Inversion-Based Nonlinear Control of Robot Arms with Flexible Links

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The design of inversion-based nonlinear control laws solving the problem of accurate trajectory tracking for robot arms having flexible links is considered. It is shown that smooth joint trajectories can always be exactly reproduced preserving internal stability of the closed-loop system. The interaction between the Lagrangian/assumed modes modeling approach and the complexity of the resulting inversion control laws is stressed. The adoption of clamped boundary conditions at the actuation side of the flexible links allows considerable simplification with respect to the case of pinned boundary conditions. The resulting control is composed of a nonlinear state feedback compensation term and of a linear feedback stabilization term. A feedforward strategy for the nonlinear part is also investigated. Simulation results are presented for a planar manipulator with two flexible links, displaying the performance of the proposed controllers also in terms of end-effector behavior.

Nomenclature

A_1, A_2	= system matrices in closed-loop flexible dynamics
a	= acceleration input vector
$B_{\delta\delta}, B_{\theta\delta}, B_{\theta\theta}$	= sub-blocks of inertia matrix
D	= modal damping matrix
$(EI)_i$	= ith link flexural rigidity
F	= input matrix in modified rigid dynamics
f_{ij}	= jth natural frequency of ith link
h_δ, h_θ	= Coriolis and centrifugal force vectors
I	= identity matrix
J_{hi}	= ith hub inertia
J_{oi}	= ith link inertia about relative joint axis
J_p	= tip payload inertia
K	= modal stiffness matrix
K_D	= derivative feedback gain matrix
K_P	= proportional feedback gain matrix
ℓ_i	= ith link length
M	= number of deflection variables
m_{hi}	= ith hub mass
m_i	= ith link mass
m_p	= tip payload mass
N	= number of joint variables
n_δ	= compound vector in flexible dynamics
O	= null matrix
Q_δ	= input weighting matrix
$S_{\delta\delta}$	= factorization matrix for h_δ
$S_{\theta\theta}$	= factorization matrix for h_θ
T	= trajectory traveling time
t	= time
u	= input torque
u_{des}	= feedforward input torque
u_r	= computed torque for modified rigid dynamics
V	= Lyapunov function
w_i	= ith link deflection
x_i	= position along ith link

δ	= vector of deflection variables
δ_{des}	= vector of desired deflection variables
δ_{ij}	= jth modal coefficient of ith link
δ_0	= vector of initial deflection variables
θ	= vector of joint variables
θ_{des}	= vector of desired joint variables
θ_i	= ith joint variable
μ_i	= number of modes of ith link
ρ_i	= ith link density
ϕ_{ij}	= jth mode shape of ith link
ψ	= forcing vector term in closed-loop flexible dynamics

Superscripts

T	= matrix (vector) transpose
-1	= matrix inverse
$'$	= spatial derivative
$\cdot$	= time derivative
$\wedge$	= estimate

I. Introduction

THE increasing use of robot manipulators in space applications has been recognized as offering both mission cost reduction and enhanced task capabilities. Lightweight materials are adopted in the construction of mechanical manipulators, as well as of large spacecraft, to have smaller in-orbit weight. In addition, lighter robots are capable of executing faster motions for a given actuator size. However, maneuvering time and accuracy are limited by the vibrations induced by structural flexibility, mainly distributed in the links. Indeed, for systems of large dimensions link flexibility prevails over elasticity of the transmission elements which may be thought of as concentrated at the joints.[1]

An effective control system for high-performance lightweight robots should necessarily consider link flexibility as well as handle the typical nonlinearities of multibody dynamics. In this respect, the modeling issues play a relevant role in the derivation of all advanced model-based control techniques.[2] Although for simulation purposes the need for accurate dynamic models of flexible manipulators is crucial, usually emphasis is given to model simplicity when designing control laws. The availability of explicit closed-form—rather than numerical—equations of motion allows a tradeoff be-

tween model completeness and compactness. In the modeling phase, general purpose symbolic manipulation packages become indispensable when the complexity of the system increases,[3] but customizing the model to the specific structure under investigation is certainly helpful for reduction of dynamic terms.[4]

The energy-based Lagrangian approach provides a natural framework for deriving the dynamic model of mechanical systems undergoing structural deformations.[5] A critical point is the method used to obtain a finite-dimensional approximation to the distributed flexibility.[6] In fact, if no model discretization is undertaken, mixed ordinary/partial differential equations would result, with considerable complexity even in simple cases.[7] The most common approximate descriptions of manipulator link deflection are based on assumed modes,[8] finite elements,[9] or Ritz-Kantorovich expansions.[10] In any case, link elasticity is usually characterized as a linear effect in that second-order deformation terms are neglected.

Early linear control laws for flexible manipulators were aimed purely at point-to-point motion, accompanied by vibration damping.[11,12] To impose a desired motion both in space and time, the trajectory tracking problem was then addressed by a number of methods, including linear pole assignment,[13] linear quadratic Gaussian control,[14] model reference adaptive control,[15] and singular perturbation techniques.[16-18]

When high accuracy in task execution is a strict requisite, exact trajectory tracking can be accomplished only by resorting to the inversion of the input/output map of the given system,[19] where nonlinear state feedback is used to compensate for coupled nonlinear terms. Controllers of this kind were developed for the cases of a one-link flexible arm,[20] of a multilink rigid robot with a single flexible boom,[21,22] and of a planar manipulator with two flexible links.[23] The same strategy was also applied to spacecraft or satellites with[24] or without[25] flexible appendages. The common feature of all of these schemes is that the joint (rigid body) variables are taken as the system outputs to which the inversion process is applied.

The ultimate goal is to control the motion of the robot end effector, without introducing additional actuation devices besides those naturally located at the manipulator joints. However, when the control objective (the output) is the end-effector location, an inversion-based strategy in the presence of link flexibility would normally lead to instability in the closed loop.[26,27] The instability problem arising in this case is the counterpart of the nonminimum-phase phenomenon occurring in linear systems when high-gain control is performed using a measured output which is not collocated with the control input.[28] On the other hand, the relevance of actuator/sensor collocation is well known in the field of control of large flexible spacecraft.[29,30]

In the nonlinear setting, the concept of zero dynamics,[19] i.e., the dynamics left in the system when the output is forced to be zero (or constant), is useful to investigate stability of tracking control. It can be shown that any meaningful output definition related to the end effector of a flexible robot arm leads to unstable zero dynamics.[31] The trajectory tracking problem for systems with unstable zero dynamics—both in the linear and the nonlinear cases—can be treated by resorting either to regulation schemes, which achieve only asymptotic output tracking,[32,33] or to noncausal solutions,[34] requiring the whole trajectory to be known in advance. In the latter case, only a feedforward solution can be generated. In the former case, feedback stabilization asks for the computation of the internal deformation associated with the desired end-effector motion; this requires the solution of a set of partial differential equations.

In view of these difficulties, we believe that in many cases it is convenient to pursue a joint-based control strategy, possibly combined with practical damping of link deflections. It will be shown that this approach always preserves stability, even in the face of the relative simplicity of the resulting nonlinear control laws. Moreover, joint-based inversion is of straightforward application to any multilink flexible structure, and the use of

nonlinear feedback overcomes the typical performance limitations of linear feedback of the joint variables.[35]

In this paper, the assumed modes method is adopted for modeling deflection of flexible arms. This approach leads to closed-form equations of motion that are general enough to accurately handle complex robotic structures. The analysis is focused on constrained modal eigenfunctions arising with two types of boundary conditions at the joint actuator locations; namely, clamped or pinned.[6,36] The structure of the dynamic model may change considerably in the two cases but, for our purposes, the relevant difference lies in the form taken by the input matrix. Improved precision could be obtained by considering unconstrained mode expansion, i.e., analyzing unconstrained mode expansion, i.e., analyzing deformation in the presence of time-dependent motion of each link base.[37-40] We point out that the control results presented here hold also when unconstrained modes replace the more commonly used constrained ones.

The consequences of the pinned/clamped alternative are exploited within the design of inversion-based controllers, rather than for model accuracy.[41] It will be shown that the clamped assumption leads to appealing simplifications in the control law, useful for real-time implementation. As opposed to more computationally demanding formats of previous works,[21-23] here inversion is required only of the sub-block of the system inertia matrix which pertains to the accelerations of the flexible variables within the flexible dynamics. Moreover, a feedforward strategy for the compensation of system nonlinearities and interactions will be discussed.

The performance of the joint-based inversion controllers will be illustrated by numerical simulation of trajectory tracking tasks for a detailed model of a two-link planar flexible manipulator.[4]

II. Dynamic Modeling

Consider a robotic manipulator composed of a serial chain of N flexible links connected by rigid rotary joints, each giving one degree of freedom to the arm. Each link is assumed to undergo bending deflections in the plane orthogonal to its driving joint axis, i.e., the plane of rigid body motion. Other types of link deflections can be considered, e.g., coupling of torsional and bending deflections[42] or spatial bending, but the model structure essentially remains the same as far as inversion control design issues are concerned.

The Lagrangian technique can be used to derive the dynamic model, through the computation of the global kinetic and potential energy of the system.[5] In view of the application of these robotic structures in space, gravity is not considered.

Because of link flexibility, the dynamic model is indeed of distributed nature. Links can be modeled as Euler-Bernoulli beams satisfying proper boundary conditions at the actuated joint and at the link tip.[6] In the case of uniform density ρ_i and constant flexural rigidity $(EI)_i$, the normal deflection $w_i(x_i, t)$ of the ith link with respect to its neutral axis, at a distance x_i from the frame placed at the ith joint, satisfies a partial differential equation of the type

$$(EI)_i \frac{\partial^4 w_i(x_i,t)}{\partial x_i^4} + \rho_i \frac{\partial^2 w_i(x_i,t)}{\partial t^2} = 0, \qquad i = 1, \ldots, N \quad (1)$$

where t denotes time.

As for the boundary conditions needed to solve Eq. (1), it is customary to consider two sets of conditions at the link base:
Clamped base:

$$w_i(0,t) = 0, \qquad w_i'(0,t) = 0, \qquad i = 1, \ldots, N \quad (2)$$

Pinned base:

$$w_i(0,t) = 0, \qquad w_i''(0,t) = 0, \qquad i = 1, \ldots, N \quad (3)$$

where the primes denote spatial derivatives with respect to x_i. Concerning the coordinate frame in which bending deforma-

tion is described, in the *clamped* case the frame is aligned with the direction of the undeformed link at the joint location (Fig. 1a), whereas in the pinned case the frame points at the instantaneous center of mass of the deformed link (Fig. 1b). It is obvious that w_i assumes a different meaning in the two cases.

Notice that the conditions are of the so-called *constrained* type and are good approximations of reality when the inertia of a lightweight link is small compared to the hub inertia. The two choices lead to different eigenfrequencies and assumed modes of deformation.[36] *Unconstrained* boundary conditions would be the correct ones,[37-40] involving the balance of shear forces and moments at the joint side of each link; this is impractical for multilink flexible manipulators, though. Indeed, Eqs. (2) or (3) have to be complemented with the proper equations representing boundary conditions at the other end of the link.

A finite-dimensional model (of order μ_i) of link flexibility can be obtained exploiting separability in time and space of the assumed modes solutions to Eq. (1),

$$w_i(x_i,t) = \sum_{j=1}^{\mu_i} \phi_{ij}(x_i)\delta_{ij}(t), \qquad i = 1,\ldots,N \qquad (4)$$

where $\delta_{ij}(t)$ are the modal coefficients associated with the assumed spatial mode shapes $\phi_{ij}(x_i)$ of link i. Accordingly, a set of Lagrangian generalized coordinates is given by $(\theta,\delta) \in \Re^{N+M}$, where

$$\theta = (\theta_1 \cdots \theta_N)^T$$

$$\delta = (\delta_{11} \cdots \delta_{1,\mu_1} \cdots \delta_{N1} \cdots \delta_{N,\mu_N})^T$$

with θ_i the ith joint variable and $M = \sum_{i=1}^N \mu_i$ the total number of flexible variables used to describe the robot deformation.

Following the usual steps of the Lagrange-Euler derivation, the closed-form equations of motion of the manipulator can be written as $N+M$ second-order nonlinear differential equations in the general form[4,41]

$$\begin{bmatrix} B_{\theta\theta}(\theta,\delta) & B_{\theta\delta}(\theta,\delta) \\ B_{\theta\delta}^T(\theta,\delta) & B_{\delta\delta}(\theta,\delta) \end{bmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\delta} \end{pmatrix} + \begin{bmatrix} h_\theta(\theta,\delta,\dot{\theta},\dot{\delta}) \\ h_\delta(\theta,\delta,\dot{\theta},\dot{\delta}) \end{bmatrix}$$
$$+ \begin{pmatrix} 0 \\ K\delta+D\dot{\delta} \end{pmatrix} = \begin{pmatrix} I \\ Q_\delta \end{pmatrix}u \qquad (5)$$

where the B are blocks of the $(N+M)\times(N+M)$ positive definite symmetric inertia matrix, partitioned according to the joint (rigid) and link (flexible) coordinates. Similarly, the h contain Coriolis and centrifugal forces, which can be computed via the Christoffel symbols,[43] i.e., via differentiation of the inertia matrix elements; each component of these terms is a quadratic form in the velocity vector $(\dot{\theta},\dot{\delta})$. The positive definite—typically diagonal—matrices K and D describe modal stiffness and damping of flexible links, respectively.

Incidentally, the equivalent rigid body system, i.e., for infinitely stiff links, is recovered by setting $\delta \equiv 0$ in the upper part of Eq. (5) leading to

$$B_{\theta\theta}(\theta)\ddot{\theta} + h_\theta(\theta,\dot{\theta}) = u \qquad (6)$$

The terms in Eq. (5) assume different analytical expressions and numerical values, depending on the choice of boundary conditions. From the model structure point of view, the $M \times N$ matrix Q_δ that weights the $N \times 1$ vector of joint input torques u in the lower equations takes on different forms. In particular, using the principle of virtual work, it can be shown that in the *clamped* case

$$Q_\delta = O \qquad (7)$$

whereas in the *pinned* case

$$Q_\delta = \text{block diag}\left\{\text{col}\left[\phi'_{i1}(0),\ldots,\phi'_{i,\mu_i}(0)\right]\right\} \qquad (8)$$
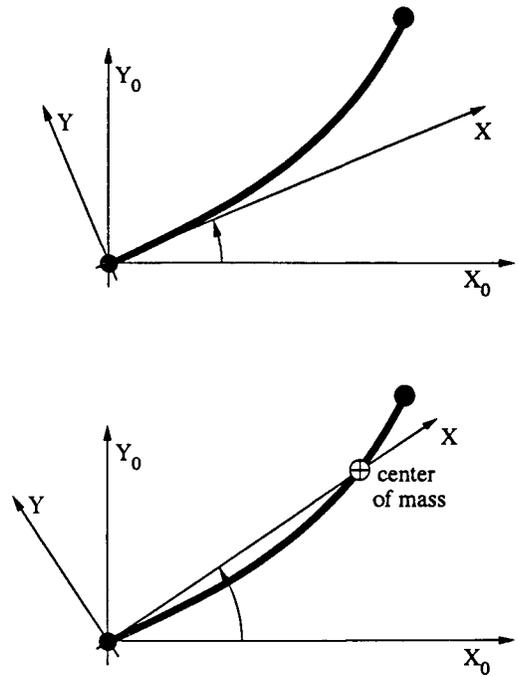

Fig. 1 Coordinate frame assignment: a) *clamped* case and b) *pinned* case.

In the following, it is understood that whenever Q_δ is chosen as in Eq. (7) or in Eq. (8), the dynamic model terms involved will be the ones corresponding to the clamped or the pinned situation, respectively.

At this point, a series of remarks are in order.

Remark 1. Although Eqs. (5) are in general highly nonlinear, it is not difficult to show that the left-hand side can be given a *linear* factorization in terms of a vector containing all precomputable parameters depending on the mechanical properties of the arm, e.g., masses, inertias, deformation moments, elasticity coefficients, etc.[4]

Remark 2. The assumption of small link deformations is usually made to derive the dynamic model. However, the equations of motion attain the same form as Eqs. (5) even when second-order terms in δ are included.

Remark 3. The off-diagonal block $B_{\theta\delta}$ represents the coupling between the rigid body and the flexible body dynamics. Various levels of simplification can be obtained neglecting terms in this block.

Remark 4. A rather common approximation is to evaluate the total kinetic energy of the system in the undeformed configuration $\delta = 0$. This implies that the inertia matrix, and thus h_θ and h_δ as well, are independent of δ (Ref. 43). Using the Christoffel symbols, it can be shown that the velocity terms h_δ will lose their quadratic dependence on $\dot{\delta}$. Moreover, if $B_{\delta\delta}$ is constant, h_θ also loses its quadratic dependence on $\dot{\delta}$, whereas each component of h_δ becomes a quadratic function of $\dot{\theta}$ only. Finally, if $B_{\theta\delta}$ is also approximated by a constant matrix then $h_\delta \equiv 0$ and $h_\theta = S_{\theta\theta}(\theta,\dot{\theta})\dot{\theta}$, i.e., a quadratic function of $\dot{\theta}$ only.[44]

Remark 5. The inclusion of gravity does not substantially modify the structure of the dynamic equations, in that only the gradient of the gravitational potential energy has to be added to Eqs. (5). In this case, however, Eq. (1) characterizing the link deformation should be properly modified.

Remark 6. If the exact orthogonal eigenfunctions ϕ_{ij} are used in Eq. (4), i.e., those satisfying the actual boundary conditions, the block $B_{\delta\delta}$ conveniently collapses into a constant diagonal one. This involves the cumbersome computation of time-varying boundary conditions[39] as well as the overall system deformation modes.[14]

Remark 7. In the widely investigated case of a single flexible link,[8,11,13,15,20,38] the small deformation assumption leads to a constant inertia matrix and thus to a linear dynamic

model. Further, diagonalization of the inertia matrix can be achieved by means of a suitable coordinate transformation.[40] If high rotational speeds are involved, however, the assumed modes expansion may lead to incorrect results even for the one-link case.

Explicit expressions of the single terms in Eqs. (5) have been recently reported for particular flexible manipulators.[4,14]

For the purpose of control derivation, it is convenient to extract the flexible accelerations from Eqs. (5) as

$$\ddot{\delta} = B_{\delta\delta}^{-1}\left[Q_\delta u - (h_\delta + K\delta + D\dot{\delta}) - B_{\theta\delta}^T\ddot{\theta}\right] \qquad (9)$$

which, substituted into the upper part of Eqs. (5), gives

$$(B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T)\ddot{\theta} + h_\theta - B_{\theta\delta}B_{\delta\delta}^{-1}(h_\delta + K\delta + D\dot{\delta}) = Fu \qquad (10)$$

with

$$F = I - B_{\theta\delta}B_{\delta\delta}^{-1}Q_\delta \qquad (11)$$

Notice that Eq. (10) describes the modification that takes place in the rigid body dynamics equation (6) due to the effects of link flexibility.

Remark 8. The $N \times N$ matrix $B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T$ has full rank as can be seen from the following identity:

$$\begin{pmatrix} B_{\theta\theta} & B_{\theta\delta} \\ B_{\theta\delta}^T & B_{\delta\delta} \end{pmatrix}\begin{pmatrix} I & O \\ -B_{\delta\delta}^{-1}B_{\theta\delta}^T & I \end{pmatrix} = \begin{pmatrix} B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T & B_{\theta\delta} \\ O & B_{\delta\delta} \end{pmatrix} \qquad (12)$$

and the positive definiteness of the inertia matrix.

Remark 9. Physical arguments can be used to show that the $N \times N$ matrix F in Eq. (11) has full rank. For instance, consider the arm in an undeformed rest configuration, i.e., $\delta = \dot{\delta} = 0$, $\dot{\theta} = 0$. If F were singular, a nonzero input torque u would exist that yields $\ddot{\theta} = 0$ thus keeping the arm at rest; this is clearly in contrast with mechanical intuition.

III. Inversion Control

Trajectory tracking in multi-input/multi-output nonlinear systems is usually achieved by input-output *inversion control* techniques.[19] Once a meaningful output has been defined for the system, a nonlinear state feedback is designed so that the resulting closed-loop system is transformed into a linear and decoupled one, with the possible appearance of unobservable internal dynamics. Under the assumption of stability of the resulting closed-loop system, exact reproduction of smooth desired output trajectories is feasible.

In the robotic case, inversion can be applied directly to the second-order differential form of the mechanical system (5), defining the vector of joint variables θ as the system output. Following the inversion algorithm, the output needs to be differentiated as many times as needed to have the input explicitly appearing. Inspection of Eq. (10) suggests that the joint accelerations $\ddot{\theta}$ (the second time derivative of the output) are at the same differential level as the torque inputs u. Therefore, the so-called *relative degree*[19] is uniform for all outputs and equal to two. Moreover, in view of Remarks 8 and 9, the input u can be fully recovered from Eq. (10).

Let a denote a joint acceleration vector. Setting $\ddot{\theta} = a$ in Eq. (10) and solving for u yields the nonlinear feedback law

$$u = F^{-1}\left[(B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T)a + h_\theta - B_{\theta\delta}B_{\delta\delta}^{-1}n_\delta\right] \qquad (13)$$

with

$$n_\delta = h_\delta + K\delta + D\dot{\delta} \qquad (14)$$

The closed-loop implementation of Eq. (13) requires the measurement of θ, $\dot{\theta}$, δ, and $\dot{\delta}$.

Remark 10. The matrix $(B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T)^{-1}F$ is the so-called *decoupling matrix*[19] of the system and is nonsingular.

In the *clamped* case $(Q_\delta = O)$ matrix F reduces to the identity, and no inversion thereof is needed in Eq. (13). On the other hand, a computationally efficient expression for F^{-1} when $Q_\delta \neq O$, e.g., in the *pinned* case, is[45]

$$F^{-1} = I + B_{\theta\delta}(B_{\delta\delta} - Q_\delta B_{\theta\delta})^{-1}Q_\delta \qquad (15)$$

requiring only the inversion of an $M \times M$ matrix.

Remark 11. The advantage of evaluating F^{-1} using Eq. (15) exists in any case, but it is more striking when $N > M$; in fact, when $N < M$, the inversion of an $M \times M$ block is needed anyway in Eq. (11).

The control (13) transforms the closed-loop system into the input-output linearized form

$$\ddot{\theta} = a \qquad (16)$$

$$\ddot{\delta} = -(I + Q_\delta F^{-1}B_{\theta\delta})\left[B_{\delta\delta}^{-1}(B_{\theta\delta}^T a + n_\delta)\right] + B_{\delta\delta}^{-1}Q_\delta F^{-1}u_r \qquad (17)$$

where

$$u_r = B_{\theta\theta}a + h_\theta \qquad (18)$$

is the well-known computed torque control[46] for the equivalent rigid system (6).

At this point, one can recognize that in the clamped case Eq. (13) becomes

$$u = (B_{\theta\theta} - B_{\theta\delta}B_{\delta\delta}^{-1}B_{\theta\delta}^T)a + h_\theta - B_{\theta\delta}B_{\delta\delta}^{-1}n_\delta \qquad (19)$$

and Eqs. (16) and (17) simplify to

$$\ddot{\theta} = a \qquad (20)$$

$$\ddot{\delta} = -B_{\delta\delta}^{-1}(B_{\theta\delta}^T a + n_\delta) \qquad (21)$$

Remark 12. It follows immediately from Eq. (19) that only the inversion of the $M \times M$ block on the diagonal of the inertia matrix relative to the flexible variables is required for control law implementation in the clamped case. This computational saving was not present in previous works on inversion-based controllers for flexible manipulators.[21-23] Therefore, the complexity of this nonlinear feedback strategy increases only with the number of flexible variables; in the limit, no inertia matrix inversion is required for the rigid case. Furthermore, if $B_{\delta\delta}$ is constant (see Remark 6) its inverse can be conveniently computed offline.

To achieve tracking of a desired joint trajectory specified by $\theta_{\text{des}}(t)$, the control design is completed by choosing the new input as

$$a = \ddot{\theta}_{\text{des}} + K_D(\dot{\theta}_{\text{des}} - \dot{\theta}) + K_P(\theta_{\text{des}} - \theta) \qquad (22)$$

where $K_P > 0$ and $K_D > 0$ are feedback gain matrices that allow pole placement in the open left-hand complex half-plane for the linear system (16). From Eq. (22) it is clear that the desired trajectory must be at least twice differentiable for having exact reproduction.

As mentioned, the applicability of the inversion controller (13) [or (19)] is based on the stability of the induced unobservable dynamics (17) [or (21)]. In the following, the discussion is concentrated on the *clamped* case only. However, similar arguments could be used in the *pinned* case, just resulting in more involved developments which are omitted.

The stability analysis can be carried out by studying the so-called *zero dynamics* associated with the system (20) and (21). This is obtained by constraining the output θ of the system to be a constant, and without loss of generality, zero. Hence, from Eq. (21) we obtain

$$\dot{\delta} = -B_{\delta\delta}^{-1}(h_\delta + K\delta + D\dot{\delta}) \qquad (23)$$

where all terms are evaluated for $\dot{\theta} = 0$.

A sufficient condition that guarantees stability of the overall closed-loop system is that the zero dynamics (23) are asymptotically stable.[19] The following result holds.

Theorem. The state $\delta = \dot{\delta} = 0$ is an asymptotically stable equilibrium point for the system (23). ☐

Proof. Applying the Lyapunov direct method, define the energy-based candidate function

$$V = \tfrac{1}{2}\delta^T K \delta + \tfrac{1}{2}\dot{\delta}^T B_{\delta\delta}\dot{\delta} \tag{24}$$

Time differentiation of Eq. (24) gives

$$\dot{V} = \delta^T K\dot{\delta} + \dot{\delta}^T B_{\delta\delta}\ddot{\delta} + \tfrac{1}{2}\dot{\delta}^T \dot{B}_{\delta\delta}\dot{\delta} \tag{25}$$

and substitution of Eq. (23) into Eq. (25) yields

$$\dot{V} = -\dot{\delta}^T(h_\delta + D\dot{\delta}) + \tfrac{1}{2}\dot{\delta}^T \dot{B}_{\delta\delta}\dot{\delta} \tag{26}$$

Observe that for the velocity dependent term h_δ there exists always a factorization $h_\delta = S_{\delta\delta}\dot{\delta}$ such that the matrix $\dot{B}_{\delta\delta} - 2S_{\delta\delta}$ is *skew symmetric*.[47] As a consequence, Eq. (26) reduces to

$$\dot{V} = -\dot{\delta}^T D\dot{\delta} \le 0 \tag{27}$$

To show asymptotic stability, note that $\dot{V} = 0$ if and only if $\dot{\delta} = 0$. In this case, the system dynamics (23) becomes $\ddot{\delta} = -B_{\delta\delta}^{-1}K\delta$, which implies that the largest invariant set in $\dot{V} = 0$ is $\delta = \dot{\delta} = 0$. Invoking LaSalle invariance set theorem,[48] the result follows. ☐

Remark 13. The foregoing result, together with the choice equation (22), also permits the conclusion that the closed-loop system is asymptotically stable under the inversion control law (19) in the trajectory tracking case ($\dot{\theta} \ne 0$). To understand this fact, consider the simpler case of an inertia matrix independent of δ (see Remark 4). When $\theta = \theta_{des}(t)$ is imposed by the control (19) and (22), the flexible variables satisfy the following *linear time-varying* equation

$$\ddot{\delta} = \psi(t) - A_1(t)\delta - A_2(t)\dot{\delta} \tag{28}$$

where

$$\psi(t) = -B_{\delta\delta}^{-1}(\theta_{des})\left[B_{\theta\delta}^T(\theta_{des}\ddot{\theta}_{des} + h_\delta(\theta_{des}, \dot{\theta}_{des})\right] \tag{29}$$

is a known function of time, and

$$A_1(t) = B_{\delta\delta}^{-1}(\theta_{des})K \tag{30}$$

$$A_2(t) = B_{\delta\delta}^{-1}(\theta_{des})D \tag{31}$$

Then, as long as all time-varying functions are bounded, stability is ensured even during trajectory tracking.

Remark 14. From Eq. (27), the rate of asymptotic convergence to zero of the flexible variables is established by the arm damping matrix D, generally resulting in a poorly damped behavior. This may be satisfactory during the large maneuvering phase of the manipulator, but it represents a major concern at the final destination. The standard remedy to this limitation is to resort to an active linear stabilizer for the deflection variables, designed for a linearized version of the system around the final configuration. It is convenient, indeed, to superimpose such a stabilizing control on the nonlinear one (19) and (22); in this way the synthesis can be advantageously performed on the system (20) and (21), which is linear in the input-output behavior, rather than on the original nonlinear system.[23] Alternatively, damping can be increased in a passive fashion by a mechanical treatment of the lightweight structure, e.g., attaching thin layers of viscoelastic material to the link surfaces.[49]

In the preceding derivation of the inversion-based control, it was assumed that full state feedback is available. The joint positions and velocities are measured via ordinary encoders and tachometers mounted on the actuators. For measuring

link deflection, different apparatus can be used ranging from strain gauges,[9] to accelerometers,[12] or optical devices.[11]

In spite of the availability of direct measurements of link flexibility, it may be convenient to avoid their use within the computation of the nonlinear part of the controller. The joint-based approach naturally lends itself to a cheap implementation in terms of joint variable measures only. In fact, one can preserve the robust linear feedback (22) and add a feedforward action to compensate for the nominal nonlinear terms. Specifically, when a twice differentiable joint trajectory $\theta_{des}(t)$ has been assigned, the forward integration of the flexible dynamics

$$\ddot{\delta} = -B_{\delta\delta}^{-1}(\theta_{des}, \delta)\left[h_\delta(\theta_{des}, \delta, \dot{\theta}_{des}, \dot{\delta}) + K\delta + D\dot{\delta} + B_{\theta\delta}^T(\theta_{des}, \delta)\ddot{\theta}_{des}\right] \tag{32}$$

from initial conditions $\delta(0) = \delta_0$, $\dot{\delta}(0) = \dot{\delta}_0$, provides the associated time evolution $\delta_{des}(t)$, $\dot{\delta}_{des}(t)$ of the flexible variables. Hence, evaluation of the nonlinearities in Eq. (19) along the computed state trajectory gives a control law in the form

$$u = u_{des}(t) + K_P(t)(\theta_{des} - \theta) + K_D(t)(\dot{\theta}_{des} - \dot{\theta}) \tag{33}$$

where Eq. (22) has been used, and

$$u_{des}(t) = B_{\theta\theta}(\theta_{des}, \delta_{des})\ddot{\theta}_{des} + h_\theta(\theta_{des}, \delta_{des}, \dot{\theta}_{des}, \dot{\delta}_{des})$$
$$- B_{\theta\delta}(\theta_{des}, \delta_{des})B_{\delta\delta}^{-1}(\theta_{des}, \delta_{des})\left[B_{\theta\delta}^T(\theta_{des}, \delta_{des})\ddot{\theta}_{des}\right.$$
$$\left. + n_\delta(\theta_{des}, \delta_{des}, \dot{\theta}_{des}, \dot{\delta}_{des})\right] \tag{34}$$

$$K_P(t) = \left[B_{\theta\theta}(\theta_{des}, \delta_{des})\right.$$
$$\left. - B_{\theta\delta}(\theta_{des}, \delta_{des})B_{\delta\delta}^{-1}(\theta_{des}, \delta_{des})B_{\theta\delta}^T(\theta_{des}, \delta_{des})\right]K_P \tag{35}$$

$$K_D(t) = \left[B_{\theta\theta}(\theta_{des}, \delta_{des})\right.$$
$$\left. - B_{\theta\delta}(\theta_{des}, \delta_{des})B_{\delta\delta}^{-1}(\theta_{des}, \delta_{des})B_{\theta\delta}^T(\theta_{des}, \delta_{des})\right]K_D \tag{36}$$

Remark 15. The initial conditions for numerical integration of Eq. (32) are typically $\delta_0 = \dot{\delta}_0 = 0$, corresponding to the undeformed rest configuration for the arm. Since the dynamics (32) are asymptotically stable, as just demonstrated, the evolution $\delta_{des}(t)$, $\dot{\delta}_{des}(t)$ generated from any initial condition will be bounded in response to a $\theta_{des}(t)$ with bounded second derivative. Indeed, the initial condition should match the actual initial deformation of the arm, otherwise the computed feedforward term (34) would not be the correct one. However, for persistent reference trajectories, an initial mismatch leads to tracking errors limited to the transient phase, since $\delta_{des}(t)$ anyhow decays toward the natural steady-state evolution.

Remark 16. An even simpler implementation of Eq. (33) is

$$u = u_{des}(t) + K_P(\theta_{des} - \theta) + K_D(\dot{\theta}_{des} - \dot{\theta}) \tag{37}$$

with constant feedback matrices. This type of control law has been tested experimentally on a two-link manipulator with a flexible forearm.[50]

IV. Case Study

The inversion-based nonlinear control laws presented earlier have been tested in simulation on a very light two-link planar flexible arm ($N = 2$), assuming two *clamped* mode shapes for each link ($\mu_1 = \mu_2 = 2$, implying $M = 4$) (Fig. 2). This reduced-order model is sufficient to encompass the relevant flexibility occurring in practical experimental control of lightweight manipulators with limited bandwidth actuators. The physical parameters characterizing the arm are the following: uniform density $\rho_1 = \rho_2 = 0.2$ kg/m; $(EI)_1 = (EI)_2 = 1$ N/m^2; link lengths $l_1 = l_2 = 0.5$ m; link masses $m_1 = m_2 = 0.1$ kg; tip payload mass $m_p = 0.1$ kg; mass of second hub $m_{h2} = 1$ kg; inertias of links about relative joint axes $J_{o1} = J_{o2} = 0.0083$ kg m^2; hub inertias $J_{h1} = J_{h2} = 0.1$ kg m^2; and tip payload inertia $J_p = 0.0005$ kg m^2.
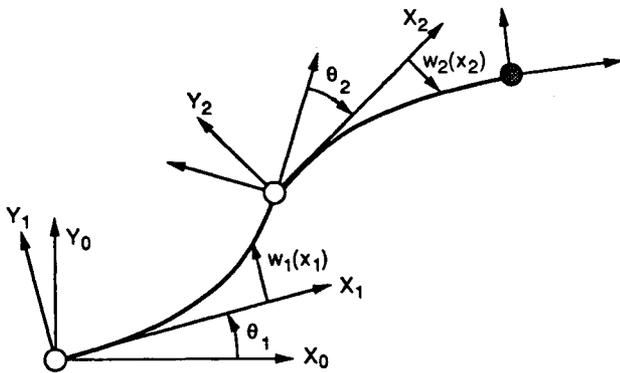
Fig. 2 Planar two-link flexible manipulator.

The first two natural eigenfrequencies of the links are

First link: $f_{11} = 0.48$ Hz, $f_{12} = 1.80$ Hz

Second link: $f_{21} = 2.18$ Hz, $f_{22} = 15.91$ Hz

which allow the computation of the stiffness matrix K as

$$K = \text{diag}\{K_i\} = \text{diag}\{0.91, 12.74, 18.73, 999.88\} \text{ kg/s}^2$$

The modal damping matrix is chosen as $D = \text{diag}\{0.1\sqrt{K_i}\}$. The complete dynamic model of the system is reported in Ref. 4.

The desired trajectory has been chosen as a sinusoidal profile with zero initial and final velocities and accelerations,

$$\theta_{\text{des},i}(t) = \theta_{\text{des},i}(0) + \left[t - \frac{T}{360} \sin\left(360 \deg \frac{t}{T}\right)\right]$$

$$\times \frac{\theta_{\text{des},i}(T) - \theta_{\text{des},i}(0)}{T}, \qquad i = 1, 2 \qquad (38)$$

where T is the traveling time. The closed-loop dynamic equations have been integrated with a fourth-order Runge-Kutta method at 1-ms sampling time.

First, the input-output feedback linearizing controller (19) and (22) with $T = 8$ s and

$$\theta_{\text{des},1}(0) = 0 \deg, \quad \theta_{\text{des},1}(8) = 45 \deg$$

$$\theta_{\text{des},2}(0) = \theta_{\text{des},2}(8) = 0 \deg \qquad (39)$$

has been simulated using

$$K_P = \text{diag}\{1, 4\} \qquad K_D = \text{diag}\{2, 4\} \qquad (40)$$

This choice corresponds to placing the poles of the resulting linear system (20) and (22) both at -1 for the first joint, and both at -2 for the second joint; in general, it was found that whenever system nonlinearities are canceled through inversion feedback, the design of suitable control gains is not a critical issue. The results in Fig. 3 indicate that the first joint evolution reproduces the desired trajectory, and the second joint is practically held fixed, with a maximum error of 0.23 deg. The tip error is limited to 1.6 cm along the y component; this error is computed as the deviation between the tip trajectory resulting from Eqs. (38) and (39)—as if the links were rigid—and the actual trajectory. The required joint input torques are indeed very small due to the lightweight nature of the arm.

The same control law, with gains as in Eq. (40), has been used for tracking a slewing motion on both joints specified by

$$\theta_{\text{des},i}(0) = 0 \deg, \quad \theta_{\text{des},i}(8) = 45 \deg, \qquad i = 1, 2 \qquad (41)$$

Satisfactory performance is obtained also in this case (Fig. 4), with a maximum tracking error of 0.33 deg on the second joint and a maximum tip error of ~ 2 cm on both Cartesian coordinates. The torque profiles are similar to the preceding ones, with a slight increase for the second joint.

Next, a linear joint feedback proportional-derivative (PD) controller has been applied for the slewing motion (40). In this case, the gains have been tuned down to

$$K_P = \text{diag}\{1, 0.04\}, \qquad K_D = \text{diag}\{2, 0.4\} \qquad (42)$$

to avoid numerical instability. In fact, since the nonlinear dynamic couplings during motion are not compensated, larger errors are induced which would result in excessive input torques applied to the flexible system. In Fig. 5, the resulting joint trajectories display a transient lag, an overshoot, and a longer settling time, which are typical of PD controllers. The maximum tracking error is 7.8 deg on the second joint, and consequently a maximum tip error of 11 cm is obtained.

To further compare the performance of the proposed nonlinear control law with a linear PD control, the reduced gains (42) have been used also for the control law (19) and (22). The results in Fig. 6 show that nonlinear compensation reduces the joint tracking errors by a factor of four, even with the same gains for the linear control part. Notice that the input torques in Figs. 4-6 are of comparable magnitude. This in turn means that the feedback gains in the inversion controller can be conveniently increased to reduce tracking errors, without affecting the overall control effort.

Finally, the results achieved with the computationally cheaper feedforward strategy (37) are shown in Fig. 7. To test
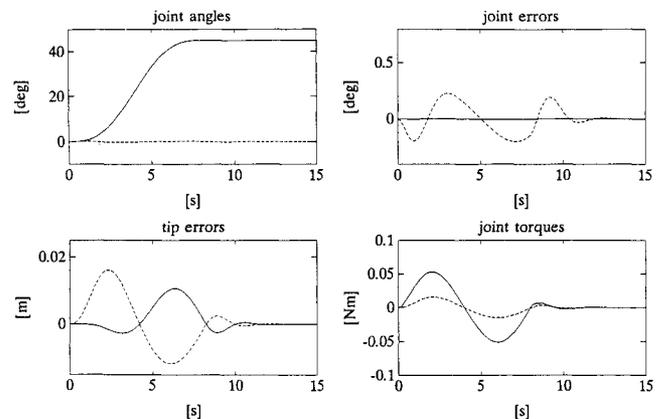

Fig. 3 Performance with inversion feedback control, trajectory (39) and gains (40); solid line = joint 1 and tip x component, dashed line = joint 2 and tip y component.
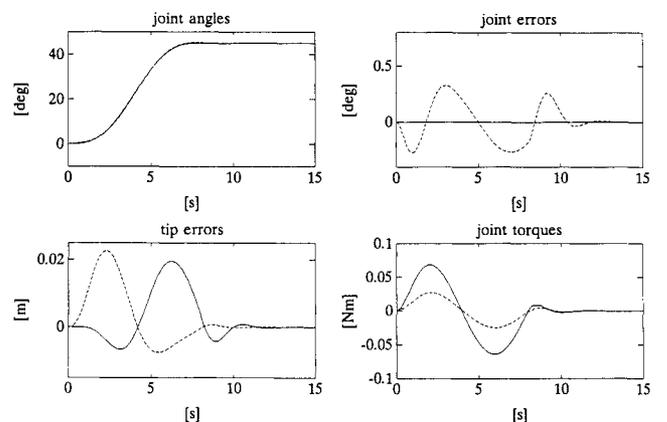

Fig. 4 Performance with inversion feedback control, trajectory (41) and gains (40); solid line = joint 1 and tip x component, dashed line = joint 2 and tip y component.

for robustness, the nominal evolution of the flexible variables to be used in Eq. (34) has been derived integrating the simplified dynamics

$$\ddot{\delta} = -\hat{B}_{\delta\delta}^{-1}(K\delta + D\dot{\delta} + \hat{B}_{\theta\delta}^{T}\ddot{\theta}_{\text{des}}) \qquad (43)$$

in place of Eq. (32), where the carets denote constant estimates of the respective terms, implying $h_\delta = 0$ (see Remark 4) and $\hat{B}_{\delta\delta}$ is diagonal. For comparison with pure PD control, the same gains (42) have been chosen. The tracking accuracy with the
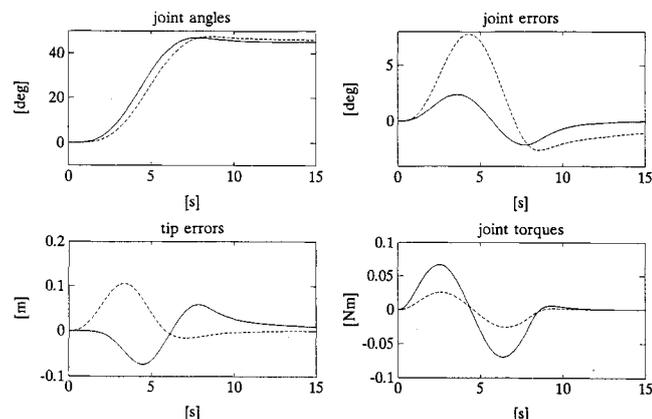

Fig. 5   Performance with PD control, trajectory (41) and gains (42); solid line = joint 1 and tip x component, dashed line = joint 2 and tip y component.
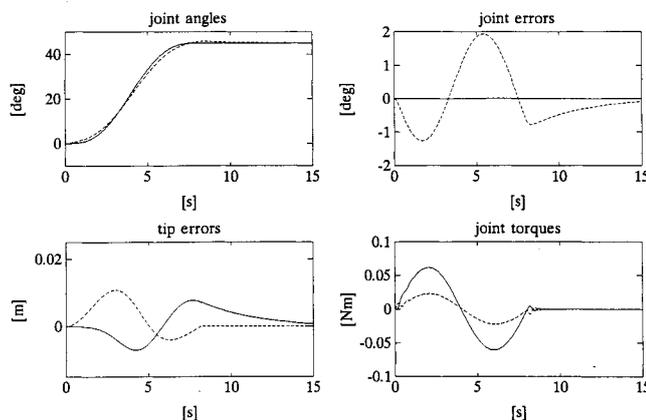

Fig. 6   Performance with inversion feedback control, trajectory (41) and gains (42); solid line = joint 1 and tip x component, dashed line = joint 2 and tip y component.
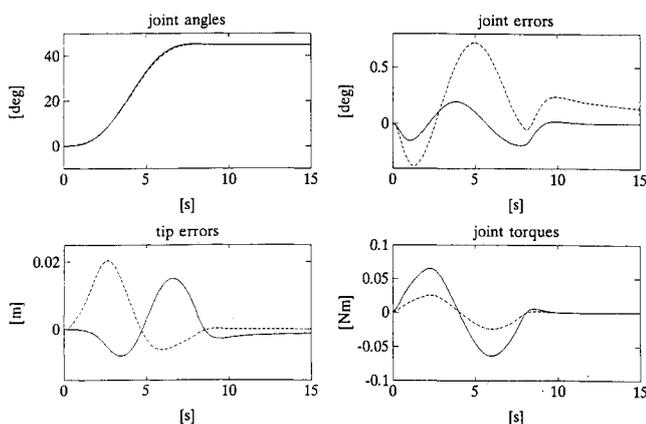

Fig. 7   Performance with inversion feedforward control, trajectory (41) and gains (42); solid line = joint 1 and tip x component, dashed line = joint 2 and tip y component.

model-based feedforward is improved both at the joint and at the tip level, although the control effort is about the same. We saw that no improvement was obtained when adding to the PD control a feedforward term based only on the desired joint acceleration.

V.   Conclusions

The design of inversion-based nonlinear control laws that guarantee stable tracking of joint trajectories for multilink flexible manipulators has been investigated. The interaction between modeling and control issues has been studied with specific concern for the effects of clamped/pinned boundary conditions on the complexity of the control law.

The theoretical and numerical results of the present work lead us to draw the following conclusions.

1) Joint-based inversion strategies are always stable for robots with flexible links.

2) Derivation of the model in the clamped case format allows remarkable simplifications in the model-based inversion controller.

3) The proposed control formulation requires inverting only the sub-block of the inertia matrix pertaining to the flexible variables, quite often a constant matrix.

4) Reduction of the computational burden is achieved by resorting to feedforward compensation of nonlinear and interaction terms, removing the need for sensing and feeding back arm deformation.

5) Satisfactory behavior is obtained for the tip motion of the flexible structure, provided that the desired joint trajectory is sufficiently smooth and enough structural damping is present.

An open research issue is to find effective control strategies for flexible manipulators that guarantee the tracking accuracy at the end-effector level similar to that obtained here at the joint level, without violating stability requirements. A recommendation toward this goal is to provide an intelligent motion planner with the capability of generating suitable reference trajectories for the joint variables in such a way that the end effector behaves as desired. Nevertheless, for current space robotics applications, it is believed that the adoption of joint-based nonlinear inversion techniques offers a performance breakthrough over classical control methods.

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