic has an annualized certainty equivalent return quoted as 6.55 percent; this corresponds to an estimated value of $\hat{r}$ in equation (33) of 1.0655. The mean standard errors are also quoted in percentage terms; the 95-percent confidence interval on $\hat{r}$ for the modified one-step strategy in this case is $1.0655 \pm 1.96 \times 0.0012$. The turnover means that an investor following the modified one-step strategy would execute trades averaging 10.6% of his initial wealth in each period.

In Table 2, we see that the modified one-step and rolling buy-and-hold heuristic strategies perform similarly and consistently outperform the cost-blind strategy and the (unmodified) one-step strategy. The rolling buy-and-hold strategy "wins" in most cases, but its performance is typically only slightly better than the modified one-step strategies. In most of these cases, the cost-blind strategies perform substantially worse than these two heuristic strategies with larger differences occurring when the transaction costs are larger and when the investor is less risk averse (has a low value of γ). Looking at the turnover, we see that, as expected, the modified one-step and rolling buy-and-hold strategies trade less than the cost-blind strategy, but more than the (unmodified) one-step strategy.

Table 2: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
12	1.5	0.5%	CE Return (%)	4.35	5.93	6.55	6.54	54.53	6.73	7.41	6.94	6.74	7.24	6.55	6.73	0.18
			Mean Std. Error (%)	0.02	0.21	0.12	0.14	0.48	0.00	0.02	0.01	0.00		Modified	Modified	0.12
			Turnover (%)	46.8	6.6	10.6	9.5	70.3	8.1	19.3	4.5	7.1	47.5	One-Step	One-Step	
12	1.5	1.0%	CE Return (%)	1.55	3.03	5.90	5.93	50.04	6.26	6.47	6.68	6.27	7.24	5.93	6.26	0.32
			Mean Std. Error (%)	0.05	0.21	0.14	0.16	0.46	0.01	0.02	0.01	0.01		Rolling Buy-	Modified	0.16
			Turnover (%)	46.1	3.1	9.1	8.4	54.3	7.2	10.1	4.1	6.9	47.5	and-Hold	One-Step	
12	1.5	2.0%	CE Return (%)	-3.85	0.52	4.91	4.93	43.42	5.42	5.36	6.22	5.41	7.24	4.93	5.36	0.42
			Mean Std. Error (%)	0.09	0.01	0.20	0.20	0.43	0.01	0.01	0.02	0.01		Rolling Buy-	Rolling Buy-	0.20
			Turnover (%)	44.7	0.0	7.1	6.7	40.2	6.5	7.9	3.5	6.5	47.5	and-Hold	and-Hold	
12	3	0.5%	CE Return (%)	2.73	3.20	3.46	3.49	50.72	3.79	4.62	3.83	3.72	4.01	3.49	3.72	0.23
			Mean Std. Error (%)	0.01	0.12	0.07	0.08	0.49	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.08
			Turnover (%)	20.9	3.5	7.5	6.3	70.3	5.3	19.5	2.7	5.6	21.0	and-Hold	Gradient	
12	3	1.0%	CE Return (%)	1.48	1.76	3.13	3.20	46.45	3.57	3.75	3.68	3.46	4.01	3.20	3.46	0.26
			Mean Std. Error (%)	0.02	0.10	0.10	0.11	0.47	0.01	0.02	0.01	0.00		Rolling Buy-	Modified	0.11
			Turnover (%)	20.7	1.5	5.6	4.8	54.3	4.8	9.1	2.5	5.4	21.0	and-Hold	Gradient	
12	3	2.0%	CE Return (%)	-0.99	0.51	2.63	2.66	40.09	3.15	3.09	3.40	3.01	4.01	2.66	3.01	0.35
			Mean Std. Error (%)	0.03	0.01	0.14	0.13	0.43	0.01	0.01	0.01	0.01		Rolling Buy-	Modified	0.13
			Turnover (%)	20.4	0.0	3.8	3.4	40.2	4.6	6.5	2.2	5.0	21.0	and-Hold	Gradient	
12	8	0.5%	CE Return (%)	1.33	1.50	1.58	1.60	40.86	1.76	2.56	1.74	1.70	1.80	1.60	1.70	0.10
			Mean Std. Error (%)	0.00	0.05	0.03	0.03	0.56	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.03
			Turnover (%)	7.8	1.3	2.9	2.4	70.3	3.3	18.7	1.0	3.2	7.8	and-Hold	Gradient	
12	8	1.0%	CE Return (%)	0.86	0.98	1.46	1.49	37.27	1.70	1.87	1.68	1.61	1.80	1.49	1.61	0.12
			Mean Std. Error (%)	0.01	0.04	0.04	0.04	0.51	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	7.8	0.6	2.2	1.8	54.3	3.0	7.6	0.9	3.1	7.8	and-Hold	Gradient	
12	8	2.0%	CE Return (%)	-0.08	0.51	1.28	1.30	31.81	1.62	1.59	1.58	1.45	1.80	1.30	1.45	0.15
			Mean Std. Error (%)	0.01	0.00	0.05	0.05	0.43	0.01	0.01	0.01	0.00		Rolling Buy-	Modified	0.05
			Turnover (%)	7.7	0.0	1.5	1.3	40.2	3.0	4.7	0.8	3.0	7.8	and-Hold	Gradient	
															Average	0.24
															Minimum	0.10
															Maximum	0.42

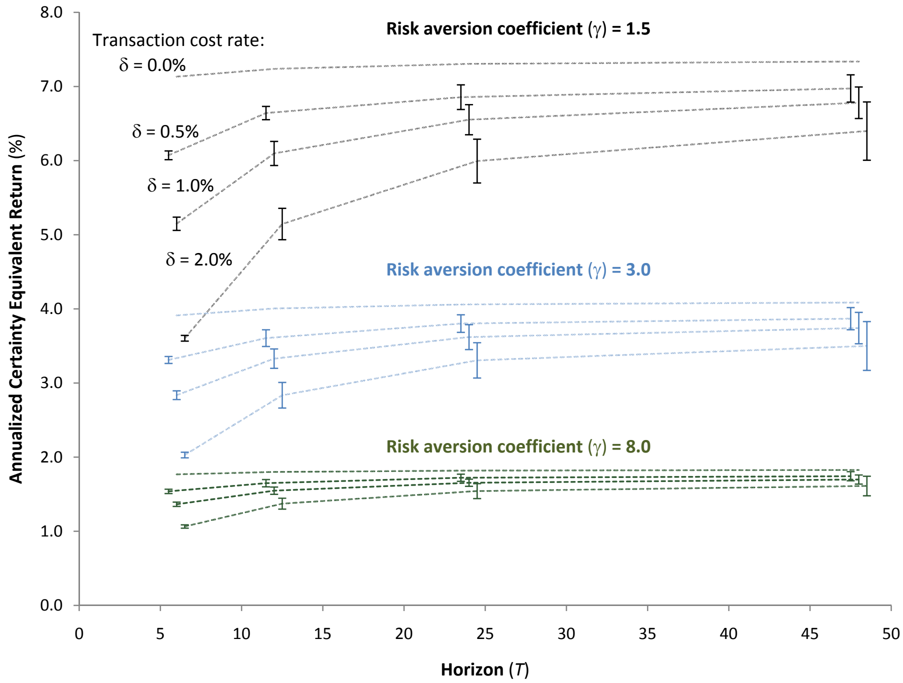

Figure 1: Bounds on optimal returns in the three-asset example.

Examining the dual bounds, we see that the zero-penalty bound performs very poorly, as expected: an investor with perfect foresight can achieve high returns, even with transaction costs. The bounds with penalties are much better and are also substantially better than the simple no-transaction-cost bound given by using the value function for the frictionless model. There is no consistent winner among the dual bounds: the modified one-step, modified gradient, and rolling buy-and-hold bounds all perform best in some cases. The modified gradient bounds win more often with higher levels of risk aversion (higher values of γ); this may be because the quality of the first-order Taylor series approximation underlying the modified-one-step and rolling buy-and-hold bounds degrades with higher levels of risk aversion. The frictionless gradient-based bound consistently outperforms the no-transaction cost bound (as it must, based on Proposition 4.2(c)), but is never the best of the dual bounds. Examining the mean standard errors, we see that the dual bounds with penalties are quite precisely estimated; the mean standard errors are typically much smaller than the mean standard errors associated with the modified one-step and rolling buy-and-hold heuristic strategies.

The duality gaps – the difference between certainty equivalent returns for the best bound and best strategy – are often quite small. In these case, there is relatively little room for improvement on the heuristic strategies. In Table 2, the gaps range from 0.10% to 0.42% and average 0.24%. The results are somewhat better for shorter time horizons and somewhat worse for the longer time horizons. With $T = 48$ months, the gaps for the three-asset case average 0.39%. The duality gaps also appear to increase with higher transaction costs and with lower risk aversion. The worst gap for all of the three-asset cases is 0.79% for the low risk aversion, large transaction cost case ($\gamma = 1.5$; $\delta = 0.02$) with the long time horizon ($T = 48$). The mean standard errors for the duality gaps are close to the mean standard errors for the heuristic strategies, as the mean standard errors for the heuristic strategies dominate those of the dual bound.

Table 3 shows the simulation results for the ten-asset model without predictability, for a time horizon (T) of 12 months; results for the other time horizons are shown in Table A4 in appendix. These results are summarized in Figure 2. Here we find that the cost-blind strategy performs much better, because, with no predictability, the cost-blind strategies trade much less: there is some rebalancing of the portfolio in response to idiosyncratic gains or losses on particular assets but no large scale changes in the asset positions in response to changes in the market state variable. For example, in the low risk aversion cases ($\gamma = 1.5$), the cost-blind strategy calls for placing all wealth in two assets, rebalancing these positions over time in response to idiosyncratic gains or losses. The modified one-step and rolling buy-and-hold heuristics place all of their wealth in these same two assets, but do not rebalance in subsequent periods. In many cases, the duality gap is very close to zero, suggesting the heuristic strategies are nearly optimal.[2]

5.4. Tuning the Heuristics

In general, we find the results of §5.3 encouraging, particularly given how little effort has been made to fine tune the heuristics and penalties used. In many cases, the duality gaps are quite small and the heuristic strategies are probably good enough for most practical applications. Where the duality gaps are larger, we may be able to improve performance by varying the heuristics and/or tighten the bounds by varying the penalties. For example, in the modified one-step strategy we reduced the transaction costs in the optimization problem (15) by dividing the costs by the smaller of 6 and the number of periods remaining. This rule apparently performs reasonably well in most cases considered here, but we could perhaps do better with different scaling rules. Similarly, we took the horizon h in the rolling buy-and-hold objective (16) to be 6; again, we could perhaps do better with a different horizon.

In Table 4, we report results with varying divisors and horizons for two cases where the duality gaps were largest. In the case with the largest duality gap for the three-asset model ($\gamma = 1.5$; $\delta = 0.02$, $T = 48$), we can cut the gap of 0.79% to 0.53% by considering a longer horizon or larger divisor. Similarly, for the case with the largest gap in Table 3 ($\gamma = 8.0$; $\delta = 0.02$, $T = 12$) with a gap of 0.52%, we can improve the performance of the heuristics and reduce the gap to 0.03%. In this case, it appears that a rolling buy-and-hold strategy with a longer time horizon is nearly optimal.

[2]There are a few cases were the estimated duality gap is slightly negative. In these cases, the estimated gaps are small compared to their mean standard errors. In these cases, we believe the true gap is very close to zero and the negative estimate is a result of sampling error.

Table 3: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
12	1.5	0.5%	CE Return (%)	11.13	11.76	13.06	13.06	53.76	13.06	13.06	13.37	13.08	13.62	13.06	13.06	0.00
			Mean Std. Error (%)	0.15	0.04	0.00	0.00	0.31	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	26.5	6.5	8.3	8.3	121.0	8.3	8.3	3.8	7.6	8.9	and-Hold	and-Hold	
12	1.5	1.0%	CE Return (%)	9.45	5.91	12.50	12.50	46.07	12.50	12.50	13.16	12.55	13.62	12.50	12.50	0.00
			Mean Std. Error (%)	0.15	0.00	0.00	0.00	0.27	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	26.3	0.0	8.3	8.3	90.0	8.3	8.3	3.2	7.6	8.9	and-Hold	and-Hold	
12	1.5	2.0%	CE Return (%)	7.23	5.91	11.39	11.39	36.40	11.39	11.39	12.83	11.48	13.62	11.39	11.39	0.00
			Mean Std. Error (%)	0.13	0.00	0.00	0.00	0.24	0.00	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	22.3	0.0	8.2	8.2	49.6	8.2	8.2	2.5	7.5	8.9	and-Hold	and-Hold	
12	3	0.5%	CE Return (%)	11.26	8.81	11.34	11.34	52.32	11.37	11.36	11.71	11.38	11.91	11.34	11.36	0.02
			Mean Std. Error (%)	0.00	0.06	0.01	0.01	0.28	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	9.8	3.3	8.3	8.3	121.0	8.3	8.3	3.2	7.6	9.8	One-Step	and-Hold	
12	3	1.0%	CE Return (%)	10.62	5.92	10.79	10.79	44.84	10.82	10.81	11.54	10.85	11.91	10.79	10.81	0.02
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.25	0.00	0.00	0.01	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	9.7	0.0	8.3	8.3	90.0	8.3	8.3	2.9	7.6	9.8	One-Step	and-Hold	
12	3	2.0%	CE Return (%)	9.35	5.91	9.24	9.23	35.36	9.81	9.81	11.23	9.80	11.91	9.35	9.80	0.45
			Mean Std. Error (%)	0.00	0.00	0.04	0.04	0.22	0.00	0.00	0.02	0.00		Cost Blind	Modified	0.00
			Turnover (%)	9.7	0.0	6.0	5.9	49.6	8.2	8.2	2.5	7.5	9.8		Gradient	
12	8	0.5%	CE Return (%)	9.09	7.00	9.16	9.16	48.05	9.25	9.22	9.60	9.21	9.74	9.16	9.21	0.05
			Mean Std. Error (%)	0.00	0.05	0.01	0.01	0.33	0.00	0.00	0.00	0.00		Rolling Buy-	Modified	0.01
			Turnover (%)	9.9	1.3	8.3	8.3	121.0	8.3	8.3	2.4	7.8	9.9	and-Hold	Gradient	
12	8	1.0%	CE Return (%)	8.45	5.91	8.49	8.49	41.26	8.75	8.74	9.47	8.68	9.74	8.49	8.68	0.19
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.29	0.00	0.00	0.01	0.00		Modified	Modified	0.03
			Turnover (%)	9.9	0.0	6.8	6.8	90.0	8.3	8.3	2.3	7.8	9.9	One-Step	Gradient	
12	8	2.0%	CE Return (%)	7.19	5.91	7.15	7.14	32.45	8.64	8.60	9.24	7.71	9.74	7.19	7.71	0.52
			Mean Std. Error (%)	0.00	0.00	0.08	0.08	0.25	0.01	0.01	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	9.8	0.0	2.3	2.3	49.6	8.1	8.2	2.1	7.7	9.9		Gradient	
															Average	0.14
															Minimum	0.00
															Maximum	0.52

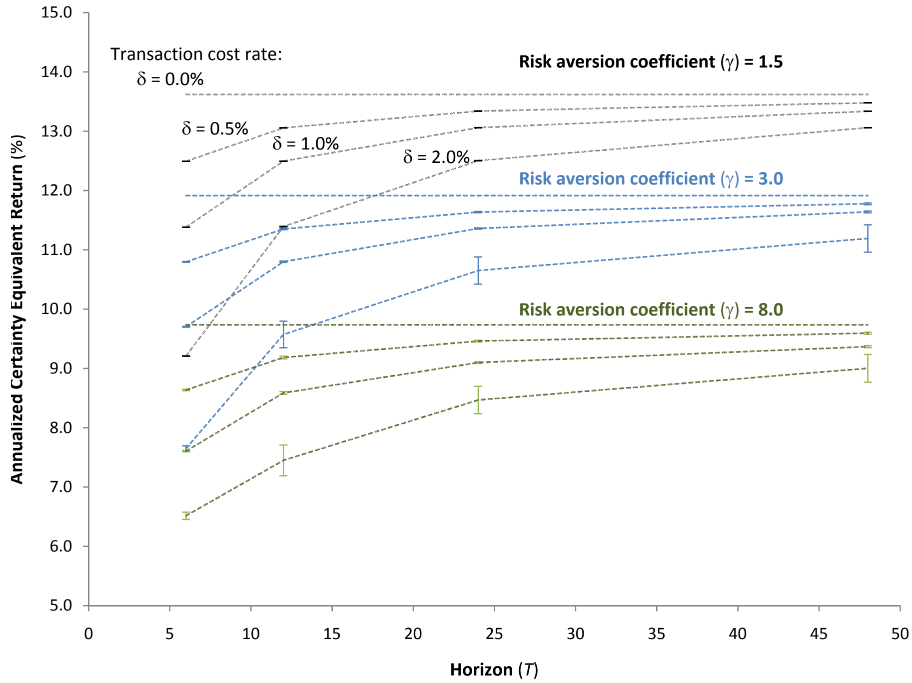

Figure 2: Bounds on optimal returns in the ten-asset example.

Table 4: Certainty Equivalent Returns (%) with Varying Parameters for Heuristics

	Horizon (h) or Divisor	3	6	9	12	15	18	21	24
		Three assets without predictability, T=48, γ=1.5, δ=0.02							
Heuristic									
	Modified One-Step	5.62	6.01	6.05	6.02	5.98	5.94	5.90	5.87
	Rolling Buy-and-hold	5.53	6.00	6.13	6.18	**6.20**	6.19	6.19	6.18
Bounds									
	Modified One-Step	7.07	6.87	6.80	6.73	**6.73**	6.73	6.75	6.75
	Rolling Buy-and-hold	7.01	6.79	6.90	7.12	7.33	7.48	7.57	7.64
Gaps									
	Mean	1.38	0.79	0.67	0.55	**0.53**	0.54	0.56	0.57
	Std. Error	0.14	0.10	0.10	0.10	0.10	0.10	0.11	0.11
		Ten assets without predictability, T=12, γ=8, δ=0.02							
Heuristic									
	Modified One-Step	6.26	7.15	7.54	7.61				
	Rolling Buy-and-hold	6.27	7.14	7.54	**7.61**				
Bounds									
	Modified One-Step	11.63	8.64	7.93	7.83				
	Rolling Buy-and-hold	11.71	8.60	7.86	**7.64**				
Gaps									
	Mean	5.36	1.45	0.32	**0.03**				
	Std. Error	0.08	0.08	0.05	0.03				

Table 5: Results for the Ten-Asset Example with Varying Leverage Limit

		Heuristic Strategies				Dual Bounds						Best Performance		
Leverage Limit (ℓ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
1.0	Best Divisor/Horion			6	6		6	6						
	CE Return (%)	10.62	5.91	10.79	10.79	74.46	10.82	10.81	11.54	10.85	11.91	10.79	10.81	0.02
	Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.46	0.00	0.00	0.01	0.00		Modified	Rolling Buy-	0.00
	Turnover (%)	9.8	0.0	8.3	8.3	166.5	8.3	8.3	2.9	7.6	9.8	One-Step	and-Hold	
2.0	Best Divisor/Horion			6	6		6	6						
	CE Return (%)	12.14	5.91	12.38	12.37	174.12	12.68	12.73	14.45	12.84	15.03	12.38	12.68	0.31
	Mean Std. Error (%)	0.00	0.00	0.01	0.01	1.42	0.00	0.00	0.01	0.00		Modified	Modified	0.01
	Turnover (%)	21.3	0.0	18.7	18.7	417.4	16.4	16.7	4.8	14.7	21.8	One-Step	One-Step	
3.0	Best Divisor/Horion			9	9		9	9						
	CE Return (%)	12.09	5.92	13.14	13.18	314.06	13.60	13.53	16.10	13.54	16.89	13.18	13.53	0.35
	Mean Std. Error (%)	0.01	0.00	0.04	0.04	3.21	0.01	0.01	0.02	0.00		Rolling Buy-	Rolling Buy-	0.05
	Turnover (%)	35.4	0.0	24.2	23.8	788.7	22.5	24.2	6.9	21.8	36.6	and-Hold	and-Hold	
4.0	Best Divisor/Horion			12	12		12	12						
	CE Return (%)	10.96	5.92	13.23	13.24	504.62	14.07	13.62	16.83	13.63	17.73	13.24	13.62	0.38
	Mean Std. Error (%)	0.01	0.00	0.11	0.10	6.27	0.03	0.01	0.03	0.00		Rolling Buy-	Rolling Buy-	0.10
	Turnover (%)	50.0	0.0	25.7	24.8	1325.6	23.3	15.8	8.0	28.6	52.4	and-Hold	and-Hold	
5.0	Best Divisor/Horion			12	12		12	12						
	CE Return (%)	9.72	5.92	13.12	13.15	757.54	14.21	13.61	17.39	13.72	18.33	13.15	13.61	0.47
	Mean Std. Error (%)	0.02	0.00	0.15	0.13	11.24	0.07	0.01	0.03	0.01		Rolling Buy-	Rolling Buy-	0.13
	Turnover (%)	63.5	0.0	25.7	24.8	2081.9	26.3	16.8	8.5	35.3	67.3	and-Hold	and-Hold	

5.5. Results with Short Sales and Borrowing

In the numerical experiments presented thus far, we have focused exclusively on the case where short positions and borrowing are not allowed. Alternatively, we can consider cases where borrowing and short positions are allowed and the investor faces a constraint on the total leverage allowed, i.e., a constraint of the form of (6) for a given leverage limit ℓ. The set of feasible trades is larger in this case (and grows larger as ℓ increases) and one might wonder how the heuristics and dual bounds perform with leverage.

To investigate this, we conducted a series of numerical experiments where we consider leverage constraints with varying limits. We focus on the ten-asset model without predictability and the case with $T = 12$, $\gamma = 3$, and $\delta = 1.0\%$ and consider leverage limits ℓ varying from 1 to 5. In this study, we varied the divisors and horizons for the modified-one step and rolling buy and hold heuristics and dual bounds (as in §5.4), considering values of 6, 9, and 12. The results are presented in Table 5.

In Table 5, we see that the certainty equivalent returns for the dual bounds and no-transaction cost bounds are all increasing with the leverage limit, reflecting the larger set of feasible trades.[3] The zero penalty bound in particular increases greatly with higher leverage limits as the larger feasible sets allow the investor with advance knowledge of asset returns to more effectively exploit the arbitrage opportunities provided by such information. The performance of the modified one-step and rolling buy-and-hold heuristic strategies need not improve with larger feasible sets; the larger feasible sets lead to higher values for the

[3]Note that with power utility and a return model with log-normal returns, it is never optimal to borrow or take short positions as these positions lead to a positive probability of an infinite negative utility. However, with our discrete approximations of the return distribution, it is optimal to borrow and take short positions in this numerical example.

heuristics' objective function (in (15) and (16)), but this may not actually lead to better performance.

Overall, the duality gaps widen with higher leverage limits; the worst case is with a leverage limit $\ell = 5.0$ which has a gap of 0.47%. Though we could perhaps do better with more sophisticated heuristics and bounds, we note that in this worst case, the modified one-step and rolling buy-and-hold heuristics (with certainty equivalent returns of approximately 13.15%) greatly outperform the cost-blind strategy (return of 9.72%) and the dual bounds with penalties (13.61%) are much tighter than the no-transaction cost bound (18.33%). Thus these heuristics and dual bounds greatly outperform strategies and bounds that simply ignore transactions costs.

6. Conclusion

In this paper, we have studied some easy-to-compute heuristics for managing portfolios with transaction costs and developed a dual approach for examining the quality of these heuristics. The approach is general in that we can consider a variety of utility functions, a variety of forms for transaction costs (provided they are convex functions), a variety of constraint sets (provided they are convex) as well as a variety of different models for returns. Our numerical experiments are promising: the run times, even without using customized optimization software, are reasonable and, in many cases, the performance of the heuristic strategy is very close to the upper bound, indicating that the heuristic strategies are very nearly optimal.

Frankly, we were surprised that these heuristics performed so well. At a high level, the key issue is to manage the tradeoff between improving asset positions and minimizing transaction costs. These heuristics capture this tradeoff in a relatively crude but apparently effective manner. In studying these heuristics, we find the dual upper bounds particularly helpful: when the bounds tell us that the performance of these heuristics is nearly optimal, we know that there is little to be gained by considering more complicated heuristics. Given the complexity of the full dynamic programming model with transaction costs, we find this quite reassuring.

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A. Appendix

A.1. Proofs and Detailed Derivations

Proof of Proposition 2.1. The proof is by induction. For the terminal case, $V_T(\boldsymbol{x}_T, c_T, \boldsymbol{z}_T) = U(\mathbf{1}'\boldsymbol{x}_T + c_T)$ is nondecreasing in c_T and concave in $(\boldsymbol{x}_T, c_T)$ because the utility function is assumed to be nondecreasing and concave in wealth. We now assume the result of the proposition holds for period t+1 and show that it must also hold for period t.

For monotonicity in c_t: Assume $c_t^1 \leq c_t^2$. Then by the induction hypothesis, for any $\boldsymbol{x}_t$ and $\boldsymbol{a}_t$,

$$\begin{aligned} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) &= \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t),\ r_f(c_t^1 - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\ &\leq \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t + \boldsymbol{a}_t),\ r_f(c_t^2 - \mathbf{1}'\boldsymbol{a}_t - \kappa(\boldsymbol{a}_t)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\ &= W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t). \end{aligned}$$

This implies

$$\begin{aligned} V_t(\boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) &= \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^1)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^1, \boldsymbol{z}_t) \\ &\leq \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^1)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t) \\ &\leq \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t, c_t^2)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t) \\ &= V_t(\boldsymbol{x}_t, c_t^2, \boldsymbol{z}_t). \end{aligned}$$

The second inequality here follows from our assumption that the set of allowed final asset positions $\mathbb{H}_t$ is nondecreasing in c_t: this implies that $\mathbb{A}_t(\boldsymbol{x}_t, c_t^1) \subseteq \mathbb{A}_t(\boldsymbol{x}_t, c_t^2)$.

For concavity: For any positions $(\boldsymbol{x}_t^1, c_t^1)$ and $(\boldsymbol{x}_t^2, c_t^2)$ and trading strategies $\boldsymbol{a}_t^1$ and $\boldsymbol{a}_t^2$, let $(\boldsymbol{x}_t^\alpha, c_t^\alpha) = \alpha(\boldsymbol{x}_t^1, c_t^1) + (1-\alpha)(\boldsymbol{x}_t^2, c_t^2)$ and let $\boldsymbol{a}_t^\alpha = \alpha\boldsymbol{a}_t^1 + (1-\alpha)\boldsymbol{a}_t^2$. Then we have

$$\begin{aligned} W_t(\boldsymbol{a}_t^\alpha, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) &= \mathbb{E}\left[V_{t+1}(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^\alpha + \boldsymbol{a}_t^\alpha),\ r_f(c_t^\alpha - \mathbf{1}'\boldsymbol{a}_t^\alpha - \kappa(\boldsymbol{a}_t^\alpha)),\ \tilde{\boldsymbol{z}}_{t+1}) \mid \boldsymbol{z}_t\right] \\ &\geq \mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^\alpha + \boldsymbol{a}_t^\alpha),\ r_f(c_t^\alpha - \mathbf{1}'\boldsymbol{a}_t^\alpha - \alpha\kappa(\boldsymbol{a}_t^1) - (1-\alpha)\kappa(\boldsymbol{a}_t^2)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\ &\geq \alpha\mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^1 + \boldsymbol{a}_t^1),\ r_f(c_t^1 - \mathbf{1}'\boldsymbol{a}_t^1 - \kappa(\boldsymbol{a}_t^1)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\ &\quad +(1-\alpha)\mathbb{E}\left[V_{t+1}\big(\tilde{\boldsymbol{r}}_{t+1} \cdot (\boldsymbol{x}_t^2 + \boldsymbol{a}_t^2),\ r_f(c_t^2 - \mathbf{1}'\boldsymbol{a}_t^2 - \kappa(\boldsymbol{a}_t^2)),\ \tilde{\boldsymbol{z}}_{t+1}\big) \mid \boldsymbol{z}_t\right] \\ &= \alpha W_t(\boldsymbol{a}_t^1, \boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha)W_t(\boldsymbol{a}_t^2, \boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t). \end{aligned}$$

The first inequality follows by using the assumption that the transaction cost function $\kappa(\boldsymbol{a}_t)$ is convex in $\boldsymbol{a}_t$ and then using the nondecreasing in c_t part of the induction hypothesis. The next inequality follows from the concavity part of the induction hypothesis.

Suppose $\boldsymbol{a}_t^{1*}$ and $\boldsymbol{a}_t^{2*}$ are optimal trades given positions $(\boldsymbol{x}_t^1, c_t^1)$ and $(\boldsymbol{x}_t^2, c_t^2)$ and let $\boldsymbol{a}_t^{\alpha*} = \alpha\boldsymbol{a}_t^{1*} + (1-\alpha)\boldsymbol{a}_t^{2*}$. The concavity result above then implies

$$\begin{aligned} V_t(\boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) &= \max_{\boldsymbol{a}_t \in \mathbb{A}_t(\boldsymbol{x}_t^\alpha, c_t^\alpha)} W_t(\boldsymbol{a}_t, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) \\ &\geq W_t(\boldsymbol{a}_t^{\alpha*}, \boldsymbol{x}_t^\alpha, c_t^\alpha, \boldsymbol{z}_t) \\ &\geq \alpha W_t(\boldsymbol{a}_t^{1*}, \boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha)W_t(\boldsymbol{a}_t^{2*}, \boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t) \\ &= \alpha V_t(\boldsymbol{x}_t^1, c_t^1, \boldsymbol{z}_t) + (1-\alpha)V_t(\boldsymbol{x}_t^2, c_t^2, \boldsymbol{z}_t) \end{aligned}$$

The first inequality above follows from our assumption that the set of allowed final asset positions $\mathbb{H}_t$ is convex: this implies that $\boldsymbol{a}_t^{\alpha*}$ is in $\mathbb{A}_t(\boldsymbol{x}_t^\alpha, c_t^\alpha)$ and is thus feasible but not necessarily optimal for the optimization problem in the first line above. □

Proof of Proposition 2.2. Proofs for the first two parts of this proposition can be constructed in much the same way as the proof of Proposition 2.1 given above.

Part 3: Given asset position $(\boldsymbol{x}_t, c_t)$ and market state $\boldsymbol{z}_t$ consider a trading strategy starting in this state that is optimal with transaction costs and achieves the value $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. This trading strategy is feasible for the model without transaction costs (because the sets of possible asset positions is assumed to be nondecreasing in cash) and, without transaction costs, would yield an expected utility μ that is at least as large as $V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. Since this strategy is feasible but not necessarily optimal in the model without transaction costs, we know $V_t^f(\mathbf{1}'\boldsymbol{x}_t + c_t, \boldsymbol{z}_t) \geq \mu \geq V_t(\boldsymbol{x}_t, c_t, \boldsymbol{z}_t)$. □

Proof of Proposition 4.1. A proof of a more general version of this result may be found in Brown, Smith, and Sun (2009); a proof for this version of the result is given in the discussion following the Proposition. □

Derivation of Rolling-Buy-and-Hold Penalty. Assuming proportional transaction costs given by equation (1), the Taylor series expansion of (23) about $\boldsymbol{a}^*$ yields a generating function of the form

$$\begin{aligned} g_t(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) \;=\; & \mathbb{E}\left[\, V_{t+h}^f\Big((\tilde{\boldsymbol{r}}_{t+h} \cdot \ldots \cdot \tilde{\boldsymbol{r}}_{t+2})'\boldsymbol{x}_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}) + r_f^{h-1} c_{t+1}(\boldsymbol{a}^*),\ \tilde{\boldsymbol{z}}_{t+h}\Big) \mid \boldsymbol{z}_{t+1}\right] \\ & + \sum_{\tau=1}^{t}\sum_{i=1}^{n} \mathbb{E}\left[\, V_{t+h}^{f\prime}(-)(\tilde{r}_{t+h,i}\ldots\tilde{r}_{t+2,i}) \mid \boldsymbol{z}_{t+1}\right] \left(\frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^+}\,(a_{\tau,i}^+ - a_{\tau,i}^{*+}) + \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^-}\,(a_{\tau,i}^- - a_{\tau,i}^{*-})\right) \\ & + \sum_{\tau=1}^{t}\sum_{i=1}^{n} \mathbb{E}\left[\, V_{t+h}^{f\prime}(-) \mid \boldsymbol{z}_{t+1}\right] r_f^{h-1} \left(\frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^+}\,(a_{\tau,i}^+ - a_{\tau,i}^{*+}) + \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^-}\,(a_{\tau,i}^- - a_{\tau,i}^{*-})\right) \end{aligned} \tag{35}$$

where

$$\begin{aligned} V_{t+h}^{f\prime}(-) &= V_{t+h}^{f\prime}\Big((\tilde{\boldsymbol{r}}_{t+h} \cdot \ldots \cdot \tilde{\boldsymbol{r}}_{t+2})'\boldsymbol{x}_{t+1}(\boldsymbol{a}^*, \boldsymbol{r}) + r_f^{h-1} c_{t+1}(\boldsymbol{a}^*),\ \tilde{\boldsymbol{z}}_{t+h}\Big) \\ \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^+} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \\ \frac{\partial x_{t+1,i}}{\partial a_{\tau,i}^-} &= \prod_{\tau'=\tau+1}^{t+1} r_{\tau',i} \\ \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^+} &= -\, r_f^{t+1-\tau}\left(1 + \delta_i^+\right) \\ \frac{\partial c_{t+1,i}}{\partial a_{\tau,i}^-} &= -\, r_f^{t+1-\tau}\left(1 - \delta_i^-\right) \end{aligned}$$

As with the modified-one-step penalty, using a generating function of the form of (35) with the formula for a "good" penalty (21), we obtain a dual feasible penalty π that is linear in the positive and negative components of $\boldsymbol{a}$ for any sequence of returns $\boldsymbol{r}$ and market states $\boldsymbol{z}$. To calculate the "good" penalty, we need to evaluate the expectations (over returns $\boldsymbol{r}_{t+1}$ and the market state $\boldsymbol{z}_{t+1}$). The equations for the generating function (35) are more complicated than the generating functions for the modified one-step penalty (22) because (35) involves expectations of the frictionless value function and its derivatives over period t to $t+h$; these will be calculated using a quadrature scheme. This additional complication leads to these bounds being somewhat more time-consuming to compute than the modified one-step bounds; see Table 1 in §5.2. □

Proof of Proposition 4.2. Part (a): To streamline our notation, we define $f(\alpha, \boldsymbol{r}, \boldsymbol{z}) = U(\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))$ and $F(\alpha) = \mathbb{E}\left[f(\alpha, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[U(\hat{w}_T(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))\right]$. Note that since $\hat{w}_T$ is a concave function and U is concave and nondecreasing, $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is concave in α for each $(\boldsymbol{r}, \boldsymbol{z})$; this implies that $F(\alpha)$ is also concave in α. The (one-sided) directional derivatives of f and F at α in direction δ are defined as

$$\begin{aligned} f'(\alpha, \delta, \boldsymbol{r}, \boldsymbol{z}) &= \lim_{\epsilon \to 0^+} \frac{f(\alpha + \epsilon\delta, \boldsymbol{r}, \boldsymbol{z}) - f(\alpha, \boldsymbol{r}, \boldsymbol{z})}{\epsilon} \\ F'(\alpha, \delta) &= \lim_{\epsilon \to 0^+} \frac{F(\alpha + \epsilon\delta) - F(\alpha)}{\epsilon} \end{aligned}$$

The concavity of $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ and $F(\alpha)$ in α implies these directional derivatives exist (in the extended real numbers). If $U(w)$ is differentiable in w at $\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r})$ and $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$ is differentiable in $\boldsymbol{a}$ at $(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r})$, then the gradient of $U \circ \hat{w}_T$ exists at this point and $f'(\alpha, \delta, \boldsymbol{r}, \boldsymbol{z}) = \nabla_{\boldsymbol{a}} U(\hat{w}_T(\alpha(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))' \ \delta$.

Let $\mathcal{A}$ denote the set of feasible trading strategies for the real model and $\hat{\mathcal{A}}$ denote the set of feasible trading strategies for the modified model. Since $\mathbb{A}_t \subseteq \hat{\mathbb{A}}_t$, the set of trading strategies in the modified model is no smaller than that of the true model, i.e., $\mathcal{A} \subseteq \hat{\mathcal{A}}$. Moreover, the sets $\hat{\mathbb{A}}_t$ are convex, so the set of feasible trading strategies $\hat{\mathcal{A}}$ is also convex.

Given the concavity of F and convexity of $\hat{A}$, a necessary and sufficient condition for $\hat{\alpha}^*$ to maximize $F(\alpha)$ over $\hat{\mathcal{A}}$ is that the directional derivatives of F at $\hat{\alpha}^*$ are nonpositive for all feasible directions:

$$F'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*) \leq 0 \tag{36}$$

for all α in $\hat{\mathcal{A}}$; see, e.g., Friedlen and Nashed (1968).[4] A result of Bertsekas (1973; Prop 2.1)[5] says that

$$F'(\alpha, \delta) \quad = \quad \mathbb{E}\left[f'(\alpha, \delta, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] \ . \tag{37}$$

Bertsekas's result relies on the fact that $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is convex in α for each $(\boldsymbol{r}, \boldsymbol{z})$ and assumes $f(\alpha, \boldsymbol{r}, \boldsymbol{z})$ is integrable for each α (i.e., $\mathbb{E}\left[|f(\alpha, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})|\right] < \infty$), but does not require any other differentiability or integrability assumptions. (Bertsekas assumes that α lies in $\mathbb{R}^n$ but his proof also goes through when α lies in an infinite-dimensional convex set.) Combining (36) and (37) and using the fact that $\mathcal{A} \subseteq \hat{\mathcal{A}}$, we have

$$\mathbb{E}\left[f'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] \leq 0 \tag{38}$$

for all α in $\mathcal{A}$.

If $U \circ \hat{w}_T$ is differentiable, then (38) is equivalent to

$$\mathbb{E}\left[\hat{\pi}(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[\nabla_{\boldsymbol{a}} U(\hat{w}_T(\hat{\alpha}^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}))' \ (\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}) - \hat{\alpha}^*(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}))\right] \leq 0 \tag{39}$$

for all α in $\mathcal{A}$. Thus, the gradient-based penalty $\hat{\pi}$ given by equation (29) is dual feasible.

If $U \circ \hat{w}_T$ is not differentiable, we can define the penalty $\hat{\pi}$ directly in terms of the directional derivative,

$$\hat{\pi}(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{z}) \quad = \quad \lim_{\epsilon \to 0^+} \frac{U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}) + \epsilon(\boldsymbol{a} - \hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z})), \boldsymbol{r})) - U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r}, \boldsymbol{z}), \boldsymbol{r}))}{\epsilon} \ . \tag{40}$$

The existence of these directional derivatives follows from concavity alone and, in this case, we have

$$\mathbb{E}\left[\hat{\pi}(\alpha(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}}), \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right] = \mathbb{E}\left[f'(\hat{\alpha}^*, \alpha - \hat{\alpha}^*, \tilde{\boldsymbol{r}}, \tilde{\boldsymbol{z}})\right], \tag{41}$$

which is less than or equal to 0 by (38). Thus this $\hat{\pi}$ is also dual feasible.

Part (b): The proof for the case where $U(w)$ and $\hat{w}_T(\boldsymbol{a}, \boldsymbol{r})$ are differentiable is given in the text before the proposition, near equations (26) and (27). Here we show the result without assuming U and $\hat{w}_T$ are differentiable,

We first focus on a $(\boldsymbol{r}, \boldsymbol{z})$ particular scenario and for ease of notation, we will suppress the dependence on $(\boldsymbol{r}, \boldsymbol{z})$: we let $\alpha^* = \alpha^*(\boldsymbol{r}, \boldsymbol{z})$ and define the function $u : \mathbb{R}^{nT} \to \mathbb{R}^1$ as $u(\boldsymbol{a}) = U(w_T(\boldsymbol{a}, \boldsymbol{r}))$ and denote its directional derivative (in direction $\boldsymbol{d}$) by $u'(\boldsymbol{a}, \boldsymbol{d})$. The penalty π in this case is $\pi(\boldsymbol{a}) = u'(\alpha^*, \boldsymbol{a} - \alpha^*)$, and the inner problem is

$$\max_{\boldsymbol{a} \in \mathbb{A}(\boldsymbol{r})} \{u(\boldsymbol{a}) - u'(\alpha^*, \boldsymbol{a} - \alpha^*)\} \ .$$

Following standard results in convex analysis (e.g., Rockafellar 1970; Thm. 23.4),[6] we can express the

[4] Friedlen, D.M. and M.Z. Nashed, 1968. A note on one-sided directional derivatives, *Mathematics Magazine* **41**(3), 147-150.

[5] Bertsekas, D.P., 1973. Stochastic optimization Problems with nondifferentiable cost functionals, *Journal of Optimization Theory and Applications* **12**(2), 218-231.

[6] Rockafellar, R.T., 1970. Convex Analysis, Princeton University Press, Princeton, NJ.

directional derivative for the concave function u as

$$u'(\boldsymbol{a},\boldsymbol{d}) \quad = \quad \inf_{\boldsymbol{h}\in\partial u(\boldsymbol{a})} \boldsymbol{d}'\boldsymbol{h},$$

where $\partial u(\boldsymbol{a})$ is the superdifferential of u at $\boldsymbol{a}$, defined as the set of all supergradients at this point:

$$\partial u(\boldsymbol{a}) \quad = \quad \left\{\boldsymbol{h}\in\mathbb{R}^{nT} \;:\; u(\boldsymbol{a})+\boldsymbol{h}'(\boldsymbol{b}-\boldsymbol{a})\geq u(\boldsymbol{b}) \;\text{ for all }\; \boldsymbol{b}\in\mathbb{R}^{nT}\right\}.$$

We thus have:

$$\begin{aligned}
\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\{u(\boldsymbol{a})-u'(\alpha^*,\boldsymbol{a}-\alpha^*)\} &= \max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\left\{u(\boldsymbol{a})-\inf_{\boldsymbol{h}\in\partial u(\alpha^*)}(\boldsymbol{a}-\alpha^*)'\boldsymbol{h}\right\}\\
&= \max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{u(\boldsymbol{a})+\boldsymbol{h}'(\alpha^*-\boldsymbol{a})\right\}\\
&= \sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{\boldsymbol{h}'\alpha^*+\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\left\{u(\boldsymbol{a})-\boldsymbol{h}'\boldsymbol{a}\right\}\right\}\\
&= \sup_{\boldsymbol{h}\in\partial u(\alpha^*)}\left\{\boldsymbol{h}'\alpha^*+u(\alpha^*)-\boldsymbol{h}'\alpha^*\right\}\\
&= u(\alpha^*)\ .
\end{aligned}$$

The fourth equality follows from the fact that for any $\boldsymbol{h}\in\partial u(\alpha^*)$, the definition of a supergradient implies

$$u(\boldsymbol{a})-\boldsymbol{h}'\boldsymbol{a} \quad\leq\quad u(\alpha^*)-\boldsymbol{h}'\alpha^*,$$

and since $\alpha^*=\alpha^*(\boldsymbol{r},\boldsymbol{z})\in\mathbb{A}(\boldsymbol{r})$, $\boldsymbol{a}=\alpha^*(\boldsymbol{r},\boldsymbol{z})$ is feasible and attains the maximum value.

Returning to the original notation, we have established that, for any $(\boldsymbol{r},\boldsymbol{z})$,

$$\max_{\boldsymbol{a}\in\mathbb{A}(\tilde{\boldsymbol{r}})}\{U(w_T(\boldsymbol{a},\boldsymbol{r}))-\pi(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\}=U(w_T(\alpha^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r}))\ .$$

Taking expectations of this, we have:

$$\mathbb{E}\left[\max_{\boldsymbol{a}\in\mathbb{A}(\tilde{\boldsymbol{r}})}\{U(w_T(\boldsymbol{a},\tilde{\boldsymbol{r}}))-\pi(\boldsymbol{a},\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}})\}\right] \quad = \quad \mathbb{E}\left[U(w_T(\alpha^*(\tilde{\boldsymbol{r}},\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{r}}))\right]\ .$$

Part(c): We have

$$\begin{aligned}
\max_{\boldsymbol{a}\in\mathbb{A}(\boldsymbol{r})}\{U(w_T(\boldsymbol{a},\boldsymbol{r}))-\hat{\pi}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\} &\leq \max_{\boldsymbol{a}\in\hat{\mathbb{A}}(\boldsymbol{r})}\{U(\hat{w}_T(\boldsymbol{a},\boldsymbol{r}))-\hat{\pi}(\boldsymbol{a},\boldsymbol{r},\boldsymbol{z})\}\\
&= U(\hat{w}_T(\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r})).
\end{aligned}$$

The inequality follows from $\mathbb{A}(\boldsymbol{r})\subseteq\hat{\mathbb{A}}(\boldsymbol{r})$ and the condition $w_T\leq\hat{w}_T$. The next equality follows by arguments analogous to that part (b) that show the choice $\boldsymbol{a}=\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z})$ must attain the maximum and the observation that $\hat{\pi}(\hat{\alpha}^*(\boldsymbol{r},\boldsymbol{z}),\boldsymbol{r},\boldsymbol{z})=0$. □

A.2. Detailed Assumptions and Results for Numerical Experiments

Assumptions for model with three risky assets and predictability, from Lynch (2001):

Table A1: Data for Three-Asset Model with Predictability

	Large-cap	Mid-cap	Small-cap	Term spread
Mean (a_r, a_z)	0.0053	0.0067	0.0072	0.0000
Regression coeff. (b_r, b_z)	0.0038	0.0058	0.0072	0.8700

Covariance (Σ_{ev})

	Large-cap	Mid-cap	Small-cap	Term spread
Large-cap	0.002894	0.003532	0.003910	-0.000135
Mid-cap		0.004886	0.005712	-0.000170
Small-cap			0.007259	-0.000191
Term spread				0.072900

Note Lynch (2001) considers four different models, each with three risky assets and a one-dimensional market state variable. His three risky assets correspond to portfolios of stocks that are either sorted by firm size (which we use) or by book-to-market ratio; the market state variable is either the term spread (which we use) or a dividend yield.

Assumptions for model with ten risky assets and no predictability:

Table A2: Data for the Ten-Asset Model without Predictability

	SP500	R1000V	RMidC	R2000V	MSCIW	NAREIT	LBUSGv	LBUSCp	LBMortBnd	USTreasBnd
Mean (a_r)	0.009850	0.010672	0.010868	0.011173	0.009073	0.009432	0.007239	0.008111	0.007828	0.006424

Covariance (Σ_e)

	SP500	R1000V	RMidC	R2000V	MSCIW	NAREIT	LBUSGv	LBUSCp	LBMortBnd	USTreasBnd
SP500	0.001887	0.001659	0.001897	0.001592	0.001531	0.000742	0.000112	0.000243	0.000194	0.000054
R1000V		0.001650	0.001721	0.001549	0.001339	0.000821	0.000109	0.000233	0.000183	0.000052
RMidC			0.002199	0.001995	0.001549	0.000968	0.000097	0.000242	0.000183	0.000039
R2000V				0.002194	0.001318	0.001166	0.000048	0.000189	0.000121	0.000011
MSCIW					0.001738	0.000634	0.000083	0.000188	0.000147	0.000041
NAREIT						0.001269	0.000110	0.000206	0.000142	0.000060
LBUSGv							0.000224	0.000259	0.000231	0.000128
LBUSCp								0.000346	0.000308	0.000150
LBMortBnd									0.000336	0.000144
USTreasBnd										0.000082

These parameters were estimated as the means and covariances of historical returns for these indices using monthly return data from 1981-2006. The indices are, from left to right, 5 stock indices: the S&P 500, the Russell 1000 Value Index, Russell MidCap Index, Russell 2000 Value, and MSCI World Gross index; Lehman Brothers' US government and corporate bond indices; Lehman Brothers' Fixed Rate Mortgage Backed Securities Index, a real estate index trust (NAREIT), and a composite index of 1-5 Year US Treasuries.

Numerical Results for Other Parameter Values:

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
6	1.5	0.5%	CE Return (%)	3.80	5.58	5.98	6.01	55.08	6.24	6.60	6.55	6.13	7.13	6.01	6.13	0.12
			Mean Std. Error (%)	0.03	0.23	0.17	0.18	0.68	0.01	0.03	0.01	0.00		Rolling Buy-	Modified	0.18
			Turnover (%)	53.2	11.6	17.6	17.2	61.2	13.5	19.7	8.3	14.0	53.7	and-Hold	Gradient	
6	1.5	1.0%	CE Return (%)	0.58	2.19	5.04	5.06	50.60	5.41	5.43	6.08	5.24	7.13	5.06	5.24	0.18
			Mean Std. Error (%)	0.05	0.21	0.19	0.20	0.65	0.02	0.03	0.02	0.01		Rolling Buy-	Modified	0.20
			Turnover (%)	52.7	4.3	15.6	15.3	48.2	11.4	13.9	7.0	13.4	53.7	and-Hold	Gradient	
6	1.5	2.0%	CE Return (%)	-5.56	0.51	3.57	3.56	43.93	3.84	3.64	5.35	3.80	7.13	3.57	3.64	0.08
			Mean Std. Error (%)	0.10	0.01	0.22	0.22	0.60	0.03	0.02	0.04	0.01		Modified	Rolling Buy-	0.22
			Turnover (%)	51.8	0.0	11.8	11.4	37.0	9.4	11.0	5.3	12.0	53.7	One-Step	and-Hold	
6	3	0.5%	CE Return (%)	2.40	3.03	3.25	3.26	51.20	3.48	3.81	3.57	3.36	3.91	3.26	3.36	0.09
			Mean Std. Error (%)	0.01	0.13	0.10	0.11	0.67	0.01	0.02	0.01	0.00		Rolling Buy-	Modified	0.11
			Turnover (%)	24.6	5.9	10.3	9.8	61.2	8.0	16.3	5.0	10.6	24.7	and-Hold	Gradient	
6	3	1.0%	CE Return (%)	0.92	1.35	2.78	2.77	46.97	3.03	3.01	3.29	2.89	3.91	2.78	2.89	0.12
			Mean Std. Error (%)	0.02	0.10	0.14	0.14	0.63	0.01	0.02	0.01	0.00		Modified	Modified	0.14
			Turnover (%)	24.5	2.1	8.2	7.9	48.2	7.2	8.7	4.3	10.0	24.7	One-Step	Gradient	
6	3	2.0%	CE Return (%)	-1.98	0.51	1.99	1.99	40.62	2.23	2.07	2.85	2.18	3.91	1.99	2.07	0.08
			Mean Std. Error (%)	0.04	0.00	0.15	0.15	0.58	0.02	0.01	0.03	0.01		Modified	Rolling Buy-	0.15
			Turnover (%)	24.2	0.0	6.0	5.7	37.0	6.3	6.0	3.3	8.8	24.7	One-Step	and-Hold	
6	8	0.5%	CE Return (%)	1.21	1.44	1.50	1.51	41.17	1.65	1.89	1.64	1.57	1.77	1.51	1.57	0.06
			Mean Std. Error (%)	0.00	0.05	0.04	0.04	0.62	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	9.3	2.2	3.9	3.7	61.2	4.8	12.3	1.9	5.9	9.3	and-Hold	Gradient	
6	8	1.0%	CE Return (%)	0.65	0.82	1.33	1.33	37.66	1.51	1.45	1.54	1.40	1.77	1.33	1.40	0.07
			Mean Std. Error (%)	0.01	0.04	0.05	0.05	0.57	0.01	0.01	0.00	0.00		Modified	Modified	0.05
			Turnover (%)	9.3	0.8	3.1	3.0	48.2	4.3	4.6	1.6	5.7	9.3	One-Step	Gradient	
6	8	2.0%	CE Return (%)	-0.46	0.51	1.04	1.04	32.33	1.23	1.09	1.38	1.14	1.77	1.04	1.09	0.04
			Mean Std. Error (%)	0.01	0.00	0.06	0.06	0.50	0.01	0.00	0.01	0.00		Modified	Rolling Buy-	0.06
			Turnover (%)	9.2	0.0	2.2	2.1	37.0	3.9	2.5	1.2	5.3	9.3	One-Step	and-Hold	
															Average	0.09
															Minimum	0.04
															Maximum	0.18

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
24	1.5	0.5%	CE Return (%)	4.69	6.25	6.68	6.69	53.64	7.02	7.87	7.17	7.06	7.30	6.69	7.02	0.33
			Mean Std. Error (%)	0.02	0.17	0.09	0.11	0.34	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.11
			Turnover (%)	44.3	3.9	7.6	5.9	90.4	4.6	20.6	2.3	3.6	45.5	and-Hold	One-Step	
24	1.5	1.0%	CE Return (%)	2.14	4.00	6.29	6.35	49.20	6.76	6.97	7.04	6.83	7.30	6.35	6.76	0.41
			Mean Std. Error (%)	0.04	0.18	0.10	0.12	0.33	0.00	0.01	0.01	0.00		Rolling Buy-	Modified	0.12
			Turnover (%)	43.1	2.1	6.1	5.0	68.0	4.2	8.9	2.2	3.5	45.5	and-Hold	One-Step	
24	1.5	2.0%	CE Return (%)	-2.79	0.57	5.68	5.70	42.66	6.36	6.29	6.80	6.38	7.30	5.70	6.29	0.59
			Mean Std. Error (%)	0.07	0.02	0.16	0.17	0.31	0.01	0.01	0.01	0.01		Rolling Buy-	Rolling Buy-	0.17
			Turnover (%)	40.9	0.0	4.5	3.9	48.6	3.9	4.4	2.0	3.4	45.5	and-Hold	and-Hold	
24	3	0.5%	CE Return (%)	2.92	3.38	3.65	3.68	50.13	3.98	5.13	3.98	3.92	4.06	3.68	3.92	0.24
			Mean Std. Error (%)	0.01	0.11	0.05	0.06	0.37	0.00	0.02	0.00	0.00		Rolling Buy-	Modified	0.06
			Turnover (%)	19.2	2.2	6.3	4.8	90.4	3.4	21.7	1.4	2.9	19.4	and-Hold	Gradient	
24	3	1.0%	CE Return (%)	1.80	2.25	3.37	3.45	45.87	3.89	4.18	3.90	3.79	4.06	3.45	3.79	0.34
			Mean Std. Error (%)	0.01	0.09	0.07	0.08	0.35	0.00	0.01	0.00	0.00		Rolling Buy-	Modified	0.08
			Turnover (%)	19.0	1.0	4.5	3.4	68.0	3.3	8.8	1.3	2.9	19.4	and-Hold	Gradient	
24	3	2.0%	CE Return (%)	-0.42	0.54	3.01	3.07	39.58	3.73	3.71	3.75	3.54	4.06	3.07	3.54	0.48
			Mean Std. Error (%)	0.03	0.01	0.11	0.11	0.33	0.01	0.01	0.01	0.01		Rolling Buy-	Modified	0.11
			Turnover (%)	18.6	0.0	2.8	2.1	48.6	3.2	4.3	1.2	2.8	19.4	and-Hold	Gradient	
24	8	0.5%	CE Return (%)	1.40	1.56	1.66	1.67	41.01	1.83	3.14	1.79	1.77	1.82	1.67	1.77	0.10
			Mean Std. Error (%)	0.00	0.04	0.02	0.02	0.50	0.00	0.03	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	7.1	0.9	2.4	1.8	90.6	2.3	21.6	0.5	1.7	7.1	and-Hold	Gradient	
24	8	1.0%	CE Return (%)	0.97	1.15	1.55	1.58	37.28	1.83	2.22	1.76	1.73	1.82	1.58	1.73	0.14
			Mean Std. Error (%)	0.00	0.04	0.03	0.03	0.48	0.01	0.02	0.00	0.00		Rolling Buy-	Modified	0.03
			Turnover (%)	7.0	0.4	1.7	1.3	68.1	2.2	8.7	0.5	1.7	7.1	and-Hold	Gradient	
24	8	2.0%	CE Return (%)	0.13	0.52	1.42	1.44	31.74	1.86	1.94	1.70	1.64	1.82	1.44	1.64	0.20
			Mean Std. Error (%)	0.01	0.00	0.04	0.04	0.42	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	7.0	0.0	1.1	0.8	48.6	2.3	3.7	0.5	1.7	7.1	and-Hold	Gradient	
															Average	0.31
															Minimum	0.10
															Maximum	0.59

Table A3: Results for the Three-Asset Model with Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
48	1.5	0.5%	CE Return (%)	4.85	6.33	6.79	6.78	53.85	7.16	8.17	7.27	7.22	7.34	6.79	7.16	0.37
			Mean Std. Error (%)	0.02	0.12	0.06	0.08	0.24	0.00	0.02	0.00	0.00		Modified	Modified	0.06
			Turnover (%)	46.1	2.5	6.6	4.4	160.0	3.3	24.7	1.2	1.9	48.5	One-Step	One-Step	
48	1.5	1.0%	CE Return (%)	2.42	4.53	6.54	6.57	49.35	6.99	7.23	7.21	7.10	7.34	6.57	6.99	0.43
			Mean Std. Error (%)	0.03	0.15	0.07	0.08	0.23	0.00	0.01	0.00	0.00		Rolling Buy-	Modified	0.08
			Turnover (%)	43.9	1.3	4.9	3.5	115.6	2.5	9.9	1.2	1.8	48.5	and-Hold	One-Step	
48	1.5	2.0%	CE Return (%)	-2.28	0.61	6.01	6.00	42.70	6.87	6.79	7.09	6.88	7.34	6.01	6.79	0.79
			Mean Std. Error (%)	0.06	0.03	0.11	0.12	0.22	0.01	0.01	0.01	0.01		Modified	Rolling Buy-	0.10
			Turnover (%)	39.7	0.0	3.3	2.4	76.6	2.3	2.6	1.1	1.8	48.5	One-Step	and-Hold	
48	3	0.5%	CE Return (%)	3.01	3.37	3.70	3.72	50.36	4.11	5.59	4.04	4.02	4.09	3.72	4.02	0.30
			Mean Std. Error (%)	0.01	0.08	0.03	0.04	0.28	0.00	0.03	0.00	0.00		Rolling Buy-	Modified	0.04
			Turnover (%)	19.4	1.6	6.0	4.2	160.0	2.9	26.8	0.7	1.6	19.8	and-Hold	Gradient	
48	3	1.0%	CE Return (%)	1.94	2.46	3.46	3.53	46.06	4.12	4.46	4.00	3.95	4.09	3.53	3.95	0.42
			Mean Std. Error (%)	0.01	0.08	0.05	0.06	0.26	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.06
			Turnover (%)	19.0	0.7	4.1	2.8	115.6	2.5	10.5	0.7	1.6	19.8	and-Hold	Gradient	
48	3	2.0%	CE Return (%)	-0.17	0.56	3.12	3.17	39.66	4.16	4.10	3.92	3.83	4.09	3.17	3.83	0.66
			Mean Std. Error (%)	0.02	0.01	0.08	0.08	0.24	0.01	0.01	0.01	0.00		Rolling Buy-	Modified	0.08
			Turnover (%)	18.2	0.0	2.3	1.5	76.6	2.4	3.1	0.7	1.5	19.8	and-Hold	Gradient	
48	8	0.5%	CE Return (%)	1.43	1.55	1.67	1.68	41.73	1.91	3.91	1.81	1.80	1.83	1.68	1.80	0.12
			Mean Std. Error (%)	0.00	0.03	0.01	0.02	0.46	0.00	0.05	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	6.9	0.7	2.2	1.6	153.2	2.8	26.4	0.3	0.9	6.9	and-Hold	Gradient	
48	8	1.0%	CE Return (%)	1.03	1.22	1.58	1.61	38.07	1.98	2.53	1.79	1.78	1.83	1.61	1.78	0.17
			Mean Std. Error (%)	0.00	0.03	0.02	0.02	0.42	0.01	0.02	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	6.8	0.3	1.6	1.1	112.9	2.4	10.9	0.3	0.9	6.9	and-Hold	Gradient	
48	8	2.0%	CE Return (%)	0.23	0.52	1.46	1.48	32.42	2.14	2.24	1.76	1.74	1.83	1.48	1.74	0.26
			Mean Std. Error (%)	0.01	0.00	0.03	0.03	0.39	0.02	0.01	0.00	0.00		Rolling Buy-	Modified	0.03
			Turnover (%)	6.7	0.0	0.9	0.6	75.9	2.3	3.5	0.3	0.9	6.9	and-Hold	Gradient	
															Average	0.39
															Minimum	0.12
															Maximum	0.79

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
6	1.5	0.5%	CE Return (%)	10.54	11.26	12.49	12.49	54.50	12.49	12.49	13.13	12.52	13.62	12.49	12.49	0.00
			Mean Std. Error (%)	0.29	0.06	0.00	0.00	0.44	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	17.1	12.8	16.6	16.6	105.7	16.6	16.6	6.3	15.1	17.2	and-Hold	and-Hold	
6	1.5	1.0%	CE Return (%)	9.42	5.91	11.38	11.38	46.90	11.38	11.38	12.79	11.44	13.62	11.38	11.38	0.00
			Mean Std. Error (%)	0.28	0.00	0.00	0.00	0.39	0.00	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	17.0	0.0	16.5	16.5	81.4	16.5	16.5	4.6	14.9	17.2	and-Hold	and-Hold	
6	1.5	2.0%	CE Return (%)	7.84	5.91	9.21	9.21	37.05	9.21	9.21	12.36	9.30	13.62	9.21	9.21	0.00
			Mean Std. Error (%)	0.20	0.00	0.01	0.00	0.33	0.00	0.00	0.04	0.01		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	16.8	0.0	16.3	16.3	47.4	16.3	16.3	2.7	14.6	17.2	and-Hold	and-Hold	
6	3	0.5%	CE Return (%)	10.72	8.49	10.79	10.79	52.99	10.81	10.80	11.51	10.83	11.91	10.79	10.80	0.01
			Mean Std. Error (%)	0.00	0.08	0.01	0.01	0.40	0.00	0.00	0.01	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	17.9	6.5	16.6	16.6	105.7	16.6	16.6	5.7	14.7	18.0	One-Step	and-Hold	
6	3	1.0%	CE Return (%)	9.54	5.91	9.69	9.70	45.62	9.72	9.71	11.20	9.76	11.91	9.70	9.71	0.01
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.36	0.00	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.01
			Turnover (%)	17.8	0.0	16.5	16.5	81.4	16.5	16.5	4.6	14.6	18.0	and-Hold	and-Hold	
6	3	2.0%	CE Return (%)	7.24	5.91	7.60	7.60	35.99	7.78	7.70	10.75	7.81	11.91	7.60	7.70	0.09
			Mean Std. Error (%)	0.00	0.00	0.06	0.06	0.31	0.00	0.00	0.03	0.01		Rolling Buy-	Rolling Buy-	0.06
			Turnover (%)	17.6	0.0	11.8	11.7	47.4	11.0	11.9	3.2	14.5	18.0	and-Hold	and-Hold	
6	8	0.5%	CE Return (%)	8.56	6.83	8.62	8.63	48.56	8.68	8.65	9.46	8.67	9.74	8.63	8.65	0.02
			Mean Std. Error (%)	0.00	0.07	0.01	0.01	0.43	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.01
			Turnover (%)	18.0	2.5	16.6	16.6	105.7	16.5	16.6	4.3	14.7	18.1	and-Hold	and-Hold	
6	8	1.0%	CE Return (%)	7.39	5.91	7.58	7.58	41.88	7.75	7.63	9.23	7.68	9.74	7.58	7.63	0.05
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.38	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.03
			Turnover (%)	17.9	0.0	13.4	13.3	81.4	10.3	7.4	3.8	14.6	18.1	and-Hold	and-Hold	
6	8	2.0%	CE Return (%)	5.12	5.91	6.45	6.45	32.97	6.71	6.58	8.87	6.70	9.74	6.45	6.58	0.12
			Mean Std. Error (%)	0.00	0.00	0.10	0.10	0.33	0.01	0.00	0.02	0.00		Rolling Buy-	Rolling Buy-	0.10
			Turnover (%)	17.8	0.0	4.4	4.4	47.4	9.0	3.3	3.0	14.8	18.1	and-Hold	and-Hold	
															Average	0.03
															Minimum	0.00
															Maximum	0.12

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
24	1.5	0.5%	CE Return (%)	12.40	12.04	13.34	13.34	53.25	13.34	13.34	13.50	13.36	13.62	13.34	13.34	0.00
			Mean Std. Error (%)	0.08	0.03	0.00	0.00	0.21	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	15.0	3.3	4.1	4.1	155.3	4.1	4.1	2.1	3.9	4.8	and-Hold	and-Hold	
24	1.5	1.0%	CE Return (%)	11.52	5.92	13.06	13.06	45.55	13.06	13.06	13.38	13.10	13.62	13.06	13.06	0.00
			Mean Std. Error (%)	0.08	0.00	0.00	0.00	0.18	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	14.9	0.0	4.1	4.1	110.7	4.1	4.1	2.0	3.9	4.8	and-Hold	and-Hold	
24	1.5	2.0%	CE Return (%)	10.39	5.91	12.50	12.51	36.00	12.50	12.50	13.18	12.58	13.62	12.51	12.50	0.00
			Mean Std. Error (%)	0.07	0.00	0.00	0.00	0.16	0.00	0.00	0.01	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	12.6	0.0	4.1	4.1	57.9	4.1	4.1	1.8	3.9	4.8	and-Hold	and-Hold	
24	3	0.5%	CE Return (%)	11.54	8.99	11.63	11.63	51.81	11.65	11.65	11.82	11.65	11.91	11.63	11.65	0.02
			Mean Std. Error (%)	0.00	0.05	0.01	0.01	0.19	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	5.8	1.7	4.1	4.1	155.3	4.2	4.1	1.8	4.0	5.9	One-Step	and-Hold	
24	3	1.0%	CE Return (%)	11.16	5.92	11.35	11.35	44.32	11.38	11.37	11.72	11.40	11.91	11.35	11.37	0.02
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.17	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	5.8	0.0	4.1	4.1	110.7	4.1	4.1	1.7	3.9	5.9	One-Step	and-Hold	
24	3	2.0%	CE Return (%)	10.42	5.91	10.07	10.04	34.96	10.90	10.91	11.55	10.88	11.91	10.42	10.88	0.46
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.15	0.00	0.00	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.7	0.0	3.1	3.0	57.9	4.1	4.1	1.6	3.9	5.9		Gradient	
24	8	0.5%	CE Return (%)	9.36	7.09	9.45	9.45	47.47	9.55	9.54	9.67	9.48	9.74	9.45	9.48	0.03
			Mean Std. Error (%)	0.00	0.04	0.01	0.01	0.29	0.00	0.00	0.00	0.00		Rolling Buy-	Modified	0.01
			Turnover (%)	5.9	0.6	4.2	4.1	155.3	4.5	4.4	1.3	4.1	6.0	and-Hold	Gradient	
24	8	1.0%	CE Return (%)	8.98	5.92	8.96	8.95	40.72	9.31	9.30	9.60	9.22	9.74	8.98	9.22	0.23
			Mean Std. Error (%)	0.00	0.00	0.02	0.02	0.25	0.00	0.00	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.9	0.0	3.5	3.4	110.7	4.2	4.1	1.3	4.0	6.0		Gradient	
24	8	2.0%	CE Return (%)	8.24	5.91	7.47	7.47	32.08	9.95	10.00	9.48	8.70	9.74	8.24	8.70	0.46
			Mean Std. Error (%)	0.00	0.00	0.06	0.06	0.20	0.02	0.02	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	5.8	0.0	1.2	1.2	57.9	4.3	4.3	1.3	4.0	6.0		Gradient	
															Average	0.13
															Minimum	0.00
															Maximum	0.46

Table A4: Results for the Ten-Asset Model without Predictability

Parameters				Heuristic Strategies				Dual Bounds						Best Performance		
Horizon (T)	Risk Aversion Coeff. (γ)	Trans. Cost Rate (δ)		Cost Blind	One-Step	Modified One-Step	Rolling Buy-and-Hold	Zero Penalty	Modified One-Step	Rolling Buy-and-Hold	Frictionless Gradient Based	Modified Gradient	No Trans. Cost Bound	Best Strategy	Best Upper Bound	Gap
48	1.5	0.5%	CE Return (%)	13.00	12.27	13.48	13.48	53.19	13.48	13.48	13.56	13.50	13.62	13.48	13.48	0.00
			Mean Std. Error (%)	0.04	0.02	0.00	0.00	0.15	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	9.6	1.7	2.1	2.1	262.5	2.1	2.1	1.2	2.0	2.9	and-Hold	and-Hold	
48	1.5	1.0%	CE Return (%)	12.54	5.92	13.34	13.34	45.45	13.34	13.34	13.51	13.37	13.62	13.34	13.34	0.00
			Mean Std. Error (%)	0.04	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	9.5	0.0	2.1	2.1	174.4	2.1	2.1	1.2	2.0	2.9	and-Hold	and-Hold	
48	1.5	2.0%	CE Return (%)	11.83	5.91	13.06	13.06	35.87	13.06	13.06	13.40	13.13	13.62	13.06	13.06	0.00
			Mean Std. Error (%)	0.03	0.00	0.00	0.00	0.11	0.00	0.00	0.00	0.00		Rolling Buy-	Rolling Buy-	0.00
			Turnover (%)	8.4	0.0	2.0	2.0	82.2	2.0	2.0	1.1	2.0	2.9	and-Hold	and-Hold	
48	3	0.5%	CE Return (%)	11.67	9.19	11.76	11.76	51.77	11.80	11.80	11.87	11.79	11.91	11.76	11.79	0.03
			Mean Std. Error (%)	0.00	0.04	0.01	0.01	0.14	0.00	0.00	0.00	0.00		Modified	Modified	0.01
			Turnover (%)	4.0	0.9	2.1	2.1	262.5	2.4	2.4	1.0	2.0	4.1	One-Step	Gradient	
48	3	1.0%	CE Return (%)	11.44	5.92	11.62	11.62	44.25	11.66	11.65	11.82	11.67	11.91	11.62	11.65	0.03
			Mean Std. Error (%)	0.00	0.00	0.01	0.01	0.13	0.00	0.00	0.00	0.00		Modified	Rolling Buy-	0.01
			Turnover (%)	4.0	0.0	2.1	2.1	174.4	2.1	2.1	1.0	2.0	4.1	One-Step	and-Hold	
48	3	2.0%	CE Return (%)	10.96	5.91	10.55	10.51	34.85	11.48	11.50	11.73	11.42	11.91	10.96	11.42	0.46
			Mean Std. Error (%)	0.00	0.00	0.03	0.03	0.11	0.00	0.00	0.01	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.0	0.0	1.6	1.5	82.2	2.0	2.0	1.0	2.0	4.1		Gradient	
48	8	0.5%	CE Return (%)	9.49	7.17	9.57	9.58	47.24	9.76	9.76	9.70	9.61	9.74	9.58	9.61	0.03
			Mean Std. Error (%)	0.00	0.04	0.02	0.02	0.24	0.01	0.01	0.00	0.00		Rolling Buy-	Modified	0.02
			Turnover (%)	4.1	0.3	2.2	2.1	249.5	3.4	3.5	0.8	2.1	4.1	and-Hold	Gradient	
48	8	1.0%	CE Return (%)	9.25	5.92	9.22	9.19	40.69	9.64	9.64	9.67	9.49	9.74	9.25	9.49	0.24
			Mean Std. Error (%)	0.00	0.00	0.02	0.02	0.20	0.00	0.00	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.1	0.0	1.9	1.8	171.6	2.7	2.7	0.8	2.1	4.1		Gradient	
48	8	2.0%	CE Return (%)	8.77	5.91	7.63	7.62	31.95	12.02	12.29	9.61	9.24	9.74	8.77	9.24	0.47
			Mean Std. Error (%)	0.00	0.00	0.05	0.05	0.17	0.07	0.07	0.00	0.00		Cost Blind	Modified	0.00
			Turnover (%)	4.1	0.0	0.6	0.6	82.0	6.0	6.1	0.8	2.0	4.1		Gradient	
															Average	0.14
															Minimum	0.00
															Maximum	0.47