On the θ-coverage and connectivity of large random networks*†

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Abstract

Wireless planar networks have been used to model wireless networks in a tradition that dates back to 1961 to the work of E. N. Gilbert. Indeed the study of connected components in wireless networks was the motivation for his pioneering work that spawned the modern field of continuum percolation theory. Given that node locations in wireless networks are not known, random planar modelling can be used to provide preliminary assessments of important quantities such as range, number of neighbors, power consumption, and connectivity, and issues such as spatial reuse and capacity.

In this paper, the problem of connectivity based on nearest neighbors is addressed. The exact threshold function for θ-coverage is found for wireless networks modelled as n points uniformly distributed in a unit square, with every node connecting to its ϕ_n nearest neighbors. A network is called θ-covered if every node, except those near the boundary, can find one of its ϕ_n nearest neighbors in any sector of angle θ. For all $\theta \in (0, 2\pi)$, if $\phi_n = (1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$, it is shown that the probability of θ-coverage goes to one as n goes to infinity, for any $\delta > 0$; on the other hand, if $\phi_n = (1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$, the probability of θ-coverage goes to zero. This sharp characterization of θ-coverage is used to show, via further geometric arguments, that the network will be connected with probability approaching one if $\phi_n = (1+\delta)\log_2 n$. Connections between these results and the performance analysis of wireless networks, especially for routing and topology control algorithms, are discussed.

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1 Introduction

In wireless "ad hoc" networks, nodes can and do need to adjust their transmission powers to connect to others, and thus carry out the network's functionalities. The strategy for each node to adjust its power, and the characteristics of the random graph structure thus generated, especially when the number of nodes is large, has attracted much attention. How many neighbors for each node is desirable is one of these questions that has been studied extensively, and these studies have been conducted under many different optimization objectives.

One of the objectives that has been studied is to maximize the expected one-hop progress of a packet in the desired direction without too much power consumption. Assuming that the node locations are generated according to a Poisson random process, Kleinrock and Silvester [1] determined that each node should connect to six nearest neighbors on average—a so called "magic number"—for different forwarding strategies. Later, the number was revised to eight in [2], and for some further forwarding strategies they found five or seven as the best numbers. Hou and Li [3] considered the scenario when each node is allowed to adjust its transmission range individually. They found that six or eight neighbors is desirable for different packet forwarding strategies. Mathar and Mattfeldt [4] analyzed the wireless network generated on a line, and also obtained some magic numbers.

Another critical objective in wireless networks is to maintain the connectivity of the overall network. To study this, networks have been modelled as n nodes uniformly distributed in a unit square. This is a tradition that goes back to E. N. Gilbert[1] [5] who may be regarded as the father of continuum percolation theory [22], and who initiated the study of random planar networks motivated precisely by study of connected components in wireless networks. Penrose [6] and Gupta and Kumar [7] have independently determined the critical range for establishing overall connectivity with probability approaching one as n goes to infinity, and this range results in each node connecting to $\log n$ neighbors on average.

If every node instead connects to the same number of neighbors, as opposed to transmitting at the same power and thus ideally covering the same distance, we showed in a previous

[1] Probably better known to information and coding theorists for the Gilbert-Varshamov bound.

work [8] that the order of the critical number of neighbors each node should connect to is $\Theta(\log n)$. Specifically, if each node connects to less than $0.074 \log n$ nearest neighbors, then the network is asymptotically disconnected, while if each node connects to greater than $5.1774 \log n$ nearest neighbors, then the network is asymptotically connected. In [9], Wan and Yi improved the bound by showing that $2.718 \log n$ neighbors for each node is enough to ensure asymptotic connectivity. Recently, Balister, Bollobas, Sarkar and Walters [10] showed that, for connectivity of networks where nodes are generated in a unit area according to Poisson process with density n, $0.3043 \log n$ is a lower bound and $0.5139 \log n$ is an upper bound.

Besides the above mentioned *distance*-based and *number-of-neighbor*-based connecting strategies, an interesting *sector*-based strategy has been proposed by Wattenhofer, Li, Bahl and Wang [11] for network topology control. In their algorithm, the transmission power of each node is augmented from zero until it can find a neighboring node within every sector of angle θ, a pre-specified parameter, or until it hits its maximum power. If the network would be connected when every node transmits at its maximum power, and θ is $2\pi/3$, they showed the overall network is connected. Later, the critical sector angle that guarantees overall connectivity was found to be $5\pi/6$ by Li, Halpern, Bahl, Wang and Wattenhofer [12].

Similar ideas involving sector-based strategies are also found in the studies of topology control using spanner graphs [13,14]. For a given positive integer k, a *Yao Graph* (also called θ-graph) is constructed as follows [15]. First partition the 2π angle around each node into k equal sectors of size $2\pi/k$, and then let each node connect to the nearest node within every sector. If $k \geq 6$, then Li, Wan and Wang [16] showed that the Yao Graph has length stretch factor $\mu = \frac{1}{1-2\sin(\pi/k)}$, i.e., the Euclidean distance along the Yao Graph between any pair of nodes is at most μ times the actual Euclidean distance.

Besides concerns such as the aforementioned forwarding strategy, overall connectivity, and topology control, understanding the geometric properties of wireless networks has also played a role in studying their communication capacity, especially in the establishment of scaling laws. In [17], the upper bound for the transport capacity is based on the fact that every successful transmission "consumes" a portion of the area of the domain. Also, in [18], in an information theoretic study of transport capacity, the upper bound is again fundamentally related to the geometric structure of the networks. In [19], geometric properties revealed by percolation theory are applied to improve the bound on transport capacity for the random physical model of [17].

The random structures formed on top of point processes have also been central subjects in random geometric graph theory [20], percolation theory [21,22], and computational geometry [23].

In this paper, inspired by the above mentioned *sector*-based topology control algorithm, we begin with the study of the θ-coverage of wireless networks. A network G_n is modelled as n nodes uniformly distributed within a unit square, with every node connecting bidirectionally to its ϕ_n nearest neighbors, where ϕ_n is a deterministic function of n to be specified. For an angle $\theta \in (0, 2\pi)$, a node is said to be *θ-covered* by its ϕ_n nearest neighbors if, among them, it can find a node in every sector of angle θ. The graph G_n is then called *θ-covered* if every node is so, except possibly for those nodes such that the disk centered at such a node, with radius equal to the distance to its ϕ_n-th nearest neighbor, is not contained within the unit square.

For all $\theta \in (0, 2\pi)$, we find the exact threshold function for θ-coverage when n goes to infinity. Specifically, we show that for any $\delta > 0$, if every node connects to its $(1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ nearest neighbors, then the probability that G_n is θ-covered goes to one as n goes to infinity. On the other hand, if every node instead connects to less than $(1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ nearest neighbors, then the probability that G_n is not θ-covered goes to one as n goes to infinity.

Furthermore, when $\theta = \pi$, we show via a geometric argument that π-coverage leads to overall connectivity of the graph. That is, if every node connects to its $(1+\delta)\log_2 n$ nearest neighbors, then both the following happen with probability going to one as n goes to infinity: i) Every node is within the convex hull formed by its chosen neighbors; ii) the network is connected. Moreover, the proof characterizes the detailed and interesting geometric structure of the resulting graph, which may be of interest in its own right.

The rest of the paper is organized as follows. In Section 2, we provide the problem formulation and the main results. In Sections 3 and 4, the proofs for the main results are provided. In Section 5, a discussion is provided on how the results may be used for the performance analysis of topology control and routing algorithms. Finally Section 6 concludes the paper.

2 Formulation and main results

Let $\mathbb{S}$ be the axis parallel unit square on the plane with the left lower corner at the origin. Suppose that n nodes $\{X_1, X_2, \cdots, X_n\}$ are placed uniformly and independently in $\mathbb{S}$. Denote by $G(n, \phi_n)$ the graph formed by the vertex set consisting of the n nodes, and letting every node connect to its ϕ_n nearest neighbors. More precisely, there exists an edge (i, j) if either j is one of the ϕ_n nearest neighbors of i, or i is one of the ϕ_n nearest neighbors of j. Denote $G(n, \phi_n)$ in short by G_n. Then the corresponding **torus graph** G_n^τ consists of points $\{X + (i, j),\ \forall X \in V(G_n),\ (i, j) \in \mathbb{Z}^2\}$ as vertex set, where $V(G_n)$ denotes all the n nodes in G_n, and the edge set of graph G_n^τ is generated by letting each vertex of G_n^τ connect to its ϕ_n nearest neighbors in $V(G_n^\tau)$. We call a point X' an **image** of a point X in G_n if $X' - X$ is a point with integer coordinates. Notice that if an edge of G_n^τ is within $\mathbb{S}$, then it is also an edge of G_n.

We are interested in the following questions:

1. *Given an angle $\theta \in (0, 2\pi)$, what is the value of ϕ_n so that every node can find a node in any sector of angle θ from among its ϕ_n nearest neighbors, with high probability?*

2. *What is the value of ϕ_n such that the resulting graph $G(n, \phi_n)$ will be connected with high probability?*

To answer these questions, we need the following definition.

Definition 2.1. *Consider graph G_n^τ on the plane. Let $Disk(X, \phi_n)$ denote the disk centered at node X with radius equal to the distance between X and its ϕ_n-th nearest neighbor. For angle $\theta \in (0, 2\pi)$, node X is called* ***θ*-covered** *by its ϕ_n nearest neighbors if there is a node in every θ sector of $Disk(X, \phi_n)$. The graphs G_n^τ and $G(n, \phi_n)$ are called* ***θ*-covered** *if every node is θ-covered in G_n^τ.*

Remark 2.1. *If $G(n, \phi_n)$ is θ-covered, then for every node $X \in G_n$, there is a node of G_n in every θ sector of $Disk(X, \phi_n)$ that is contained in $\mathbb{S}$. Furthermore, if $Disk(X, \phi_n)$ is* *contained within $\mathbb{S}$, then node X being π-covered by its ϕ_n nearest neighbors is equivalent to saying that node X is within the convex hull of its ϕ_n nearest neighbors in G_n.*

The main results of the paper are as follows:

Theorem 1. *For all $\delta > 0$ and $\theta \in (0, 2\pi)$,*

(i) $\lim_{n\to\infty} Pr\left\{G(n, (1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n) \text{ is } \theta\text{-covered}\right\} = 1;$

(ii) $\lim_{n\to\infty} Pr\left\{G(n, (1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n) \text{ is } \theta\text{-covered}\right\} = 0.$

Remark 2.2. *For θ small, $\log_{\frac{2\pi}{2\pi-\theta}} n = \frac{\log n}{-\log(1-\frac{\theta}{2\pi})} \approx \frac{2\pi}{\theta}\log n$. Note that even under optimal arrangement a node needs at least $\frac{2\pi}{\theta}$ neighboring nodes for θ-coverage. So the factor $\log n$ is the price caused by randomness in order to get θ-coverage for every node in the network of n nodes.*

Theorem 2. *For all $\delta > 0$,*

(i) $\lim_{n\to\infty} Pr\{G^{\tau}(n, (1+\delta)\log_2 n) \text{ is connected}\} = 1;$

(ii) $\lim_{n\to\infty} Pr\{G(n, (1+\delta)\log_2 n) \text{ is connected}\} = 1.$

3 $\log_{\frac{2\pi}{2\pi-\theta}} n$ is the threshold function for θ-coverage of the network

In this section we prove Theorem 1, i.e., $\log_{\frac{2\pi}{2\pi-\theta}} n$ is the threshold function for θ-coverage of the network. In the first subsection, for all $\delta > 0$, we show that letting each node connect to $(1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ nearest neighbors results in θ-coverage of the network with probability approaching one as n goes to infinity. In the second subsection we show that the probability that there exists a node which is not θ-covered approaches one, if each node only connects to its $(1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ nearest neighbors.

3.1 $(1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ neighbors are sufficient for θ-coverage of the network

We first examine the situation for node X_1; then we will show that the union bound suffices to prove the result. In this subsection we fix $\phi_n := (1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$, and for brevity we denote $Disk(X_1, \phi_n)$ as D_{ϕ_n}.

Let M be a positive integer, and let $\mathcal{S} := \{S_1, \cdots, S_{\lceil \frac{2\pi}{\theta} \rceil M}\}$ be $\lceil \frac{2\pi}{\theta} \rceil M$ sectors (of $Disk(X_1, \phi_n)$) of angle $(2\pi - \theta) + \frac{\theta}{M}$ each. They are positioned in the following way. S_1 begins from the left hand of the x-axis and spans counterclockwise. S_{i+1} begins an angle of θ/M away from S_i, counterclockwise, for $i = 1, \cdots, \lceil \frac{2\pi}{\theta} \rceil M - 1$; see Figure 1. Note that any radius of D_{ϕ_n} can find counterclockwise (as well as clockwise) a starting radius of such a sector that is no more than an angle of θ/M away.

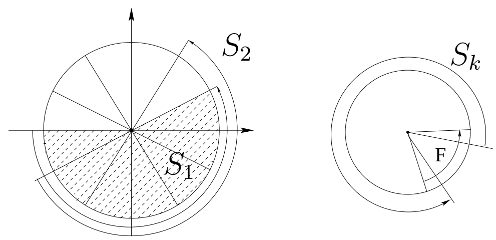

Figure 1: Left: The sectors of $\mathcal{S}$ for node 1 when $\theta = \pi$ and $M = 6$. The shaded area is S_1. Right: Figure used in the proof of Lemma 3.1.

With a little abuse of notation between a disk and its area, we have the following lemma.

Lemma 3.1. *If node X_1 is not θ-covered by its ϕ_n nearest neighbors, then there exists a sector $S \in \mathcal{S}$ such that it contains all of node X_1's ϕ_n nearest neighbors.*

Proof. Node X_1 not being θ-covered by its ϕ_n nearest neighbors means there exists a sector F of angle θ such that no node is in F, and F is contained in D_{ϕ_n}. F's ending side (suppose a sector always spans counterclockwise) can find clockwise a starting side of a sector in $\mathcal{S}$, S_k say, such that S_k's ending side is within F; see Figure 1. □

Lemma 3.2. *Define event A_k, $1 \leq k \leq n$, as follows:*

$$A_k := \{\textit{Node } X_k \textit{ is NOT } \theta\textit{-covered by its } \phi_n \textit{ nearest neighbors in } G_n^\tau\}.$$

Then for all θ and all n, we have

$$Pr\{A_k\} \leq \lceil \frac{2\pi}{\theta} \rceil M \left(\left(\frac{2\pi - \theta + \theta/M}{2\pi} \right)^{\phi_n} + c_1 e^{-c_2 n} \right),$$

for some fixed positive constants c_1 and c_2.

Proof. By symmetry, we only need to consider $Pr(A_1)$. By Lemma 3.1, we have

$$\begin{aligned}
Pr\{A_1\} &\leq Pr\{\text{All of } X_1\text{'s } \phi_n \text{ nearest neighbors are within some sector } S \in \mathcal{S}\} \\
&\leq \sum_{i=1}^{\lceil \frac{2\pi}{\theta} \rceil M} Pr\{\text{All of } X_1\text{'s } \phi_n \text{ nearest neighbors are within sector } S_i \in \mathcal{S}\} \\
&:= \sum_{i=1}^{\lceil \frac{2\pi}{\theta} \rceil M} Pr\{B_i\}.
\end{aligned}$$

Once given D_{ϕ_n}, if its diameter is less than 1, then the $(\phi_n - 1)$ nearest neighbors of X_1 are uniformly and independently distributed in D_{ϕ_n}, and the ϕ_n-th nearest neighbor is uniformly distributed along the boundary of D_{ϕ_n}. So for $i \leq \lceil \frac{2\pi}{\theta} \rceil M$, we have

$$\begin{aligned}
Pr\{B_i\} &= Pr\{B_i;\, Diameter(D_{\phi_n}) < 1\} + Pr\{B_i;\, Diameter(D_{\phi_n}) \geq 1\} \\
&\leq Pr\{B_i \,|\, Diameter(D_{\phi_n}) < 1\} + Pr\{Diameter(D_{\phi_n}) \geq 1\} \\
&= \left(\frac{2\pi - \theta + \theta/M}{2\pi}\right)^{\phi_n} + Pr\{Diameter(D_{\phi_n}) \geq 1\}.
\end{aligned}$$

Now we upper bound $Pr\{Diameter(D_{\phi_n}) \geq 1\}$. Draw a unit square, $\mathbb{S}'$, centered at node X_1, and parallel to $\mathbb{S}$. Because of the construction of the torus graph G_n^τ, there are exactly n vertices in $\mathbb{S}'$, and the $n-1$ vertices other than X_1 are uniformly and independently distributed in $\mathbb{S}'$. Denote them by $\{Y_2, \cdots, Y_n\}$, and the distance between X_1 and Y_k by $d(X_1, Y_k)$. Then $\{I_{[d(X_1,Y_k) \leq 1/2]}, k = 2, \cdots, n\}$ are $(n-1)$ iid random variables with Bernoulli distribution, and so $Pr(I_{[d(X_1,Y_k) \leq 1/2]} = 1) = \pi(1/2)^2 = \pi/4$. Since $\phi_n - 1 < (n-1)\pi/8$ for n large, by the Chernoff bound (see page 12 of [24]), we have

$$\begin{aligned}
Pr(Diameter(D_{\phi_n}) \geq 1) &= Pr\left(\sum_{k=2}^{n} I_{[d(X_1,Y_k) \leq 1/2]} \leq \phi_n - 1\right) \\
&\leq Pr\left(\sum_{k=2}^{n} I_{[d(X_1,Y_k) \leq 1/2]} \leq (n-1) \cdot \frac{\pi}{8}\right) \\
&\leq \exp\left\{-(n-1) \cdot (-\frac{\pi}{8}\log 2 - \frac{7\pi}{8}\log\frac{6}{7})\right\}.
\end{aligned}$$

So letting $c_2 := -\frac{\pi}{8}\log 2 - \frac{7\pi}{8}\log\frac{6}{7}$, which is positive, and $c_1 := e^{c_2}$, we have

$$Pr\{A_1\} \leq \left\lceil \frac{2\pi}{\theta} \right\rceil M \left((\frac{2\pi - \theta + \theta/M}{2\pi})^{\phi_n} + c_1 e^{-c_2 n} \right). \qquad \square$$

Proof of part (i) of Theorem 1.

Let us define event A_k, for all $k \in N := \{1, 2, \cdots, n\}$, as in Lemma 3.2. Then because of symmetry we have,

$$
\begin{aligned}
&Pr\{G(n, \phi_n) \text{ is } \theta\text{-covered}\}\\
&\quad = 1 - Pr\{\cup_{k \in N} A_k\}\\
&\quad \geq 1 - \sum_{k=1}^{n} Pr\{A_k\}\\
&\quad = 1 - n \cdot Pr\{A_1\}\\
&\quad \geq 1 - n \left\lceil \frac{2\pi}{\theta} \right\rceil M \left(\left(\frac{2\pi - \theta + \theta/M}{2\pi} \right)^{\phi_n} + c_1 e^{-c_2 n} \right)\\
&\quad = 1 - \left\lceil \frac{2\pi}{\theta} \right\rceil M \cdot n c_1 e^{-c_2 n} -\\
&\quad \left\lceil \frac{2\pi}{\theta} \right\rceil M \cdot \exp\left\{ \log n - \frac{(1+\delta)\log n}{\log \frac{2\pi}{2\pi - \theta}} \log \frac{2\pi}{2\pi - \theta} + \frac{(1+\delta)\log n}{\log \frac{2\pi}{2\pi-\theta}} \log(1 + \frac{\theta}{(2\pi - \theta)M}) \right\}\\
&\quad = 1 - o(1) - \left\lceil \frac{2\pi}{\theta} \right\rceil M \cdot \exp\left\{ \left(-\delta + (1+\delta) \frac{\log(1 + \frac{\theta}{(2\pi-\theta)M})}{\log \frac{2\pi}{2\pi-\theta}} \right) \log n \right\}.
\end{aligned}
$$

So, if M is sufficiently large such that $(1+\delta)\frac{\log(1+\frac{\theta}{(2\pi-\theta)M})}{\log \frac{2\pi}{2\pi-\theta}} < \delta$, then

$$
\lim_n Pr\left\{ G(n, (1+\delta)\log_{\frac{2\pi}{2\pi-\theta}} n) \text{ is } \theta\text{-covered} \right\} = 1.
$$

$\square$

3.2 $(1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ neighbors are necessary for θ-coverage of the network

For convenience, set $\phi_\theta(n) := (1-\delta)\log_{\frac{2\pi}{2\pi-\theta}} n$ in this subsection. Also define $Disk(X, \phi_\theta(n))$ as the disk centered at node X with radius equal to the distance to its $\phi_\theta(n)$-th nearest neighbor.

We need the following lemmas.

Lemma 3.3. *Tessellate the unit square $\mathbb{S}$ with small axis-parallel squares of side $d_n := \sqrt{\frac{K \log n}{n}}$, where K is a tunable parameter; see Figure 2. Label the squares as S_k^n, and the disk inscribing each square S_k^n as C_k^n, $k = 1, \cdots, \frac{n}{K \log n}$. Let $N(S_k^n)$ denote the number of* *nodes in S_k^n, and similarly for $N(C_k^n)$. Then, for all $K^- > 0$, there exist positive constants K and K^+ such that*

$$\lim_n Pr\left\{\max_{1\le k\le n/K\log n} N(S_k^n) \quad \le K^+\log n\right\} = 1, \text{ and}$$
$$\lim_n Pr\left\{\min_{1\le k\le n/K\log n} N(C_k^n) \quad \ge K^-\log n\right\} = 1. \tag{1}$$

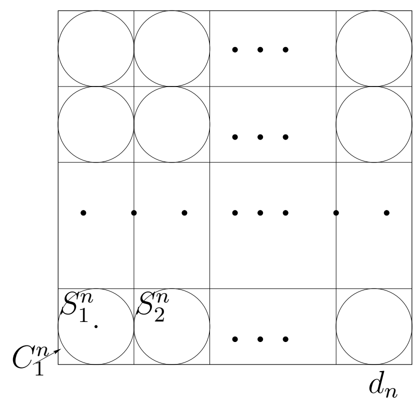

Figure 2: The square tessellation.

Proof. We introduce the following notations:

$$p_1 := Pr\{X_i \in S_k^n\} = \frac{K\log n}{n}, \forall i,k;$$
$$p_2 := Pr\{X_i \in C_k^n\} = \pi\left(\frac{1}{2}\sqrt{\frac{K\log n}{n}}\right)^2 = \frac{\pi K\log n}{4n}, \forall i,k.$$

First we show that, for all $K > 0$, if one chooses $K^+ := K(e-1)+1$, then $\lim_n Pr\{\max_{1\le k\le n/K\log n} N(S_k^n) \quad \le K^+\log n\} = 1$.

Consider $N(S_1^n)$. It has a binomial distribution with parameters (p_1, n). So by the Chernoff bound, we have

$$Pr\left(N(S_1^n) > K^+\log n\right) \le \frac{E\exp(N(S_1^n))}{\exp(K^+\log n)} = \frac{E\exp(N(S_1^n))}{n^{K^+}}.$$

Since $E\exp(N(S_1^n)) = (1+(e-1)p_1)^n \le n^{K(e-1)}$ (because $1+x\le e^x$), we have

$$Pr\left(N(S_1^n) > K^+\log n\right) \le n^{K(e-1)-K^+} = n^{-1}.$$

Thus by the union bound, we have

$$
\begin{aligned}
&Pr\left\{\text{For some } k\text{: } N(S_k^n) > (K(e-1)+1)\log n\right\} \\
&\leq \sum_{1}^{n/K\log n} Pr\left\{N(S_k^n) > (K(e-1)+1)\log n\right\} \\
&\leq \frac{n}{K\log n} n^{-1} \to 0, \text{ as } n \to \infty. \qquad (2)
\end{aligned}
$$

Now we show that, for all $K^- > 0$, if one chooses $K = \frac{4(K^-+1)}{\pi(1-e^{-1})}$, then $\lim_n Pr\{\min_{1\leq k\leq n/K\log n} N(C_k^n) \ \geq K^-\log n\} = 1$.

This can be shown similarly as for the upper bound. Notice that $N(C_1^n)$ has a binomial distribution with parameters (p_2, n). So by the Chernoff bound, we have

$$
\begin{aligned}
Pr\left(N(C_1^n) < K^-\log n\right) &= Pr\left(-N(C_1^n) > -K^-\log n\right) \\
&\leq \frac{E\exp(-N(S_1^n))}{\exp(-K^-\log n)} = n^{K^-} E\exp(-N(C_1^n)).
\end{aligned}
$$

Since $E\exp(-N(C_1^n)) = (1-(1-e^{-1})p_2)^n \leq n^{-\frac{(1-e^{-1})\pi K}{4}}$ (because $1-x \leq e^{-x}$), we have

$$
Pr(N(C_1^n) < K^-\log n) \leq n^{K^- - \frac{\pi(1-e^{-1})}{4}K} = n^{-1}.
$$

Thus by the union bound, we have

$$
\begin{aligned}
&Pr\left\{\text{For some } k\text{: } N(C_k^n) < K^-\log n\right\} \\
&\leq \sum_{1}^{n/K\log n} Pr\left\{N(C_k^n) < K^-\log n\right\} \\
&\leq \frac{n}{K\log n} n^{-1} \to 0, \text{ as } n \to \infty. \qquad (3)
\end{aligned}
$$

Combining (2) and (3), the lemma is proven. □

Given Lemma 3.3, the following corollary follows immediately, which will be used in Section 4 for the proof of Theorem 2(ii).

Corollary 3.1. *For any $\delta > 0$, there exists $\tilde{K} > 0$ such that in the torus graph G_n^τ,*

$$
\lim_{n\to\infty} Pr\left\{\textit{Every node can find its } (1+\delta)\log_2 n \textit{ nearest neighbors within distance } \sqrt{\frac{\tilde{K}\log n}{n}}\right\} = 1.
$$

Lemma 3.4. *Suppose* $\mathbb{S}$ *is tessellated by small axis-parallel squares* S_k^n, $k = 1, \cdots, n/K \log n$, *as in Lemma 3.3, and* C_k^n *is the disk inscribing* S_k^n. *For all* $0 < \theta < 2\pi$, *define* A_k^n *as follows:*

$$A_k^n \;:=\; \{\textit{There is a node } X \textit{ in } C_k^n, \textit{ with } Disk(X, \phi_\theta(n)) \subseteq C_k^n, \textit{ which}$$
$$\textit{is not } \theta\textit{-covered by its } \phi_\theta(n) \textit{ nearest neighbors.}\}$$

Then there exists $K^- = K^-(\theta)$, *such that, for all* $n_1 \geq K^- \log n$,

$$Pr\{A_k^n | N(C_k^n) = n_1\} \geq c_3 n^{-(1-\delta)},$$

where $c_3 > 0$ *is a constant.*

Proof. Given $N(C_k^n) = n_1$, the n_1 nodes can be considered as iid random variables with uniform distribution. Denote them as $Y_1, \cdots, Y_{n_1}$. We first show that, for large enough K^-, there exists constant $c_3 > 0$ such that Pr$\{$ $Disk(Y_1, \phi_\theta(n))$ is within $C_k^n\} > c_3$, for all n large.

Denote C_k^n's center and radius as O_k and r_n respectively. By the law of large numbers, for K^- such that $\frac{K^- \log n}{\phi_\theta(n)} > 9$, i.e., $K^- > \frac{9(1-\delta)}{\log(2\pi/(2\pi-\theta))}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} Pr\{\text{There are at least } \phi_\theta(n) \text{ nodes within distance } r_n/3 \text{ of } O_k \,|\, N(C_k^n) \geq K^- \log n\} \\
&= \lim_{n\to\infty} Pr\{\text{At least a fraction } \tfrac{\phi_\theta(n)}{K^- \log n} (< 1/9) \text{ of nodes} \\
&\qquad\qquad \text{fall within distance } r_n/3 \text{ of } O_k \,|\, N(C_k^n) \geq K^- \log n\} = 1.
\end{aligned}$$

Once there are at least $\phi_\theta(n)$ nodes within distance $r_n/3$ of O_k, then the event $\{Y_1$ is within that range$\}$ implies that $Disk(Y_1, \phi_\theta(n)) \subseteq C_k^n$. Since $Pr\{Y_1$ is within distance $r_n/3$ of $O_k\} = 1/9$, we know there exists $c_3 > 0$ such that $Pr\{Disk(Y_1, \phi_\theta(n))$ is within C_k^n $\} > c_3$.

Denote $Sec_0(\theta)$ as the sector of C_1^n with angle θ which begins from the North and spans clockwise. Then we have,

$$\begin{aligned}
&Pr\{A_k^n | N(C_k^n) = n_1\} \\
&\geq Pr\{A_k^n; Disk(Y_1, \phi_\theta(n)) \subseteq C_k^n \,|\, N(C_k^n) = n_1\} \\
&= Pr\{Disk(Y_1, \phi_\theta(n)) \subseteq C_k^n \,|\, N(C_k^n) = n_1\} \cdot Pr\{A_k^n \,|\, Disk(Y_1, \phi_\theta(n)) \subseteq C_k^n\} \\
&\geq c_3 \cdot Pr\{\text{All } Y_1\text{'s } \phi_\theta(n) \text{ nearest neighbors are outside } Sec_0(\theta)\} \\
&= c_3 \left(\frac{2\pi - \theta}{2\pi}\right)^{(1-\delta)\log_{2\pi/(2\pi-\theta)} n} \\
&= c_3 n^{-(1-\delta)}.
\end{aligned}$$

□

Proof of part (ii) of Theorem 1.

For the $K^-(\theta)$ required in Lemma 3.4, we can select K large enough according to Lemma 3.3, such that, if one tessellates $\mathbb{S}$ according to this K, (1) holds. Now let $a := n/(K \log n)$, the total number of the small squares in the tessellation of $\mathbb{S}$. Define a set of integer vectors as follows,

$$D := \left\{ (n_1^{(1)}, n_1^{(2)}, \cdots, n_1^{(a)}) : \; n_1^{(k)} \geq K^- \log n, \forall k; \sum_{k=1}^{a} n_1^{(k)} \leq n \right\}.$$

Then

$$\begin{aligned}
&Pr(\text{Every node } X \text{ with } Disk(X, \phi_\theta(n)) \subseteq \mathbb{S} \text{ is } \theta\text{-covered}) \\
&\quad \leq Pr(\bigcap_{k=1}^{a} \bar{A}_k^n) \\
&\quad = \left(\sum_{(n_1^{(1)}, \cdots, n_1^{(a)}) \in D} + \sum_{otherwise} \right) Pr\left(\bigcap_{k=1}^{a} \bar{A}_k^n ; N(C_k^n) = n_1^{(k)}, 1 \leq k \leq a \right) \\
&\quad \overset{(b)}{=} o(1) + \\
&\qquad \sum_{(n_1^{(1)}, \cdots, n_1^{(a)}) \in D} Pr\left(N(C_k^n) = n_1^{(k)}, 1 \leq k \leq a \right) \cdot Pr\left(\bigcap_{k=1}^{a} \bar{A}_k^n | N(C_k^n) = n_1^{(k)}, 1 \leq k \leq a \right),
\end{aligned}$$

where (b) holds because of Lemma 3.3.

For every k, event A_k^n is completely determined by the locations of the nodes *within* disk C_k^n. If $\{N(C_k^n)\}$ is given, i.e. $N(C_k^n) = n_k$ for $1 \leq k \leq a$, then the nodes within one small disk are uniformly distributed in it, and are independent of all other small disks. Thus A_k^n, $1 \leq k \leq a$, are conditionally independent of each other given $\{N(C_k^n), 1 \leq k \leq a\}$. So we have

$$\begin{aligned}
&Pr(\text{Every node } X \text{ with } Disk(X, \phi_\theta(n)) \subseteq \mathbb{S} \text{ is } \theta\text{-covered}) \\
&\quad = o(1) + \sum_{(n_1^{(1)}, \cdots, n_1^{(a)}) \in D} Pr\left(N(C_k^n) = n_1^{(k)}, 1 \leq k \leq a \right) \cdot \prod_{k=1}^{a} Pr\left(\bar{A}_k^n \,|\, N(C_k^n) = n_1^{(k)} \right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(d)}{\leq} o(1) + \sum_{(n_1^{(1)},\cdots,n_1^{(a)})\in D} Pr\left(N(C_k^n) = n_1^{(k)}, 1 \leq k \leq a\right) \cdot \prod_{k=1}^{a}(1 - c_3 n^{-(1-\delta)}) \\
&\leq o(1) + (1 + o(1)) \cdot \prod_{k=1}^{a}(1 - c_3 n^{-(1-\delta)}) \\
&= o(1) + (1 + o(1)) \cdot \exp\left\{a\left(-c_3 n^{-(1-\delta)} + o(n^{-(1-\delta)})\right)\right\} \\
&= o(1) + (1 + o(1)) \cdot \exp\left\{\frac{n}{K\log n}\left(-c_3 n^{-(1-\delta)} + o(n^{-(1-\delta)})\right)\right\} \\
&= o(1) + (1 + o(1)) \cdot \exp\left\{\frac{-c_3 n^{\delta}}{K\log n}(1 + o(1))\right\} \\
&\to 0, \text{ as } n \to \infty,
\end{aligned}
$$

where (d) holds because of Lemma 3.4. $\square$

4 $(1+\delta)\log_2 n$ neighbors are sufficient for connectedness of the network

In this section we prove Theorem 2. Within this section, denote $G(n, (1+\delta)\log_2 n)$ in short by G_n, and $G^\tau(n, (1+\delta)\log_2 n)$ by G_n^τ.

We need the following lemma.

Lemma 4.1. *Suppose U_1, V_1, U_2, V_2 are four nodes in G_n^τ, and V_1 is one of U_1's ϕ_n nearest neighbors in G_n^τ, while V_2 is U_2's. Furthermore, U_1V_1 and U_2V_2 cross each other at point W; see Figure 3. Then, nodes U_1, V_1, U_2, V_2 belong to the same connected component of G_n^τ.*

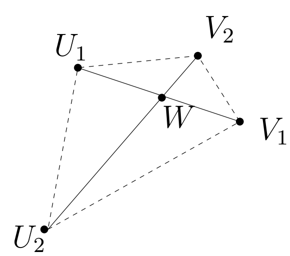

Figure 3: Crossing edges of G_n^τ.

Proof. In order to show that U_1V_1 and U_2V_2 belong to the same connected component of G_n^τ, it suffices to show that either $\min\{|U_2V_1|, |U_2U_1|\} < |U_2V_2|$, or $\min\{|U_1U_2|, |U_1V_2|\} < |U_1V_1|$. Otherwise, we would have $|U_2V_1| \leq |U_2V_2|$, $|U_2U_1| \leq |U_2V_2|$, $|U_1U_2| \geq |U_1V_1|$, and

$|U_1V_2| \geq |U_1V_1|$. This implies that $\angle V_2V_1U_1 \geq \angle U_1V_2V_1$ and $\angle U_2V_2V_1 \geq \angle V_2V_1U_2$. Since $\angle U_1V_2V_1 > \angle U_2V_2V_1$, we will have $\angle V_2V_1U_1 > \angle U_2V_2V_1 \geq \angle V_2V_1U_2$. This is a contradiction, and so U_1V_1 and U_2V_2 belong to the same connected component of G_n^τ. $\square$

Remark 4.1. *It is easy to verify that the result still holds if one replaces G_n^τ with G_n.*

Now we prove Theorem 2(i), i.e. $\lim_n\{G_n^\tau \text{ is connected}\} = 1$.

Proof of part (i) of Theorem 2.

Since δ is positive, by applying Theorem 1 (i) we know that there exists an angle $\theta^* \in (0, \pi)$ such that $\lim_n Pr\{G_n^\tau \text{ is } \theta^*\text{-covered}\} = 1$. From now on, we assume that G_n^τ is θ^*-covered, and we prove its connectedness by a geometric contradiction argument.

First notice that, for each realization of G_n^τ, both the set of degrees of all the angles between edges, and the set of lengths of all the edges, are finite. This is because G_n^τ is constructed as the periotic repetition of a finite variation.

If G_n^τ is not connected, then there are two *separate* components $\mathcal{C}_\mathcal{A}$ and $\mathcal{C}_\mathcal{B}$. By Lemma 4.1, no edge of $\mathcal{C}_\mathcal{A}$ can cross an edge of $\mathcal{C}_\mathcal{B}$. So because of the finiteness of the sets of both the angle degrees and the edge lengths, the minimum distance between a point in $\mathcal{C}_\mathcal{A}$ and a point in $\mathcal{C}_\mathcal{B}$ is strictly positive. Note that a point here need not be a vertex (one of the original nodes); it could be a point on an edge. Now we show that one can find a pair of new points, one in $\mathcal{C}_\mathcal{A}$ and one in $\mathcal{C}_\mathcal{B}$, with strictly smaller distance, thus leading to a contradiction.

Assume the two points achieving the minimum distance are $X \in \mathcal{C}_\mathcal{A}$ and $Y \in \mathcal{C}_\mathcal{B}$; see Figure 4. Draw the line L_X crossing X and perpendicular to XY. Draw line L_Y similarly. There are three possible cases: 1) If either X or Y is a node (vertex), say X is, then X must have an edge XZ to the lower side of L_X towards L_Y, because G_n^τ is θ^*-covered; see Case (i) of Figure 4. Then one can find a point Z' on the edge such that $|Z'Y| < |XY|$. 2) If either X or Y is a crossing point of two edges of G_n^τ, say X is, then one of the two edges must cross L_X; see Case (ii) of Figure 4. Thus one can also find a point Z' such that $|Z'Y| < |XY|$. 3) If neither X nor Y is a node or crossing point, then the two edges in which X and Y lie, respectively, must be parallel to each other and perpendicular to XY; see Case (iii) of Figure 4. Consider the end points of the two edges to the right of XY. Suppose the nearer one is X'. Let Y' be the point on the edge in which Y lies such that $X'Y'$ is parallel to XY.

Then $|X'Y'| = |XY|$. Applying the same argument as earlier for Cases 1) and 2) to $|X'Y'|$, since X' is now a node, we again get a contradiction. Thus, assuming a strictly positive distance between two components of G_n^τ leads to a contradiction; so G_n^τ is connected. $\square$

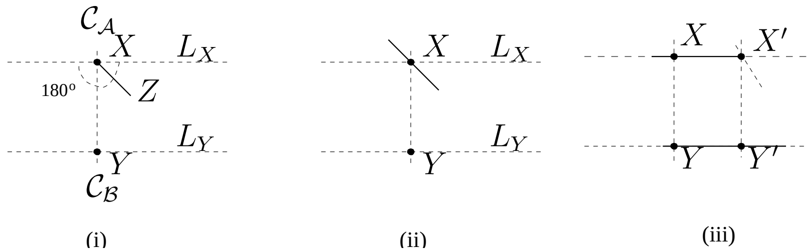

Figure 4: Case (i) X or Y is a node; Case (ii) X or Y is a crossing point; Case (iii) Neither X nor Y is a node or crossing point.

In order to prove Theorem 2(ii), we need to study the geometric properties of the graph G_n^τ. The following definition and lemmas are needed.

Definition 4.1. *For a point X belonging to an edge of G_n^τ, a sequence of points $\{Y_k,$ $1 \le k \le m+1\}$, is called an* **enclosing circle of** *X if (i) Y_kY_{k+1}, $1 \le k \le m$, is an edge or part of an edge of G_n^τ, with $Y_{m+1} = Y_1$; (ii) $\{Y_kY_{k+1},\ 1 \le k \le m\}$ forms a simple curve on the plane; (iii) X is within that curve and belongs to the same component of G_n^τ containing $\{Y_k,\ 1 \le k \le m+1\}$.*

Lemma 4.2. *If G_n^τ is θ-covered for $\theta < \pi$, and no edge of G_n^τ has length more than $L := \sqrt{\frac{\tilde{K}\log n}{\pi n}}$ for $\tilde{K} > 0$, then there exists an enclosing circle for any point X belonging to an edge of G_n^τ. Furthermore, the Euclidean diameter (the maximum Euclidean distance between every pair of points) of the enclosing circle is no more than $M_1 L$, where M_1 only depends on θ.*

Proof. Suppose X is on an edge of G_n^τ with A_1 as an end node. Denote the line containing XA_1 as Γ; see Figure 5. Since G_n^τ is θ-covered, there is a node A_2 with $\angle XA_1A_2 \le \theta$ on one side ("the upper side") of Γ. Again because of the θ-coverage, there is a node A_3 with $\angle XA_2A_3 \le \theta$. So continuing, we can find a sequence of nodes $\{A_k,\ k \ge 1\}$, which satisfy: (i) A_kA_{k+1} is an edge of G_n^τ; and (ii) $\angle A_kXA_1$, $k \ge 1$, keeps expanding counterclockwise, i.e., $\angle A_{k+1}XA_1 > \angle A_kXA_1$ for $k \ge 1$. Considering the other side of line Γ similarly, we can find another sequence of nodes $\{A'_k,\ k \ge 1\}$, spanning clockwise (letting $A'_1 := A_1$).

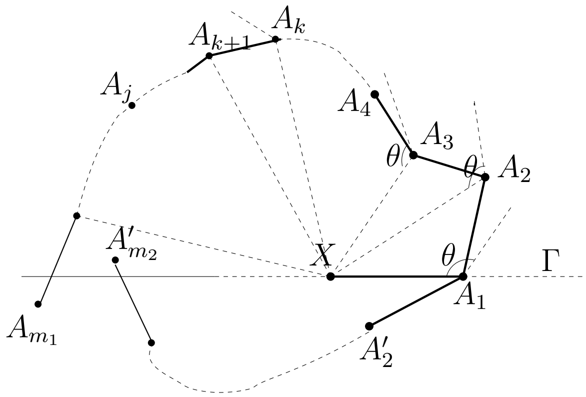

Figure 5: Finding an enclosing circle.

Without loss of generality, assume $\theta \in (\pi/2, \pi)$. Let β^* be an angle such that $\theta < \beta^* + \theta < \pi$. Now we show that there exists constant $\epsilon^* > 0$, only depending on θ and β^*, such that

$$|A_jX| \leq \exp(\frac{\pi}{\sin(\theta+\beta^*)}) \cdot (L + \frac{L}{\epsilon^*}), \ \forall j \text{ such that } \angle A_jXA_1 < \pi. \tag{4}$$

For $k < j-1$, consider the triangle formed by X, A_{k+1}, and A_k; see Figure 6. For brevity, let $a_k := |A_kX|$, $a_{k+1} := |A_{k+1}X|$, $\delta_k := |A_kA_{k+1}|$, $\alpha_k := \angle A_{k+1}XA_k$.

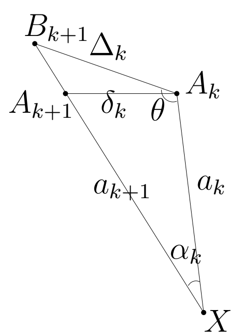

Figure 6: The triangle formed by XA_kA_{k+1}.

Since no edge of G_n^τ is longer than L, we have $L/a_k \geq \delta_k/a_k \geq \sin\alpha_k$. So there exists $\epsilon^* > 0$ such that $\alpha_k \leq \beta^*$ (less than $\pi - \theta$) whenever $L/a_k \leq \epsilon^*$. Moreover, once $\alpha_k < \pi - \theta$, if one draws an angle of θ at A_k with one side coinciding with A_kX, the line containing XA_{k+1} will cross the other side at a point B_{k+1} with $|B_{k+1}X| \geq |A_{k+1}X|$; see Figure 6. So, in summary, we now have

$$|B_{k+1}A_k| \ = \ a_k \cdot \frac{\sin\alpha_k}{\sin(\theta+\alpha_k)} \leq a_k \cdot \frac{\sin\alpha_k}{\sin(\theta+\beta^*)}, \text{ if } a_k \geq L/\epsilon^*.$$

Denoting $\Delta_k := |B_{k+1}A_k|$, we thus obtain the following relations:

$$a_{k+1} \leq \begin{cases} a_k + \Delta_k \leq \left(1 + \frac{\sin\alpha_k}{\sin(\theta+\beta^*)}\right) a_k, \text{ if } a_k \geq L/\epsilon^*; \\ a_k + L, \text{ otherwise.} \end{cases}$$

It is easy to verify that the sequence $\{a_k\}$ is dominated by the following sequence:

$$b_{k+1} := b_k \left(1 + \frac{\sin\alpha_k}{\sin(\theta+\beta^*)}\right), \text{ with } b_1 = L + \frac{L}{\epsilon^*}.$$

We then have,

$$\begin{aligned} a_{k+1} &\leq b_{k+1} \\ &= \prod_{i=1}^{k+1}\left(1 + \frac{\sin\alpha_i}{\sin(\theta+\beta^*)}\right) \cdot \left(L + \frac{L}{\epsilon^*}\right) \\ &= \exp\left\{\sum_{i=1}^{k+1} \log\left(1 + \frac{\sin\alpha_i}{\sin(\theta+\beta^*)}\right)\right\}\left(L + \frac{L}{\epsilon^*}\right) \\ &\leq \exp\left\{\sum_{i=1}^{k+1} \frac{\sin\alpha_i}{\sin(\theta+\beta^*)}\right\}\left(L + \frac{L}{\epsilon^*}\right) \\ &\leq \exp\left\{\sum_{i=1}^{k+1} \frac{\alpha_i}{\sin(\theta+\beta^*)}\right\}\left(L + \frac{L}{\epsilon^*}\right) \\ &\leq \exp\left\{\frac{\pi}{\sin(\theta+\beta^*)}\right\}\left(L + \frac{L}{\epsilon^*}\right), \end{aligned}$$

for all $k \leq j-1$. Hence we have proved (4), which means that all the points along the sequence $A_1A_2\cdots A_j$ are within distance $M'L$ of X, for some constant $M' > 0$ depending only on θ.

Notice now that since G_n^τ is a period repetition of a finite set, $\{\angle A_kXA_1,\ s.t.\ \angle A_kXA_1 < \pi\}$ can only include a finite number of possible values because nodes in $\{A_k, s.t.\ \angle A_kXA_1 < \pi\}$ are within a finite range of X. Since every such value is strictly positive, we know there exists $m_1 > 0$ such that $\angle A_{m_1}XA_1 > \pi \geq \angle A_{m_1-1}XA_1$; see Figure 5.

Similarly, we can show that there exists $M'' > 0$ such that all the points along the sequence $\{A_1'A_2'\cdots A_l'\}$ are within distance $M''L$ of X, as long as $\angle A_l'XA_1 < \pi$. Moreover, there exists $m_2 > 0$ such that $\angle A'_{m_2}XA_1 > \pi \geq \angle A'_{m_2-1}XA_1$; see Figure 5.

Without loss of generality, assume that the crossing point between $A'_{m_2-1}A'_{m_2}$ and Γ is nearer to X than the crossing point between $A_{m_1-1}A_{m_1}$ and Γ; see Figure 5. Then sequence $A'_k A'_{k+1}$, $k = 1, 2, \cdots$, must cross $XA_1A_2 \cdots A_{m_1}$ at some point, say Y. Thus we have found an enclosing circle of X, and all the points on the circle are within distance $2(\max\{M', M''\} + 1) \cdot L$ of each other. $\square$

Lemma 4.3. *For $\alpha > 0$ and $i, j \in \{0, 1\}$, let $\mathbb{S}_n^{i,j}(\alpha)$ be the axis-parallel square of size $n^{-\alpha}$ within $\mathbb{S}$ and with one corner at (i, j); see Figure 8. Define the event*

$$E_n^{i,j}(\alpha) := \{\textit{Every node in } \mathbb{S}_n^{i,j}(\alpha) \textit{ is } \pi/4\textit{-covered in } G_n^\tau\}.$$

Then there exists $\alpha^* \in (0, 1/2)$ such that $\lim_n Pr\{E_n^{i,j}(\alpha^*)\} = 1$, for all $i, j \in \{0, 1\}$.

Proof. By symmetry, we only need to consider $E_n^{0,0}(\alpha)$. Define $A_k(\pi/4) := \{$Node X_k is NOT $\pi/4$-covered by its ϕ_n nearest neighbors in $G_n^\tau\}$, $k = 1, \cdots, n$. It is easy to show, as in Lemma 3.2, that for any integer $M > 0$,

$$\begin{aligned} Pr\{A_k(\pi/4)\} &\leq \left\lceil \frac{2\pi}{\pi/4} \right\rceil M \left(\left(\frac{2\pi - \frac{\pi}{4} + \frac{\pi}{4}/M}{2\pi} \right)^{\phi_n} + c_1 e^{-c_2 n} \right) \\ &= 8M \left(\frac{7}{8} + \frac{1}{8M} \right)^{\phi_n} + 8M c_1 e^{-c_2 n}, \end{aligned}$$

where c_1, c_2 are two positive constants.

Define $E_k(\alpha) := \{$Node X_k is in $\mathbb{S}_n^{0,0}(\alpha)\}$, $k = 1, \cdots, n$. Then

$$\begin{aligned} Pr\left\{\overline{E_n^{0,0}(\alpha)}\right\} &\leq Pr\left\{\cup_{k=1}^n A_k(\frac{\pi}{4}) \text{ and } E_k(\alpha)\right\} \\ &\leq \sum_{k=1}^n Pr\left\{A_k(\frac{\pi}{4}) \text{ and } E_k(\alpha)\right\} \\ &= n \cdot Pr\left\{A_1(\frac{\pi}{4}) \text{ and } E_1(\alpha)\right\} \\ &= nPr\{A_1(\frac{\pi}{4})\} \cdot Pr\{E_1(\alpha)\} \\ &\leq n \cdot n^{-2\alpha} \cdot \left(8M \exp\left\{\phi_n \log\left(\frac{7}{8} + \frac{1}{8M}\right)\right\} + c_1 e^{-c_2 n}\right) \\ &= 8M \exp\left\{(1 - 2\alpha)\log n - \frac{(1+\delta)}{\log 2} \log n \cdot \log\left(\frac{8}{7}\frac{7M}{7M+1}\right)\right\} + o(1). \end{aligned}$$

Since we can select M to be sufficiently large, it therefore suffices to show that there exists an $\alpha^* \in (0, 1/2)$, such that $(1 - 2\alpha^*) - \frac{(1+\delta)}{\log 2}\log(\frac{8}{7}) < 0$. Actually this is just equivalent to requiring $\alpha^* > \frac{1}{2}(1 - \frac{(1+\delta)\log\frac{8}{7}}{\log 2})$, and so we see easily the existence of such an α^*. $\square$

Now we prove Theorem 2(ii), that, for all $\delta > 0$,

$$\lim_n Pr\{G(n, (1+\delta)\log_2 n) \text{ is connected}\} = 1.$$

Proof of Theorem 2 (ii).

By Theorems 1(i) and 2(i), we know that there is a $\theta^* < \pi$ such that

$$Pr\{G_n^\tau \text{ is } \theta^*\text{-covered and connected}\} \to 1, \text{ as } n \to \infty.$$

By Corollary 3.1, we know that there exists $K^* > 0$, such that, with probability converging to one, every edge of G_n^τ has length no more than $L^* := \sqrt{\frac{K^* \log n}{\pi n}}$.

By Lemma 4.3, we know there exists $\alpha^* \in (0, 1/2)$, such that, with probability converging to one as n goes to infinity, every node in $\mathbb{S}_n^{i,j}(\alpha^*)$, $i, j \in \{0, 1\}$, is $\pi/4$-covered.

So now we assume that: (i) G_n^τ is connected and θ^*-covered; (ii) Every edge in G_n^τ has length no more than $L^* := \sqrt{\frac{K^* \log n}{\pi n}}$; and (iii) Every node in $\mathbb{S}_n^{i,j}(\alpha^*)$, $i, j \in \{0, 1\}$, is $\pi/4$-covered.

The proof then proceeds in three steps.

Step 1: *By Lemma 4.2, there exists $M_1 > 0$ such that any point on an edge of G_n^τ can find an enclosing circle with diameter no more than $M_1 L^*$. For any pair of nodes $X, Y \in G_n$, such that they are at a distance of at least $2M_1 L^*$ away from the boundary of $\mathbb{S}$, we now prove that they belong to the same connected component in G_n.*

In graph G_n^τ, draw the line segment connecting X and Y; see Figure 7. Then XY will cross an enclosing circle of X at point Y_1, and the diameter is no more than $M_1 L^*$. $Y_1 Y$ will then cross an enclosing circle of Y_1 at some point Y_2. So continuing, we obtain a sequence of points $X, Y_1, \cdots, Y_{J-1}, Y_J$, with the following properties: (i) Y_j, $1 \le j \le J$, are points on edges of G_n^τ; (ii) X and $\{Y_j\}$ are within the same component of G_n^τ; and (iii) node Y is within an enclosing circle of Y_J.

Since G_n^τ is connected, there is a path of G_n^τ that starts from Y and ends at an edge of the enclosing circle of Y_J. Because both X and Y are at least at a distance $2M_1 L^*$ away from the boundary of $\mathbb{S}$, every edge of the enclosing circles is within the boundary of $\mathbb{S}$. By the definitions of G_n and G_n^τ, they will remain in graph G_n.

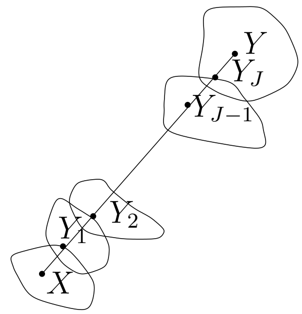

Figure 7: X and Y are connected by enclosing circles.

By Remark 4.1 and Lemma 4.1, we know that two crossing edges of G_n must belong to the same (connected) component of G_n. So we have proved that X and Y belong to the same (connected) component of G_n. Hence we conclude that all the nodes lying in the center square, $\mathbb{S}^*$, which is at a distance $2M_1L^*$ from the boundary of $\mathbb{S}$, belong to the same (connected) component of G_n; see Figure 8.

Step 2: *Let $\gamma \in (\alpha^*, 1/2)$, and now we prove that every node in $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$, is within the same component of G_n as that of the nodes in $\mathbb{S}^*$.*

By symmetry, we only need to consider the case for $\mathbb{S}_n^{0,0}(\gamma)$. Suppose node X is in $\mathbb{S}_n^{0,0}(\gamma)$; see Figure 8. Draw an angle of $\pi/4$, $\angle AXB$, towards O, the center of $\mathbb{S}$, with XO dividing the angle into equal parts. Since $n^{-\gamma}$ is much less than $n^{-\alpha^*}$, XA and XB will be entirely contained in $\mathbb{S}_n^{0,0}(\alpha^*)$ before they enter $\mathbb{S}^*$.

Because X is $\pi/4$-covered, it can find a neighboring node Y_1 in $\angle AXB$ towards $\mathbb{S}^*$. Draw line Y_1A_1 parallel to XA, and line Y_1B_1 parallel to XB. Then $\angle A_1Y_1B_1$ is contained in $\angle AXB$, and node Y_1 can find a neighbor Y_2 in $\angle A_1Y_1B_1$. Since both of the two boundary lines of $\angle A_1Y_1B_1$ have a segment in $\mathbb{S}^*$ with length at least $1/2$, node Y_2 cannot be such that Y_1Y_2 crosses $\mathbb{S}^*$ and yet Y_2 is on the opposite side of Y_1. This is because $|Y_1Y_2|$ is no larger than $L^* = \sqrt{\frac{K^* \log n}{\pi n}}$, which is far less than $1/2$. This procedure can be continued till a node in $\mathbb{S}^*$ is finally reached. Hence X is connected to the nodes in $\mathbb{S}^*$. So we conclude that every node in $\mathbb{S}_n^{0,0}(\gamma)$ is in the same (connected) component of G_n as that of a node in $\mathbb{S}^*$, and so is every node in $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$.

Step 3: *Now we prove that every node of G_n that is neither in $\mathbb{S}^*$ nor in $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$,*

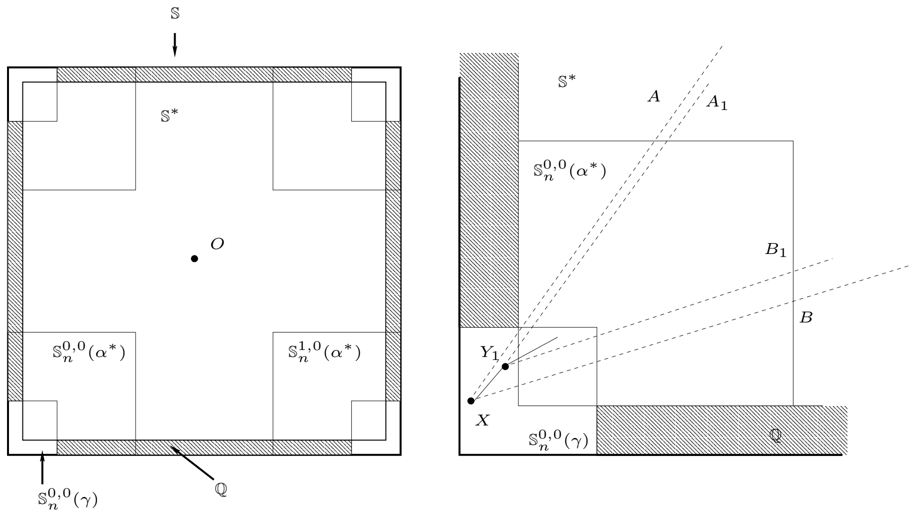

Figure 8: Left: Squares $\mathbb{S}$, $\mathbb{S}^*$, $\mathbb{S}_n^{i,j}(\alpha^*)$ and $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$, and $\mathbb{Q}$. Right: A zoom-in of the left lower corner of $\mathbb{S}$.

is in the same (connected) component of G_n as that of a node in $\mathbb{S}^*$ or $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$.

Again by symmetry, we only need to show this for the nodes in the rectangle, say $\mathbb{Q}$, defined by points $(n^{-\gamma}, 0)$, $(n^{-\gamma}, 2M_1L^*)$, $(1 - n^{-\gamma}, 2M_1L^*)$, and $(1 - n^{-\gamma}, 0)$; see Figure 8. Suppose Z is such a node in $\mathbb{Q}$. Since Z is θ^*-covered and $\theta^* < \pi$, Z can find a neighbor, say Z_1, that has a strictly larger y-coordinate than itself. Continuing this, we can find a path of G_n consisting of nodes Z_k, $k \geq 1$, with strictly increasing y-coordinates. Because $|Z_kZ_{k+1}| \leq M_1L^*$, which is much less than the sizes of $\mathbb{S}_n^{0,0}(\gamma)$, $\mathbb{S}_n^{1,0}(\gamma)$ and $\mathbb{S}^*$, the path must end up reaching a node in one of these three squares. This means every node in $\mathbb{Q}$ belongs to the same component of G_n to which all the nodes in $\mathbb{S}^*$ and $\mathbb{S}_n^{i,j}(\gamma)$, $i, j \in \{0, 1\}$ belong.

Summing the above three steps, we have proved that the graph G_n is connected. □

5 $\boldsymbol{\theta}$-coverage and its application to the performance analysis of wireless networks

The uniform distribution is a simple model for conducting performance analysis, which, as noted earlier, dates back to the pioneering work of E. N. Gilbert [5]. It has continued to be an important model for the study of wireless networks, especially for topology control and routing algorithms. Our results, which address properties of the network, can be used to analyze the performance of such algorithms.

The concept of θ-coverage was used in [11, 12] as the key idea to establish an efficient topology for wireless networks. In these works, the model is deterministic, and it is shown that $\theta = 5\pi/6$ is the critical sector angle for ensuring connectivity. Based on the analysis and results in our paper, we can assess the performance of their algorithm for uniformly distributed networks with many nodes. First, we see that each node only needs to guarantee π-coverage, as opposed to $5\pi/6$-coverage, in order to ensure that the whole network is connected. Such π-coverage can be established, at least asymptotically, by just letting every node connect to its $\log_2 n$ nearest neighbors, which does not require the capability for each node to estimate angles, as used in those algorithms. Secondly, the average degree of a node in such θ-covered networks is about $O(\log n)$, with high probability.

Greedy forwarding is a key idea in many graphical or geographical routing algorithms, e.g., MFR (Most Forward within Radius) [2], Cartesian routing [25], GPSR (Greedy Perimeter Stateless Routing) [26], GRA (Geographical Routing Algorithm) [27], and GOAFR (Greedy Other Adaptive Face Routing) [28]. Because pure greedy forwarding could result in stopping at an intermediate node when a packet cannot find a node closer than itself, many of the routing algorithms implement strategies to circumvent this. If every node in the network is connected to sufficiently many neighbors, at the scale of $O(\log n)$, so that the resulting network is θ-covered, then we can determine the size of the void that a packet needs to detour. This is because now every node is surrounded by an enclosing circle (see Definition 4.1), and its size is as given by Lemma 4.2. This provides some insight into designing new routing algorithms.

The results on θ-coverage can also be used to obtain an estimate of stretching factors for certain topology control and routing algorithms. For example, in [29], a protocol to let every node maintain a specified degree is proposed. Since a node needs to connect to $\Theta(\log n)$ nearest neighbors [8] to maintain connectivity, the network will be θ-covered with high probability. Once θ is less than π, the result in [12] yields the stretching factor of the resulting graph.

6 Concluding Remarks

To study the behavior of the geometric structure of wireless networks with a large number of nodes, we have formally defined and examined the notion of θ-coverage, a concept that has appeared in several previous research works. In this paper, a network is generated as n nodes uniformly distributed within a unit square, with every node connecting to the same number of nearest neighbors. We have shown that the exact threshold function for θ-coverage, including even the pre-constant, is $\log_{\frac{2\pi}{2\pi-\theta}} n$, for any $\theta \in (0, 2\pi)$. Using a new argument solely involving geometry, we have also shown that π-coverage with high probability implies overall connectivity with high probability.

As discussed in the paper, since the uniform distribution is a simple and favorite model for analysis and simulation studies, these results provide some insights into the performance analysis of wireless networks. However, how quick the probability converges with different pre-constant remains unknown, and this will be useful to study for the behavior of networks of modest size. Also of interest is the area coverage problem, i.e., how to cover some area such that every point in the area is covered by several nodes. This has been studied for the distance-based connection model in [30], while for the number of neighbors based connection model, or the sector-based connection model, it remains open.

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