

***CAN WE MAKE A ROBOT BALLERINA  
PERFORM A PIROUETTE?***  
**ORBITAL STABILIZATION OF PERIODIC  
MOTIONS OF UNDERACTUATED  
MECHANICAL SYSTEMS<sup>1</sup>**

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Abstract: This paper provides an introduction to several problems and techniques related to controlling periodic motions of dynamical systems. In particular, we define and discuss problems of motion planning and orbit planning, analysis methods such as the classical Poincaré first-return map and the transverse linearization, and exponentially orbitally stabilizing control designs. We begin with general nonlinear systems, and then specialize to a class of underactuated mechanical systems for which a particularly rich structure allows many of the problems to be solved analytically. The paper concludes with a discussion of numerical issues related to control design via periodic Riccati equations.

Keywords: Transverse Linearization, Orbital Stabilization, Periodic Motions, Underactuated Mechanical Systems, Virtual Holonomic Constraints

## 1. INTRODUCTION

Most people will agree that performing a pirouette is intrinsically challenging: for humans it takes both natural talent and years of training. Looking at it from the perspective of a control-systems scientist doesn't necessarily make it any easier, but does allow us to be more specific about what

the problem is. For one, the motion is periodic, and it is well known that stabilization of periodic motions provides many challenges over and above those found when stabilizing an equilibrium point. A second difficulty is that, standing on tip-toes, the dancer cannot directly maintain their upright position, that is, the system is underactuated: there are less independent control inputs than dynamical degrees of freedom.

When studying the orbital stabilization of periodic motions of underactuated mechanical sys-

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<sup>1</sup> The work has been partly supported by the Swedish Research Council (the grants: 2005-4182, 2006-5243) and the Kempe Foundation.

tems, we use a tool known as the transverse linearization. Roughly speaking, the transverse linearization is a periodic linear system of dimension one less than the nonlinear system such that stabilization of this system is equivalent to exponential orbital stabilization of a desired periodic motion of the original nonlinear system. We consider a large class of mechanical systems that includes many popular research set-ups (the Furuta pendulum, the Pendubot, the Acrobot, a pendulum on a cart, a spherical pendulum) and applications (bipeds, ocean-going vessels). Remarkably, for this class of nonlinear controlled systems the transverse linearization around any feasible orbit can be introduced analytically. This opens up a wide range of opportunities for using linear control theory to design nonlinear control systems.

This approach can be considered as an alternative to the standard Poincaré first-return map, the most frequently used tool for analysis of existence and stability of periodic trajectories. Calculation of the Poincaré map of a nonlinear system typically requires numerical solution of the system dynamics for a large number of initial conditions, which is computationally expensive and motivates investigation of alternative strategies.

The most prominent challenges which remain (and restrict wider application of this technique) are numerical. In the end, one must find a stabilizing controller for a periodic linear system. For example, the theory behind the LQR approach for periodic systems is well established, but requires finding the unique periodic stabilizing solution of a matrix Riccati differential equation with periodic coefficients. In certain cases this is achievable however there is a strong need for more reliable numerical methods.

The structure of the paper is as follows: in Section 2, we describe a number of mathematical tools which are useful for the analysis and stabilization of periodic motions of general nonlinear systems; in Section 3, we specialize to the case of a mechanical systems of underactuation degree one; in Section 4 we provide references to a number of recent applications of the techniques described; finally, we give some brief conclusions in Section 5.

## 2. ANALYSIS OF PERIODIC MOTIONS OF DYNAMICAL SYSTEMS

In this section we provide an introductory overview of a number of techniques that can be used to study periodic motions of dynamical systems, in particular: the Poincaré map, transverse dynamics, and the controlled transverse linearization.

### 2.1 Problem Formulation

Consider a general nonlinear control system, dynamics of which can be described by

$$\dot{x} = f(x, u), \quad (1)$$

where  $x$  is a state vector,  $f(\cdot, \cdot)$  is a continuously differential vector function, and  $u$  is a vector of control inputs.

For this system one can formulate the following task.

**Problem 1.** (Periodic motion planning). *Find two vector functions of time  $u_*(t)$  and  $x_*(t)$ , such that  $x_*(t)$  is a solution of (1) with  $u = u_*(t)$ ,*

$$x_*(t) = x_*(t + T) \quad \forall t \geq 0$$

*for some  $T > 0$ , and satisfies certain pre-defined specifications.*

Sometimes, the specifications include a particular period  $T$  and certain desired ranges for the components of  $x_*(t)$ .

After the desired motion is planned, as assumed e.g. in (Byrnes *et al.*, 1991; Spong, 1997; Bloch *et al.*, 2000; Ortega *et al.*, 2002), the problem of feedback stabilization can be formulated.

However, in some practical situations it is more natural to force the systems trajectories not to track a particular motion but to stay as close as possible to a particular orbit, which may be defined geometrically, in terms of an independent “phase” variable  $\varphi$ , which may or may not be defined with specific reference to time:

$$\mathcal{M} = \{x : \exists \varphi \in \mathbb{S} \text{ s.t. } x = \bar{\Phi}(\varphi)\}, \quad (2)$$

where  $\mathbb{S}$  is a one-dimensional manifold topologically equivalent to a circle and  $\bar{\Phi}(\cdot)$  is continuously differentiable.

In such cases, it makes sense to consider, not stabilization to a particular trajectory as in the motion-planning problem, but stabilization to an orbit:

**Problem 2.** (Orbit planning). *Find a feedback transformation  $g(x, v)$  and a closed orbit  $\mathcal{M}$  in the state space such that all the solutions of*

$$\dot{x} = f(x, g(x, 0)), \quad x(0) \in \mathcal{M} \quad (3)$$

*remain on  $\mathcal{M}$  and are periodic.*

This problem appears to be more challenging than the previous one, since as soon as Problem 2 is solved,

$$u_*(t) = g(x_*(t), 0),$$

where  $x_*(t)$  satisfies (3), provides a solution for Problem 1. It is worth noting, however, that if Problem 1 is solved, the feedback transformation

$$g(x, v) = u_*(x_*(T(x))) + v$$

based on a projecting function

$$\mathcal{T} : \text{a neighborhood of } \mathcal{M} \rightarrow \mathbb{R}/[0, T],$$

which is smooth and satisfies the relation

$$\mathcal{T}(x_*(t)) = t \quad \forall t \in \mathbb{R}/[0, T], \quad (4)$$

provides the solution for Problem 2.

After the desired orbit is planned, as assumed e.g. in (Hauser and Chung, 1994; Banaszuk and Hauser, 1995; Nielsen and Maggiore, 2006), the following task is of interest.

**Problem 3.** (Exponential orbital stabilization).

Find a function  $k(x)$  such that the solutions of (1) with  $u = g(x, k(x))$  initiated at  $t = 0$  in a neighborhood of the desired orbit (2) exponentially approach the orbit, i.e. there exist  $c_1 > 0$  and  $c_2 > 0$  such that

$$d(x(t), \mathcal{M}) \leq c_1 d(x(0), \mathcal{M}) \exp\{-c_2 t\}, \quad (5)$$

where

$$d(x, \mathcal{M}) = \min \{\|x - \bar{\Phi}(\varphi)\| : \varphi \in \mathbb{S}\} \quad (6)$$

is the standard distance based on the Euclidean norm.

We would like now to discuss some standard tools that allow one to verify orbital exponential stability for an autonomous system of the form

$$\dot{x} = \mathcal{F}(x), \quad (7)$$

which in the present context can be defined by

$$\mathcal{F}(x) = f\left(x, g(x, k(x))\right).$$

## 2.2 Poincaré map

One of the classical tools for verifying existence and stability of nontrivial periodic orbits is Poincaré first-return map analysis (Poincaré, 1916-1954).

Let a surface  $S$  be transversal to the flow of a periodic orbit  $\mathcal{M}$  of (7) and consider a sufficiently small region  $S_0 \subset S$  which is open relative to  $S$  and contains the intersection of  $S$  and  $\mathcal{M}$ . The Poincaré map  $P : S_0 \rightarrow S$  is defined by the first hit rule, i.e. it maps the initial points of the solutions of (7) belonging to  $S_0$  into the points where these solutions hit  $S$  again for the first time, see Fig. 1.

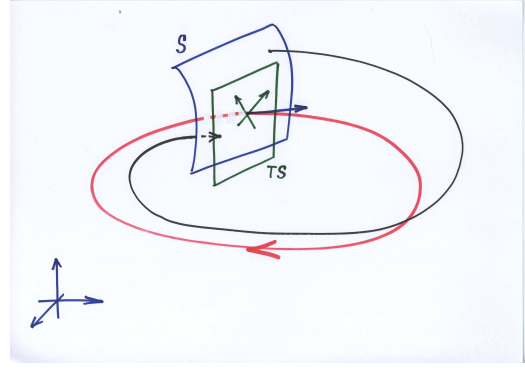


Fig. 1. Poincaré map  $P : S \rightarrow S$  for the periodic trajectory  $x_*(t)$  (red) is defined by the first hit rule.

The map is well-defined because of continuous dependence of solutions on initial conditions (see, e.g. (Khalil, 2002)) and the trajectory  $x^*(t)$  corresponds to the fixed point of  $P(\cdot)$ . If the Poincaré map is contracting, then the orbit is asymptotically stable. Exponential orbital stability can be verified using linearization of the Poincaré map  $dP : TS \rightarrow TS$ , which acts on the tangent space to  $S$  at the point of intersection of  $S$  and  $\mathcal{M}$ .

It is well-known (Andronov and Vitt, 1933; Urabe, 1967; Hale, 1980; Yoshizawa, 1966; Rouche and Mawhin, 1980; Leonov, 2006) that (5) is satisfied if and only if all the eigenvalues of the linear operator  $dP$  are strictly inside the unit circle. Moreover, the rate of contraction toward the orbit over the period, which is estimated by  $\exp\{-c_1 T\}$ , is defined by the absolute value of the eigenvalue of  $dP$  closest to the unit circle.

Control design exploiting Poincaré map analysis is hard and only a few successful applications are known, see e.g. (Grizzle *et al.*, 1999; Grizzle *et al.*, 2001; Chevallereau *et al.*, 2005) for some results in the context of bipedal walking robots, where it has proved possible to reduce analysis to a one-dimensional Poincaré map, making a computational approach feasible.

## 2.3 Transverse dynamics

It is not hard to see that in order to introduce a Poincaré map, it is sufficient to have a surface  $S$  that is transversal to the orbit at a single point. However, sometimes it is useful to introduce moving Poincaré sections (Leonov, 2006), a family  $\{S(t)\}_{t \in [0, T]}$  of surfaces each of which is transversal to the orbit (2) and intersects it at  $x_*(t)$ , see Fig. 2.

Suppose  $S(T) = S(0)$  and the union of all the surfaces in the family covers a neighborhood of

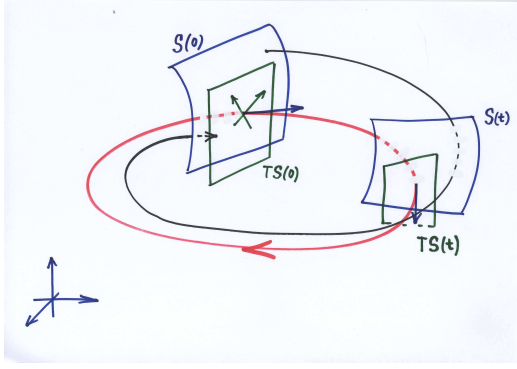


Fig. 2. Moving Poincaré section for the periodic trajectory  $x_*(t)$  (red) of (7).  $\mathcal{F}(x_*(t)) \notin TS(t)$ . In a vicinity of  $\mathcal{M}$ , defined by (2), we have  $\mathcal{M} \cap S(t) = \{x_*(t)\}$ .

the orbit. Then, one can define a new set of coordinates  $x_\perp(t)$  and  $\varphi(t)$  in a vicinity of each point of the orbit, where  $x_\perp(t)$  are also coordinates on  $S(t)$  with the origin in  $x_*(t)$  and  $\varphi(t)$  is a scalar coordinate that defines the corresponding surface and travels along the orbit (Urabe, 1967; Hale, 1980; Hauser and Chung, 1994).

We have  $x(t) = U(t) [\varphi(t), x_\perp(t)]^T$ . Assuming that  $U(t)$  is continuously differentiable together with its inverse, it is easy to rewrite (7) in terms of the new coordinates.

Linearizing the dynamics for  $x_\perp(t)$  around the desired trajectory  $x_*(t) = U(t) [\varphi(t), 0]^T$ , one can define a linear comparison (first approximation) system

$$\dot{z} = A(t)z, \quad (8)$$

where  $A(t) = A(t+T)$  and  $z(t)$  is the vector of the transversal coordinates, which belongs to the tangent space  $TS(t)$ . For another way of describing these dynamics see (Leonov, 2006).

Exponential stability of the zero solution for (8) is equivalent to exponential orbital stability of the solution  $x_*(t)$  of (7) (Leonov, 2006). This system is called transverse linearization, short for linearization of the transverse dynamics. The concept has been used for feedback control of various classes of systems, see e.g. (Nam and Arapostathis, 1992; Samson, 1995; Gillespie *et al.*, 2001; Altafini, 2002; Coelho and Nunes, 2003; Banaszuk and Hauser, 1995; Chung and Hauser, 1997).

It is worth noting that

$$\dim\{z(t)\} = \dim\{x(t)\} - 1.$$

Correspondingly, the comparison system (8) is different from the classical linearization of (7) around the trajectory  $x_*(t)$

$$\dot{\bar{z}} = \bar{A}(t)\bar{z}, \quad \bar{A}(t) = \left( \frac{\partial \mathcal{F}(x)}{\partial x} \right) \Big|_{x=x_*(t)}$$

with  $\dim\{\bar{z}(t)\} = \dim\{x(t)\}$ . Moreover, the later system can not be exponentially stable unless  $x_*(t)$  is an isolated equilibrium of (7). To prove this it is enough to notice that  $\bar{z}(t) = \dot{x}_*(t)$  is a non-vanishing solution (Andronov and Vitt, 1933; Yoshizawa, 1966).

In summary: if one can find appropriate transverse coordinates at all points of the orbit, then proving exponential orbital stability (or instability) is reduced to analysis of a particular time-periodic linear system, the coefficients of which can be calculated analytically.

#### 2.4 Controlled transverse linearization

Above, we have discussed transverse-linearization-based approaches to stability analysis. Here, however, we are interested in making this technique useful for control design.

The local exponential stabilization task (see Problem 3 above) in the case when the desired orbit is just an equilibrium can be approached via analysis of the standard controlled linearization:

$$\dot{\bar{z}} = \bar{A}\bar{z} + \bar{B}\bar{v},$$

where

$$\bar{A} = \frac{\partial f(x, g(x, v))}{\partial x} \Big|_{\substack{x=x_* \\ v=0}}, \quad \bar{B} = \frac{\partial f(x, g(x, v))}{\partial v} \Big|_{\substack{x=x_* \\ v=0}}$$

and  $\dim\{\bar{z}(t)\} = \dim\{x(t)\}$ .

In the case when  $x_*(t)$  is a nontrivial periodic trajectory, such an approach clearly fails. However, defining a good family of Poincaré sections  $\{S(t)\}_{t \in [0, T]}$  as above, and linearizing the transverse dynamics corresponding to  $x_\perp(t)$  for appropriately rewritten  $\dot{x} = f(x, g(x, v))$ , one obtains a controlled transverse linearization (Hauser and Chung, 1994; Nielsen and Maggiore, 2006) in the following form

$$\dot{z} = A(t)z + B(t)\bar{v}, \quad (9)$$

where  $A(t) = A(t+T)$ ,  $B(t) = B(t+T)$ ,  $z(t)$  is the vector of the transversal coordinates, which belongs to the tangent space  $TS(t)$ , and  $\dim\{z(t)\} = \dim\{x(t)\} - 1$ .

The main idea is to design a feedback controller for the linear time-periodic comparison system (9) and transform it into an orbitally exponentially stabilizing time-invariant state feedback controller for the nonlinear system using a projection operator as in (4), see (Banaszuk and Hauser, 1995; Shiriaev *et al.*, n.d.).

### 2.5 LQR design for the controlled transverse linearization

One approach to stabilizing the periodic linear system (9) is based on the solution to the linear quadratic regulation problem (LQR).

**Proposition 1.** (Stabilization via LQR).

[(Yakubovich, 1986; Bittanti *et al.*, 1991)]

Suppose the pair of matrices  $A(t)$  and  $B(t)$  is completely controllable over the period.

Then, there exists a unique solution  $R(t) = R^T(t) = R(t+T) > 0$  of the periodic differential Riccati equation

$$\begin{aligned} \dot{R}(t) + A^T(t)R(t) + R(t)A(t) + G \\ = R(t)B(t)\Gamma^{-1}B^T(t)R(t) \end{aligned} \quad (10)$$

where  $G = G^T \geq 0$  and  $\Gamma > 0$ , such that the equilibrium  $z = 0$  of the linear system (32) with

$$w = K(t)z, \quad K(t) = -\Gamma^{-1}B^T(t)R(t)$$

is exponentially stable.

A solution of (10) can be computed as follows (Yakubovich, 1986):

- Solve the following initial value problem:

$$\dot{Z}(t) = \begin{bmatrix} 0_{N-1} & I_{N-1} \\ -I_{N-1} & 0_{N-1} \end{bmatrix} \begin{bmatrix} -G & A^T(t) \\ A(t) & B(t)\Gamma^{-1}B^T(t) \end{bmatrix} Z(t),$$

where  $N = \dim\{x\}$ , with  $Z(0) = I_{2N-2}$  over the period.

- Let the columns of the  $(2N-2) \times (N-1)$  matrix  $Z_0$  be the basis vectors of the invariant stable subspace in the kernel of  $Z(T)$  and solve the linear matrix ordinary differential equation

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0_{N-1} & I_{N-1} \\ -I_{N-1} & 0_{N-1} \end{bmatrix} \begin{bmatrix} -G & A^T(t) \\ A(t) & B(t)\Gamma^{-1}B^T(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

initiated at  $\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = Z_0$ .

- Compute the stabilizing solution of (10) as

$$R(t) = -X_2(t)X_1^{-1}(t).$$

This algorithm requires twice solving a matrix periodic differential equation with the same number of stable and unstable eigenvalues in its transition matrix. Standard solution methods work in some cases, however there is a strong need for more reliable methods. Some recent developments are reported in (Gusev *et al.*, 2007; Johansson *et al.*, 2007).

## 3. THEORY FOR A CLASS OF UNDERACTUATED MECHANICAL SYSTEMS

In this section, we develop the ideas presented above for the class of underactuated nonlinear mechanical systems. Controlling mechanical systems with a limited number of actuators is a challenging task (Byrnes *et al.*, 1991; Spong, 1997; Bloch *et al.*, 2000; Ortega *et al.*, 2002). When the target behavior is more complicated than a simple equilibrium, e.g. a periodic trajectory, the challenges become greater still (Fradkov and Pogromsky, 1998, Chapter 6), and even establishing existence of periodic motions in a nonlinear system is often difficult (Rouche and Mawhin, 1980; Yoshizawa, 1966).

In particular, we consider orbital stabilization for systems that have one fewer independent control inputs than mechanical degrees of freedom, i.e. systems of *underactuation degree one*. Even with this restricted focus, the problems are sufficiently challenging, and the class of motivating examples is sufficiently rich, to warrant a detailed study. This class includes popular “control challenge” systems such as the inverted pendulum on a cart, the Furuta pendulum, the Pendubot, and the Acrobot, and can also represent the dynamics of practical systems such as humanoid robots, surface vessels and others, see e.g. (Grizzle *et al.*, 2001; Chevallereau *et al.*, 2003; Chevallereau *et al.*, 2004; Chevallereau *et al.*, 2005; Grizzle *et al.*, 2005; Miossec and Aoustin, 2005; Fossen and Strand, 2001; Fossen, 2002; Manchester *et al.*, 2007; Skjetne *et al.*, 2004; Nakaura *et al.*, 2004; Shimizu *et al.*, 2006; Mazenc and Bowong, 2003; Canudas-de-Wit *et al.*, 2002; Aracil *et al.*, 2002).

Consider an  $n$ -degree-of-freedom controlled Euler-Lagrange system (Ortega *et al.*, 1998):

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = B(q)u. \quad (11)$$

Here  $q \in \mathbb{R}^n$  is a vector of generalized coordinates,  $u \in \mathbb{R}^m$  is a vector of independent control inputs, the function

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q) \quad (12)$$

is the Lagrangian of the system (11),  $M(q)$  is a positive definite matrix of inertia,  $V(q)$  is the potential energy of the system, and  $B(q)$  is a full-rank matrix function of appropriate dimensions, which defines applications of generalized controlled forces and is often constant.

We will assume that

$$\dim\{u\} = m < n = \dim\{q\},$$

so that our mechanical system is underactuated.

The system (11), (12) can be rewritten (Ortega *et al.*, 1998; Spong *et al.*, 2006) as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u, \quad (13)$$

where  $G(q) = \left[ \frac{\partial V(q)}{\partial q_1}, \dots, \frac{\partial V(q)}{\partial q_n} \right]$  and  $C(q, \dot{q})$  is a matrix of Coriolis and generalized centrifugal forces.

For the class of systems described by (13) it is possible to suggest approaches for solving the problems introduced above. We start with the orbit planning task.

### 3.1 Planning periodic orbits

Before suggesting a procedure for planning an orbit, it is useful to make some general observations about the mechanical systems for which such a problem is already solved.

So, suppose someone managed to find a nontrivial periodic trajectory  $q_*(t) = q_*(t+T)$  and a control transformation  $g(q, \dot{q}, v)$  such that the orbit (2) with  $x_*(t) = [q_*^T(t), \dot{q}_*^T(t)]^T$  is invariant under (13) with  $u = g(q, \dot{q}, 0)$ .

It is not hard to see that the planned trajectory, being a function of time, can be reparametrized by another independent variable  $\theta$  instead of time. In many cases (when the motion is symmetric in certain sense) the new independent variable can be one of the generalized coordinates. In all situations it can be the arc-length along the path in state space. As a result, one obtains the following description of the desired trajectory<sup>2</sup>

$$q_*(t) = \Phi(\theta_*(t)), \quad \theta_*(t) \in \mathbb{R} \quad \text{for } t \in [0, T] \quad (14)$$

in terms of the desired behavior of the new variable  $\theta$ , and the following description of the desired orbit (2)

$$\mathcal{M} = \left\{ (q, \dot{q}) = \left( \Phi(\theta_*(t)), \Phi'(\theta_*(t))\dot{\theta}_*(t) \right) \text{ for } t \in [0, T] \right\}, \quad (15)$$

where  $\Phi'(\cdot)$  denotes the vector of component-wise derivatives of  $\Phi(\cdot)$ .

<sup>2</sup> It should be noted that there is a subtle, but important, difference between the trajectory specification  $q_*(t) = \Phi(\theta_*(t))$  and the specification  $x = \Phi(\varphi)$  in (2): the former is a specification of the generalized coordinates, but not their velocities, whereas the latter is a specification of the entire state, which for a mechanical system includes the generalized velocities.

Imposing the virtual holonomic constraints (Grizzle *et al.*, 2001; Shiriaev *et al.*, 2005)

$$q = \Phi(\theta) = [\phi_1(\theta), \dots, \phi_n(\theta)]^T$$

on the dynamics (13), one obtains a two-dimensional reduced system in the following particular form (Perram *et al.*, 2003; Shiriaev *et al.*, 2005; Shiriaev *et al.*, 2006b)

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = 0, \quad (16)$$

where (Shiriaev *et al.*, 2005)

$$\begin{aligned} \alpha(\theta) &= B^\perp(q) M(\Phi(\theta)) \Phi'(\theta), \\ \beta(\theta) &= B^\perp(q) \left[ C(\Phi(\theta), \Phi'(\theta)) \Phi'(\theta) \right. \\ &\quad \left. + M(\Phi(\theta)) \Phi''(\theta) \right], \\ \gamma(\theta) &= B^\perp(q) G(\Phi(\theta)), \end{aligned}$$

and  $B^\perp(q)$  is a full rank matrix such that  $B^\perp(q)B(q) = 0$ .

The system (16) admits  $\theta_*(t)$  as one of the solutions and possesses the following conserved quantity (Perram *et al.*, 2003; Shiriaev *et al.*, 2006b)

$$\begin{aligned} I(\theta, \dot{\theta}, \theta(0), \dot{\theta}(0)) &= \dot{\theta}^2 - \psi(\theta_*(0), \theta)\dot{\theta}_*^2(0) \\ &\quad + \int_{\theta_*(0)}^{\theta} \psi(s, \theta) \frac{2\gamma(s)}{\alpha(s)} ds \end{aligned} \quad (17)$$

with

$$\psi(\theta_1, \theta_2) = \exp \left\{ - \int_{\theta_1}^{\theta_2} \frac{2\beta(\tau)}{\alpha(\tau)} d\tau \right\}, \quad (18)$$

which is zero along every solution initiated at  $(\theta(0), \dot{\theta}(0))$  as long as it exists. In particular,

$$I(\theta_*(t), \dot{\theta}_*(t), \theta_*(0), \dot{\theta}_*(0)) \equiv 0.$$

Existence of a conserved quantity for the reduced dynamics, called virtual limit systems, and the fact that every periodic motion can be defined by a vector-function  $\Phi(\theta)$ , describing synchronizations among the generalized coordinates, and a periodic solution of the system (16) inspires the following procedure for planning a periodic orbit:

**Step 1** Define a vector function  $\Phi(p, \theta)$  parametrized by a vector of parameters  $p$ .

**Step 2** Compute the corresponding family of virtual limit systems

$$\alpha(p, \theta)\ddot{\theta} + \beta(p, \theta)\dot{\theta}^2 + \gamma(p, \theta) = 0. \quad (19)$$

**Step 3** Find an appropriate value of  $p$  such that there exists a periodic solution  $\theta = \theta_*(t)$  for (19) implying the desired characteristics of the periodic trajectory  $q = \Phi(\theta_*(t))$  for (13).

**Step 4** Compute the needed feedback transformation  $u = g(q, \dot{q}, v)$ .

The following result is useful for **Step 3** in the case when the desired orbit is small.

**Theorem 1.** (Existence of a center). [(Shiriaev *et al.*, 2006b)]

Let  $\theta_0$  be an equilibrium of the system (16), i.e.  $\gamma(\theta_0) = 0$ .

Suppose that:

- (1) There is a vicinity  $\mathcal{V}$  of  $\theta_0$  such that the scalar functions  $\alpha(\cdot)$ ,  $\beta(\cdot)$  and  $\gamma(\cdot)$  are continuous on  $\mathcal{V}$ , i.e.  $\alpha(\theta)$ ,  $\beta(\theta)$ ,  $\gamma(\theta) \in C^0(\mathcal{V})$ ;
- (2) The function  $\gamma(\theta)/\alpha(\theta)$  is continuously differentiable at  $\theta = \theta_0$ .
- (3) For any  $\theta_i \in \mathcal{V}$ , there exists  $\delta > 0$  such that for any  $\dot{\theta}_i$  with  $|\dot{\theta}_i| < \delta$ , the solution of the nonlinear system (16) initiated at  $(\theta(0), \dot{\theta}(0)) = (\theta_i, \dot{\theta}_i)$  exists for all  $t \geq 0$  and is unique.

If the linear system

$$\frac{d^2}{dt^2}z + \left[ \frac{d}{d\theta} \frac{\gamma(\theta)}{\alpha(\theta)} \right] \Big|_{\theta=\theta_0} \cdot z = 0.$$

has a center at  $z = 0$ , then the nonlinear system (16) has a center at the equilibrium  $\theta_0$ .  $\square$

In order to succeed in **Step 4** in the special case when the degree of underactuation is one, i.e. when

$$\dim\{u\} = m = n - 1 = \dim\{q\} - 1, \quad (20)$$

we can proceed as follows (Shiriaev *et al.*, 2005; Shiriaev *et al.*, n.d.).

Let us introduce the following new coordinates for (11):

$$y_1 = q_1 - \phi_1(\theta), \quad \dots, \quad y_n = q_n - \phi_n(\theta). \quad (21)$$

In an open subset of  $\mathbb{R}^n$  one can consider the  $n+1$  scalar quantities  $y_1, y_2, \dots, y_n$ , and  $\theta$  as excessive coordinates for the controlled  $n$ -degree-of-freedom Euler-Lagrange system (11). Therefore, one coordinate can be expressed as a function of the other coordinates.

Without loss of generality, let us assume that this is the case for  $y_n$ , so the new independent coordinates are

$$y = (y_1, \dots, y_{n-1})^T \in \mathbb{R}^{n-1} \quad \text{and} \quad \theta \in \mathbb{R}. \quad (22)$$

and the last equality in (21) can be rewritten as

$$q_n = \phi_n(\theta) + h(y, \theta),$$

where  $h(\cdot)$  is a scalar smooth function, so that

$$q = \Phi(\theta) + [y^T, h(y, \theta)]^T. \quad (23)$$

The following statement is useful in the case of underactuation degree one, i.e. (20).

**Proposition 2.** (Feedback transformation).

[(Shiriaev *et al.*, 2005)]

Let

$$L(q) = \begin{bmatrix} I_{n-1}, & 0_{(n-1) \times 1} \\ \text{grad } h(q) \end{bmatrix} + [0_{n \times (n-1)}, \Phi'(\theta)],$$

where

$$\text{grad } h(q) = \left[ \frac{\partial h(\cdot)}{\partial y_1}, \dots, \frac{\partial h(\cdot)}{\partial y_{n-1}}, \frac{\partial h(\cdot)}{\partial \theta} \right]$$

with  $y$  and  $\theta$  substituted in terms of  $q$  using the inverse transformation to (23).

Suppose the  $n \times n$  matrix function  $L(q)$  and the  $(n-1) \times (n-1)$  matrix function

$$N(q) = [I_{n-1}, 0_{(n-1) \times 1}]L(q)^{-1} [M(q)^{-1}B(q)]$$

are both non-singular in a vicinity of the orbit  $\mathcal{M}$ , defined by (15).

Then, there exists the feedback transformation in the form

$$g(q, \dot{q}, v) = N(q)^{-1} [v - R(q, \dot{q})], \quad (24)$$

well-defined in this vicinity, which makes the orbit  $\mathcal{M}$  of (13) with  $u = g(q, \dot{q}, 0)$  invariant.  $\square$

Slightly relaxed version of this result can be found in (Shiriaev *et al.*, n.d.).

Assuming the the problem of planning a nontrivial periodic orbit is solved either using the four-step procedure above or differently, but with recovering the virtual constraint function  $\Phi(\cdot)$  from (14), one can compute a transverse linearization to be used for stabilization.

### 3.2 Analytical construction of a transverse linearization in the case of underactuation degree one

It can be shown that the system (13) under feedback transformation (24), based on the idea of partial feedback linearization (Spong, 2004), can be rewritten in the new coordinates (22) as follows

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = g_y(\cdot)y + g_{\dot{y}}(\cdot)\dot{y} + g_v(\cdot)v \quad (25)$$

$$\ddot{y} = v, \quad (26)$$

where the left hand side of (25) matches the structure of the *virtual limit system* (16) and  $g_y(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})$ ,  $g_{\dot{y}}(\theta, \dot{\theta}, \ddot{\theta}, y, \dot{y})$ ,  $g_v(\theta, \dot{\theta}, y, \dot{y})$  are smooth functions of appropriate dimensions.

It appears that one of the transverse coordinates  $x_{\perp}$  can be taken as  $I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0))$ , defined by (17) and (18). To see this, the following two properties of this function are essential.

**Property 1.** (independence on initial point). [(Shiriaev et al., n.d.)]

For any  $x_1$  and  $x_2$  the function (17) satisfies the identity (see Fig. 3)

$$I(x_1, x_2, \theta_*(0), \dot{\theta}_*(0)) \equiv I(x_1, x_2, \theta_*(\rho_0), \dot{\theta}_*(\rho_0)) \quad (27)$$

for all  $\rho_0 \in [0, T]$ .  $\square$

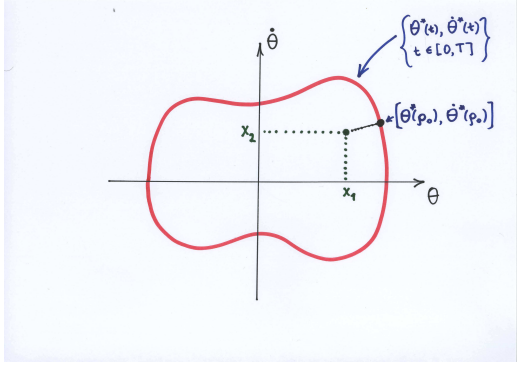


Fig. 3. An illustration for Properties 1 and 2 of (17).

**Property 2.** ( $I \sim$  distance to the orbit).

[(Shiriaev et al., n.d.)]

In a vicinity of the orbit

$$I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0)) = 4 \left[ \dot{\theta}_*(\rho_0)^2 + \ddot{\theta}_*(\rho_0)^2 \right] \times D(\theta, \dot{\theta}) + O(|\theta - \theta_*(\rho_0)|^3) + O(|\dot{\theta} - \dot{\theta}_*(\rho_0)|^3), \quad (28)$$

where

$$D(\theta, \dot{\theta}) = \min_{0 \leq \rho < T} \left\{ |\theta - \theta_*(\rho)|^2 + |\dot{\theta} - \dot{\theta}_*(\rho)|^2 \right\}$$

and

$$\rho_0 = \arg \min_{0 \leq \rho < T} \left\{ |\theta - \theta_*(\rho)|^2 + |\dot{\theta} - \dot{\theta}_*(\rho)|^2 \right\}. \quad \square$$

Now, it is clear that the new coordinates (22) together with their derivatives define the distance to the two-dimensional manifold of the constrained dynamics (16), which contains the desired trajectory  $\theta_*(t)$ , as illustrated on Fig. 4.

Since  $I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0))$  defines the distance to the desired orbit along this manifold, we can make the following choice for the transverse coordinates

$$x_{\perp} = \left[ I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0)), y_1, \dots, y_{n-1}, \dot{y}_1, \dots, \dot{y}_{n-1} \right]^T. \quad (29)$$

In order to compute linearization of dynamics for  $x_{\perp}$  analytically, we need the following property of  $I(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0))$ .

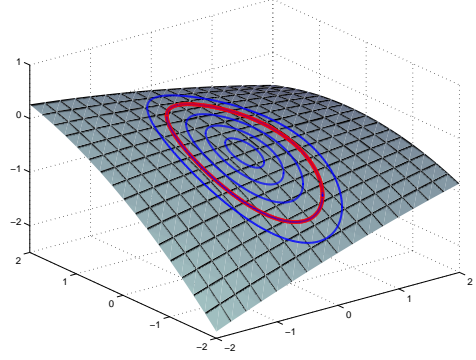


Fig. 4. The two dimensional manifold dynamics on which is defined by (16) and the desired orbit (15) (red).

**Property 3.** ( $I(\cdot)$  away from the cycle).

[(Shiriaev et al., 2005)]

With  $\theta_0$  and  $\dot{\theta}_0$  being some constants, the time derivative of the function  $I(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0)$  defined by (17), calculated along a solution  $[\theta(t), \dot{\theta}(t)]$  of the system

$$\alpha(\theta)\ddot{\theta} + \beta(\theta)\dot{\theta}^2 + \gamma(\theta) = W, \quad (30)$$

can be computed as

$$\frac{d}{dt} I = \dot{\theta} \left\{ \frac{2}{\alpha(\theta)} W - \frac{2\beta(\theta)}{\alpha(\theta)} I \right\}. \quad (31)$$

$\square$

Now, exploiting the structure of (25) and with the help of Property 3, we can compute linearization of dynamics for the transverse coordinates  $x_{\perp}$  in the form (9)

$$\dot{z} = A(t)z + B(t)w, \quad (32)$$

where

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ 0_{(n-1) \times 1} & 0_{(n-1) \times (n-1)} & I_{(n-1) \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times (n-1)} & 0_{(n-1) \times (n-1)} \end{bmatrix},$$

$$B(t) = \begin{bmatrix} b_1(t) \\ 0_{(n-1) \times (n-1)} \\ I_{(n-1) \times (n-1)} \end{bmatrix} \quad (33)$$

with



$$\begin{aligned}
b_1(t) &= \dot{\theta}_*(t) \frac{2g_v(\theta_*(t), \dot{\theta}_*(t), 0, 0)}{\alpha(\theta_*(t))}, \\
a_{11}(t) &= -\dot{\theta}_*(t) \frac{2\beta(\theta_*(t))}{\alpha(\theta_*(t))}, \\
a_{12}(t) &= \dot{\theta}_*(t) \frac{2g_y(\theta_*(t), \dot{\theta}_*(t), \ddot{\theta}_*(t), 0, 0)}{\alpha(\theta_*(t))}, \\
a_{13}(t) &= \dot{\theta}_*(t) \frac{2g_{\dot{y}}(\theta_*(t), \dot{\theta}_*(t), \ddot{\theta}_*(t), 0, 0)}{\alpha(\theta_*(t))}.
\end{aligned}$$

### 3.3 Exponential orbital stabilization in the case of underactuation degree one

The following theorem provides a method for constructing an orbitally exponentially stabilizing feedback controller based on a controller for the linear comparison system.

**Theorem 2.** *The following statements are equivalent.*

Statement 1: *There exists a periodic matrix gain  $K(t) = K(t + T)$  such that the feedback controller*

$$w = K(t)z \quad (34)$$

*exponentially stabilizes the equilibrium  $z = 0$  of the linear control system (32)–(33).*

Statement 2: *There exists a controller of the form*

$$v = f(\theta, \dot{\theta}, y, \dot{y}) \quad (35)$$

*that makes the target orbit (15) exponentially stable in the transformed system (25)–(26).*

Furthermore, the feedback controllers can be constructed as follows:

- Given (34), a possible choice for (35) is

$$u = K \left( T(\theta, \dot{\theta}) \right) x_{\perp}, \quad (36)$$

where  $x_{\perp}$  is given by (29) with  $I(\cdot)$  defined by (17)–(18).

- Given (35), a possible choice for (34) is

$$K(t) = \left[ \begin{array}{c} \frac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial y} \\ \frac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial \dot{y}} \\ \frac{\frac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial \theta} \dot{\theta}_*(t) - \frac{\partial f(\theta, \dot{\theta}, y, \dot{y})}{\partial \dot{\theta}} \ddot{\theta}_*(t)}{2 \left( \dot{\theta}_*(t) \right)^2 + 2 \left( \ddot{\theta}_*(t) \right)^2} \end{array} \right] \Bigg|_{\substack{y=\dot{y}=0 \\ \theta=\dot{\theta}_*(t) \\ \dot{\theta}=\dot{\theta}_*(t)}} \quad (37)$$

where  $I(\cdot)$  is given by (17)–(18).

□

This result is a particular case of a more general theorem, proved in (Shiriaev *et al.*, n.d.), where the assumption of Proposition 2, which are needed to conclude exponential orbital stability of the desired motion of the original system (13) (not only the transformed one) are relaxed.

To solve the problem of orbital exponential stabilizations it is enough to design a controller for the linear time-periodic system (32)–(33). The latter can be done as shown in Section 2.5.

## 4. APPLICATION EXAMPLES

The methods described in the previous section for motion planning, transverse linearization, and exponential orbital stability, have been applied by the authors to a number practical and “control challenge” problems.

The Furuta pendulum, an unactuated pendulum mounted on the end of an actuated horizontal-plane rotary arm, has been used as a testbed for the transverse linearization method (Shiriaev *et al.*, 2007) and other approaches to orbital stabilization of periodic motions (Aguilar *et al.*, 2006; Freidovich *et al.*, 2007d).

The transverse linearization technique has also been applied to the inertial wheel pendulum (Freidovich *et al.*, 2007a), the Pendubot (Freidovich *et al.*, 2008), and the Devil stick (Shiriaev *et al.*, 2006a). In (Shiriaev *et al.*, 2007; Freidovich *et al.*, 2007b) and (Freidovich *et al.*, 2008) experimental results are described and the implementation issues are discussed.

The techniques for orbit planning have been applied to a model of an underactuated ship in order to analyze the feasibility of certain motion-planning tasks (Manchester *et al.*, 2007).

In (Freidovich *et al.*, 2007c) a method has been proposed to smoothly transition between different feasible periodic trajectories.

## 5. CONCLUSION

Stabilization of periodic motions is a challenging task, considerably more difficult than stabilization of an equilibrium point. In this paper, we have given an overview of some classical and some more recent mathematical tools, which can be brought to bear on the problem.

The notion of a transverse linearization has been defined, and developed in detail, for the class of mechanical systems with the number of actuators

one fewer than the number of degrees of freedom. In this class, the transverse coordinates can be explicitly calculated. This means that one can orbitally exponentially stabilize any feasible periodic orbit if one can stabilize a linear periodic system. The latter can be done if one solves a matrix periodic differential Riccati equation. Methods for obtaining reliable numerical solutions of such equations are currently lacking, and an present an important opportunity for future development.

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