2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010



# Passive Linear Time-Varying Systems: State-Space Realizations, Stability in Feedback, and Controller Synthesis

James Richard Forbes and Christopher John Damaren

*Abstract*— **In this paper we consider linear time-varying passive systems. We state various theorems, which rely on the state-space matrices of the system, that identify when a linear-time varying system is purely passive, input strictly passive, output strictly passive, or input-state strictly passive which is a nonstandard notion of passivity defined in this paper. Two of our theorems resemble the Kalman-Yakubovich-Popov Lemma, one applicable to time-varying systems with a feedthrough matrix and the other for linear time-varying systems without one. The negative feedback interconnection of various systems is considered. We show that an output strictly passive system negatively interconnected with an input-state strictly passive system is globally asymptotically stable. We also show that both linear time-varying input-state and output strictly passive systems when connected in negative feedback with a sector bounded, memoryless nonlinearity are also globally asymptotically stable. The optimal design of a time-varying output strictly passive controller is also considered. We present an example: the position and velocity control of a time-varying mass controlled via a dynamic time-varying compensator and a sector bounded, memoryless nonlinearity.**

## I. INTRODUCTION

Over the past several decades passive systems, their stability in feedback, and their synthesis have been rigorously investigated. Passive systems have been analyzed, synthesized, and controlled in very different fields of engineering: electrical circuits, mechanical and aerospace systems, and especially in robotics [1], [2]. Passivity has also played a crucial role in adaptive control formulations [3], [4].

The Positive Real Lemma and the Kalman-Yakubovich-Popov (KYP) Lemma are both well known [5]–[9]. Each lemma specifies conditions that when met indicate a particular linear time-invariant (LTI) system with appropriate state-space matrices is positive real (PR) or strictly positive real (SPR). Identifying PR and SPR systems is significant because a positive real system can be robustly stabilized when connected in negative feedback with a SPR controller. With this in mind, various authors have investigated designing optimal SPR controllers, often employing numerical optimization methods [10]–[13].

This work is concerned with linear time-varying (LTV) systems which possess some sort of passive characteristic such as pure passivity, input strict passivity, output strict passivity, etc. We are concerned with the closed-loop stability of LTV systems and sector bounded, memoryless nonlinearities interconnected in negative feedback. After stating some preliminaries we present a series of theorems identifying passivity, output strict passivity, input strict passivity, and a new notion of passivity, input-state strict passivity. Although theorems pertaining to passive LTV systems are presented in the literature (see [14], [15]), the theorems pertaining to input, output, and input-state strictly passive LTV systems are unique to this paper. This work is partially motivated by the desire to state a theorem for LTV systems that is equivalent to the KYP Lemma for LTI systems. In addition to identifying passive systems via state-space equations, we show that the negative feedback interconnection of an input strictly passive LTV system and an input-state strictly passive LTV system is globally asymptotically stable. We also show that an input-state strictly passive LTV system, as well as an output strictly passive LTV system, connected in a negative feedback loop with a sector bounded, memoryless nonlinearity is globally asymptotically stable. Motivated by optimal control theory, we will present a method to optimally design a LTV output strictly passive feedback controller. Last, we give an example: position and velocity control of a time-varying mass. We control the system via a LTV controller optimally designed, and via a sector bounded, memoryless nonlinearity.

## II. PRELIMINARIES

To start, a function $\mathbf{u} \in L_2$ if $\|\mathbf{u}\|_2 = \sqrt{\int_0^\infty \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)dt} < \infty$ and $\mathbf{u} \in L_{2e}$ if $\|\mathbf{u}\|_{2T} = \sqrt{\int_0^T \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)dt} < \infty$, $0 \le T < \infty$ where $\mathbf{u}^\mathsf{T}(\cdot)$ is the transpose of the vector $\mathbf{u}(\cdot)$.

### A. Linear Time-Varying Systems

In this paper we will be concerned with square linear-time varying systems of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{1b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$ and the time-varying matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{C}(\cdot)$, and $\mathbf{D}(\cdot)$ are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. The nominal input-output equations are specified by (1a) and (1b), while an alternate output is $\mathbf{z}(t) = \mathbf{L}(t)\mathbf{x}(t) + \mathbf{W}(t)\mathbf{u}(t)$ where $\mathbf{z} \in \mathbb{R}^m$. We will assume complete controllability of $(\mathbf{A}(\cdot), \mathbf{B}(\cdot))$, and complete observability of $(\mathbf{C}(\cdot), \mathbf{A}(\cdot))$ and $(\mathbf{L}(\cdot), \mathbf{A}(\cdot))$ [16], [17].

J.R. Forbes, Ph.D. Candidate: forbes@utias.utoronto.ca. C.J. Damaren, Associate Professor: damaren@utias.utoronto.ca. Both authors are associated with the University of Toronto Institute for Aerospace Studies, 4925 Dufferin Street, Toronto, Ontario, Canada, M3H 5T6.

978-1-4244-7425-7/10/$26.00 ©2010 AACC

In a mechanical context (1) may represent a system with time-varying mass and damping, where the inputs are forces and the outputs are velocities. In an electrical circuit context, (1) may represent the interconnection of time-varying passive circuit components, such as resistors, capacitors, inductors, gyrators, and transformers, where the inputs are port currents and the outputs are port voltages.

The solution to (1) is

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{C}(t)\mathbf{\Phi}(t,t_0)\mathbf{x}_0 + \mathbf{C}(t)\int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\ d\tau \\ &\quad + \mathbf{D}(t)\mathbf{u}(t) \\ &= (\boldsymbol{G}\mathbf{u})(t) \end{aligned} \tag{2}$$

where $\mathbf{\Phi}(\cdot,\cdot)$ is the state transition matrix and $\mathbf{x}_0 = \mathbf{x}(t_0)$ is the initial system state. The state transition matrix satisfies

$$\dot{\mathbf{\Phi}}(t,t_0) = \mathbf{A}(t)\mathbf{\Phi}(t,t_0)\ , \qquad \mathbf{\Phi}(t_0,t_0) = \mathbf{1}\ ,$$
$$\mathbf{\Phi}(t_2,t_0) = \mathbf{\Phi}(t_2,t_1)\mathbf{\Phi}(t_1,t_0)\ , \qquad \mathbf{\Phi}^{-1}(t,\tau) = \mathbf{\Phi}(\tau,t).$$

### B. Passive System Characteristics

An operator $\boldsymbol{G} : L_{2e} \to L_{2e}$ is [18], [19]

- passive if $\exists\beta$ such that (s.t.) $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \beta,\ \forall \mathbf{u} \in L_{2e}, \forall T \geq 0$,
- input strictly passive if $\exists\delta > 0$ and $\exists\beta$ s.t. $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \delta\int_0^T \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)dt + \beta,\ \forall\mathbf{u} \in L_{2e}, \forall T \geq 0$,
- output strictly passive if $\exists\epsilon > 0$ and $\exists\beta$ s.t. $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \epsilon\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)dt + \beta,\ \forall\mathbf{u} \in L_{2e}, \forall T \geq 0$,
- input-output, or very strictly passive if $\exists\delta > 0$, $\exists\epsilon > 0$, and $\exists\beta$ s.t. $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \epsilon\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{y}(t)dt + \delta\int_0^T \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)dt + \beta,\ \forall\mathbf{u} \in L_{2e}, \forall T \geq 0$,
- state-strictly passive if $\exists\psi(\cdot) > 0$ and $\exists\beta$ s.t. $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \int_0^T \psi(\mathbf{x}(t))dt + \beta,\ \forall\mathbf{u} \in L_{2e}, \forall T \geq 0$,
- input-state strictly passive if $\exists\delta > 0$, $\exists\psi(\cdot) > 0$, and $\exists\beta$ s.t. $\int_0^T \mathbf{y}^\mathsf{T}(t)\mathbf{u}(t)dt \geq \int_0^T \psi(\mathbf{x}(t))dt + \delta\int_0^T \mathbf{u}^\mathsf{T}(t)\mathbf{u}(t)dt + \beta,\ \forall\mathbf{u} \in L_{2e}, \forall T \geq 0$ .

The constant $\beta$ is nonzero for nonzero initial conditions, and zero for quiescent initial conditions. The units of $\delta$ are gain and the units of $\epsilon$ are one over gain.

A state strictly passive system is not necessarily equivalent to an output strictly passive system. Some references, for example [20], call a state strictly passive system a strictly passive system, which we feel is confusing when compared with, for example, [18]. In [18] an input strictly passive system is called a strictly passive system. Hence, we utilize a more verbose nomenclature to distinguish between input, output, very, state, and input-state strictly passive systems.

In the sections to follow we will discuss when a LTV system of the form (1) is passive, input-state strictly passive, etc. We will first consider the case when $\mathbf{D}(t) \neq \mathbf{0}$ (Section III), and second when $\mathbf{D}(t) = \mathbf{0}$ (Section IV).

## III. State-Space Realization of Passive Linear Time-Varying Systems Possessing a Feedthrough Matrix

We will first consider the case where $\mathbf{D}(t) \neq \mathbf{0}$. Rather then having to evaluate one of the passivity integrals presented in Section II-B, it would be advantageous to be able to determine if a system is passive or input strictly passive by evaluating an expression related to the system's state-space matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{C}(\cdot)$, and $\mathbf{D}(\cdot)$, much like the Positive Real Lemma and the KYP Lemma.

### A. State-Space Realization of Passive Linear Time-Varying Systems

We will first consider passive LTV systems with a state-space realization equivalent to (1). The following theorem is similar to those presented in [14] and [15], and is of the same flavor as the the Positive Real Lemma associated with LTI systems. We prove the following theorem differently than [14], [15].

**Theorem III.1.** *A LTV system described by* (1) *and* (2) *that is completely controllable and completely observable is passive if there exists continuous, bounded* $\mathbf{P}(t) = \mathbf{P}^\mathsf{T}(t) > 0$, $\mathbf{L}(\cdot)$, *and* $\mathbf{W}(\cdot)$ *s.t.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t) \tag{3a}$$
$$\mathbf{C}^\mathsf{T}(t) - \mathbf{P}(t)\mathbf{B}(t) = \mathbf{L}^\mathsf{T}(t)\mathbf{W}(t) \tag{3b}$$
$$\mathbf{D}(t) + \mathbf{D}^\mathsf{T}(t) = \mathbf{W}^\mathsf{T}(t)\mathbf{W}(t). \tag{3c}$$

*Proof:* (Sufficiency) To be concise we will neglect writing the temporal argument of the input and output signals, and time-varying matrices. Consider the following Lyapunov function and its temporal derivative:

$$\begin{aligned} V &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}\ , \\ \dot{V} &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\dot{\mathbf{x}} + \tfrac{1}{2}\dot{\mathbf{x}}^\mathsf{T}\mathbf{P}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^\mathsf{T}\dot{\mathbf{P}}\mathbf{x} \\ &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P}\right)\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{B}\mathbf{u}. \end{aligned}$$

Integrating $\dot{V}$ from $0$ to $T$ gives

$$\begin{aligned} &\int_0^T \dot{V}dt = V(T)\overbrace{-V(0)}^{\beta} \geq \beta, \\ &\int_0^T \left[\tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P}\right)\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{B}\mathbf{u}\right]dt \\ &= \int_0^T \left[-\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{L}\mathbf{x} + \mathbf{x}^\mathsf{T}\left(\mathbf{C}^\mathsf{T} - \mathbf{L}^\mathsf{T}\mathbf{W}\right)\mathbf{u}\right]dt \geq \beta, \\ &\int_0^T \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u}dt \geq \int_0^T \left(\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{L}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{W}\mathbf{u}\right)dt + \beta. \end{aligned}$$

Recall that $\mathbf{D}$ is square, and can be broken up into symmetric and skew-symmetric parts: $\mathbf{D} = \frac{1}{2}\left(\mathbf{D} + \mathbf{D}^\mathsf{T}\right) + \frac{1}{2}\left(\mathbf{D} - \mathbf{D}^\mathsf{T}\right)$ where $\mathbf{u}^\mathsf{T}\left(\mathbf{D} - \mathbf{D}^\mathsf{T}\right)\mathbf{u} = 0$. Given that $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ and

$\mathbf{y}^\mathsf{T}\mathbf{u} = \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u} + \frac{1}{2}\mathbf{u}^\mathsf{T}\left(\mathbf{D}^\mathsf{T} + \mathbf{D}\right)\mathbf{u}$ we arrive at

$$\begin{aligned}&\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt \\ &\geq \int_0^T \left(\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{L}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{W}\mathbf{u} + \tfrac{1}{2}\mathbf{u}^\mathsf{T}\mathbf{W}^\mathsf{T}\mathbf{W}\mathbf{u}\right) dt + \beta \\ &= \int_0^T \tfrac{1}{2}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right)^\mathsf{T}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right) dt + \beta \geq \beta\end{aligned}$$

which completes the proof.

Often (3a) is replaced by $\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{Q}(t)$ where $\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t) = \mathbf{Q}(t) = \mathbf{Q}^\mathsf{T}(t) \geq 0$. Equation (3a) is a first order, ordinary matrix differential equation in $\mathbf{P}(\cdot)$. To be solved, a boundary condition must be known. From [14] and [15] we know $\mathbf{P}(T) = \mathbf{P}^\mathsf{T}(T) > 0$ must be specified. Equation (3a) could be solved either numerically or analytically for $\mathbf{P}(\cdot)$.

What is useful about Theorem III.1 is that passivity of a system can be tested by checking that (3) is satisfied, rather than evaluating the passivity integral presented in Section II-B.

### *B. State-Space Realization of Input Strictly Passive Linear Time-Varying Systems*

We will now consider input strictly passive LTV systems.

**Theorem III.2.** *A LTV system described by* (1) *and* (2) *that is completely controllable and completely observable with* $\mathbf{D}(t) = \tilde{\mathbf{D}}(t) + \delta\mathbf{1}$ *where* $\delta > 0$ *is input strictly passive if there exists continuous, bounded* $\mathbf{P}(t) = \mathbf{P}^\mathsf{T}(t) > 0$*,* $\mathbf{L}(\cdot)$ *and* $\mathbf{W}(\cdot)$ *s.t.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t) \quad (4a)$$

$$\mathbf{C}^\mathsf{T}(t) - \mathbf{P}(t)\mathbf{B}(t) = \mathbf{L}^\mathsf{T}(t)\mathbf{W}(t) \quad (4b)$$

$$\tilde{\mathbf{D}}(t) + \tilde{\mathbf{D}}^\mathsf{T}(t) = \mathbf{W}^\mathsf{T}(t)\mathbf{W}(t). \quad (4c)$$

*Proof:* (Sufficiency) Following the proof of Theorem III.1 we have

$$\int_0^T \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u}dt \geq \int_0^T \left(\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{L}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{W}\mathbf{u}\right) dt + \beta.$$

With $\mathbf{y}^\mathsf{T}\mathbf{u} = \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u} + \frac{1}{2}\mathbf{u}^\mathsf{T}\left(\tilde{\mathbf{D}}^\mathsf{T} + \tilde{\mathbf{D}}\right)\mathbf{u} + \delta\mathbf{u}^\mathsf{T}\mathbf{u} = \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u} + \frac{1}{2}\mathbf{u}^\mathsf{T}\mathbf{W}^\mathsf{T}\mathbf{W}\mathbf{u} + \delta\mathbf{u}^\mathsf{T}\mathbf{u}$ we arrive at

$$\begin{aligned}&\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt \\ &\geq \int_0^T \left(\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{L}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{W}\mathbf{u} + \tfrac{1}{2}\mathbf{u}^\mathsf{T}\mathbf{W}^\mathsf{T}\mathbf{W}\mathbf{u} + \delta\mathbf{u}^\mathsf{T}\mathbf{u}\right) dt \\ &\quad +\beta \\ &= \int_0^T \tfrac{1}{2}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right)^\mathsf{T}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right) dt + \delta\int_0^T \mathbf{u}^\mathsf{T}\mathbf{u}dt + \beta \\ &\geq \delta\int_0^T \mathbf{u}^\mathsf{T}\mathbf{u}dt + \beta\end{aligned}$$

which completes the proof.

### *C. State-Space Realization of Input-State Strictly Passive Linear Time-Varying Systems*

Next, we will consider input-state strictly passive LTV systems.

**Theorem III.3.** *A LTV system described by* (1) *and* (2) *that is completely controllable and completely observable with* $\mathbf{D}(t) = \tilde{\mathbf{D}}(t) + \delta\mathbf{1}$ *where* $\delta > 0$ *is input-state strictly passive if there exists* $\nu > 0$ *and continuous, bounded* $\mathbf{P}(t) = \mathbf{P}^\mathsf{T}(t) > 0$*,* $\mathbf{L}(\cdot)$ *and* $\mathbf{W}(\cdot)$ *s.t.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t) - 2\nu\mathbf{P} \quad (5a)$$

$$\mathbf{C}^\mathsf{T}(t) - \mathbf{P}(t)\mathbf{B}(t) = \mathbf{L}^\mathsf{T}(t)\mathbf{W}(t) \quad (5b)$$

$$\tilde{\mathbf{D}}(t) + \tilde{\mathbf{D}}^\mathsf{T}(t) = \mathbf{W}^\mathsf{T}(t)\mathbf{W}(t). \quad (5c)$$

*Proof:* (Sufficiency) Modifying the proof of Theorem III.2 slightly gives

$$\begin{aligned}\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt \;&\geq\; \int_0^T \tfrac{1}{2}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right)^\mathsf{T}\left(\mathbf{L}\mathbf{x} + \mathbf{W}\mathbf{u}\right) dt \\ &\quad + \delta\int_0^T \mathbf{u}^\mathsf{T}\mathbf{u}dt + \nu\int_0^T \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}dt + \beta \\ &\geq\; \delta\int_0^T \mathbf{u}^\mathsf{T}\mathbf{u}dt + \nu\int_0^T \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}dt + \beta\end{aligned}$$

which completes the proof.

Comparing (3a) and (4a), $-2\nu\mathbf{P}(\cdot)$ has been added to the right hand side so that in terms of (5a) we could equivalently write $\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{Q}(t)$ where $\mathbf{Q}(t) = \mathbf{Q}^\mathsf{T}(t) > 0$.

### *D. Comments*

One might ask why there is no output strictly passive theorem for LTV systems, nor an input-output strictly passive theorem. It seems that showing that a system is output strictly passive is prohibited by the presence of a $\mathbf{D}(\cdot)$ matrix.

## IV. State-Space Realization of Passive Linear Time-Varying Systems without a Feedthrough Matrix

We will now consider systems with $\mathbf{D}(t) = \mathbf{0}$. Doing so is important because the vast majority of real, physical systems (in particular mechanical, aerospace, and electrical systems) do not have a feedthrough matrix.

### *A. State-Space Realization of Passive Linear Time-Varying Systems*

Given a system with state-space matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, and $\mathbf{C}(\cdot)$ where $\mathbf{D}(t) = \mathbf{0}$, we will now derive conditions that indicate if a system is passive. Much like Theorem III.1, the following corollary is in the spirit of the Positive Real Lemma.

**Corollary IV.1.** *A LTV system described by* (1) *and* (2) *that is completely controllable and completely observable with*

$\mathbf{D}(t) = \mathbf{0}$ *is passive if there exists continuous, bounded* $\mathbf{P}(t) = \mathbf{P}^\mathsf{T}(t) > 0$ *and* $\mathbf{L}(\cdot)$ *s.t.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t) \quad (6a)$$
$$\mathbf{P}(t)\mathbf{B}(t) = \mathbf{C}^\mathsf{T}(t). \quad (6b)$$

*Proof:* (Sufficiency) Using the Lyapunov function presented in the proof of Theorem III.1 and the fact that $\mathbf{y} = \mathbf{C}\mathbf{x}$ we have

$$\int_0^T \left[\tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P}\right)\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{B}\mathbf{u}\right]dt \geq \beta,$$
$$\int_0^T \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u}dt \geq \tfrac{1}{2}\int_0^T \mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}(t)\mathbf{L}(t)\mathbf{x}dt + \beta,$$
$$\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt \geq \beta.$$

*B. State-Space Realization of Output Strictly Passive Linear Time-Varying Systems*

We will now derive a state-space expression based on a systems time-varying matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, and $\mathbf{C}(\cdot)$ where $\mathbf{D}(t) = \mathbf{0}$ which when satisfied informs us that the system in question is output strictly passive, much like the Kalman-Yakubovich-Popov Lemma for strictly positive real systems presented in [19].

**Corollary IV.2.** *A LTV system described by* (1) *and* (2) *that is completely controllable and completely observable with* $\mathbf{D}(t) = \mathbf{0}$ *is output strictly passive if there exists continuous, bounded* $\mathbf{P}(t) = \mathbf{P}^\mathsf{T}(t) > 0$ *and* $\mathbf{Q}(t) = \mathbf{Q}^\mathsf{T}(t) > 0$ *s.t.*

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^\mathsf{T}(t)\mathbf{P}(t) = -\mathbf{Q}(t) \quad (7a)$$
$$\mathbf{P}(t)\mathbf{B}(t) = \mathbf{C}^\mathsf{T}(t). \quad (7b)$$

*Proof:* (Sufficiency) Following the proof of Corollary IV.1 and the fact that $\mathbf{y} = \mathbf{C}\mathbf{x}$ we have

$$\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt = \int_0^T \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{u}dt \geq \tfrac{1}{2}\int_0^T \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x}dt + \beta.$$

Under the assumption that $\mathbf{C}$ and $\mathbf{Q}$ are bounded we can write

$$\mathbf{C}^\mathsf{T}\mathbf{C} \leq \overline{c}\mathbf{1} < \infty\ , \quad 0 < \underline{q}\mathbf{1} \leq \mathbf{Q} \leq \overline{q}\mathbf{1} < \infty.$$

It follows that

$$\mathbf{y}^\mathsf{T}\mathbf{y} = \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{x} \leq \overline{c}\mathbf{x}^\mathsf{T}\mathbf{x}\ , \quad \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \geq \underline{q}\mathbf{x}^\mathsf{T}\mathbf{x}$$
$$\Rightarrow \quad \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \geq \frac{\underline{q}}{\overline{c}}\mathbf{y}^\mathsf{T}\mathbf{y}.$$

Using the above we arrive at

$$\int_0^T \mathbf{y}^\mathsf{T}\mathbf{u}dt \geq \underbrace{\frac{\underline{q}}{2\overline{c}}}_{\epsilon}\int_0^T \mathbf{y}^\mathsf{T}\mathbf{y}dt + \beta$$

which completes the proof.

Note that for systems with no feedthrough matrix, if the system is state strictly passive it is implied that the system is output strictly passive. Also recall that an output strictly passive system has finite gain, $\frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} \leq \gamma$ where $\gamma > 0$ is the system gain. When $\mathbf{D}(t) = \mathbf{0}$ and the system is output strictly passive $\gamma = \frac{1}{\epsilon}$.

## V. Stability Theory

Consider the negative feedback interconnection of two systems, $\boldsymbol{G}_1 : L_{2e} \to L_{2e}$ and $\boldsymbol{G}_2 : L_{2e} \to L_{2e}$, as presented in Fig. 1. Each system $\boldsymbol{G}_i$ has associated with it an input strictly passive parameter, $\delta_i$, an output strictly passive parameter, $\epsilon_i$, or a positive definite function $\psi_i(\cdot)$ as defined in Section II-B where $i = 1, 2$. Both the weak and strong versions of the passivity theorem are well known [18]. The weak version of the passivity theorem states that when $\mathbf{u}_2 = \mathbf{0}$ if $\boldsymbol{G}_1$ is passive while $\boldsymbol{G}_2$ is input strictly passive then $\mathbf{u}_1 \in L_2$ implies $\mathbf{y}_1 \in L_2$. The strong version states that if $\delta_1 + \epsilon_2 > 0$ and $\delta_2 + \epsilon_1 > 0$ then $\mathbf{u}_1, \mathbf{u}_2 \in L_2$ implies $\mathbf{y}_1, \mathbf{y}_2 \in L_2$.

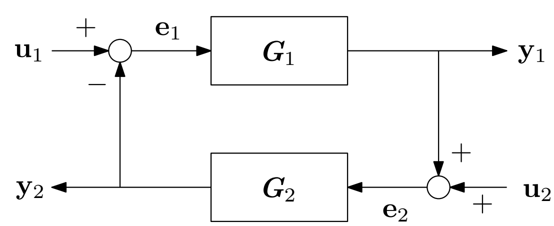


Fig. 1. Negative feedback interconnection.

*A. Stability Involving Input-State Strictly Passive Systems*

In this work we have defined input-state strictly passive LTV systems. We will now show that the negative feedback interconnection of an output strictly passive LTV system without a feedthrough matrix and an input-state strictly passive LTV system is globally asymptotically stable in the sense of Lyapunov.

**Theorem V.1.** *Assuming no external inputs, the negative feedback interconnection of an output strictly passive LTV system with* $\mathbf{D}_1(t) = \mathbf{0}$ *and an input-state strictly passive LTV system is globally asymptotically stable.*

*Proof:* Consider the following Lyapunov function and its temporal derivative:

$$V = \tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{P}_1\mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2\ ,$$
$$\dot{V} = \tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\left(\dot{\mathbf{P}}_1 + \mathbf{P}_1\mathbf{A}_1 + \mathbf{A}_1^\mathsf{T}\mathbf{P}_1\right)\mathbf{x}_1 + \mathbf{x}_1^\mathsf{T}\mathbf{P}_1\mathbf{B}_1\mathbf{e}_1$$
$$+\tfrac{1}{2}\mathbf{x}_2^\mathsf{T}\left(\dot{\mathbf{P}}_2 + \mathbf{P}_2\mathbf{A}_2 + \mathbf{A}_2^\mathsf{T}\mathbf{P}_2\right)\mathbf{x}_2 + \mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{B}_2\mathbf{e}_2.$$

Using Theorem III.3 and Corollary IV.2 (see (5) and (7)) and the fact that $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}_1$ and $\mathbf{x}_2^\mathsf{T}\mathbf{C}_2^\mathsf{T}\mathbf{e}_2 = \mathbf{y}_2^\mathsf{T}\mathbf{e}_2 -$

$\frac{1}{2}\mathbf{e}_2^\mathsf{T}\left(\tilde{\mathbf{D}}_2^\mathsf{T}+\tilde{\mathbf{D}}_2\right)\mathbf{e}_2-\delta_2\mathbf{e}_2^\mathsf{T}\mathbf{e}_2$ we arrive at

$$\begin{aligned}\dot{V} &= -\tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{Q}_1\mathbf{x}_1+\overbrace{\mathbf{x}_1^\mathsf{T}\mathbf{C}_1^\mathsf{T}}^{\mathbf{y}_1^\mathsf{T}}\mathbf{e}_1-\nu\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2-\tfrac{1}{2}\mathbf{x}_2^\mathsf{T}\mathbf{L}_2^\mathsf{T}\mathbf{L}_2\mathbf{x}_2\\ &\quad+\mathbf{x}_2^\mathsf{T}\left(\mathbf{C}_2^\mathsf{T}-\mathbf{L}_2^\mathsf{T}\mathbf{W}_2\right)\mathbf{e}_2\\ &\le -\tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{Q}_1\mathbf{x}_1+\mathbf{y}_1^\mathsf{T}\mathbf{e}_1-\nu\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2-\tfrac{1}{2}\mathbf{x}_2^\mathsf{T}\mathbf{L}_2^\mathsf{T}\mathbf{L}_2\mathbf{x}_2\\ &\quad-\mathbf{x}_2^\mathsf{T}\mathbf{L}_2^\mathsf{T}\mathbf{W}_2\mathbf{e}_2-\tfrac{1}{2}\mathbf{x}_2^\mathsf{T}\mathbf{W}_2^\mathsf{T}\mathbf{W}_2\mathbf{x}_2-\delta_2\mathbf{e}_2^\mathsf{T}\mathbf{e}_2+\mathbf{y}_2^\mathsf{T}\mathbf{e}_2.\end{aligned}$$

The $\mathbf{y}_1^\mathsf{T}\mathbf{e}_1$ and $\mathbf{y}_2^\mathsf{T}\mathbf{e}_2$ terms sum to zero owing to the fact that $\mathbf{e}_1=-\mathbf{y}_2$ and $\mathbf{e}_2=\mathbf{y}_1$ when external inputs are zero. We then have

$$\begin{aligned}\dot{V} &\le -\tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{Q}_1\mathbf{x}_1-\nu\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2\\ &\quad-\tfrac{1}{2}\left(\mathbf{L}_2\mathbf{x}_2+\mathbf{W}_2\mathbf{e}_2\right)^\mathsf{T}\left(\mathbf{L}_2\mathbf{x}_2+\mathbf{W}_2\mathbf{e}_2\right)-\delta_2\mathbf{y}_2^\mathsf{T}\mathbf{y}_2\\ &\le -\tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{Q}_1\mathbf{x}_1-\nu\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2-\delta_2\mathbf{x}_1^\mathsf{T}\mathbf{C}_1^\mathsf{T}\mathbf{C}_1\mathbf{x}_1\\ &\le -\tfrac{1}{2}\mathbf{x}_1^\mathsf{T}\mathbf{Q}_1\mathbf{x}_1-\nu\mathbf{x}_2^\mathsf{T}\mathbf{P}_2\mathbf{x}_2\\ &\le -\tfrac{1}{2}\left(\underline{q_1}\mathbf{x}_1^\mathsf{T}\mathbf{x}_1+\nu\underline{p_2}\mathbf{x}_2^\mathsf{T}\mathbf{x}_2\right)\\ &< 0\end{aligned}$$

where

$$0<\underline{q_1}\mathbf{1}\le\mathbf{Q}_1\le\overline{q}_1\mathbf{1}<\infty,\quad 0<\underline{p_2}\mathbf{1}\le\mathbf{P}_2\le\overline{p}_2\mathbf{1}<\infty.$$

Thus the system is globally asymptotically stable.

The symmetry of the traditional strong passivity theorem is maintained. It doesn't matter which system, $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$, is output strictly passive as long as the other system is input-state strictly passive. It can easily be shown that the negative feedback interconnection of two input-state strictly passive systems is also globally asymptotically stable using a similar proof.

In a LTI context it can be shown that a passive (i.e., PR) system can be stabilized by a system that is SPR. To do so, a Lyapunov function similar to the one presented above is used along with the Positive Real Lemma and the KYP Lemma, but the Krasovskii-LaSalle theorem must be used to prove global asymptotic stability. The negative feedback interconnection of a passive LTV system (Corollary IV.1) and an input-state strictly passive system can only be shown to be stable because the associated $\dot{V}$ expression is negative semidefinite, not negative definite, and one can not use the Krasovskii-LaSalle theorem while dealing with nonautonomous systems.

*B. Stability Involving Sector Bounded, Memoryless Nonlinearities*

Consider the negative feedback interconnection of a dynamic linear time-varying system and a sector bounded, memoryless nonlinearity as shown in Fig. 2. The sector bounded, memoryless nonlinearities we are considering are those that satisfy

$$\boldsymbol{\phi}(\mathbf{0},t)=\mathbf{0},\ \forall t\ge 0, \tag{8a}$$
$$\mathbf{y}^\mathsf{T}(t)\boldsymbol{\phi}(\mathbf{y}(t),t)\ge 0,\ \forall\mathbf{y}\in\mathbb{R}^m\ \forall t\ge 0. \tag{8b}$$

We will now show that the negative feedback interconnection of an input-state strictly passive LTV system and a sector bounded, memoryless nonlinearity is globally asymptotically stable.

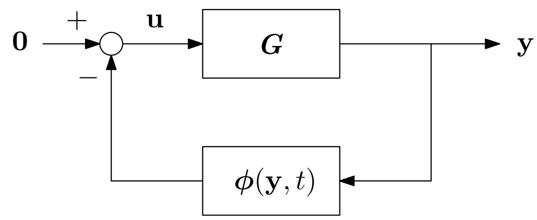


Fig. 2. Negative feedback interconnection of a dynamic, linear time-varying system and a sector bounded, memoryless nonlinearity

**Theorem V.2.** *Assuming no external inputs, the negative feedback interconnection of an input-state strictly passive LTV system and a sector bounded, memoryless nonlinearity satisfying the properties of* (8) *is globally asymptotically stable.*

*Proof:* We will make use of (5) from Theorem III.3. The control is simply $\mathbf{u}(t)=-\boldsymbol{\phi}(\mathbf{y},t)$. Consider the following Lyapunov function and its temporal derivative:

$$\begin{aligned}V &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x},\\ \dot{V} &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\dot{\mathbf{P}}+\mathbf{P}\mathbf{A}+\mathbf{A}^\mathsf{T}\mathbf{P}\right)\mathbf{x}+\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{B}\mathbf{u}\\ &= -\tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\mathbf{L}^\mathsf{T}\mathbf{L}+2\nu\mathbf{P}\right)\mathbf{x}+\mathbf{x}^\mathsf{T}\left(\mathbf{C}^\mathsf{T}-\mathbf{L}^\mathsf{T}\mathbf{W}\right)\mathbf{u}\\ &= -\tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\mathbf{L}^\mathsf{T}\mathbf{L}+2\nu\mathbf{P}\right)\mathbf{x}+\left(\mathbf{y}^\mathsf{T}-\tfrac{1}{2}\mathbf{u}^\mathsf{T}\left(\mathbf{D}^\mathsf{T}+\mathbf{D}\right)\right)\mathbf{u}\\ &\quad-\mathbf{x}^\mathsf{T}\mathbf{L}^\mathsf{T}\mathbf{W}\mathbf{u}\\ &= -\nu\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}+\mathbf{y}^\mathsf{T}\mathbf{u}-\tfrac{1}{2}\left(\mathbf{L}\mathbf{x}+\mathbf{W}\mathbf{u}\right)^\mathsf{T}\left(\mathbf{L}\mathbf{x}+\mathbf{W}\mathbf{u}\right)\\ &\quad-\delta\mathbf{u}^\mathsf{T}\mathbf{u}\\ &\le -\nu\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}-\mathbf{y}^\mathsf{T}\boldsymbol{\phi}(\mathbf{y},t)\\ &\le -\nu\underline{p}\mathbf{x}^\mathsf{T}\mathbf{x}\qquad(\text{Note}: 0<\underline{p}\mathbf{1}\le\mathbf{P}\le\overline{p}\mathbf{1}<\infty)\\ &< 0.\end{aligned}$$

Thus, the system is globally asymptotically stable.

Next we will show that the negative feedback interconnection of an input strictly passive LTV system and a sector bounded, memoryless nonlinearity is globally asymptotically stable.

**Theorem V.3.** *Assuming no external inputs, the negative feedback interconnection of an output strictly passive LTV system with* $\mathbf{D}(t)=\mathbf{0}$ *and a sector bounded, memoryless nonlinearity satisfying the properties of* (8) *is globally asymptotically stable.*

*Proof:* We are assuming no feedthrough matrix is present, thus we will make use of (7) from Corollary IV.2. Consider the following Lyapunov function and its temporal derivative:

$$\begin{aligned} V &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{x}, \\ \dot{V} &= \tfrac{1}{2}\mathbf{x}^\mathsf{T}\left(\dot{\mathbf{P}} + \mathbf{P}\mathbf{A} + \mathbf{A}^\mathsf{T}\mathbf{P}\right)\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{P}\mathbf{B}\mathbf{u} \\ &= -\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} - \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\boldsymbol{\phi}(\mathbf{y},t) \\ &= -\tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} - \mathbf{y}^\mathsf{T}\boldsymbol{\phi}(\mathbf{y},t) \\ &\leq -\tfrac{1}{2}\underline{q}\mathbf{x}^\mathsf{T}\mathbf{x} \qquad (\text{Note}: 0 < \underline{q}\mathbf{1} \leq \mathbf{Q} \leq \overline{q}\mathbf{1} < \infty) \\ &< 0. \end{aligned}$$

Thus, the system is globally asymptotically stable.

## VI. Control of a Time-Varying Mechanical System

### A. The Differential Equations and Output Strictly Passive Nature of the Plant

Consider the following simple control problem: regulate the position and velocity of a mass to zero, where the mass decreases with respect to time. This problem is not unlike so called "rocket problems" where a rocket initially has a finite mass, yet the mass decrease as a function of time as fuel is expelled during launch. Let the time-varying mass be described by

$$m(t) = m_f e^{-\alpha t} + m_i$$

where $m_i$ is the mass with "no fuel", $m_f$ is the mass of the "fuel", and $\alpha$ governs the decay rate of the mass. Let the position of the mass be $z(\cdot)$, and the velocity be $\dot{z}(\cdot)$. Using Newton's second law, the system differential equation can be determined:

$$\frac{d}{dt}(m(t)\dot{z}(t)) = u(t) - c\dot{z} \Leftrightarrow m(t)\ddot{z}(t) + (\dot{m}(t)+c)\dot{z}(t) = u(t)$$

where $c$ is the viscous friction coefficient (which is assumed to be very small). First, we prewrap the system with proportional control to be able to regulate to the zero position:

$$m(t)\ddot{z}(t) + (\dot{m}(t)+c)\dot{z}(t) + kz(t) = u(t).$$

The above system can be placed into a first-order form with the velocity as the system output:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \overbrace{\begin{bmatrix} 0 & 1 \\ \frac{-k}{m(t)} & \frac{\dot{m}(t)+c}{m(t)} \end{bmatrix}}^{\mathbf{A}(t)} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{m(t)} \end{bmatrix}}^{\mathbf{B}(t)} u(t) \\ y(t) = \dot{z}(t) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

We wish to stabilize the system via the passivity theorem using an LTV controller. We will now show that the plant is output strictly passive. Consider the system Hamiltonian and its temporal derivative:

$$\begin{aligned} H(t) &= \tfrac{1}{2}m(t)\dot{z}^2(t) + \tfrac{1}{2}kz^2(t), \\ \dot{H}(t) &= m(t)\dot{z}(t)\ddot{z}(t) + \tfrac{1}{2}\dot{m}(t)\dot{z}^2(t) + kz(t)\dot{z}(t) \\ &= u(t)\dot{z}(t) - \tfrac{1}{2}\dot{m}(t)\dot{z}^2(t) - c\dot{z}^2(t). \end{aligned}$$

Assuming quiescent initial conditions, integrating $\dot{H}(\cdot)$ over $t \in [0,T]$ yields:

$$\begin{aligned} \int_0^T \dot{H}(t)dt &= H(T) - H(0) \geq 0, \\ \int_0^T u(t)\dot{z}(t)dt &\geq \int_0^T \left(\tfrac{1}{2}\dot{m}(t)\dot{z}^2(t) + c\dot{z}^2(t)\right)dt \\ \Rightarrow \int_0^T u(t)\dot{z}(t)dt &\geq c\int_0^T \dot{z}^2(t)dt. \end{aligned}$$

Thus, the system is output strictly passive owing to the small amount of damping present.

### B. Controller Design Inspired by Optimal Control

Because the plant is output strictly passive an $\epsilon_1$ exists, which also means the plant has finite gain. Recall the strong version of the passivity theorem states that for the closed-loop system to be $L_2$ stable, $\delta_1 + \epsilon_2 > 0$ and $\delta_2 + \epsilon_1 > 0$. We will specify a priori that the controller will not possess a feedthrough matrix. Thus, we must design a LTV controller such that $\epsilon_2 > 0$, that is the controller is output strictly passive and satisfies Corollary IV.2. Alternatively, we could specify that the controller does possess a feedthrough matrix, and then design a controller such that Theorem III.3 is satisfied, which would ensure via Theorem V.1 the closed-loop system would be globally asymptotically stable. We elect to follow the former approach, designing the controller to be output strictly passive.

Consider the following controller:

$$\begin{aligned} \dot{\mathbf{x}}_c(t) &= \mathbf{A}_c(t)\mathbf{x}_c(t) + \mathbf{B}_c(t)\mathbf{y}(t) \\ \mathbf{u}(t) &= -\mathbf{C}_c(t)\mathbf{x}_c(t) \end{aligned}$$

where $\mathbf{x}_c \in \mathbb{R}^{n_c}$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$ and the time-varying matrices $\mathbf{A}_c(\cdot)$, $\mathbf{B}_c(\cdot)$, and $\mathbf{C}_c(\cdot)$ are appropriately dimensioned real matrices that are continuous and bounded over the time interval of interest. As with the plant, complete controllability and complete observability are assumed. We could arbitrarily assign the state-space matrices associated with the control, and then iteratively change parameters until the controller satisfies (7), and a reasonable system response is achieved. This, however, is not an intelligent approach.

It is well known that given the performance index [21]

$$\mathcal{J} = \mathbf{x}^\mathsf{T}(T)\mathbf{S}\mathbf{x}(T) + \int_0^T \left(\mathbf{x}^\mathsf{T}(t)\mathbf{M}\mathbf{x}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{N}\mathbf{u}(t)\right)dt,$$

where $\mathbf{S} = \mathbf{S}^\mathsf{T} > 0$, $\mathbf{M} \geq 0$, and $\mathbf{N} > 0$, one can derive an optimal state feedback $\mathbf{C}_c(t) = \mathbf{N}^{-1}\mathbf{B}^\mathsf{T}(t)\mathbf{X}(t)$. The matrix $\mathbf{X}(\cdot)$ is positive definite and can be found by solving the matrix Riccati equation

$$\begin{aligned} -\dot{\mathbf{X}}(t) &= \mathbf{M} + \mathbf{A}^\mathsf{T}(t)\mathbf{X}(t) + \mathbf{X}(t)\mathbf{A}(t) \\ &\quad -\mathbf{X}(t)\mathbf{B}(t)\mathbf{N}^{-1}\mathbf{B}^\mathsf{T}(t)\mathbf{X}(t), \quad \mathbf{X}(T) = \mathbf{S} \end{aligned}$$

backward in time from $t = T$ to $t = 0$ s. This is the well known Linear Quadratic Regulator (LQR) solution for time-varying systems.

Given that we have designed $\mathbf{C}_c(\cdot)$ via a LQR formulation, we must now design $\mathbf{A}_c(\cdot)$ and $\mathbf{B}_c(\cdot)$ such that the controller is output strictly passive. To do so, we will let $\mathbf{A}_c(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{C}_c(t)$. It now remains to find $\mathbf{B}_c(\cdot)$. By employing Corollary IV.2 we have $\mathbf{B}_c(t) = \mathbf{P}^{-1}(t)\mathbf{C}_c^{\mathsf{T}}(t)$ where $\mathbf{P}(\cdot)$ is found by solving

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}_c(t) + \mathbf{A}_c^{\mathsf{T}}(t)\mathbf{P}(t) = -\mathbf{Q}(t)$$

backwards in time from $t = T$ to $t = 0$ s given the boundary condition $\mathbf{P}(T)$.

We will solve for $\mathbf{P}(\cdot)$ numerically. To do so, pick $\mathbf{Q}(T)$ and $\mathbf{P}(T)$ and then iteratively solve

$$\begin{aligned}\mathbf{P}(t_{k-1}) &= \mathbf{P}(t_k) \\ &+ (t_k - t_{k-1})\left(\mathbf{P}(t_k)\mathbf{A}_c(t_k) + \mathbf{A}_c^{\mathsf{T}}(t_k)\mathbf{P}(t_k) + \mathbf{Q}(t_k)\right)\end{aligned}$$

backward from $t = T = 15$ s to $t = 0$ s (which, in this case, is our time interval of interest). We will set $\mathbf{Q}(t) = (10e^{-\alpha t} + m_i)\mathbf{1}$. In a similar fashion, $\mathbf{X}(\cdot)$ can be solved backward in time.

We will numerically execute the above control scheme using the parameters presented in Table I. The maximum

TABLE I

SYSTEM PARAMETERS USED IN SIMULATION.

| | |
|---|---|
| $m_i$ | 1.5 kg |
| $m_f$ | 1 kg |
| $\alpha$ | $\frac{1}{2}$ t$^{-1}$ |
| $c$ | $10^{-5}$ Ns/m |
| $k$ | 5 N/m |

and minimum eigenvalues of $\mathbf{X}(\cdot)$ are shown in Fig. 3, as are those of $\mathbf{P}(\cdot)$ in Fig. 4. All eigenvalues are positive as expected (although they are very small at $t = 15$ s, they are positive). The position and velocity of the closed-loop

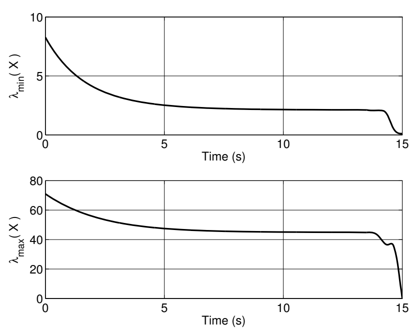


Fig. 3. Maximum and minimum eigenvalues of $\mathbf{X}(\cdot)$ vs. time.

system versus time as released from initial conditions of 1 m and 1 m/s are shown in Figs. 5 and 6.

## C. Memoryless Nonlinear Control

Rather then using a dynamic compensator, we will now employ Theorem V.3 and stabilize the plant with a sector bounded, memoryless nonlinear control of the form

$$u(t) = -\phi(\dot{z}(t), t) = -\phi(y(t), t) = -\tanh(y(t))(1+4e^{-\alpha t}).$$

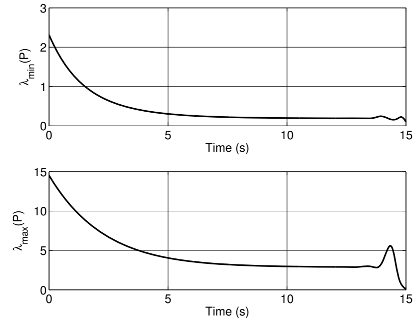


Fig. 4. Maximum and minimum eigenvalues of $\mathbf{P}(\cdot)$ vs. time.

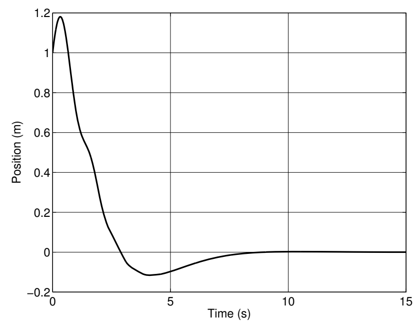


Fig. 5. Position vs. time response using dynamic, OSP time-varying control.

This memoryless nonlinearity is confined to the first and third quadrant of the $(y(t), \phi(y(t), t))$ plane for all positive time, and $\phi(0, t) = 0$, $\forall t \geq 0$ as well. Fig. 7 shows the profile of the nonlinearity at times $t = 0, 2.5, 5, 7.5, 10$ s.

The position and velocity of the system versus time as released from initial conditions of 1 m and 1 m/s are shown in Figs. 8 and 9.

# VII. Closing Remarks

This paper is concerned with identifying passive, input strictly passive, output strictly passive, and state strictly passive LTV systems, and determining stability of various negative feedback interconnections. In particular, Theorem III.2 associated with input strictly passive LTV systems, as well as Theorem III.3 associated with input-state strictly passive LTV systems which include $\mathbf{D}(\cdot)$, and Corollary IV.2

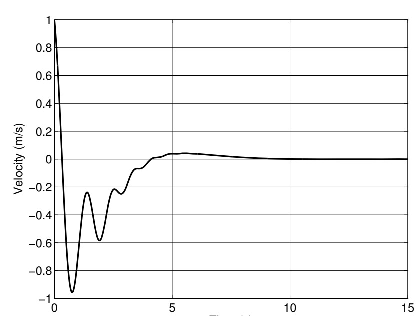


Fig. 6. Velocity vs. time response using dynamic, OSP time-varying control.

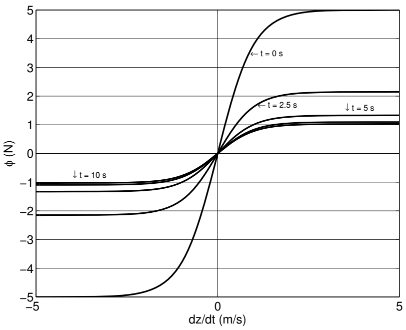


Fig. 7. Profile of memoryless nonlinearity.

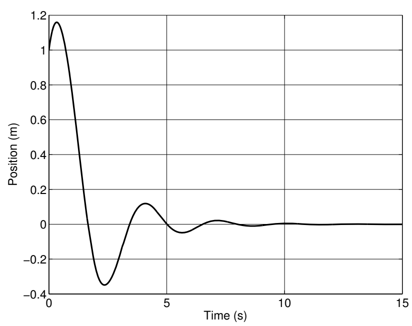


Fig. 8. Position vs. time response using memoryless nonlinear control.

associated with output strictly passive LTV systems which exclude $\mathbf{D}(\cdot)$ are (we believe) unique to this paper. Theorem III.3 and Corollary IV.2 can be thought of as KYP Lemmas for time-varying systems. We show that the negative feedback interconnection of an output strictly passive system and an input-state strictly passive system is globally asymptotically stable in the sense of Lyapunov. Similarly, we show that an input-state strictly passive system and an output strictly passive system negatively interconnected with a sector bounded, memoryless nonlinearity are both globally asymptotically stable. We present a numerical example, the control of a time-varying mass, where the position and velocity of the mass is controlled via a dynamic, output strictly passive LTV controller, and a sector bounded, memoryless nonlinearity.

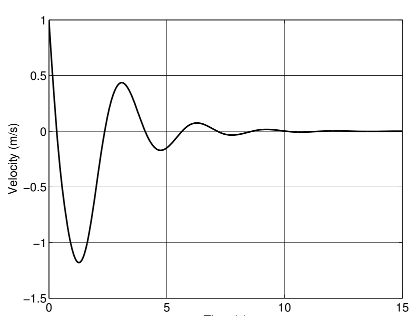


Fig. 9. Velocity vs. time response using memoryless nonlinear control.

## REFERENCES


[1] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez, *Passivity-Based Control of Euler-Lagrange Systems*. London: Springer, 1998.

[2] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1973.

[3] R. Ortega and M. W. Spong, "Adaptive Motion Control of Rigid Robots: A Tutorial," *Automatica*, vol. 25, November 1989.

[4] O. Egeland and J.-M. Godhavn, "Passivity Based Adaptive Attitude Control of a Rigid Spacecraft," *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 842–846, 1994.

[5] B. D. O. Anderson, "A System Theory Criterion for Positive Real Matrices," *SIAM Journal of Control*, vol. 5, no. 2, pp. 171–182, 1967.

[6] B. D. O. Anderson, "Dual Form of a Positive Real Lemma," *Proceedings of the IEEE*, vol. 55, pp. 1749–1750, October 1967.

[7] J. T. Wen, "Time Domain and Frequency Domain Conditions for Strict Positive Realness," *IEEE Transactions on Automatic Control*, vol. 33, pp. 988–992, October 1988.

[8] J. H. Taylor, "Strictly Positive Real Functions and the Lefschetz-Kalman-Yakubovich (LKY) Lemma," *IEEE Transactions on Circuits and Systems*, vol. 21, pp. 310–311, March 1974.

[9] G. Tao and P. Ioannou, "Strictly Positive Real Matrices and the Lefschetz-Kalman-Yakubovich Lemma," *IEEE Transactions on Automatic Control*, vol. 33, pp. 1183–1185, December 1988.

[10] J. C. Geromel and P. B. Gapsik, "Synthisis of Positive Real $\mathcal{H}_2$ Controllers," *IEEE Transactions on Automatic Control*, vol. 42, pp. 988–992, July 1997.

[11] T. Shimomura and S. P. Pullen, "Strictly Positive Real $\mathcal{H}_2$ Controller Synthesis via Iterative Algorithms for Convex Optimization," *Journal of Guidance, Control and Dynamics*, vol. 25, pp. 1003–1011, November-December 2002.

[12] C. J. Damaren, "Optimal Strictly Positive Real Controllers Using Direct Optimization," *Journal of the Franklin Institute*, vol. 343, pp. 271–278, 2006.

[13] J. R. Forbes and C. J. Damaren, "Design of Gain-Scheduled Strictly Positive Real Controllers Using Numerical Optimization for Flexible Robotic Systems," *ASME Journal of Dynamic Systems, Measurement and Control*, Accepted for Publication.

[14] B. D. O. Anderson and J. B. Moore, "Procedures for Time-varying Impedance Synthesis," *Proceedings of the 11th Midwest Symposium on Circuit Theory*, pp. 17–27, May 1968.

[15] B. D. O. Anderson and P. J. Moylan, "Synthesis of Linear Time-Varying Passive Networks," *IEEE Transactions on Circuits and Systems*, vol. CAS-21, pp. 678–687, September 1974.

[16] H. D'Angelo, *Linear Time-Varying Systems: Analysis and Synthesis*. Boston, MA: Allyn and Bacon, Inc., 1970.

[17] L. A. Zadeh and C. A. Desoer, *Linear System Theory: The State Space Approach*. New York, NY: McGraw-Hill Book Company, Inc., 1963.

[18] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York, NY: Academic Press., 1975. Reprinted by Society for Industrial and Applied Mathematics. Philadelphia, PA: 2009.

[19] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, Inc., 2nd ed., 1993. Reprinted by Society for Industrial and Applied Mathematics. Philadelphia, PA; 2002.

[20] H. Khalil, *Nonlinear Systems*. Pearson Prentice Hall, Third Ed., 2002.

[21] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. Toronto: Wiley-Interscience, a Division of John Wiley and Sons, Inc., 1972.