Continuous-Time Controllers for Stabilizing Periodic Orbits of Hybrid Systems: Application to an Underactuated 3D Bipedal Robot

Kaveh Akbari Hamed, Brian G. Buss, and Jessy W. Grizzle

Abstract— **This paper presents a systematic approach to exponentially stabilize periodic orbits in nonlinear systems with impulse effects, a special class of hybrid systems. Stabilization is achieved with a time invariant continuous-time controller. The presented method assumes a parametrized family of continuous-time controllers has been designed so that (1) a periodic orbit is induced, and (2) the orbit itself is invariant under the choice of parameters in the controllers. By investigating the properties of the Poincaré return map, a sensitivity analysis is presented that translates the stabilization problem into a set of Bilinear Matrix Inequalities (BMIs). A BMI optimization problem is set up to select the parameters of the continuous-time controller to achieve exponential stability. We illustrate the power of the approach by finding new stabilizing solutions for periodic orbits of an underactuated 3D bipedal robot.**

I. Introduction

This paper addresses the problem of designing continuous-time feedback controllers to exponentially stabilize periodic orbits for hybrid systems [1]. Our motivation comes from the desire to exponentially stabilize periodic walking gaits in bipedal robots, but the results we present apply to hybrid as well as non-hybrid systems [2]-[8]. Hybrid systems model many important processes, including power systems [6] and mechanical systems with impacts [9]-[25]. The primary tool for analyzing the stability of periodic orbits for hybrid systems is the method of Poincaré sections, in which the flow of the system is replaced by the Poincaré return map, which is a discrete-time system evolving on the Poincaré section [8], [26]-[29].

Stabilization of periodic orbits in hybrid systems is often achieved with multi-layered feedback control architectures in which an event-based controller updates parameters of a lower-level continuous-time controller. In several concrete applications, a continuous-time controller has been designed that creates a periodic orbit, but does not manage to render it exponentially stable [21], [35]. In these cases, a set of adjustable parameters has been introduced into the continuous-time controller, which are then updated when the state of the hybrid system intersects a Poincaré section [30], [9, Chap. 4], [15]. This event-based control action is often designed with the objective of rendering the Jacobian of the Poincaré return map around the fixed point a Hurwitz matrix. This approach has been successfully used in [15], [23], [22], [31] to design event-based stabilizing controllers for bipedal robots. One drawback of achieving stability via event-based actions is the potentially large delay between the occurrence of a perturbation and the compensating effect of the event-based controller. Diehl *et al.* [1] presented a method for stabilizing periodic orbits of hybrid systems by solving a nonlinear optimization problem to minimize a smoothed version of the spectral radius of the monodromy matrix.

The contribution of this paper is to present a method based on bilinear matrix inequalities (BMIs) to design continuous-time controllers that provide exponential stability without relying on event-based controllers. A family of parametrized continuous-time controllers is presented under which the periodic orbits are invariant. Unlike [30], [9], [31], the parameters are constant, not updated by event-based controllers. Thus the goal of the optimization is to choose *a priori* a fixed set of parameter values which renders the periodic orbit stable under the corresponding closed loop dynamics. The properties of the Poincaré return map are studied under the invariance condition and a sensitivity analysis is presented. On the basis of the sensitivity analysis, the problem of stabilization of the periodic orbits is then translated into a set of bilinear matrix inequalities. A BMI optimization problem is set up to tune the constant parameters of the continuous-time controllers to stabilize the periodic orbits. Finally, this approach is illustrated to design stabilizing continuous-time controllers for underactuated 3D bipedal robots.

The remainder of this paper is organized as follows. In Section II we provide formal definitions related to hybrid systems and the Poincaré return map. Required conditions on the periodic orbit and the feedback law are presented to set up the sensitivity analysis. Two families of feedback laws satisfying the conditions are presented. Section III describes the formulation of an optimization problem with BMI constraints to guarantee the stability of the linearized Poincaré map based on the sensitivity analysis. In Section IV we illustrate the method to design continuous-time controllers to stabilize a periodic orbit of an underactuated 3D bipedal robot. Conclusions are summarized in Section V.

II. Stabilization Problem of Periodic Orbits for Hybrid Systems

This section addresses the stabilization problem of periodic orbits for hybrid systems by employing a class of *parametrized* and *continuous-time controllers*. Next, the properties of the Poincaré return map are studied to present a *sensitivity analysis* to exponentially stabilize the periodic

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K. Akbari Hamed, B. G. Buss, and J. W. Grizzle are with the Department of Electrical Engineering and Computer Science, University Michigan, Ann Arbor, MI, USA, {kavehah,bgbuss,grizzle}@umich.edu

orbits by tuning the constant parameters of the continuous-time controllers. To simplify the analysis, we study a hybrid model consisting of one continuous phase and one discrete transition. In particular, the hybrid model is expressed as

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)\,u, & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-), & x^- \in \mathcal{S}, \end{cases} \tag{1}$$

in which $x \in \mathcal{X}$ and $\mathcal{X} \subset \mathbb{R}^n$ are the *state vector* and *state manifold*, respectively. Furthermore, $u \in \mathcal{U}$ is the *continuous-time control input*, where $\mathcal{U} \subset \mathbb{R}^m$ represents the *set of admissible control inputs*. $f : \mathcal{X} \to \mathrm{T}\mathcal{X}$ and columns of g are smooth (i.e., $\mathcal{C}^\infty$) vector fields, in which $\mathrm{T}\mathcal{X}$ denotes the *tangent bundle* of the state manifold $\mathcal{X}$. The *switching manifold* is represented by $\mathcal{S}$ on which, the state solutions undergo a sudden jump according to the switching map as $x^+ = \Delta(x^-)$. Here, $x^-(t)$ and $x^+(t)$ denote the left and right limits of the state trajectory at the time instant t, respectively. Throughout this paper, we shall assume that the switching manifold can be expressed as

$$\mathcal{S} := \{x \in \mathcal{X} \,|\, s(x) = 0\}, \tag{2}$$

where $s : \mathcal{X} \to \mathbb{R}$ is the $\mathcal{C}^\infty$ *switching function*. Moreover, $\Delta : \mathcal{S} \to \mathcal{X}$ represents the $\mathcal{C}^\infty$ *switching map*.

A. Family of parametrized controllers and closed-loop hybrid model

In this subsection, we present a family of parametrized and continuous-time controllers under which there is an *invariant* periodic orbit for the closed-loop hybrid model. The parametrized feedback laws of this paper are different from those presented in [30], [9], [31]. In particular, the event-based update law preserves the orbit for the closed-loop system of [30], [9], [31], while in this paper, the periodic orbit is invariant under the family of parametrized feedback laws. In particular, in Subsection II-D, it will be shown that according to the invariance property, the event-based control action cannot be employed to stabilize the orbit. To achieve these goals, the continuous-time control input u is taken as the $\mathcal{C}^\infty$ and parametrized feedback law $\Gamma(x,\xi) \in \mathcal{U}$, in which $\xi \in \Xi$ represents the p-dimensional *constant parameter vector* and $\Xi \subset \mathbb{R}^p$ is the *set of admissible parameters*. By employing the feedback law $\Gamma(x,\xi)$, the closed-loop hybrid model would be parametrized and can be given by

$$\Sigma_\xi^{\mathrm{cl}} : \begin{cases} \dot{x} = f^{\mathrm{cl}}(x,\xi), & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-), & x^- \in \mathcal{S}, \end{cases} \tag{3}$$

in which the superscript "cl" stands for the closed-loop dynamics and $f^{\mathrm{cl}}(x,\xi) := f(x) + g(x)\,\Gamma(x,\xi)$. For a given parameter vector $\xi \in \Xi$, the unique solution of the differential equation $\dot{x} = f^{\mathrm{cl}}(x,\xi)$ with the initial condition $x(0) = x_0$ over the maximal interval of existence is then denoted by $\varphi(t,x_0,\xi)$, $t \geq 0$. Furthermore, the *time-to-switching function* $T : \mathcal{X} \times \Xi \to \mathbb{R}_{\geq 0}$ is defined as the first time at which the flow $\varphi(t,x_0,\xi)$ intersects the switching manifold $\mathcal{S}$, i.e.,

$$T(x_0,\xi) := \inf\{t \geq 0 \,|\, s(\varphi(t,x_0,\xi)) = 0\}. \tag{4}$$

B. Invariant periodic orbit

This subsection presents the fundamental assumptions of the parametrized and continuous-time feedback laws employed in this paper. In particular, we shall assume that the following assumptions are satisfied.

Assumption 1 (Invariant Periodic Orbit): There exists an initial condition $x_i^* \in \mathcal{X} \setminus \mathcal{S}$ such that for all parameter vectors $\xi \in \Xi$, the orbit

$$\mathcal{O} := \{x = \varphi(t,x_i^*,\xi) \,|\, 0 \leq t < T^*\} \tag{5}$$

is a *periodic* orbit of the closed-loop hybrid system (3), in which $T^* := T(x_i^*,\xi) > 0$ is the *bounded* and *minimal period* of the orbit.

Assumption 1 states the existence of an *invariant* periodic orbit $\mathcal{O}$ for the family of parametrized and closed-loop hybrid models in (3). In particular, the solution of the differential equation $\dot{x} = f^{\mathrm{cl}}(x,\xi)$ with the initial condition $x(0) = x_i^*$ does not depend on the parameter vector ξ and for simplicity, we denote it by $\varphi^*(t) := \varphi(t,x_i^*,\xi)$. In Section II-C, two examples of feedback laws satisfying Assumption 1 will be presented.

Assumption 2 (Transversality): The periodic orbit $\mathcal{O}$ in (5) is *transversal* to the switching manifold $\mathcal{S}$. In particular, (i) $\overline{\mathcal{O}} \cap \mathcal{S}$ is a singleton, i.e.,

$$\overline{\mathcal{O}} \cap \mathcal{S} = \{x_f^*\}, \tag{6}$$

where $\overline{\mathcal{O}}$ represents the set closure of $\mathcal{O}$, and (ii),

$$\frac{\partial s}{\partial x}(x_f^*)\, f^{\mathrm{cl}}(x_f^*,\xi) \neq 0. \tag{7}$$

Assumption 2 implies that the orbit $\mathcal{O}$ is not tangent to the switching manifold $\mathcal{S}$ at the point x_f^*. In addition, from the periodicity condition, it can be concluded that $x_i^* = \Delta(x_f^*)$.

C. Examples of continuous-time feedback controllers satisfying the invariance assumption

In this subsection we present two examples of continuous-time feedback laws satisfying the invariance condition in Assumption 1. For this goal, we first present the following assumption.

Assumption 3 (Phasing Variable): There exists a $\mathcal{C}^\infty$ scalar quantity $\theta(x)$ as a function of the state vector x, referred to as the *phasing variable*, which is strictly monotonic (strictly increasing or decreasing) on the periodic orbit $\mathcal{O}$, i.e.,

$$\dot{\theta}(x) = \frac{\partial \theta}{\partial x}(x)\, f^{\mathrm{cl}}(x,\xi) \neq 0, \quad \forall x \in \overline{\mathcal{O}}. \tag{8}$$

Under Assumption 3, the desired evolution of the state variables on the orbit $\mathcal{O}$ can be expressed in terms of the phasing variable θ rather than the time. In particular, let $\Theta(t)$ represent the time evolution of the phasing variable θ on the orbit $\mathcal{O}$. Then, one can define the *desired evolution* of the state vector on the orbit $\mathcal{O}$ in terms of θ as follows

$$x_{\mathrm{d}}(\theta) := \varphi^*(t)\Big|_{t=\Theta^{-1}(\theta)}, \tag{9}$$

in which $t = \Theta^{-1}(\theta)$ denotes the inverse of the function $\theta = \Theta(t)$. Now we are in a position to present two families of parametrized controllers which satisfy Assumption 1.

Example 1 (Feedforward and Linear State Feedback): The first family of continuous-time controllers can be expressed as

$$\Gamma(x, \xi) := \Gamma^*(x) - K\,(x - x_{\mathrm{d}}(\theta)), \quad (10)$$

in which $\Gamma^*(x)$ is a *feedforward term* corresponding to the orbit $\mathcal{O}$. The parameter vector ξ is taken as the columns of the gain matrix $K \in \mathbb{R}^{m\times n}$, i.e., $\xi := \mathrm{vec}(K)$, where vec(.) represents the vectorization operator. It can be concluded that for all ξ, $\mathcal{O}$ is a periodic orbit of the closed-loop hybrid model and hence, Assumption 1 is satisfied. More generally, if the gain matrix K is a function of x parametrized by a finite-dimensional vector ξ, the assumption is still satisfied.

Example 2 (Input-Output Linearizing Controller): For the second family of controllers, let us define an output vector with dimension equal to the dimension of the input vector, i.e., $\dim y = \dim u = m$. In particular, we define the output function

$$y(x) := H\,(x - x_{\mathrm{d}}(\theta)), \quad (11)$$

where $H \in \mathbb{R}^{m\times n}$ is the *output matrix* and $\xi := \mathrm{vec}(H)$. The output function $y(x)$ in (11) vanishes on the orbit $\mathcal{O}$, and we assume that it has uniform vector relative degree r on an open neighborhood of $\mathcal{O}$. The family of input-output linearizing controllers is then given by

$$\Gamma(x, \xi) := -(L_g\, L_f^{r-1}\, y(x))^{-1} \left(L_f^r\, y(x) + \sum_{i=0}^{r-1} k_i\, L_f^i\, y(x) \right) \quad (12)$$

in which the scalars $k_0, \cdots, k_{r-1}$ are chosen such that the polynomial $\lambda^r + k_{r-1}\,\lambda^{r-1} + \cdots + k_0 = 0$ is Hurwitz. The controller (12) results in the output dynamics

$$y^{(r)} + k_{r-1}\, y^{(r-1)} + \cdots + k_0\, y = 0 \quad (13)$$

for which the origin $(y, \dot{y}, \cdots, y^{(r-1)}) = (0, 0, \cdots, 0)$ is exponentially stable. Furthermore, corresponding to the output function y in (11), one can define the following parameterized zero dynamics manifold

$$\mathcal{Z}(\xi) := \{x \in \mathcal{X} \,|\, y(x) = L_f y(x) = \cdots = L_f^{r-1} y(x) = 0\}$$

on which y is identically zero. Since the *decoupling matrix* $L_g L_f^{r-1} y(x)$ is square and invertible on the orbit, the feedback law driving y to zero is *unique* on each zero dynamics manifold $\mathcal{Z}(\xi)$ [32]. In addition, the orbit $\mathcal{O}$ is *common* to all the zero dynamics manifolds $\mathcal{Z}(\xi)$. Hence, the feedback law, restricted to the orbit, is independent of ξ.

In general, one could include the controller gains $k_i, i = 0, \cdots, r-1$ as well as the elements of the output matrix H in the parameter vector ξ. With this choice of ξ the closed loop system still satisfies Assumption 1. However, in this example, we assume that the controller gains are designed to stabilize the output dynamics (13). The objective is then how to design output functions (11) to guarantee the stability of the *internal system*, that is, of the maximal dynamics of closed-loop system compatible with the output functions being zero.

D. Poincaré return map and sensitivity analysis

This subsection addresses the properties of the Poincaré return map for the closed-loop hybrid system under the invariance condition. The parametrized Poincaré return map for the closed-loop hybrid system (3) is defined as $P : \mathcal{S} \times \Xi \to \mathcal{S}$ by

$$P(x, \xi) := \varphi\,(T(\Delta(x), \xi), \Delta(x), \xi) \quad (14)$$

which results in the following discrete-time system

$$x_{k+1} = P(x_k, \xi), \quad (15)$$

defined on the Poincaré section $\mathcal{S}$. According to the invariance condition, x_f^* is a fixed point of the Poincaré return map for all $\xi \in \Xi$, i.e.,

$$P(x_f^*, \xi) = x_f^*, \quad \forall \xi \in \Xi. \quad (16)$$

One immediate consequence of (16) is that

$$\frac{\partial P}{\partial \xi}(x_f^*, \xi) = 0, \quad \forall \xi \in \Xi, \quad (17)$$

and hence, the pair $(\frac{\partial P}{\partial x}(x_f^*, \xi), \frac{\partial P}{\partial \xi}(x_f^*, \xi))$ is not controllable. Therefore, the linear event-based controller design approach of [30], [9], [31] cannot be employed to stabilize the orbit $\mathcal{O}$ by updating parameter vector ξ in a step-to-step manner. Instead, we assume that the parameter vector ξ is constant and the objective is to design constant parameters of the continuous-time controller to exponentially stabilize the orbit $\mathcal{O}$ for the closed-loop system without employing event-based update loops. To achieve this goal, we study the discrete-time system (15) linearized around the fixed-point x_f^* which is given by

$$\delta x_{k+1} = \frac{\partial P}{\partial x}(x_f^*, \xi)\,\delta x_k, \quad (18)$$

where $\delta x_k := x_k - x_f^*$. Next, the objective is to find the parameter vector ξ such that the Jacobian matrix $\frac{\partial P}{\partial x}(x_f^*, \xi)$ becomes Hurwitz. As the Poincaré return map in general is calculated by numerical integration of the hybrid model, there is no closed-form expression for $\frac{\partial P}{\partial x}(x_f^*, \xi)$. This problems is more critical in mechanical systems of bipedal robots with high degrees of freedom. To resolve this problem, we turn our attention to the *sensitivity analysis*. In particular, by writing down the Taylor series expansion of the Jacobian matrix $\frac{\partial P}{\partial x}(x_f^*, \xi)$ around some *nominal parameter vector* ξ^* for sufficiently small $\|\xi - \xi^*\|$, (18) becomes

$$\delta x_{k+1} = \left(\frac{\partial P}{\partial x}(x_f^*, \xi^*) + \sum_{i=1}^{p} \frac{\partial^2 P}{\partial \xi_i \partial x}(x_f^*, \xi^*)\,\Delta\xi_i \right) \delta x_k, \quad (19)$$

where $\Delta\xi_i := \xi_i - \xi_i^*$ for $i = 1, \cdots, p$. Next, the objective is to find the constant perturbation value $\Delta\xi := (\Delta\xi_1, \cdots, \Delta\xi_p)^\top$ such that the origin $\delta x = 0$ is exponentially stable for (19). To simplify the calculation of the first- and second-order Jacobian matrices in (19), we present the following theorem as a numerical approach to calculate the Jacobian matrices on the basis of the trajectory sensitivity matrix.

Theorem 1 (Calculations of Jacobian Matrices): Let $\Phi(t,x_0,\xi) := \frac{\partial \varphi}{\partial x_0}(t,x_0,\xi)$ represent the trajectory sensitivity matrix and define

$$\Phi_f^*(\xi) := \Phi(T^*, x_i^*, \xi).$$

Then, under Assumptions 1 and 2, the Jacobian matrices in (19) can be expressed as[1]

$$\frac{\partial P}{\partial x}(x_f^*,\xi^*) = \Pi(x_f^*,\xi^*)\,\Phi_f^*(\xi^*)\,\Upsilon(x_f^*) \tag{20}$$

$$\frac{\partial^2 P}{\partial \xi_i \partial x}(x_f^*,\xi^*) = \Pi(x_f^*,\xi^*)\,\frac{\partial \Phi_f^*}{\partial \xi_i}(\xi^*)\,\Upsilon(x_f^*), \tag{21}$$

for $i = 1,\cdots,p$, in which

$$\Pi(x_f^*,\xi^*) := I - \frac{f^{\mathrm{cl}}(x_f^*,\xi^*)\frac{\partial s}{\partial x}(x_f^*)}{\frac{\partial s}{\partial x}(x_f^*) f^{\mathrm{cl}}(x_f^*,\xi^*)}$$
$$\Upsilon(x_f^*) := \frac{\partial \Delta}{\partial x}(x_f^*).$$

Proof: See [36]. ■

Remark 1: Theorem 1 simplifies the calculation of the Jacobian matrices in (19) by relating them to the final value of the trajectory sensitivity matrix, i.e., $\Phi_f^*(\xi)$. Furthermore, $\Phi_f^*(\xi)$ can be obtained by numerical integration of the well-known *variational equation* [27], that is,

$$\dot{\Phi}(t,x_i^*,\xi) = \frac{\partial f^{\mathrm{cl}}}{\partial x}(\varphi^*(t),\xi)\,\Phi(t,x_i^*,\xi), \quad 0 \le t \le T^*$$
$$\Phi(0,x_i^*,\xi) = I.$$

Next, numerical differentiation approaches like the two-point symmetric difference can be applied to calculate $\frac{\partial \Phi_f^*}{\partial \xi_i}(\xi^*)$ in (21) as follows

$$\frac{\partial \Phi_f^*}{\partial \xi_i}(\xi^*) = \frac{1}{2\delta}\left(\Phi_f^*(\xi^* + \delta\, e_i) - \Phi_f^*(\xi^* - \delta\, e_i)\right),$$

where $\{e_1,\cdots,e_p\}$ is the set of standard bases for $\mathbb{R}^p$ and $\delta > 0$ is a small perturbation value.

III. Translation of the Stabilization Problem into Bilinear Matrix Inequalities

A bilinear matrix inequality is a relation of the form

$$F_0 + \sum_{i=1}^m F_i u_i + \sum_{j=1}^n G_j v_j + \sum_{i=1}^m \sum_{j=1}^n H_{ij} u_i v_j > 0$$

where $F_0,\ldots,F_m, G_1,\ldots,G_n, H_{11},\ldots,H_{mn}$ are given symmetric matrices and $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ are real vectors. The notation $M > 0$, where M is a symmetric matrix, means that M is positive definite. A good introduction to BMIs is given in [28].

[1] It is important to consider the Jacobian matrix $\frac{\partial P}{\partial x}(x_f^*,\xi)$ as a mapping from $\mathcal{S}$ into $\mathcal{S}$ by pre and post multiplying it by projection and lift matrices. However, to simplify the notation, we do not consider the projection and lift matrices here.

The objective of this section is to translate the problem of exponential stabilization of the origin $\delta x = 0$ for the discrete-system (19) into a set of BMIs. This optimization problem can be solved offline to find a stabilizing set of parameter values for the closed loop system. To achieve this goal, we first define

$$A_0 := \frac{\partial P}{\partial x}(x_f^*,\xi^*)$$
$$A_i := \frac{\partial^2 P}{\partial \xi_i \partial x}(x_f^*,\xi^*), \quad i = 1,\cdots,p. \tag{22}$$

Next, we present the following theorem to find the constant perturbation vector $\Delta\xi$.

Theorem 2 (BMIs for Stabilizations of the Orbit): The following statements are correct.

1) There exists a B matrix such that

$$A_0 + \sum_{i=1}^p A_i\,\Delta\xi_i = A_0 + B\,(I \otimes \Delta\xi), \tag{23}$$

where "⊗" represents the Kronecker product.

2) The origin is exponentially stable for the system (19) if there exist matrices $W = W^\top$ and $\Delta\xi$, and a scalar $\mu \ge 0$ such that the following BMI is satisfied

$$\begin{bmatrix} W & A_0\,W + B\,(I\otimes\Delta\xi)\,W \\ \star & (1-\mu)\,W \end{bmatrix} > 0. \tag{24}$$

Proof: See [36] for Part 1. For Part 2, let us consider the Lyapunov function $V_k := V(\delta x_k) := \delta x_k^\top W^{-1} \delta x_k$. Then, from BMI (24), $W > 0$ and $(1-\mu)\,W > 0$ which together with $\mu \ge 0$ yield $\mu \in [0,1)$. Furthermore, using Schur's Lemma,

$$W\,\left(A_0 + B\,(I\otimes\Delta\xi)\right)^\top W^{-1}\left(A_0 + B\,(I\otimes\Delta\xi)\right) W - W < -\mu\, W. \tag{25}$$

Pre and post multiplying (25) with W^{-1} results in $\Delta V_k := V_{k+1} - V_k < -\mu\, V_k$ and

$$\|\delta x_k\|_2 < \sqrt{\frac{\lambda_{\max}(W^{-1})}{\lambda_{\min}(W^{-1})}(1-\mu)^k}\,\|\delta x_0\|_2 \tag{26}$$

for $k = 1, 2, \cdots$, where $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ are the minimum and maximum eigenvalues, respectively. This completes the proof of Part 2. ■

PENBMI[2] is a general-purpose solver for BMIs which guarantees the convergence to a critical point satisfying the first-order KKT optimality conditions [37]. The solver PENBMI integrated with the MATLAB environment through the YALMIP[3] interface can then be used to solve the BMI of Theorem 2. We are interested in solutions of (24) with a small perturbation vector $\Delta\xi$ to have a good approximation based on Taylor series expansion in (19). In addition, from (26), we would like to maximize μ, or equivalently minimize

[2] http://www.penopt.com/penbmi.html

[3] http://users.isy.liu.se/johanl/yalmip/

$-\mu$ to improve the convergence rate. Consequently, we present the following optimization problem

$$\begin{aligned} \min_{W,\Delta\xi,\mu,\gamma} \quad & -w\mu + \gamma \\ \text{s.t.} \quad & \begin{bmatrix} W & A_0\,W + B\,(I \otimes \Delta\xi)\,W \\ \star & (1-\mu)\,W \end{bmatrix} > 0 \\ & \|\Delta\xi\|_2^2 < \gamma \\ & \mu \geq 0, \end{aligned} \tag{27}$$

to tune the constant parameter vector $\xi = \xi^* + \Delta\xi$, where $w > 0$ is a positive scalar as a tradoff between improving the convergence rate and minimizing the norm of $\|\Delta\xi\|_2$. Using Schur's Lemma, $\|\Delta\xi\|_2^2 < \gamma$ is also equivalent to the following LMI constraint

$$\begin{bmatrix} I & \Delta\xi \\ \Delta\xi^\top & \gamma \end{bmatrix} > 0.$$

Hence, the optimization problem (27) becomes

$$\begin{aligned} \min_{W,\Delta\xi,\mu,\gamma} \quad & -w\mu + \gamma \\ \text{s.t.} \quad & \begin{bmatrix} W & A_0\,W + B\,(I \otimes \Delta\xi)\,W \\ \star & (1-\mu)\,W \end{bmatrix} > 0 \\ & \begin{bmatrix} I & \Delta\xi \\ \star & \gamma \end{bmatrix} > 0 \\ & \mu \geq 0, \end{aligned} \tag{28}$$

which can be handled by the solver PENBMI.

Remark 2: The approach of sensitivity analysis and BMI optimization in (28) can be extended to non-hybrid systems, described by ordinary differential equations. In this case, one can assume that $\Delta(x) = \mathrm{id}(x)$, where $\mathrm{id}(x)$ represents the identity map. Furthermore, this approach and Theorem 1 can be extended to hybrid systems with multiple continuous phases.

IV. Application to Underactuated 3D Bipedal Robots

In this section we illustrate the results of this paper to systematically stabilize a walking gait of an underactuated 3D bipedal robot with 8 degrees of freedom and 2 degrees of underactuation. The biped model and several walking controllers based on virtual constraints and hybrid zero dynamics were previously described in [21]. The studied robot consists of a torso and two identical legs with revolute knees and point feet. Each hip has two degrees of freedom. All of the internal joints are actuated, whereas the roll (i.e., q_1) and pitch (i.e., q_2) angles are unactuated. The structure and coordinates are shown in Fig. 1.

Virtual constraints are kinematic relations among the generalized coordinates enforced asymptotically by feedback control [8], [9], [21], [33], [34]. It has been shown that for mechanical systems with more than one degree of underactuation, the choice of virtual constraints affects the stability of the periodic orbit [21]. In [21] it was shown that controlling the actuated coordinates failed to stabilize a periodic walking gait. *Physical intuition* led to a different choice of virtual

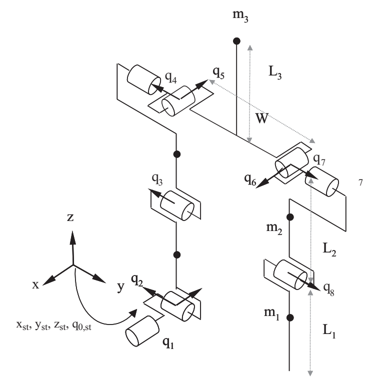

Fig. 1. A five-link 3D bipedal robot during the right stance phase with point feet and the associated configuration variables [21].

constraints which stabilized the same orbit; however, for a related robot with additional degrees of freedom due to series elastic actuators, the same intuition did not lead to a stable periodic orbit [35]. This underlines the importance of having a *systematic* method of choosing these constraints. Any attempt, however, to apply the method of [1] to design virtual constraints for this robot would require recomputation of the 15×15 Jacobian matrix of the Poincaré map at each iteration of the nonlinear optimization algorithm, making the algorithm impractical for this type of problem.

As in [21], we consider virtual constraints of the form

$$y = h_0(q) - h_d(\theta) = H\,(q - q_{\mathrm{d}}(\theta)), \tag{29}$$

where $h_0(q) = Hq$ is a set of *controlled variables* and $h_d(\theta) = Hq_{\mathrm{d}}(\theta)$ gives the desired evolutions of h_0 on the orbit $\mathcal{O}$, and θ is a gait phase variable satisfying Assumption 3. The input-output linearizing controller of Example 2 is used to enforce the virtual constraints. A periodic orbit $\mathcal{O}$ was designed via the optimization algorithm of [21]. With the controlled variables taken to be simply the actuated coordinates

$$H^*\,q := (q_3, q_4, q_5, q_6, q_7, q_8)^\top, \tag{30}$$

the dominant eigenvalues of the 15-dimensional Poincaré map become $\{-2.1076, 0.8733, -0.3888\}$.

We now apply the algorithm developed in this paper to design virtual constraints which stabilize the periodic orbit $\mathcal{O}$. For this, we let $\xi = \mathrm{vec}(H) \in \mathbb{R}^{48}$, where $H \in \mathbb{R}^{6\times 8}$ is the output matrix of (29). Numerical estimation of the resulting linearized Poincaré map yields values of the matrices $A_0, \cdots, A_p$, $p = 48$.

Figure 2 depicts the 2-norm of A_i versus the elements of the H matrix. As shown in the figure, the most important sensitivity matrices around the nominal output function correspond to the first column of the H matrix, which is related to the roll angle q_1. Based on this observation, we reduce the dimension of the BMI optimization problem (28) by letting $\Delta\xi$ parameterize only the first column of H; that is, we redefine $\Delta\xi$ by $H = H^* + \begin{bmatrix}\Delta\xi & 0_{6\times 1} & \cdots & 0_{6\times 1}\end{bmatrix}$. Solving this optimization problem with $w = 20$ results in

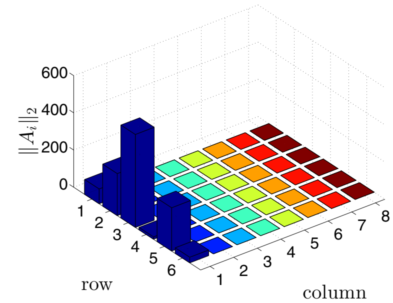

Fig. 2. Plot of the 2-norm of the sensitivity matrices versus the components of the 6×8 H matrix around the nominal output function. Here, $i = \text{row} + 6(\text{column} - 1)$.

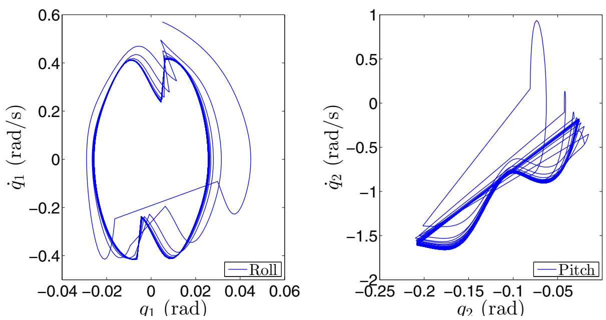

Fig. 3. Phase portraits of the closed-loop system during 50 consecutive steps of walking corresponding to the optimal solutions of (28) around the nominal output function.

the following controlled variables

$$H\,q = \begin{bmatrix} q_3 + 0.4460\,q_1 \\ q_4 + 0.4474\,q_1 \\ q_5 + 0.6976\,q_1 \\ q_6 - 0.5948\,q_1 \\ q_7 - 0.3012\,q_1 \\ q_8 - 0.5726\,q_1 \end{bmatrix}. \quad (31)$$

Using this output function, the dominant eigenvalues of the approximate Jacobian matrix based on Taylor series expansion in (19) and real Jacobian matrix become $\{-0.8622, 0.8623, 0.1719\}$ and $\{-0.8486, 0.8595, 0.2381\}$, respectively. Next, the simulation of the closed-loop system is started from a point off of the orbit with an error of 10 (deg/s) in the velocity coordinates. Figure 3 depicts the phase portraits of the closed-loop system during 50 consecutive steps of walking. Convergence to a stable limit cycle is clear.

Finally, we let $\Delta\xi$ parameterize only the $(4, 1)$ element of H, i.e., $H = H^* + \Delta\xi\, E_{4,1}$, in which $E_{4,1} \in \mathbb{R}^{6\times 8}$ is a matrix whose elements are zero except the $(4, 1)$ element which is 1. Solving the optimization problem (28) with $w = 20$ then results in the following controlled variables

$$H\,q := (q_3, q_4, q_5, q_6 - 1.8265\,q_1, q_7, q_8)^\top. \quad (32)$$

Furthermore, the dominant eigenvalues of the approximate and real Jacobian matrices are $\{-0.9023, 0.9015, 0.8740\}$ and $\{-0.9354, 0.8228, 0.8693\}$, respectively.

V. Conclusion

We have introduced a method for designing continuous-time controllers to exponentially stabilize periodic orbits of a class of hybrid systems. The key assumption is that a parametrized family of continuous-time controllers is known for which a periodic orbit is induced, and the orbit is invariant under the choice of parameters. The properties of the Poincaré map under the invariance condition were studied to present a sensitivity analysis. This analysis allows the problem of stabilization of periodic orbits to be translated into a BMI optimization problem, which can be solved easily with available software packages. The power of this approach was demonstrated by redesigning virtual constraints for a 3D bipedal robot so as to stabilize a periodic walking gait which previously required event-based stabilization. The resulting virtual constraints offer new insights into the gait stabilization problem for bipedal robots.

The algorithm we have presented can be extended to more general classes of hybrid and non-hybrid systems, including hybrid systems with multiple continuous phases. We will be reporting the results on a 3D underactuated bipedal robot that has 26 states and 6 actuators, four of which have series compliance. We will also investigate the results with simultaneous continuous-time and discrete-time control actions for increasing the robustness of walking gaits.

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