Hexagonal Constellations for Adaptive Physical-Layer Network Coding 2-Way Relaying

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***Abstract*—We focus on a constellation design for adaptive physical-layer network coding strategy in a wireless 2-way relay channel. It is well known that 4QAM constellations require extended-cardinality network coding adaptation to avoid all singular channel parameters at the Multiple-Access (MA) stage. The cardinality extension is undesirable since it introduces redundancy decreasing the data rates at the broadcast stage. In this paper, we target a design of constellations avoiding all the singularities without the cardinality extension. We show that such a constellation is 4-ary constellation taken from hexagonal lattice (4HEX) which keeps comparable error performance at the MA stage as 4QAM, however without the cardinality extension. The similar properties has been found also by unconventional 3HEX and 7HEX constellations.**

***Index Terms*—Physical-layer network coding, constellation design, wireless two-way relaying.**

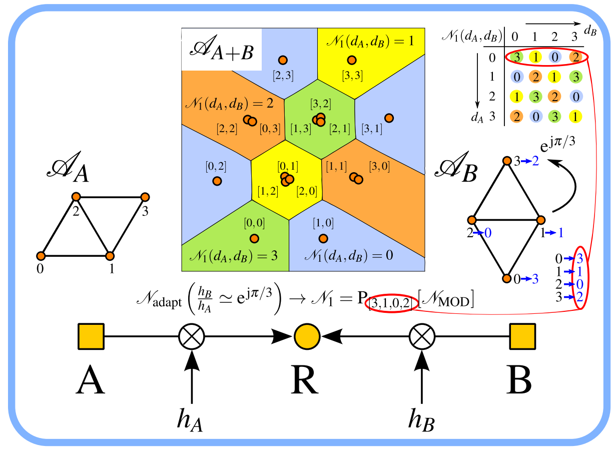

Fig. 1. Multiple-access stage of PLNC in a 2-WRC using adaptive network coding $\mathscr{N}_{\text{adapt}}(h_B/h_A)$ and 4ary hexagonal constellations.

I. Introduction

PHYSICAL-LAYER Network Coding (PLNC) has received much attention in the research community, see [1] and its references. It offers theoretically the highest achievable throughput in a wireless 2-Way Relay Channel (2-WRC) assuming perfect Channel State Information (CSI) at all the nodes [2]. Considering more practical situation with CSI at the receivers, PLNC performance emerges a new type of fading phenomenon. This fading appears when a ratio of channel coefficients equals to certain critical values (denoted as singularities) and typical constellations are used by the terminals. The singularities force a minimal distance of network coding relay decoding to 0, regardless of magnitudes of the channel coefficients. Even if the channel parameters are Rayleigh/Rice distributed, the average performance is remarkably degraded by the presence of the singularities. The authors in [3] propose an adaptive PLNC strategy eliminating this performance degradation. It adapts network coding function according to the actual channel parameter ratio so as to maximize the minimal distance (it is equivalent to avoiding all the singularities). The authors in [4] designed multi-dimensional constellations which avoid the singularities even without network coding adaptation, however its spectral efficiency seems to be upper-limited by 1 bit-per-complex-dimension. The existence of singularities for constellations with higher spectral efficiency is without the adaptation apparently inevitable. Paper [5] presents a simplified search for network coding functions based on theory of Latin squares. Both [3] and [5] conclude that 4QAM requires extended-cardinality network coding adaptation to avoid all the singularities. The cardinality extension is undesirable since it introduces redundancy decreasing the data rates at the BroadCast (BC) stage.

In this letter, we target a constellation design for adaptive PLNC avoiding the singularities without the cardinality extension. Based on results from [6], we show that singularities of 4QAM with absolute value 1 which are mutually rotated by 90° can be avoided by minimum-cardinality network coding functions. This is due to 90° rotational symmetry of rectangular lattice and 4QAM constellation-shape. Therefore, we analyse constellations taken from hexagonal lattice with 60° rotation symmetry which may avoid more singularities than rectangular lattice-constellations. As a contribution, we present a 4-ary hexagonal constellation (4HEX), illustrated in Fig. 1, which avoid all the singularities without the cardinality extension and with comparable performance at the Multiple Access (MA) stage as 4QAM. The similar properties has been found also by unconventional 3HEX and 7HEX constellations.

II. System Model

A. Signal Space Model and Used Notation

Let both terminals A and B use the same constellation (including the same constellation-indexing) $\mathscr{A}_A=\mathscr{A}_B=\mathscr{A}$ which is assumed to be linear ($\mathscr{A}\subseteq\mathbb{C}$) and taken from a common lattice. The notation from the perspective of terminal A is following. Baseband signal points in the constellation space s_A forming the alphabet $\mathscr{A}=\{s_A^{(i)}\}_{i=0}^{M-1}$ are assumed to be normalised to the unit mean symbol energy, where M is a constellation cardinality $M=|\mathscr{A}|$. Upper-indices $\star^{(i)}$ are used when a concrete value of the variable is to be stressed. We

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define constellation mapper $\mathscr{M}:\mathbb{Z}_M \to \mathscr{A}$ such that the alphabet indices directly correspond to data symbols $\mathscr{M}(d_A) = s_A^{(d_A)}$, where data symbols are $d_A \in \mathbb{Z}_M$ and $\mathbb{Z}_M = \{0,1,\ldots,(M-1)\}$ denotes the set of non-negative integers lower than M. Using lattice-generator matrix $\mathbf{G}$, we unambiguously describe constellation signals by lattice-coordinate vectors $\mathbf{a}$ as $s_A^{(i)} = \mathbf{G}\mathbf{a}^{(i)} - \overline{m}, \forall i \in \mathbb{Z}_M$, where $\overline{m} = 1/M \sum_{i=1}^{M} \mathbf{G}\mathbf{a}^{(i)}$ ensures that $\mathscr{A}$ has a zero-mean. Vector $\mathbf{a} = [a_0, a_1]^T \in \mathscr{S} \subseteq \mathbb{Z}^2$ is taken from a set of lattice coordinates $\mathscr{S}$ (determining the constellation shape), where $\mathbb{Z} = \{\ldots,-1,0,1,\ldots\}$ denotes the ring of integers. Generator matrices of rectangular Z^2 and hexagonal A_2 lattice are $\mathbf{G}_{Z^2} = [1,\mathrm{j}]$ and $\mathbf{G}_{A_2} = [1, 1/2 + \mathrm{j}\sqrt{3}/2]$. One-to-one indexing mapper between constellation-indices and lattice-coordinates is defined as $\mathscr{I}:\mathbb{Z}_M \mapsto \mathscr{S}$, $\mathscr{I}(d_A) = \mathbf{a}^{(d_A)}$.

B. 2-Way Relay Channel and Model Assumptions

A 2-WRC consists of two terminals A and B bi-directionally communicating via a supporting relay R in a half-duplex manner (each node cannot send and receive at the same time). We assume an idealised time-synchronised scenario and Rayleigh/Rice flat fading with CSI at the receivers. We analyse an uncoded per-symbol PLNC relaying and we expect similar performance trends as with concatenated channel coding, see [7], [8] for more details about such a receiver processing.

C. Adaptive Physical-Layer Network Coding in the 2-WRC

PLNC 2-way relaying consists of a MA stage and a BC stage. At the first MA stage, both terminals transmit simultaneously to the relay which receives a signal superposition

$$x = h_A s_A + h_B s_B + w = u + w, \tag{1}$$

where $u \in \mathscr{A}_{A+B}$ denotes a superimposed signal $u^{(d_A,d_B)} = h_A s_A^{(d_A)} + h_B s_B^{(d_B)}$, where w is a complex AWGN noise with variance $2N_0$ and h_A, h_B are fading channel coefficients. The relay decodes a network coded data symbol d_{AB} as $\widehat{d}_{AB} = \arg\max_{d_{AB}} p(x|d_{AB})$, where likelihood function is

$$p(x|d_{AB}) = \frac{1}{M} \sum_{\mathscr{N}(d_A,d_B)=d_{AB}} \frac{1}{2\pi N_0} e^{-\frac{\left|x-u^{(d_A,d_B)}\right|^2}{2N_0}}. \tag{2}$$

The summation in (2) runs over all $[d_A, d_B] \in \mathbb{Z}_M^2$ such that $\mathscr{N}(d_A,d_B) = d_{AB}$. The network coding function $\mathscr{N}$ fulfils an exclusive law of network coding [3]

$$\begin{aligned} \mathscr{N}(d_A,d_B) &\neq \mathscr{N}(d'_A,d_B), \quad d_A \neq d'_A, \\ \mathscr{N}(d_A,d_B) &\neq \mathscr{N}(d_A,d'_B), \quad d_B \neq d'_B \end{aligned} \tag{3}$$

in order to ensure decodability at the destinations when one of the terminal data symbols is provided. We denote cardinality of network coded symbols as M_{AB} ($d_{AB} \in \mathbb{Z}_{M_{AB}}$). It is generally $M_{AB} \geq M$, when $M_{AB} = M$ (resp. $M_{AB} > M$), we denote such $\mathscr{N}$ as a *minimum-* (resp. *extended-*) cardinality one. Maximal capacity gain due to the network coding-based information compression is achieved when cardinality M_{AB} is minimal. Function $\mathscr{N}$ is uniquely specified by its Latin square [5] which we denote as a matrix $\mathbf{N}$, where $\mathscr{N}(d_A,d_B) = [\mathbf{N}]_{d_A,d_B}$. We use modulo-sum $\mathscr{N}_{\mathrm{MOD}}(d_A,d_B) = (d_A+d_B)_{\mathrm{mod}M}$ and bit-wise XOR $\mathscr{N}_{\mathrm{XOR}}(d_A,d_B) = d_A \oplus d_B$ minimum-cardinality functions which possess the following 4-ary Latin squares

$$\mathbf{N}_{\mathrm{XOR}} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \ \mathbf{N}_{\mathrm{MOD}} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}. \tag{4}$$

In the adaptive PLNC strategy [3], the relay adaptively selects network coding function $\mathscr{N}_{\mathrm{adapt}}$ according to instantaneous CSI (channel ratio h_B/h_A) in order to maximize the minimal distance of the relay processing. It can be shown that the minimal distance is maximized if all so-called singular channel parameters are *avoided* [5].

Definition 1. A *singular* channel parameter $\alpha = h_B/h_A$, where $h_A, h_B \neq 0$ is such a channel ratio for which

$$s_A^{(d_A)} + \alpha s_B^{(d_B)} = s_A^{(d'_A)} + \alpha s_B^{(d'_B)}, [d_A, d_B] \neq [d'_A, d'_B]. \tag{5}$$

Definition 2. A function $\mathscr{N}$ *avoids a singularity* α when (5) holds and $\mathscr{N}(d_A,d_B) = \mathscr{N}(d'_A,d'_B)$ and thus situation (5) is not a source of errors in the network coded data decoding.

At the second BC stage, the relay broadcasts network coded data symbol d_{AB} and the final destinations subsequently perform successful detection exploiting knowledge of its own data symbols and network coding function invertibility (3).

III. Analysis of 4QAM Singularities

We analyse the singularities of 4QAM with a help of Proposition 3 originally introduced in [6].

Proposition 3. *Assume a superposition of two constellations taken from a common lattice. If both constellation-indices form a modulo-arithmetic progression along each lattice dimension (inverse indexing mapper $\mathscr{I}^{-1}$ is modulo-affine (denoted as Affine Indexing (AI)), then all equal superimposed-constellation points correspond to an identical modulo-sum data symbol.*

Proof: Two superimposed-constellation points taken from a common lattice which correspond to distinct data pairs $s_A^{(d_A)} + s_B^{(d_B)} = s_A^{(d'_A)} + s_B^{(d'_B)}$, $[d_A,d_B] \neq [d'_A,d'_B]$ are equal when $\mathbf{G}(\mathbf{a}+\mathbf{b}) - 2\overline{m} = \mathbf{G}(\mathbf{a}'+\mathbf{b}') - 2\overline{m}$ and so when

$$\mathbf{a}+\mathbf{b} = \mathbf{a}'+\mathbf{b}' \tag{6}$$

where $\mathbf{a} = \mathscr{I}(d_A), \mathbf{b} = \mathscr{I}(d_B), \mathbf{a}' = \mathscr{I}(d'_A), \mathbf{b}' = \mathscr{I}(d'_B)$. Let the precondition $\mathscr{I}^{-1}$ to be modulo-affine is fulfilled, so $\mathscr{I}^{-1}(\mathbf{a}) = \left(\mathbf{c}^T\mathbf{a} + z\right)_{\mathrm{mod}M}$, where $c_i \in \mathbb{Z}$ denotes a common increment of modulo-arithmetic progression in the ith dimension and $z \in \mathbb{Z}_M$ is an arbitrary constant. The modulo-sum data symbol is then

$$d_{AB} = (d_A+d_B)_{\mathrm{mod}M} = \left(\mathbf{c}^T(\mathbf{a}+\mathbf{b}) + 2z\right)_{\mathrm{mod}M}. \tag{7}$$

By the same manipulations, we obtain $d'_{AB} = \left(\mathbf{c}^T(\mathbf{a}'+\mathbf{b}') + 2z\right)_{\mathrm{mod}M}$. Clearly, when superimposed-constellation points are equal ((6) holds), then $d_{AB} = d'_{AB}$ which proves the claim. ■

Proposition 3 shows that singularity $\alpha = 1$ ($h_A = h_B = 1$) can be avoided by minimum-cardinality modulo-sum function $\mathscr{N}_{\mathrm{MOD}}$ providing AI indexed constellations. 4QAM can be indexed by AI with $\mathbf{c} = [1,2]^T$ as found in [6].

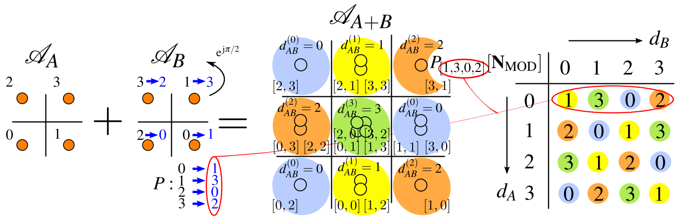

Fig. 2. The column permutation (that re-index $\mathscr{A}_B$ to be jointly AI with indexing $\mathscr{A}_A$) of modulo-sum function avoids singularity $\alpha = \mathrm{e}^{\mathrm{j}\pi/2}$.

Definition 4. A vector permutation $P_{\mathbf{p}}$ which maps vector $\mathbf{n} = [0, 1, \ldots (M-1)]$ on a vector $\mathbf{p} = [p_0, p_1, \ldots p_{(M-1)}]$ is denoted as $P_{\mathbf{p}}[\mathbf{n}] = \mathbf{p}$. In a similar way, we define a column matrix permutation as $P_{\mathbf{p}}[\mathbf{N}] = [\mathbf{N}_{\star p_0}, \mathbf{N}_{\star p_1}, \ldots \mathbf{N}_{\star p_{(M-1)}}]$ where $\mathbf{N}_{\star i}$ denotes the ith column of matrix $\mathbf{N}$. We describe a row permutation as $P_{\mathbf{p}^T}[\mathbf{N}] = P_{\mathbf{p}}[\mathbf{N}^T]^T$ and an exponent of permutation as $P_{\mathbf{p}}^2[\mathbf{N}] = P_{\mathbf{p}}[P_{\mathbf{p}}[\mathbf{N}]]$ where P^0 is an identity.

Lemma 5. *Let network coding function $\mathscr{N}$ with Latin square $\mathbf{N}$ avoid a singularity α using constellations $\mathscr{A}_A, \mathscr{A}_B$. If we re-index constellation $\mathscr{A}_B$ (resp. $\mathscr{A}_A$) according to some permutation $P_{\mathbf{p}}$, then the singularity α is avoided by the network coding function with Latin square $P_{\mathbf{p}}[\mathbf{N}]$ (resp. $P_{\mathbf{p}^T}[\mathbf{N}]$).*

Lemma 5 is rather obvious and instead of the proof we rather clearly demonstrate the principle on the example in Fig. 2. Here, the channel parameters $h_A = 1$ and $h_B = \mathrm{e}^{\mathrm{j}\pi/2}$ ($\alpha = \mathrm{e}^{\mathrm{j}\pi/2}$) cause 90° rotation of the constellation $\mathscr{A}_B$. The rotation effectively means only constellation re-indexing due to the symmetry of rectangular lattice and 4QAM constellation shape. Now, if we find some new indices (denoted by blue colour in Fig. 2) forming AI jointly with indexing of $\mathscr{A}_A$, then the singularity is avoided by modulo-sum function. There always exists a permutation which maps old indices to the new AI indices $P_{\mathbf{p}}$ and according to Lemma 5 the avoiding singularity network coding function has Latin square $P_{\mathbf{p}}[\mathbf{N}_{\mathrm{MOD}}]$. Similar example is depicted in Fig. 1 for $\alpha = \mathrm{e}^{\mathrm{j}\pi/3}$.

Lemma 6. *If network coding function $\mathscr{N}$ avoids a singularity α, $\alpha \neq 0$, then the same $\mathscr{N}$ avoids also $1/\alpha$.*

Proof: We obtain the claim simply, when we multiply both sides of (5) by $1/\alpha$ on condition that s_A, s_B are from the same alphabet $\mathscr{A}$. ∎

Let summarize analysis of 4QAM singularities. According to Lemma 5 and 6, once we avoid singularities with angle in the range $[0, 90^\circ]$ and with $|\alpha| \leq 1$, then we avoid all of them. Particularly, they are $\{1, (1+\mathrm{j})/2\}$. According to Proposition 3, we avoid singularity 1 by minimum-cardinality modulo-sum function or according to [3] by bit-wise XOR. Unfortunately, works [3], [5] show that avoidance of singularity $(1+\mathrm{j})/2$ requires necessarily *extended-cardinality* Latin square

$$\mathbf{N}_2 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 3 & 0 & 4 & 2 \\ 4 & 2 & 0 & 1 \end{bmatrix}. \tag{8}$$

To verify that all singularities have been successfully avoided, we numerically evaluate parametric minimal distance

$$\Delta^2_{\min}(\alpha) = \min_{\mathscr{N}_{\mathrm{adapt}}(d_A, d_B) \neq \mathscr{N}_{\mathrm{adapt}}(d'_A, d'_B)} \left\| \Delta s_A^{(d_A, d'_A)} + \alpha \Delta s_B^{(d_B, d'_B)} \right\|^2,$$

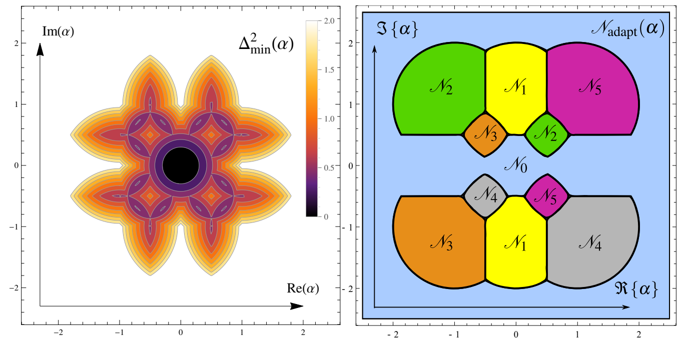

Fig. 3. Parametric minimal distance $\Delta^2_{\min}(\alpha)$ and adaptive network coding function $\mathscr{N}_{\mathrm{adapt}}(\alpha)$ of 4QAM constellation.

where $\Delta s_A^{(d_A, d'_A)} = s_A^{(d_A)} - s_A^{(d'_A)}$, $\Delta s_B^{(d_B, d'_B)} = s_B^{(d_B)} - s_B^{(d'_B)}$. All singularities are avoided when the minimal distance is non-zero for all non-zero α.

Figure 3 shows that adaptive PLNC using 4QAM avoids all singularities when the relay adaptively selects $\mathscr{N}_i$ according to adaptive network coding function $\mathscr{N}_{\mathrm{adapt}}(\alpha)$ where $\mathbf{N}_2$ is defined by (8) and

$$\mathbf{N}_0 = \mathbf{N}_{\mathrm{XOR}}, \quad \mathbf{N}_1 = P_{[1,3,0,2]}[\mathbf{N}_0], \quad \mathbf{N}_3 = P_{[1,3,0,2]}[\mathbf{N}_2],$$
$$\mathbf{N}_4 = P^2_{[1,3,0,2]}[\mathbf{N}_2], \quad \mathbf{N}_5 = P^3_{[1,3,0,2]}[\mathbf{N}_2],$$

where permutation $P_{[1,3,0,2]}$ corresponds to 90° reindexing of $\mathscr{A}_B$ as shown in Fig. 2.

IV. Proposed Hexagonal Constellations

We have found that all unit-length singularities $|\alpha| = 1$ of 4QAM can be avoided by *minimum-cardinality* network coding functions. Particularly, there are four unit-length singularities $\left\{e^{\mathrm{j}2\pi k/4}\right\}_{k=0}^{3}$ due to the 90° symmetry of rectangular lattice. This motivates us to consider constellations taken from hexagonal lattice since there are possibly six unit-length singularities $\left\{e^{\mathrm{j}2\pi k/6}\right\}_{k=0}^{5}$ due to its 60° symmetry. The number of singularities is limited and depends only on the alphabet cardinality so we may hope that if we avoid more singularities by minimum-cardinality functions we could conceivably avoid all of them. As we will confirm in the case of 3, 4 and 7HEX constellations.

A. 4HEX Constellation and Modulo-Sum Based Adaptation

Let us assume 4-ary hexagonal constellation (4HEX)

$$\mathscr{A}_{4\mathrm{HEX}} = \sqrt{2}/4 \left\{ -3 - \mathrm{j}\sqrt{3}, 1 - \mathrm{j}\sqrt{3}, -1 + \mathrm{j}\sqrt{3}, 3 + \mathrm{j}\sqrt{3} \right\},$$

depicted in Fig. 1. The shape of 4HEX is symmetric to 180° rotation. Therefore we suffice to analyse singularities with angle in the range $[0, 180^\circ]$ and with $|\alpha| \leq 1$. They are $\left\{1, e^{\mathrm{j}\pi/3}, e^{\mathrm{j}2\pi/3}, \sqrt{3}/3\, e^{\mathrm{j}\pi/6}, \mathrm{j}\sqrt{3}/3, \sqrt{3}/3\, e^{\mathrm{j}5\pi/6}\right\}$. We avoid unit-length singularities $\left\{1, e^{\mathrm{j}\pi/3}, e^{\mathrm{j}2\pi/3}\right\}$ by the same procedure as in the case of 4QAM:

Procedure: *For a given singularity, we search for such a new indexing of $h_A\mathscr{A}_A$ and $h_B\mathscr{A}_B$ that would be jointly AI and then the permutation (which maps old indices to the new indices) of modulo-sum function avoid this singularity.*

Surprisingly, there does not exist an indexing that would be jointly AI for $\left\{\sqrt{3}/3\, e^{\mathrm{j}\pi/6}, \mathrm{j}\sqrt{3}/3, \sqrt{3}/3\, e^{\mathrm{j}5\pi/6}\right\}$. But a more detailed analysis shows that two super-imposed points which fall to the same position are composed of signal points taken

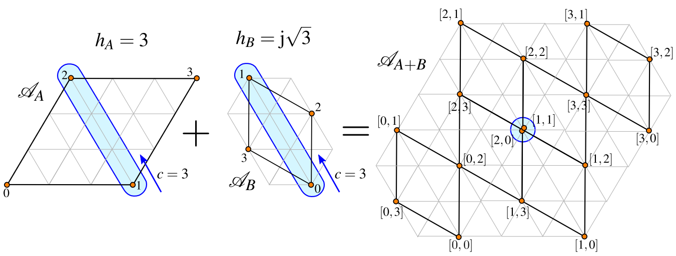

Fig. 4. Singularity $\alpha = \mathrm{j}\sqrt{3}/3$ is avoided by modulo-sum network coding function if all indices in the critical lattice-dimension (emphasized) are indexed by affine-indexing (here with coefficient $c = 3$).

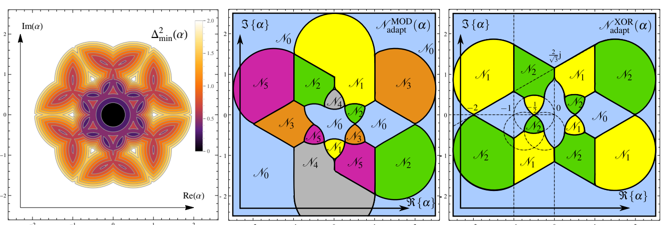

Fig. 5. Parametric minimal distance $\Delta^2_{\min}(\alpha)$, adaptive network coding based on modulo-sum $\mathscr{N}^{\mathrm{MOD}}_{\mathrm{adapt}}(\alpha)$ and bit-wise XOR $\mathscr{N}^{\mathrm{XOR}}_{\mathrm{adapt}}(\alpha)$ for 4HEX.

from a single lattice dimension. Thus, if constellation-indices are jointly AI in this critical dimension, then the singularity is also avoided by permuted modulo-sum as shown in Fig. 4 for $\alpha = \mathrm{j}\sqrt{3}/3$. All singularities of 4HEX can be avoided by *minimum-cardinality* adaptive $\mathscr{N}^{\mathrm{MOD}}_{\mathrm{adapt}}(\alpha)$ based on the following modulo-sum functions

$$\mathbf{N}_0 = \mathbf{N}_{\mathrm{MOD}}, \mathbf{N}_1 = P_{[3,1,0,2]}[\mathbf{N}_0], \mathbf{N}_2 = P_{[0,2,1,3]^T}[P_{[2,1,0,3]}[\mathbf{N}_0]],$$
$$\mathbf{N}_3 = P_{[3,2,1,0]}[\mathbf{N}_0], \mathbf{N}_4 = P_{[3,2,1,0]}[\mathbf{N}_1], \mathbf{N}_5 = P_{[3,2,1,0]}[\mathbf{N}_2].$$

B. 4HEX Constellation and XOR Based Adaptation

There is a numerically manageable number of 4ary minimum-cardinality Latin squares. By brute-force search, we have found that bit-wise XOR based network coding adaptation $\mathscr{N}^{\mathrm{XOR}}_{\mathrm{adapt}}(\alpha)$ requires adaptation to only 3 functions ($\mathscr{N}^{\mathrm{MOD}}_{\mathrm{adapt}}(\alpha)$ requires 6) $\mathbf{N}_i = P^i_{[0,2,3,1]}[\mathbf{N}_{\mathrm{XOR}}], i \in \mathbb{Z}_3$ as shown in Fig. 5. In addition, any permuted XOR Latin square can be described as a linear operation which enables simpler channel coding concatenation approach [8].

C. 3-ary and 7-ary Hexagonal Constellations

Let us consider $\mathscr{A}_{3\mathrm{HEX}} = \{e^{-\mathrm{j}5\pi/6}, e^{-\mathrm{j}\pi/6}, \mathrm{j}\}$ and $\mathscr{A}_{7\mathrm{HEX}} = \sqrt{7/6}\{e^{-\mathrm{j}2\pi/3}, e^{-\mathrm{j}\pi/3}, -1, 0, 1, e^{\mathrm{j}2\pi/3}, e^{\mathrm{j}\pi/3}\}$ constellations with 60° rotationally symmetric shapes. The singularities with angle in the range $[0, 60^\circ]$ and $|\alpha| \leq 1$ are $\{1\}$ and $\{1/2, 1, 1/\sqrt{3}e^{\mathrm{j}\pi/6}, 2/\sqrt{3}e^{\mathrm{j}\pi/6}\}$, respectively. Following the same procedure as for 4HEX, we found that the singularities are avoided by minimum-cardinality modulo-sum network coding adaptation depicted in Fig. 6 where $\mathbf{N}_i = P^i_{[0,2,1]}[\mathbf{N}_{\mathrm{MOD}}], i \in \mathbb{Z}_2$ and $\mathbf{N}_i = P^i_{[1,4,0,3,6,2,5]}[\mathbf{N}_{\mathrm{MOD}}], i \in \mathbb{Z}_6$, respectively. Note, that bitwise XOR function is not a minimum-cardinality one any more for odd alphabet cardinality M. Unfortunately, we have not found similar properties by 8HEX constellation which has a more practical power of 2 cardinality.

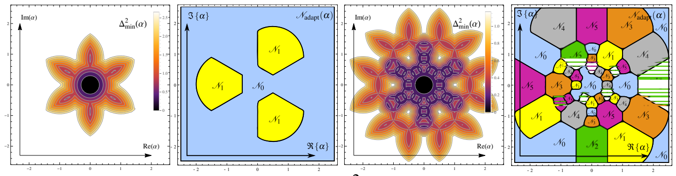

Fig. 6. Parametric minimal distance $\Delta^2_{\min}(\alpha)$ and adaptive network coding function $\mathscr{N}_{\mathrm{adapt}}(\alpha)$ of 3HEX and 7HEX constellations.

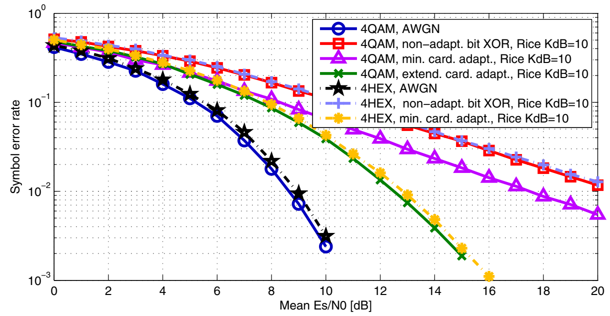

Fig. 7. Network coded symbol error rate at the MA stage of adaptive PLNC in Rician $K = 10$dB channel using 4QAM and 4HEX constellations.

V. Performance Evaluation

Performance of uncoded network coded symbol decoding at the MA stage is depicted in Fig. 7. We conclude that the performance of minimum-cardinality network coding adaptation using 4HEX is considerably better than the minimum-cardinality adaptation using 4QAM (because the singularity $(1+\mathrm{j})/2$ cannot be avoided.) and it is comparable to extended-cardinality adaptation using 4QAM.

VI. Conclusion

We present 3, 4, and 7HEX constellations which avoid all singularities in adaptive physical-layer network coding 2-way relaying without the network coding cardinality extension which increases redundancy at the BC stage (as in the case of 4QAM). Proposed 4HEX clearly outperforms 4QAM in the considered scenario. Our approach is potentially applicable to constellations taken from general lattice.

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