A Study of the Routing Capacity Regions of Networks

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***Abstract*—The routing capacity region of networks with multiple unicast sessions can be characterized using Farkas' lemma as an infinite set of linear inequalities. In this paper this result is sharpened by exploiting properties of the solution satisfied by each rate-tuple on the boundary of the capacity region, and a finite description of the routing capacity region which depends on network parameters is offered. For the special case of undirected ring networks additional results on the complexity of the description are provided.**

***Index Terms*—routing, network capacity, multicast sessions, linear programming.**

I. Introduction

ROUTING protocols underlie the traditional strategies for communicating information in data networks. The newer paradigm of *network coding* (see, e.g., [1], [9]) offers potentially more reliable coding schemes with higher throughput and error correcting capabilities, but it is costlier to implement (see, e.g., [8]). It is important to better understand routing because of its significance to most practical networks. Furthermore, routing capacity regions provide inner bounds for the corresponding network coding capacity regions, and there are cases where the two capacity regions for the same networking problem are identical (e.g., [7], [11], [16], [15]).

We here focus on the routing capacity regions for a general class of networks supporting multiple multicast sessions. Much of the routing literature focuses on the *multicommodity flow* problem in which every message in the network is transmitted from a source to a unique destination. The famous *max-flow min-cut* theorem provides bounds on the rates of the different messages being simultaneously transmitted between the different source-destination pairs. [12, Part VII] surveys many of the cases where this bound is tight. The paper [5] is an early reference which provides an example where the bound is not tight.

The papers [6], [10] establish a special case of Farkas' lemma (see, e.g., [13, §7.3]) sometimes called the "Japanese theorem;" this result provides necessary and sufficient conditions for determining if an arbitrary set of rates has a feasible routing solution for a networking problem with multiple unicast sessions. A shortcoming of this result is that the description of the routing capacity region for the multicommodity flow problem involves the intersection of an infinite set of inequalities.

While the assumption of a unique destination is natural for many application areas of network optimization, for communication problems we want to allow for the possibility of messages from a single transmitter to multiple receivers. Using standard terminology from communications, we further refer to *unicast* or *multicast* messages to indicate if the set of destinations is a single terminal or a set of multiple terminals. We will use the terms unicast and multicommodity flow interchangeably. In the network coding literature, [2] considers a routing problem similar to the maximum concurrent flow problem [14] for directed, acyclic graphs; that paper introduces a notion of a scalar routing capacity for a network and specifies a linear program to find it.

Just as one can form a system of linear inequalities to describe a multicommodity flow problem, one can likewise study the general multiple multicast problem where every terminal in the network potentially has messages for every non-empty subset of the other accessible terminals. For a multicommodity flow or unicast session the flow for a session which enters an intermediate vertex along the path is identical to the flow for that session emanating from that vertex. The natural generalization for multicast sessions constrains each spanning subtree carrying flow to have all of the edges or nodes of that subtree transmit the same flow. The set of flows along the various paths and subtrees are jointly constrained by the capacities of the edges or nodes in the graph, and the corresponding fractional routing capacity region can in principle be determined by Fourier-Motzkin elimination [13]. However, as the results of Fourier-Motzkin elimination are specific to the set of constraints for a particular networking problem, our objective is to offer a characterization which will apply to many networking problems.

The papers [15] and [16] extend the Japanese theorem to networks supporting multiple multicast sessions and describe an inequality elimination technique to help study the network coding capacity region of special cases of the multiple multicast problem on an undirected ring network. The technique determines the minimal necessary and sufficient set of inequalities among the infinite set of inequalities specified by the Japanese theorem and is a consequence of properties of the routing solution for any rate-tuple on the boundary of the routing capacity region; this technique appears to be new even

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for the special case of multicommodity flow problems. We use it to further characterize the minimal set of inequalities for general directed or undirected networks and for undirected ring networks.

Our focus in this paper is on the *size* of the coefficients of the inequalities that appear in the minimal description of the routing rate region of an undirected network. We combine the inequality elimination technique with complexity results (see, e.g., [4], [13]) on the description of a rational system of linear inequalities to bound the coefficients of the linear inequalities that describe the routing rate region. We further discuss an average case analysis of the size of linear inequalities for undirected ring networks. The outline of the paper is as follows. In Section II we formulate the problem and review some of the results of [15] and [16]. In Section III we present our results on the complexity of routing capacity regions of networks.

II. Preliminaries

A. Network Model

Consider a network that is represented by a graph $G(V,E)$, where V and E respectively denote the set of vertices and edges in the network graph. The edges are either all undirected, meaning that the sum of flow along both directions of an edge is bounded by the capacity of the edge, or all directed. Furthermore, for any subgraph S of the network let $V(S)$ and $E(S)$ respectively denote its set of vertices and edges. In a general communication setting, every vertex $v \in V$ can simultaneously send messages to arbitrary nonempty subsets of accessible vertices in $V \setminus \{v\}$. Every message M with source vertex v_s and set of destination nodes $\{v_1, \cdots, v_k\}$, is associated with a rate R_M and with a set of $t(M)$ spanning subtrees, $\{T_M^1, \cdots, T_M^{t(M)}\}$, that connect v_s to $\{v_1, \cdots, v_k\}$. For message M, let r_M^j be the amount of flow for that message that passes through spanning subtree T_M^j, $j \in \{1, \ldots, t(M)\}$. We then have $\sum_{j=1}^{t(M)} r_M^j = R_M$. In an edge-constrained network the flows passing through every edge satisfy its capacity constraint, i.e., $\sum_{M,j:\, e\in E(T_M^j)} r_M^j \le C_e$, for all $e \in E$, where C_e denotes the capacity of edge e. A rate-tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ corresponding to the sessions $M_1, \cdots, M_N$ is said to be *feasible* if for each $i \in \{1, \ldots, N\}$ there exists a non-negative assignment of $\{r_{M_i}^1, \cdots, r_{M_i}^{t(M_i)}\}$ that simultaneously satisfies $\sum_{j=1}^{t(M_i)} r_{M_i}^j = R_{M_i}$ and the edge constraints.

B. Multiple Multicast Capacity Region of Networks

The Japanese theorem characterizes the set of feasible routing rates-tuples for edge-constrained networks in problems where there are only multiple unicast sessions. Each inequality is based upon a collection of edge "distances" and is in terms of the shortest path lengths for each session. It is easy to establish an extension of the Japanese theorem to networks with multiple multicast sessions and with edge constraints. In what follows, $\mathbb{Z}^+$ denotes the set of nonnegative integers.

Theorem 1 ([16], [15])**.** *Consider the edge-constrained network $G(V,E)$. For function $f : E \to \mathbb{Z}^+$, define $L_f(T) = \sum_{e\in E(T)} f(e)$ and $\ell_f(M) = \min_{j\in\{1,\ \ldots,\ t(M)\}} L_f(T_M^j)$. The rate-tuple $\mathcal{R} = (R_{M_1}, \cdots, R_{M_N})$ is feasible in $G(V,E)$ if and only if for every function $f : E \to \mathbb{Z}^+$, the following inequality holds:*

$$\sum_{i=1}^{N} \ell_f(M_i) R_{M_i} \le \sum_{e\in E} f(e) C_e. \tag{1}$$

Theorem 1 is unsatisfying because it describes a routing capacity region with infinitely many inequalities; since Fourier-Motzkin elimination is a finite procedure we know that the collection of feasible rate-tuples is a polytope defined by a finite set of inequalities. We next discuss an approach to strengthening Theorem 1. We focus here on distance functions f that are *nontrivial* in the sense that there is at least one session M with $\ell_f(M) > 0$. We say that the resulting Japanese theorem inequality is *redundant* if for any assignment of capacities the feasible rate-tuples on the corresponding defining hyperplane all lie on the hyperplane bounding another nontrivial Japanese theorem inequality. The following result establishes that the true significance of the distance function is summarized by the collections of shortest paths for the unicast sessions and shortest subtrees for the multicast sessions corresponding to that function.

Theorem 2 (Inequality Elimination Theorem [16], [15])**.** *Consider a network $G(V,E)$ with a set of messages $\{M_1, \cdots, M_N\}$, and two distinct nontrivial distance functions f and g. If*

1) *for every $e \in E$, $f(e) = 0$ whenever $g(e) = 0$, and*
2) *for every session $M_i, i \in \{1, \cdots, N\}$ and for all $j \in \{1, \cdots, t(M_i)\}$ the property $L_g(T_{M_i}^j) = \ell_g(M_i)$ implies $L_f(T_{M_i}^j) = \ell_f(M_i)$ (but not necessarily the converse),*

then the inequality (1) corresponding to g is redundant in the description of the fractional routing capacity region given the inequality corresponding to f.

We will illustrate the result of Theorem 2 with an example. Consider an undirected triangle network with $V = \{1, 2, 3\}$ and set of messages $\{1 \to 2, 2 \to 1, 2 \to 3, 3 \to 2, 3 \to 1, 1 \to 3, 1 \to \{2,3\}, 2 \to \{1,3\}, 3 \to \{1,2\}\}$, and suppose $C_{(1,2)} = C_{(2,3)} = C_{(3,1)} = 1$. Take $g((1,2)) = 2$, $g((2,3)) = 1$, and $g((3,1)) = 3$. It is easy to verify

- $\ell_g(1 \to 2) = \ell_g(2 \to 1) = 2$ and the shortest path is $(1,2)$,
- $\ell_g(2 \to 3) = \ell_g(3 \to 2) = 1$ and the shortest path is $(2,3)$,
- $\ell_g(3 \to 1) = \ell_g(1 \to 3) = 3$ and both paths are shortest, and
- $\ell_g(1 \to \{2,3\}) = \ell_g(2 \to \{1,3\}) = \ell_g(3 \to \{1,2\}) = 3$ and the shortest subtree is $\{(1,2),(2,3)\}$.

Therefore, the halfspace corresponding to distance function g

is

$$
\begin{aligned}
&2(R_{1\to 2}+R_{2\to 1})+(R_{2\to 3}+R_{3\to 2})\\
&+3(R_{3\to 1}+R_{1\to 3}+R_{1\to\{2,3\}}+R_{2\to\{1,3\}}+R_{3\to\{1,2\}})\\
&\leq 2C_{(1,2)}+C_{(2,3)}+3C_{(3,1)}=6. \quad (2)
\end{aligned}
$$

Next take $f((1,2))=1,\ f((2,3))=0$, and $f((3,1))=1$. Notice that the shortest paths and shortest subtrees for each session under distance function g remain shortest paths and shortest subtrees for the sessions under f, although f has a second shortest path for unicast sessions $1\to 2$ and $2\to 1$ and a second shortest subtree for the multicast sessions. The halfspace corresponding to f is

$$
\begin{aligned}
&(R_{1\to 2}+R_{2\to 1})+(R_{3\to 1}+R_{1\to 3})\\
&+(R_{1\to\{2,3\}}+R_{2\to\{1,3\}}+R_{3\to\{1,2\}})\\
&\leq C_{(1,2)}+C_{(3,1)}=2. \quad (3)
\end{aligned}
$$

The theorem states that (2) is redundant for defining the routing capacity region in the presence of (3). The reason is that a polytope is defined by a collection of hyperplanes, and every *feasible* rate-tuple like $R_{1\to 2}=R_{2\to 3}=R_{3\to 1}=1,\ R_M=0,\ M\notin\{1\to 2,\ 2\to 3,\ 3\to 1\}$ which satisfies (2) with equality must also satisfy (3) with equality. The rate-tuple $R_{1\to\{2,3\}}=2,\ R_M=0,\ M\notin\{1\to\{2,3\}\}$ is an example of an *infeasible* rate-tuple which satisfies (2) with equality; it is infeasible because four units of capacity are needed to transmit two units of multicast traffic, and the network has only three units of capacity. For the problem of characterizing the routing capacity region of a network we can ignore the infeasible rate-tuples.

The papers [16] and [15] consider two special cases of the multiple multicast problem on undirected ring networks. These papers introduce the inequality elimination technique and use it to prove that distance functions with range $\{0,1\}$ are sufficient for characterizing the capacity regions. We next present new consequences of the inequality elimination theorem.

III. On the Complexity of the Routing Capacity Region

We next consider the complexity of the routing capacity region for an undirected graph. Let p and q be relatively prime integers and let $\alpha=p/q$. Define

$$size(\alpha)=1+\lceil\log_2(1+|p|)\rceil+\lceil\log_2(1+|q|)\rceil.$$

For the rational vector $\mathbf{c}=(\gamma_1,\cdots,\gamma_n)$ and the rational matrix $A=(\alpha_{i,j})_{1\leq i\leq m,1\leq j\leq n}$ we have:

$$
\begin{aligned}
size(\mathbf{c})&=n+size(\gamma_1)+\cdots+size(\gamma_n),\\
size(A)&=mn+\sum_{m,n}size(\alpha_{i,j}). \quad (4)
\end{aligned}
$$

Let $\mathbf{x}=(x_1,\cdots,x_n)^T$. Then the size of linear inequality $\mathbf{ax}\leq\alpha$ is defined as $1+size(\mathbf{a})+size(\alpha)$. The size of a system $A\mathbf{x}\leq\mathbf{b}$ of linear inequalities is defined as $1+size(A)+size(\mathbf{b})$. Next let $P\subset\mathbb{R}^n$ be a rational polyhedron. The *facet complexity* of P defined as the smallest number $\phi\geq n$ for which there exists a system $A\mathbf{x}\leq\mathbf{b}$ of rational linear inequalities defining P and each inequality in $A\mathbf{x}\leq\mathbf{b}$ has size at most ϕ.

Consider an undirected network $G(V,E)$ and the rate-tuple $\mathcal{R}=(R_{M_1},\cdots,R_{M_N})$. Let P denote the set of achievable rate-tuples in $\mathbb{R}^N$. Theorem 2 provides a systematic method to characterize the minimal description of P for the general multiple multicast problem. Here we wish to establish upper and lower bounds on the maximum values of the functions that appear in the minimal description of P. The following theorem establishes a lower bound in the special case of undirected ring networks.

Theorem 3. *Let $G(V,E)$ be an undirected ring network with vertices $1,\ 2,\ \cdots,\ |E|$ in a clockwise direction. For $i\in\{1,\ 2,\ \ldots,\ |E|-1\}$, let edge i connect vertices i and $i+1$, and let edge $|E|$ connect vertices $|E|$ and 1. There exist a distance function g that cannot be eliminated by any nontrivial distance function f with $\max_{e\in E}f(e)<2^{\lfloor(|E|-2)/3\rfloor}$.*

Proof: Consider a multicast session with $k-1$ destinations, and suppose the source and destination vertices form the set $\{v_1,\ v_2,\ \ldots,\ v_k\}$, where $1\leq v_1<v_2<\cdots<v_k\leq|E|$. Observe that a minimal spanning subtree is the subgraph consisting of the original network except for the vertices $v_j+1,\ldots,v_{j+1}-1$ and edges $v_j,\ldots,v_{j+1}-1$ for some $j\in\{1,\ldots,k\}$ (with $v_{k+1}=v_1$). Therefore, for any distance function the shortest paths or shortest subtrees for this collection of sessions will correspond to the longest paths $v_j,\ldots,v_{j+1}-1,\ j\in\{1,\ldots,k\}$.

Let $\beta=2^{\lfloor(|E|-2)/3\rfloor}$. Suppose we consider the distance function

$$g(e)=\begin{cases}\beta & e\equiv 1\pmod 3,\\ 2^{\lfloor(e-2)/3\rfloor}, & \text{otherwise}\end{cases}$$

and try to find another distance function f that eliminates g. Since the shortest broadcast trees are preserved under f, it follows that

$$\max_{i\in E}f(i)=f(e),\ e\equiv 1\pmod 3. \quad (5)$$

Furthermore, for $s\in\{2,\ \ldots,\ \lfloor|E|/3\rfloor\}$ consider the multicast session consisting of all vertices except $3s-4,\ 3s-3$, and $3s-1$. Under g, the path consisting of edges $3s-5,\ 3s-4,\ 3s-3$, and the path consisting of edges $3s-2$ and $3s-1$ are both longest, and therefore (5) implies

$$f(3s-4)+f(3s-3)=f(3s-1),\ s\in\{2,\ \ldots,\ \lfloor|E|/3\rfloor\}. \quad (6)$$

Finally, for $s\in\{2,\ \ldots,\ \lfloor(|E|+1)/3\rfloor\}$ consider the multicast session consisting of all vertices except $3s-4$ and $3s-2$. Under g, the path consisting of edges $3s-5$ and $3s-4$ and the path consisting of edges $3s-3$ and $3s-2$ are both longest, and therefore (5) implies

$$f(3s-4)=f(3s-3),\ s\in\{2,\ \ldots,\ \lfloor(|E|+1)/3\rfloor\}. \quad (7)$$

By (6) and (7), we see that

$$2f(3s-4)=f(3s-1),\ s\in\{2,\ \ldots,\ \lfloor|E|/3\rfloor\}. \quad (8)$$

Equations (5)-(8) imply that $f(e)=f(2)\cdot g(e)$ for all $e\in E$. ■

Let ϕ^* denote the maximum distance among distance functions used for a shortest description of P. Theorem 3 establishes that $\phi^*\geq 2^{\lfloor(|E|-2)/3\rfloor}$. We next extend Theorem 3 to any undirected graph.

Corollary 4. *Given undirected graph $G(V,E)$ with maximum cycle length m, for the networking problem in which all possible multicast sessions are supported, the minimum description of the corresponding routing rate region requires a distance function with $\phi^*\geq 2^{\lfloor(m-2)/3\rfloor}$.*

Proof: Let $\mathcal{C}$ denote a maximum cycle of G. We extend the proof of Theorem 3 by using the same distance function f along $\mathcal{C}$ and setting $f(e), e\notin\mathcal{C}$, to be sufficiently large. ■

Next we bound ϕ^* from above.

Theorem 5. *For an undirected or a directed network $G(V,E)$, $\phi^*\leq 2^{24|E|^3+8|E|^2}$.*

Proof: Suppose that the distance vector $\mathbf{f}=(f(1),\cdots,f(|E|))$ belongs to the minimal description of P. We form the homogeneous set of inequalities $A\mathbf{g}\leq 0$ such that $\{\mathbf{g}:A\mathbf{g}\leq 0,\mathbf{g}\in\mathbb{Z}^{|E|}\}$ is the set of all distance vectors that can eliminate $\mathbf{f}$ by the criteria given in Theorem 2. This includes all inequalities that describe the shortest subtrees for every session corresponding to function f, and also the nonnegativity of elements of $\mathbf{g}$. Notice that this set is non-empty since $\mathbf{f}$ is a solution to it. Furthermore, all elements of matrix A are in $\{0,+1,-1\}$. This implies the upper bound $3|E|+1$ on the size of the inequalities in $A\mathbf{g}\leq 0$. Therefore the facet complexity of $A\mathbf{g}\leq 0$, is at most $\phi_A=3|E|+1$. [13, Theorem 10.2] implies that $A\mathbf{g}\leq 0$ has a rational solution of size at most $4|E|^2\phi_A=12|E|^3+4|E|^2$. Let $\mathbf{g_r}=(p_1/q_1,\cdots,p_{|E|}/q_{|E|})$ denote such a solution. Since $A\mathbf{g}\leq 0$ is a homogeneous set of inequalities, any integral multiple of $\mathbf{g_r}$ is also a solution to $A\mathbf{g}\leq 0$. Now consider the vector $\mathbf{g_z}=(q_1\cdots q_{|E|})\mathbf{g_r}$. Clearly $\mathbf{g_z}\in\{\mathbf{g}:A\mathbf{g}\leq 0,\mathbf{g}\in\mathbb{Z}^{|E|}\}$, so it can eliminate $\mathbf{f}$. Let $\mathbf{g_z}(i)$ be the maximum entry of $\mathbf{g_z}$. Then $size(\mathbf{g_z}(i))\leq size(q_1\cdots q_{|E|})+size(\mathbf{g_r}(i))$. Since $size(q_1\cdots q_{|E|})\leq size(\mathbf{g_r})$ and $size(\mathbf{g_r}(i))\leq size(\mathbf{g_r})$, then $size(\mathbf{g_z}(i))\leq 24|E|^3+8|E|^2$. This yields the result. ■

The following result suggests that a small fraction of the distance functions in our characterization of the fractional routing capacity region are truly needed and that most distance functions can be eliminated by distance functions where the maximum entry grows polynomially with $|E|$.

Theorem 6. *Let $G(V,E)$ be an undirected ring network with edges labeled $1,\ 2,\ \cdots,\ |E|$ in a clockwise order. Choose any integer $m\geq 6$, and suppose $g_{\max}=\max_{e\in E}g(e)>g^*\doteq |E|^m/(1-|E|^m/g_{\max})$. Assume without loss of generality that $g(|E|)=g_{\max}$; for $e\in E\setminus|E|$ let $g(e)$ be randomly and uniformly chosen among the nonnegative integers less than or equal to $g_{\max}$. Then with probability at least $1-(4/|E|^{m-5})-(1/|E|^{m|E|})-(|E|/|E|^{m(|E|-1)})$ we can find a nontrivial distance function f with $f_{\max}\leq g^*$ that eliminates g.*

Proof: Given distance function g with $g_{\max}>g^*$, let $\eta=\lfloor g_{\max}/|E|^m\rfloor$, and define

$$f(e)=g(e)-(g(e)\pmod{\eta}),\ e\in E.$$

Distance function f eliminates distance function g if for every pair of edge-disjoint subsets $E_1\subset E$ and $E_2\subset E$ occurring along paths of G, the condition $\sum_{e\in E_1}g(e)\leq\sum_{e\in E_2}g(e)$ implies $\sum_{e\in E_1}f(e)\leq\sum_{e\in E_2}f(e)$. Let $\mathcal{E}_{E_1,E_2}$ be the event that $\sum_{e\in E_1}g(e)\leq\sum_{e\in E_2}g(e)$ and $\sum_{e\in E_1}f(e)>\sum_{e\in E_2}f(e)$. Define

$$\begin{aligned}\Delta_g &= \sum_{e\in E_1}g(e)-\sum_{e\in E_2}g(e)\\ \text{and }\ \Delta_f &= \sum_{e\in E_1}f(e)-\sum_{e\in E_2}f(e).\end{aligned}$$

Since $0\leq g(e)-f(e)<\eta$ for all $e\in E$, it follows that

$$|\Delta_g-\Delta_f|\ \leq\ \sum_{e\in E}|g(e)-f(e)|\ <\ \eta\cdot|E|.$$

We know that $\Delta_g\leq 0$ and $\Delta_f>0$, and therefore $|\Delta_g|<\eta\cdot|E|$. Let $E_{\min}=\min_{e\in E_1\cup E_2}e$. Observe that $E_{\min}\neq|E|$. Given $g(e),e\in E\setminus E_{\min}$, there are at most $2\eta\cdot|E|$ choices for $g(E_{\min})$ that result in $-\eta\cdot|E|<\Delta_g\leq 0$. Furthermore, $g(E_{\min})$ is a random variable uniformly distributed over the integers between 0 and $g_{\max}$. Therefore,

$$\mathbb{P}(\mathcal{E}_{E_1,E_2})\ \leq\ \frac{2\eta\cdot|E|}{g_{\max}+1}\ <\ \frac{2\cdot\frac{g_{\max}}{|E|^m}\cdot|E|}{g_{\max}}\ =\ \frac{2}{|E|^{m-1}}.$$

The number of pairs of edge-disjoint subsets E_1 and E_2 we need to consider can be determined by the possibilities for $\min_{e\in E_1}e,\max_{e\in E_1}e,\min_{e\in E_2}e$ and $\max_{e\in E_2}e$ and is therefore less than $2|E|^4$. Hence,

$$\mathbb{P}\left(\bigcup_{E_1,E_2}\mathcal{E}_{E_1,E_2}\right)\ <\ 2|E|^4\cdot\frac{2}{|E|^{m-1}}\ =\ \frac{4}{|E|^{m-5}}.$$

We have not yet considered if f is a trivial distance function. f is nontrivial if and only if there exist edges e and $e'\neq e$ such that $f(e)>0$ and $f(e')>0$. For all $e\in E$, $\mathbb{P}(f(e)=0)=\mathbb{P}(g(e)\equiv g(e')\pmod{\eta})=1/g_{\max}<1/|E|^m$. Thus, with probability at least $1-(4/|E|^{m-5})-(1/|E|^{m|E|})-(|E|/|E|^{m(|E|-1)})$ we can use distance function f to eliminate g. Since $f(e)\pmod{\eta}=0$ for all $e\in E$, we can define distance function f^* with $f^*(e)=f(e)/\eta,\ e\in E$, and eliminate f by f^*. Notice that for all $e\in E$,

$$\begin{aligned}f^*(e)\ &=\ \tfrac{f(e)}{\eta}\ \leq\ \tfrac{g(e)}{\eta}\ \leq\ \tfrac{g_{\max}}{\lfloor g_{\max}/|E|^m\rfloor}\ <\ \tfrac{g_{\max}}{\frac{g_{\max}}{|E|^m}-1}\\ &=\ \tfrac{|E|^m}{1-\frac{|E|^m}{g_{\max}}}.\end{aligned}\tag{9}$$

■

ACKNOWLEDGMENT

The inequality elimination technique which is the foundation of this work was developed as part of the derivation of two network coding capacity regions. Gerhard Kramer made significant contributions to that work, and we are grateful to him for years of stimulating, productive and informative discussions.

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