The Outlier Process: Unifying Line Processes and Robust Statistics

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Abstract

This paper unifies "line-process" approaches for regularization with discontinuities and robust estimation techniques. We generalize the notion of a "line process" to that of an analog "outlier process" and show that a problem formulated in terms of outlier processes can be viewed in terms of robust statistics. We also characterize a class of robust statistical problems for which an equivalent outlier-process formulation exists and give a straightforward method for converting a robust estimation problem into an outlier-process formulation. This outlier-processes approach provides a general framework which subsumes the traditional line-process approaches as well as a wide class of robust estimation problems. Examples in image reconstruction and optical flow are used to illustrate the approach.

1 Introduction

The modeling of spatial discontinuities for problems such as surface recovery, segmentation, image reconstruction, and optical flow has been intensely studied. In particular "line-process" models of discontinuities have been popular due, in part, to their intuitive and physical appeal, as well as their ability to model spatial properties of discontinuities. More recently, the use of robust statistics in computer vision has become popular and, at first glance, it is not at all clear that line-process approaches and robust statistics have anything in common. The goal of this paper is to show that they are closely related and that, by bringing this relationship to light, each approach can benefit from the other. Moreover, we propose a new framework based on analog or binary "outlier processes" which subsumes traditional line process approaches and a wide class of robust estimation approaches.

We first generalize the notion of a "line process" to that of an "outlier process" and show that a problem formulated in terms of outlier processes can be viewed in terms of robust statistics. While line processes have been used to account for spatial discontinuities, outlier processes are intended to be more general and can also be used to cope with gross measurement errors encountered in problems like stereo and optical flow.

Then we characterize a class of robust statistical problems for which an equivalent outlier-process formulation exists and derive a straightforward mechanism for converting the robust estimation problem to the outlier-process problem. The resulting formulation, with explicit outlier processes, is more general than the original robust estimation problem. For example, since the outlier processes are explicit we can formulate constraints on their spatial organization. Moreover, the deterministic continuation methods used for minimizing the robust formulation (eg. deterministic annealing [5] or Graduated Non-Convexity [4]) can be directly applied to the explicit outlier processes formulation with spatial organization constraints.

The remainder of the paper reviews previous work on line processes and robust statistics and shows how to convert between the two formulations. Examples illustrate the use of outlier processes and the conversion of a robust estimation problem into an optimization problem with an explicit analog outlier process.

2 Previous Work

Geman and Geman [9] introduced the notion of a binary "line process" for modeling spatial discontinuities in image brightness and formulated constraints on the local spatial organization of discontinuities. Unfortunately, the introduction of line processes results in a non-convex optimization problem.

Blake and Zisserman [4] showed that binary line processes can be eliminated from an optimization problem when no spatial constraints are imposed on the discontinuities. The result is an objective function containing an energy functional, or "weak constraint", which enforces spatial smoothness as long as neighboring points are "similar enough".

*Portions of this work were performed at the NASA Ames Research Center, Yale University, and the University of Toronto with support from NASA (NGT-50749), ONR(N00014-91-J-1577), and NSERC.

†This work was supported by grants from AFOSR (AFOSR-90-0224) and ARPA-ONR (N00014-92-J-4048).

The weak constraint approach still requires minimizing a non-convex objective function, but Blake and Zisserman showed that their energy functional could be generalized by the introduction of a "control parameter" which can be used to adjust the "shape" of the function. Using this parameter, they devised a *continuation method* called Graduated Non-Convexity (GNC).

Separately, the field of robust statistics [10] has developed methods to address the fact that the parametric models of classical statistics are often approximations of the phenomena being modeled. In particular, the field addresses how to handle *outliers*, or gross errors, that do not conform to the statistical assumptions.

Robust estimators, like line processes, have been used to account for spatial discontinuities in early vision problems [2, 12]. But, while line-process formulations typically are concerned with violations of the spatial smoothness assumption, measurements (of depth, optical flow, etc.) may also contain outliers. Black and Anandan [3] use robust estimation to account for both spatial discontinuities and measurement errors in the recovery of dense optical flow fields. Similarly, Geiger and Pereira [6] add a "sparse process" on image measurements in the context of image compression. Geiger and Yuille [7] show that mean-field techniques [5] can be used to integrate out both sparse and line processes.

Eliminating the line processes by minimizing over them [4] or integrating them out [5] produces an objective function with an energy functional which is similar to the *redescending estimators* [10] used in robust statistics. A number of authors have noted this similarity [2, 3, 6, 7, 8]. The connection becomes clearer when we consider analog line processes with general penalty functions. Geman and Reynolds [8] show that minimizing over the analog processes produces particular estimators and they also specify conditions on an estimator that must be satisfied if it is to have an equivalent line-process formulation. We provide a constructive proof of these conditions in [1] and, in this paper, make explicit the mechanism for recovering the analog process and extend their results to include a measurement process.

Finally, Rangarajan and Chellappa [11] show how an analog line process can be recovered for a general class of estimators, but do not address the robustness of the data term or connect the approach to robust estimation. Black [2] introduces analog "outlier processes" and exploits the results of Rangarajan and Chellappa [11] to convert between robust estimation problems and outlier process formulations.

3 Line Processes

To introduce the idea of binary line processes we will consider a simple example of reconstructing a smooth surface $\mathbf{u}$ from noisy depth data $\mathbf{d}$. Assume that the data is an $n \times n$ image of sites S, and each site (or pixel), $s \in S$, has a set of neighbors $t \in \mathcal{G}_s$. For a first-order neighborhood system, $\mathcal{G}_s$, these are just the sites to the North, South, East, and West of site s. We also define a dual $n \times n$ lattice, $S^L = (s,t)$, of all nearest neighbor pairs (s,t) in S. This lattice is coupled to the original in such a way that the best interpretation of the data will be one in which the data is piecewise smooth. An analog line process $l_{s,t} \in S^L$ takes on values $0 \leq l_{s,t} \leq C$, for some positive constant C (for the remainder of the paper we take $C = 1$). The line process indicates the presence ($l_{s,t} \to 0$) or absence ($l_{s,t} \to 1$) of a discontinuity between neighboring sites s and t. We also define a penalty $0 \leq \Psi(l_{s,t}) \leq 1$ which is paid for introducing a discontinuity. The penalty function goes to 1 as $l_{s,t}$ tends to 0 and $\Psi(l_{s,t}) \to 0$ when there is no discontinuity ($l_{s,t} \to 1$). For these experiments we take

$$\Psi(z) = z - 1 - \log z,$$

which is derived from the Lorentzian estimator [7].

To recover the surface $\mathbf{u}$ and the line processes $\mathbf{l}$ we minimize the objective function $E(\mathbf{u}, \mathbf{d}, \mathbf{l})$:

$$\sum_{s \in S} \left((u_s - d_s)^2 + \sum_{t \in \mathcal{G}_s} [(u_s - u_t)^2 \, l_{s,t} + \Psi(l_{s,t})] \right). \quad (1)$$

The first term ensures that the recovered surface is faithful to the data, while the second term encodes our prior assumption about piecewise smooth nature of the surface. When no discontinuity is present ($l_{s,t} \to 1$), the smoothness term has the original least-squares form and no penalty is paid, but when a discontinuity is introduced ($l_{s,t} \to 0$) the penalty term $\Psi(l_{s,t})$ dominates. Minimizing this new objective function with respect to $\mathbf{u}$ and $\mathbf{l}$ gives a piecewise smooth surface with breaks where the spatial gradient is too large.

4 Robust Statistics

As identified by Hampel *et al.* [10, page 11] the main goals of robust statistics are: "(*i*) To describe the structure best fitting the bulk of the data, (*ii*) To identify deviating data points (outliers) or deviating substructures for further treatment, if desired."

Least-squares estimation is notoriously sensitive to outliers; the problem being that outliers contribute "too much" to the overall solution. Outlying points are assigned a high weight by the quadratic estimator

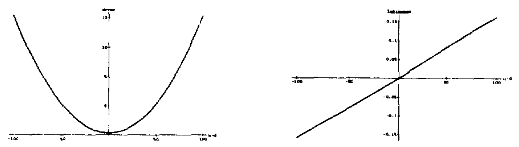


Figure 1: Quadratic estimator (a) and ψ-function (b).

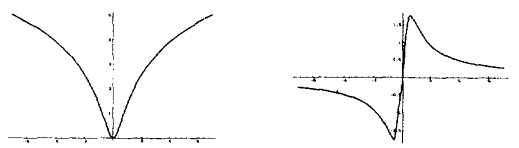


Figure 2: Lorentzian Estimator. (a) Estimator, (b) ψ-function.

(see Figure 1a). To analyze the behavior of an estimator, we take the approach of Hampel *et al.* [10] based on ***influence functions.*** The influence function characterizes the bias that a particular measurement has on the solution and is proportional to the derivative, ψ, of the estimator [10]. Consider, for example, the quadratic estimator:

$$\rho(x) = x^2, \qquad \psi(x) = 2x. \tag{2}$$

For least-squares estimation, the influence of outliers increases linearly and without bound (Figure 1b).

To increase robustness, an estimator must be more forgiving about outlying measurements. We will consider *redescending* estimators [10] for which the influence of outliers tends to zero.[3] One such estimator is the *Lorentzian*:

$$\rho(x,\sigma) = \log\left(1 + \frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right), \quad \psi(x,\sigma) = \frac{2x}{2\sigma^2 + x^2}.$$

The estimator is plotted along with its ψ-function in Figure 2. Examination of the ψ-functions reveals that the estimator has a *saturating* property; that is, the influence of outliers tends to zero. The remainder of the paper will examine the relationship between these redescending estimators and outlier processes.

We now apply robust estimation to the least-squares formulation of the surface recovery problem:

$$E(\mathbf{u},\mathbf{d}) = \sum_{s\in S}[(u_s - d_s)^2 + \sum_{t\in\mathcal{G}_s}(u_s - u_t)^2].$$

[3] Hampel *et al.* chose a more conservative definition of redescending estimators that requires $\psi(x) = 0$, $|x| \geq r$ for some positive r.

Both the data and spatial smoothness assumptions may be violated causing gross errors in $(u_s - d_s)$ and $(u_s - u_t)$ respectively. Our approach is to replace the quadratic estimator with robust estimators ρ_D and ρ_S for the data and spatial terms. This gives the following objective function $E(\mathbf{u},\mathbf{d})$:

$$\sum_{s\in S}[\rho_D(u_s - d_s, \sigma_D) + \sum_{t\in\mathcal{G}_s}\rho_S(u_s - u_t, \sigma_S)], \tag{3}$$

where the scale parameters σ_* may or may not be present depending on the estimator. As with the line-process approaches the objective function may be non-convex depending on the choice of estimator.

5 Unifying Robust Estimation and Outlier Processes

This section unifies the robust estimation approaches and traditional line-process approaches. First we introduce the notion of an outlier process which is a generalization of the line process. We then show how these binary or analog outlier processes can be eliminated in the same way that line processes are eliminated and how this results in a robust estimation problem. The connection is made complete by deriving a mechanism for converting robust estimators into outlier processes.

5.1 Outlier Processes

The line-process formulation of the surface recovery problem (Eqn. 1) accounts for violations of the spatial smoothness term, but does not account for violations of the data term. In many situations the data term, like the spatial term, is only an approximate model of the data process. This prompts us to generalize the notion of a "line process" to that of an "outlier process" that can be applied to both data and spatial terms. The motivation behind such a generalization is to formulate a process that performs *outlier rejection* in the same spirit as the robust estimators do. The recovery problem is then reformulated as the minimization of $E(\mathbf{u},\mathbf{d},\mathbf{l},\mathbf{m})$ using outlier processes as follows:

$$\sum_{s\in S}[[(u_s - d_s)^2\, m_s + \Psi_D(m_s)] + \sum_{t\in\mathcal{G}_s}[(u_s - u_t)^2\, l_{s,t} + \Psi_S(l_{s,t})]], \tag{4}$$

where we have simply introduced a ***measurement*** process m_s and a new penalty term Ψ_D for rejecting the measurement. This process allows us to ignore erroneous information from the data term.

5.2 From Outlier Processes to Robust Estimation

The outlier-process formulation leads to a joint estimation problem where one not only has to estimate $\mathbf{u}$ but also the outlier processes $\mathbf{l}$ and $\mathbf{m}$. In the case of the simple binary line-process formulation, Blake and Zisserman [4] show that the line variables can be removed from the equation by first minimizing over them. They obtain a new objective function that is solely a function of $\mathbf{u}$. Exactly the same treatment can be applied to the general analog outlier-process version.

Since the measurement term does not depend on $\mathbf{l}$ and the smoothness term does not depend on $\mathbf{m}$ we can write the optimization problem as

$$\min_{\mathbf{u}} \left[\left[\min_{\mathbf{m}} \sum_{s \in S} (u_s - d_s)^2 \, m_s + \Psi_D(m_s) \right] + \left[\min_{\mathbf{l}} \sum_{s \in S} \sum_{t \in \mathcal{G}_s} (u_s - u_t)^2 \, l_{s,t} + \Psi_S(l_{s,t}) \right] \right]. \quad (5)$$

We can now minimize with respect to each process separately; that is for each term we compute:

$$\rho(x) = \inf_{0 \le z \le 1} (x^2 \, z + \Psi(z)), \quad (6)$$

where z is the outlier process. Finally, we can rewrite the minimization problem as

$$\min_{\mathbf{u}} \left[\sum_{s \in S} \rho_D(u_s - d_s) + \sum_{s \in S} \sum_{t \in \mathcal{G}_s} \rho_S(u_s - u_t) \right]. \quad (7)$$

Geiger and Yuille [7] propose a similar formulation with a binary (as opposed to analog) process for the data and spatial terms and, using mean-field theory techniques, they integrate out the binary processes giving a robust estimation problem with the mean-field function as the robust estimator.

5.3 From Robust Estimators to Outlier Processes

Finally, to close the loop, we must take an objective function written in terms of robust estimators and derive a new objective function which is written in terms of analog outlier processes. Consider a simple robust objective function

$$E(x) = \rho(x) \quad (8)$$

defined in terms of some "error" x, where x, for example, might be the spatial gradient $u_s - u_t$ or the data error $u_s - d_s$. We want to construct a new objective function:

$$E(x, z) = x^2 z + \Psi(z) \quad (9)$$

where $0 \le z \le 1$ is an outlier process, and Ψ is a "penalty function", such that the minimum of $E(x, z)$ with respect to x is the same as $E(x)$.

1. Define $\phi(w) = \rho(\sqrt{w/\tau})$ where w is possibly scaled by a parameter τ in ρ.
2. Compute the first and second partial derivatives ($\phi'(w)$ and $\phi''(w)$) of ϕ with respect to w.
3. If $\lim_{w \to 0} \phi'(w) = 1$, and
   $\lim_{w \to \infty} \phi'(w) = 0$, and
   $\phi''(w) < 0$,

   then proceed, otherwise stop: ρ does not have a simple outlier process formulation.
4. Define the outlier process $z = \phi'(w)$.
5. Solve $z = \phi'(w)$ for w giving $w = (\phi')^{-1}(z)$.
6. Define:

   $$\Psi(z) = \phi(w) - zw = \phi((\phi')^{-1}(z)) - z(\phi')^{-1}(z).$$

7. The new objective function is:

   $$E(x, z) = \tau x^2 z + \Psi(z)$$

Figure 3: A simple mechanism for recovering the outlier process from a robust estimator.

The reader is referred to [1] for a derivation of $\Psi(z)$ which is a simplification of the approach of Rangarajan and Chellappa [11]. The result is the same as that of Geman and Reynolds [8] but the derivation provides a constructive proof of the result. The results of the derivation are summarized in Figure 3 which provides a straightforward mechanism for converting robust formulations to outlier process formulations. A catalog of common estimators and their outlier processes is provided in [1].

Example: To illustrate the mechanism we consider the robust surface reconstruction example:

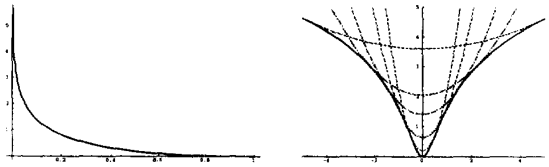
$$E(\mathbf{u}) = \sum_{s \in S} \rho(u_s - d_s, \sigma_D) + \sum_{t \in \mathcal{G}_s} \rho(u_s - u_t, \sigma_S), \quad (10)$$

where σ_D and σ_S are constant scale parameters and where ρ is the Lorentzian estimator:

$$\rho(x, \sigma) = \log\left(1 + \frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right). \quad (11)$$

The first step is to define $\phi(w) = \log(1 + w)$. We then compute the first and second partial derivatives w.r.t. w:

$$\phi'(w) = \frac{1}{1 + w}, \quad \phi''(w) = -\frac{1}{(1 + w)^2}.$$



a b

Figure 4: Lorentzian outlier process. *(a)* Penalty function. *(b)* Infimum of $E(x, z, \sigma)$.

We observe that these satisfy the conditions in Step 3 of the mechanism and we can define $z = \phi'(w)$. Solving for w gives:

$$w = (\phi')^{-1}(z) = 1/z - 1.$$

Given z and w we can write the penalty function as

$$\Psi(z) = -1 + z + \log(1/z) = z - 1 - \log z.$$

The penalty function is plotted in Figure 4*a*. The outlier-process formulation for the Lorentzian is then:

$$E(x, z, \sigma) = \frac{1}{2}\left(\frac{x}{\sigma}\right)^2 z + \Psi(z). \tag{12}$$

The infimum of this family of quadratics is the Lorentzian estimator which is shown in bold in Figure 4*b*. The figure also shows (Eqn. 12) plotted (in gray) for various values of z.

We can now write the surface recovery problem using the Lorentzian outlier process. We minimize $E(\mathbf{u}, \mathbf{m}, \mathbf{l}, \sigma_D, \sigma_S)$:

$$\sum_{s \in S}\left[\left(\frac{1}{2\sigma_D^2}(u_s - d_s)^2 m_s + \Psi(m_s)\right) + \sum_{t \in \mathcal{G}_s}\left(\frac{1}{2\sigma_S^2}(u_s - u_t)^2 l_{s,t} + \Psi(l_{s,t})\right)\right], \tag{13}$$

where Ψ is the penalty function defined above.

6 Exploiting the Relationship

The recovery of the analog outlier process allows the formulation of prior assumptions on the spatial organization of outliers without sacrificing the deterministic continuation methods like GNC.

6.1 Adding Spatial Interactions

One motivation for recovering the outlier process is that it allows us to incorporate into the objective function prior assumptions on the nature of discontinuities. We generalize standard spatial coherence constraints to the case of analog spatial outlier processes.

C_{hist} :
so | to
uo | vo

C_{supp} : so | to | uo

Figure 5: Cliques (up to rotation) for spatial constraints.

We consider two kinds of interaction terms; ***hysteresis*** which assists in the formation of unbroken contours and *non-maximum suppression* which inhibits multiple responses to a single edge present in the data. We define a new term $E_I(\mathbf{l})$ which encodes our prior assumptions about the organization of spatial discontinuities:

$$\epsilon_1 \sum_{C_{hyst}} (l_{s,t} l_{u,v}) - \epsilon_2 \sum_{C_{supp}} (l_{s,t} l_{t,u}), \tag{14}$$

where the cliques are defined in Figure 5. The parameters ϵ_1 and ϵ_2 assume values in the interval $[0, 1]$. We now minimize the objective function $E(\mathbf{u}, \mathbf{d}, \mathbf{m}, \mathbf{l})$ which contains the data term E_D and spatial smoothness term E_S as before with the new spatial interaction term:

$$\lambda_D E_D(\mathbf{u}, \mathbf{d}, \mathbf{m}) + \lambda_S E_S(\mathbf{u}, \mathbf{l}) + \lambda_I E_I(\mathbf{l}), \tag{15}$$

where the λ_* control the relative importance of the various terms.

6.2 Continuation Methods

Continuation methods provide one popular class of approaches for minimizing non-convex functions such as the robust formulation described above. The idea is to choose an estimator which has a control parameter that can be used to change the shape of the estimator. This parameter is exploited to construct a convex approximation to the objective function which can be readily minimized. The minimum is then *tracked* as the control parameter is adjusted so that the objective function increasingly approximates the original non-convex estimation problem.

We can recover an analog outlier process for this type of estimator and the penalty function in the outlier-process formulation retains the control parameter of the original estimator. This allows us to apply continuation methods to the explicit outlier-process formulations, and to do so even in the presence of spatial interactions.

By deriving penalty functions with continuation parameters we can apply standard continuation methods to problems that involve spatial interaction of outlier processes. By adjusting the control parameter we can begin minimizing an objective function that gives high

a b c

Figure 6: Random Noise Example. *a)* First random noise image in the sequence. *b)* True horizontal motion (black = -1 pixel, white = 1 pixel, gray = 0 pixels). *c)* True vertical motion.

penalties for introducing outliers. This will mean that initially, no outliers will be introduced. Then by adjusting the control parameter, outliers begin to appear and interact.

7 Outlier Processes: Experimental Results

The first experiment shows how a robust treatment can improve optical flow estimates by accounting for gross measurement errors as well as spatial discontinuities. The second experiment shows how to take a robust estimation problem, recover an explicit outlier process, and then add constraints on the spatial organization of the outliers.

7.1 Optical Flow

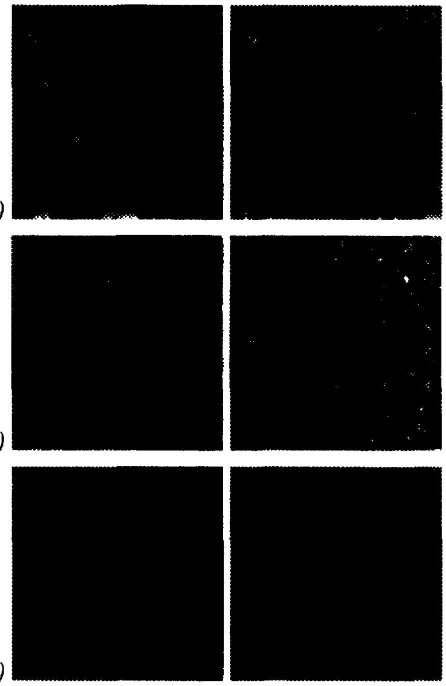
Depth discontinuities in the scene, or the independent motion of objects, gives rise to optical flow fields that are piecewise smooth. While line processes and weak-continuity methods have been used to preserve flow discontinuities (eg. [3, 12]), less attention has been paid to violations of the ***brightness constancy assumption***:

$$I(x,y,t) \;=\; I(x+u\delta t, y+v\delta t, t+\delta t), \qquad (16)$$

where $I(x,y,t)$ is the image brightness at a point (x,y) at time t, (u,v) is the horizontal and vertical image velocity at a point, and δt is small. The assumption is violated in cases of transparency, shadows, reflections, and motion discontinuities. In these cases, erroneous measurements can be treated as outliers. Black and Anandan [3] formulate the optical flow problem as robust estimation to account for measurement and spatial outliers:

$$\begin{aligned} E(\mathbf{u}) \;=\; & \sum_{s\in S}[\rho((I_x u_s + I_y v_s + I_t), \sigma_D) \\ & + \lambda \sum_{n\in\mathcal{G}_s} \rho(\|\mathbf{u}_s - \mathbf{u}_n\|, \sigma_S)], \qquad (17) \end{aligned}$$

where ρ is the Lorentzian estimator.



(a)

(b)

(c)

Figure 7: Effect of robust data term, (10% uniform noise). *a)* Least-squares (quadratic) solution. *b)* Quadratic data term and robust smoothness term. *c)* Fully robust formulation.

Here we briefly summarize some illustrative experiments with synthetic images (see [2, 3] for details). Consider the randomly textured image sequence shown in Figure 6 in which the right half of the image is translating one pixel to the left. The second image in the sequence has been corrupted with 10% uniform random noise. We compare the performance of three common approaches: a least-squares formulation, a version with a quadratic measurement term and robust smoothness term, and the robust formulation.

The results are illustrated in Figure 7. The left column shows the horizontal motion and the right column shows the vertical motion recovered by each of the approaches. Figure 7*a* shows the noisy, but smooth, results obtained by least-squares. In Figures 7*b* the introduction of a line process results in a piecewise smooth field, but the gross errors in the data produce spurious motion discontinuities. Figure 7*c* shows the improvement realized when outliers are rejected in both the measurement and spatial smoothness terms. Figure 8 shows where the spatial and measurement errors were treated as outliers.

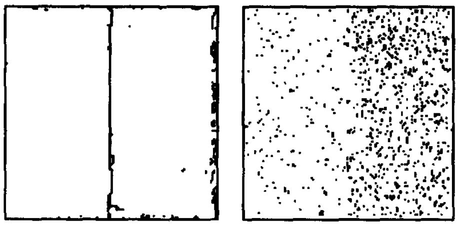


Figure 8: Outliers in the smoothness and data terms. *a)* Spatial discontinuities. *b)* Data outliers.

7.2 Image Reconstruction

We now turn the problem of fitting a piecewise smooth brightness model $\mathbf{u}_s$ to image data $\mathbf{d}_s$ where we account for spatial discontinuities in brightness as well as measurement discontinuities due to texture or noise. For illustration we consider the image in Figure 9*b* which is obtained by degrading Figure 9*a* with additive white Gaussian noise (variance=175).

Robust Regularization: In the robust formulation (Eqn. 3) outlying spatial and data measurements are rejected using a robust estimator which is taken to be the Lorentzian (Eqn. 11) for these experiments. The function was minimized using simultaneous over-relaxation (SOR) [4] with a two step continuation method and the parameters were as follows: $\lambda_D = \lambda_S = 1.0$, $\sigma_D(1) = 75.0$ $\sigma_D(2) = 35.0$, $\sigma_S(1) = 30.0$, $\sigma_S(2) = 4.5$, and 30 iterations were used at each step in the continuation method.

Column (*c*) in Figure 9 shows the results of the optimization. The top figure shows the recovered piecewise smooth brightness model. The middle figure shows that a large number of the noisy data points that were treated as outliers and rejected. The bottom figure illustrates the spatial outliers corresponding to brightness changes in the recovered image.

Recovered Outlier Process: In the next experiment we used the mechanism in Section 5.3 to recover the outlier-process formulation of the Lorentzian estimator (Eqn. 11). We then minimize Eqn. 13 by alternatively solving for the outlier processes in closed form and then minimizing with respect to $\mathbf{u}_s$ using SOR. The parameters are exactly the same as before; in particular we use exactly the same continuation method with the explicit outlier-process formulation as was used in the robust formulation.

The results are shown in Figure 9 Column (*d*). Notice that the reconstructed image, the data outliers, and the spatial outliers are nearly identical to the robust formulation as expected. Small differences can be expected due to the slightly different optimization techniques.

Introducing Spatial Coherence Constraints: Now that we have an explicit spatial outlier process we can introduce spatial constraints (Eqn. 14). We now minimize (Eqn. 15) in exactly the same way as in the case without spatial constraints, with all the parameters the same and, in particultar, $\lambda_I = \lambda_S = \lambda_D = 1.0$. The results in Figure 9 Column (*e*) show that the introduction of the spatial coherence constraints result in more extended and completed contours and fewer "thick" edges.

8 Conclusion

This paper has shown that the unifying concept underlying the line-process and robust-statistical approaches is the notion of outlier rejection. The generalization of line processes to outlier processes makes the connection to robust statistics clear. Moreover, the elimination of the outlier processes by minimization (eg. Blake and Zisserman) or by integration (eg. Mean-Field approaches) provides the connection to robust estimation approaches based on influence functions [10].

The real power in this connection lies in the ability to go in the other direction; that is, to recover an outlier process from a robust estimation problem. What this permits is the straightforward extension of results from robust statistics to problems in vision. In particular, by recovering an analog outlier process we can enforce constraints on spatial continuity which are crucial for many problems. Moreover, we can continue to use the continuation methods developed for solving nonconvex optimization problems to solve the robust formulations with explicit outlier processes. Finally, the connection between robust statistics and line-processes techniques provides a physical interpretation for the former and a host of new outlier processes with known outlier rejection properties for the latter.

Acknowledgements

We thank P. Anandan, R. Chellappa, D. Geiger G. Gindi A. Jepson, and E. Mjolsness for their helpful comments.

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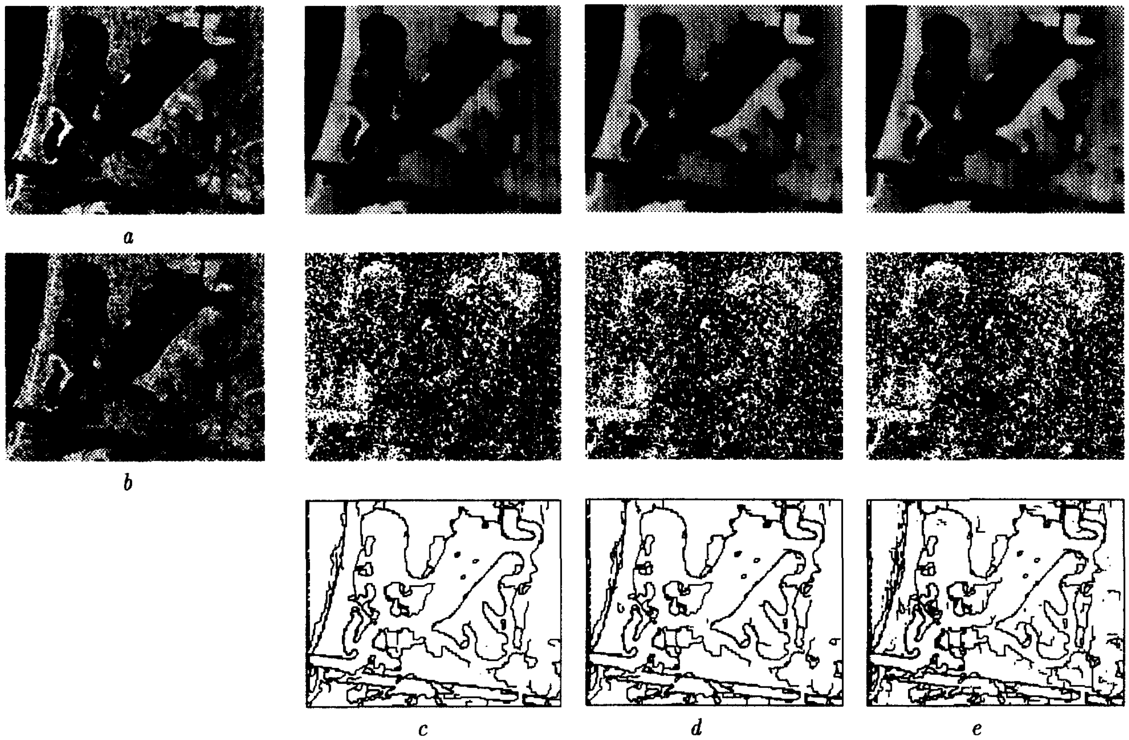

Figure 9: Image Reconstruction Experiment. (a) Original image; (b) Degraded image. *Top:* Reconstructed image, *Middle:* Data outliers (black); *Bottom:* Spatial outliers (black). (c) Robust results; (d) Outlier process – no spatial interactions; (e) Outlier process – with spatial interactions.

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