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ASYMPTOTIC ANALYSIS OF THE NÖRLUND AND STIRLING POLYNOMIALS

Mark Daniel Ward

We provide a full asymptotic analysis of the Nörlund polynomials and Stirling polynomials. We give a general asymptotic expansion—to any desired degree of accuracy—when the parameter is not an integer. We use singularity analysis, Hankel contours, and transfer theory. This investigation was motivated by a need for such a complete asymptotic description, with parameter 1/2, during this author's recent solution of Wilf's 3rd (previously) Unsolved Problem.

Dedicated to the memory of Philippe Flajolet (1948–2011)
and to the memory of Herbert S. Wilf (1931–2012).

1. BACKGROUND

Let $B(z) = z/(e^z - 1)$. The Nörlund polynomials $b_n^{\langle\alpha\rangle}$ are defined by

$$(B(z))^\alpha = \Big(\frac{z}{e^z-1}\Big)^\alpha = \sum_{n=0}^{\infty} b_n^{\langle\alpha\rangle}\frac{z^n}{n!}. \tag{1}$$

The Nörlund polynomials have been studied in many contexts. They were introduced by Nörlund [**10**]. Many connections have been identified with Bernoulli and Stirling numbers; see, e.g., [**2**], [**3**], [**5**], [**6**]. Properties of the Nörlund polynomials have also been studied directly; see, for instance, [**1**] and [**9**].

The function $B(z) = z/(e^z - 1)$ is well known as the exponential generating function of the Bernoulli numbers. In the notation of Nörlund polynomials, the Bernoulli numbers are exactly $b_n^{\langle 1\rangle}$. I.e., the Bernoulli numbers are exactly the

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Nörlund polynomials $b_n^{\langle\alpha\rangle}$ evaluated at $\alpha = 1$. Thus, the Nörlund polynomials evaluated at a general positive real number α are generalized Bernoulli numbers.

The Stirling numbers (of both the first and second kind) and their many identities have been extensively studied and are among the main objects of study in combinatorial analysis. Unfortunately, the notation "Stirling polynomials" has different meanings in different contexts. Don Knuth wrote to the author recently [**8**] (upon seeing an abstract of the author's talk based on a preprint of this paper), urging the use of Stirling polynomials $\sigma_n(x)$ as described on page 272 of [**7**]:

$$\left(\frac{ze^z}{e^z - 1}\right)^x = x \sum_n \sigma_n(x) z^n.$$

With this definition, we associate Nörlund and Stirling polynomials by the identity

$$\frac{b_n^{\langle\alpha\rangle}}{n!} = \alpha(-1)^n \sigma_n(\alpha).$$

Because of the extensive use of the Bernoulli numbers in the Euler-MacLaurin formula and in many other asymptotic expansions, it is natural to investigate the asymptotic properties of the Nörlund and Stirling polynomials, when α is fixed, and as $n \to \infty$.

2. MOTIVATION

Apparently, a *complete asymptotic description* of the Nörlund and Stirling polynomials has not yet appeared in the literature. The asymptotic properties of the coefficients $\frac{b_n^{\langle 1/2\rangle}}{n!}$ of $\sqrt{B(z)}$ played a key role in [**11**] (a solution of Wilf's 3rd previously Unsolved Problem [**12**]). Here, we provide a general analysis of $\frac{b_n^{\langle\alpha\rangle}}{n!}$ and $\sigma_n(\alpha)$ for any positive $\alpha \in \mathbb{R}$, as $n \to \infty$. The case of integer-valued α is not complicated, but the analysis with non-integer α's is quite intricate. The general form of $\frac{b_n^{\langle\alpha\rangle}}{n!}$, for any α, is given in Table 1. We use the notation

$$\alpha^{\underline{j}} = (\alpha)(\alpha - 1)(\alpha - 2)\cdots(\alpha - j + 1)$$

for the jth falling power of α. We utilize analytic combinatorics, in the style of Flajolet and Sedgewick [**4**]; see especially p. 381–384, which discusses the asymptotic properties of $[z^n](1-z)^\alpha$. Our analysis requires an understanding of $[z^n]z^\alpha(1-z)^\alpha$. When α is an integer, this is a trivial modification; however, when α is not an integer, the coefficients e_k (from

n	$\frac{b_n^{\langle\alpha\rangle}}{n!}$
0	1
1	$-\frac{\alpha}{2}$
2	$\frac{\alpha}{12} + \frac{\alpha^{\underline{2}}}{8}$
3	$-\frac{\alpha^{\underline{2}}}{24} - \frac{\alpha^{\underline{3}}}{48}$
4	$-\frac{\alpha}{720} + \frac{\alpha^{\underline{2}}}{288} + \frac{\alpha^{\underline{3}}}{96} + \frac{\alpha^{\underline{4}}}{384}$
5	$\frac{\alpha^{\underline{2}}}{1440} - \frac{\alpha^{\underline{3}}}{576} - \frac{\alpha^{\underline{4}}}{576} - \frac{\alpha^{\underline{5}}}{3840}$

Table 1. The coefficients of $(B(z))^\alpha$.

[4]) must be modified and will depend on α too. We refer to these more general constants here as $e_k(\alpha, j)$.

3. MAIN RESULTS

One feature of the following singularity analysis is the contrast between the structure of $(B(z))^\alpha$ when α is an integer versus a non-integer. In the first case (Remark 1), where α is an integer, countably many poles provide a full asymptotic description. In the second case (Theorem 2 and Corollary 3), where α is a non-integer, the asymptotics rely on contributions from two algebraic singularities.

The case where α is an integer is very straightforward, so we simply refer to this as a "Remark."

REMARK 1. Let $(B(z))^\alpha = \left(\frac{z}{e^z-1}\right)^\alpha = \sum_{n=0}^{\infty} \frac{b_n^{\langle\alpha\rangle}}{n!} z^n$, where α is any positive integer. Then for $n > \alpha$,

$$\frac{b_n^{\langle\alpha\rangle}}{n!} = \sum_{\substack{0\le j\le \alpha-1\\ j\equiv n \bmod 2}} \binom{n-j-1}{n-\alpha} \frac{b_j^{\langle\alpha\rangle}}{j!} \frac{2\,\zeta(n-j)}{(2\pi)^{n-j}} (-1)^{\alpha+(n+j)/2}. \tag{2}$$

Also, let $\left(\frac{ze^z}{e^z-1}\right)^\alpha = \alpha \sum_n \sigma_n(\alpha) z^n$, where α is again a positive integer. Then for $n > \alpha$, since $\frac{b_n^{\langle\alpha\rangle}}{n!} = \alpha(-1)^n \sigma_n(\alpha)$, we have

$$\sigma_n(\alpha) = \sum_{\substack{0\le j\le \alpha-1\\ j\equiv n \bmod 2}} \binom{n-j-1}{n-\alpha} \sigma_j(\alpha) \frac{2\,\zeta(n-j)}{(2\pi)^{n-j}} (-1)^{\alpha-(n+j)/2}. \tag{3}$$

Note. Although $\frac{b_0^{\langle\alpha\rangle}}{0!}, \frac{b_1^{\langle\alpha\rangle}}{1!}, \ldots, \frac{b_{\alpha-1}^{\langle\alpha\rangle}}{(\alpha-1)!}$ are found on the right hand side of (2), and $\sigma_0(\alpha), \sigma_1(\alpha), \ldots, \sigma_{\alpha-1}(\alpha)$ are found on the right hand side of (3), these terms can just be treated as constants, since they do not depend on n.

The case where α is a non-integer is more intricate:

Theorem 2. *Let* $(B(z))^\alpha = \left(\frac{z}{e^z-1}\right)^\alpha = \sum_{n=0}^{\infty} \frac{b_n^{\langle\alpha\rangle}}{n!} z^n$, *where α is any positive non-integer. Then for each positive integer T, the asymptotic expansion of* $\frac{b_n^{\langle\alpha\rangle}}{n!}$ *is*

$$\frac{b_n^{\langle\alpha\rangle}}{n!} = \frac{n^\alpha}{\Gamma(\alpha)(2\pi)^n} \sum_{j=0}^{T-1} \frac{b_j^{\langle\alpha\rangle}}{j!} \frac{c_{n,j}^{\langle\alpha\rangle}(2\pi)^j (\alpha-1)^{\underline{j}}}{n^{j+1}} \left(1 + \sum_{k=1}^{T-j-1} \frac{e_k(\alpha, j)}{n^k}\right) + O\left(\frac{1}{(2\pi)^n n^{T+1-\alpha}}\right), \tag{4}$$

where

$$c_{n,j}^{\langle\alpha\rangle} := \begin{cases} 2(-1)^{(n-j)/2}\cos(\pi(j-\alpha)) & \textit{for } j \equiv n \bmod 2, \\ 2(-1)^{(n-j-1)/2}\sin(\pi(j-\alpha)) & \textit{for } j \not\equiv n \bmod 2, \end{cases}$$

and

$$e_k(\alpha, j) = \sum_{\ell=k}^{2k} (-1)^\ell \lambda_{k,\ell}^{\langle\alpha\rangle} (\alpha - j - 1)^{\underline{\ell}}, \quad \textit{and} \quad \lambda_{k,\ell}^{\langle\alpha\rangle} = [v^k t^\ell] e^t (1+vt)^{\alpha-1/v-1}.$$

Corollary 3. *Let* $\left(\frac{ze^z}{e^z-1}\right)^\alpha = \alpha \sum_n \sigma_n(\alpha) z^n$, *where* α *is again a positive non-integer. Then for* $n > \alpha$, *since* $\frac{b_n^{\langle\alpha\rangle}}{n!} = \alpha(-1)^n \sigma_n(\alpha)$, *it follows from Theorem 2 that, for each positive integer* T, *the asymptotic expansion of* $\sigma_n(\alpha)$ *is*

$$\sigma_n(\alpha) = \frac{n^\alpha}{\Gamma(\alpha)(2\pi)^n} \sum_{j=0}^{T-1} (-1)^{n+j} \sigma_n(\alpha) \frac{c_{n,j}^{\langle\alpha\rangle} (2\pi)^j (\alpha-1)^{\underline{j}}}{n^{j+1}} \left(1 + \sum_{k=1}^{T-j-1} \frac{e_k(\alpha, j)}{n^k}\right) + O\left(\frac{1}{(2\pi)^n n^{T+1-\alpha}}\right), \tag{5}$$

where, as in Theorem 2,

$$c_{n,j}^{\langle\alpha\rangle} := \begin{cases} 2(-1)^{(n-j)/2}\cos(\pi(j-\alpha)) & \textit{for } j \equiv n \bmod 2, \\ 2(-1)^{(n-j-1)/2}\sin(\pi(j-\alpha)) & \textit{for } j \not\equiv n \bmod 2, \end{cases}$$

and

$$e_k(\alpha, j) = \sum_{\ell=k}^{2k} (-1)^\ell \lambda_{k,\ell}^{\langle\alpha\rangle} (\alpha - j - 1)^{\underline{\ell}}, \quad \textit{and} \quad \lambda_{k,\ell}^{\langle\alpha\rangle} = [v^k t^\ell] e^t (1+vt)^{\alpha-1/v-1}.$$

4. PROOF OF REMARK 1

The remark can be seen as a partial fraction decomposition. We cast the proof using singularity analysis, so the reader can contrast the singularity analysis used to prove the remark with the singularity analysis used in the proof of Theorem 2.

We use singularity analysis in the style of [**4**]. Since α is an integer in Remark 1, then $(B(z))^\alpha = \left(\frac{z}{e^z-1}\right)^\alpha$ has a removable singularity at $z = 0$ and has a pole of order α at each point $\zeta_k := 2\pi i k$, for nonzero integers k. There are no other singularities for the function $(B(z))^\alpha$.

The series representation of $(B(z))^\alpha = \left(\frac{z}{e^z-1}\right)^\alpha$ from (1) is valid in an open disc of radius 2π around the origin. Thus, for each nonzero integer k, an analogous series representation is available in an open disc of radius 2π around ζ_k:

$$\left(\frac{z-\zeta_k}{e^{z-\zeta_k}-1}\right)^\alpha = \sum_{n=0}^\infty \frac{b_n^{\langle\alpha\rangle}}{n!}(z-\zeta_k)^n.$$

Simplifying this expression using $e^{z-\zeta_k} = e^z$ (since k is an integer), and multiplying throughout by $\frac{z^\alpha}{(z-\zeta_k)^\alpha}$, yields

$$\left(\frac{z}{e^z-1}\right)^\alpha = z^\alpha \sum_{n=0}^{\infty} \frac{b_n^{\langle\alpha\rangle}}{n!}(z-\zeta_k)^{n-\alpha}.$$

Now we collect the contributions from each pole of order α, i.e., from $\zeta_k = 2\pi i k$ for nonzero integers k. For $n > \alpha$,

$$\begin{aligned}(6) \qquad [z^n](B(z))^\alpha &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \zeta_k^{-n}[z^n]z^\alpha \sum_{j=0}^{\alpha-1} \frac{b_j^{\langle\alpha\rangle}}{j!}\zeta_k^j(z-1)^{j-\alpha} \\ &= \sum_{k\geq 1}\sum_{j=0}^{\alpha-1}\left(\frac{1}{\zeta_k^{n-j}}+\frac{1}{\zeta_{-k}^{n-j}}\right)\frac{b_j^{\langle\alpha\rangle}}{j!}[z^n]z^\alpha(z-1)^{j-\alpha}.\end{aligned}$$

Since α is an integer, then

$$[z^n]z^\alpha(z-1)^{j-\alpha} = (-1)^{j-\alpha}[z^{n-\alpha}](1-z)^{j-\alpha} = (-1)^{j-\alpha}\binom{n-j-1}{n-\alpha}.$$

Also, since j, k, n are all integers,

$$\frac{1}{\zeta_k^{n-j}} + \frac{1}{\zeta_{-k}^{n-j}} = \frac{2(-1)^{(n-j)/2}}{(2\pi k)^{n-j}}, \qquad \text{if } j \equiv n \bmod 2,$$

and $\frac{1}{\zeta_k^{n-j}} + \frac{1}{\zeta_{-k}^{n-j}} = 0$ otherwise. Thus, (6) simplifies to

$$[z^n](B(z))^\alpha = \sum_{k\geq 1}\sum_{\substack{0\leq j\leq \alpha-1\\ j\equiv n \bmod 2}} \frac{2(-1)^{(n-j)/2}(-1)^{j-\alpha}}{(2\pi k)^{n-j}}\frac{b_j^{\langle\alpha\rangle}}{j!}\binom{n-j-1}{n-\alpha}.$$

After one final simplification using $\zeta(n-j) = \sum_{k\geq 1}\frac{1}{k^{n-j}}$, the proof of Remark 1 is complete.

5. PROOF OF THEOREM 2

In the proof of Remark 1, since α was a positive integer, then $(B(z))^\alpha$ had a pole of order α at $\zeta_k := 2\pi i k$ for every nonzero integer k. In contrast, in Theorem 2, α is a positive non-integer. Thus $(B(z))^\alpha$ has an algebraic singularity at $2\pi i k$ for every nonzero integer k, as depicted in Figure 1a. The singularity at 0 is removable since $z/(e^z-1)$ has a removable singularity at 0. Thus, in the analysis below, we want to ensure that the branch cuts are selected in such a way that $(B(z))^\alpha$ is analytic in a doubly-indented disc, centered at the origin, with radius strictly

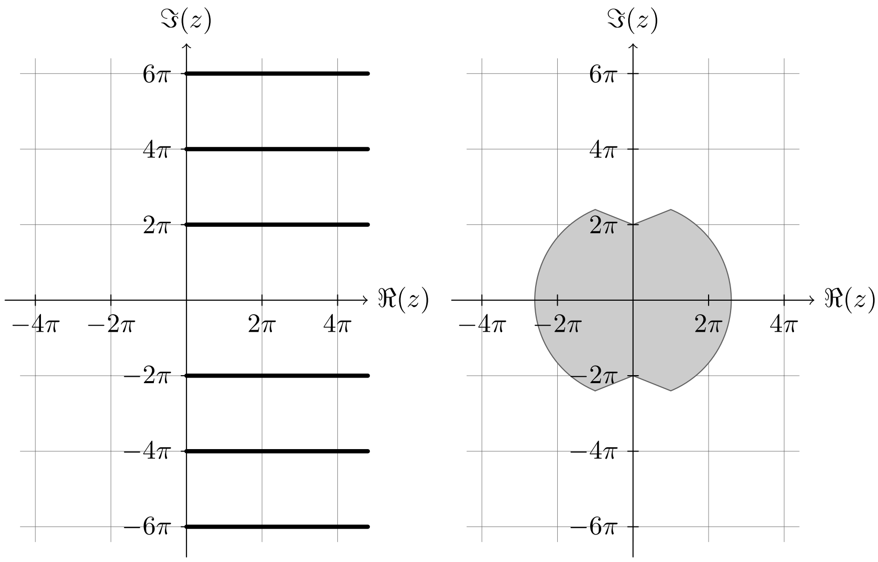

Figure 1. (a.) One possible choice of branch cuts corresponding to the algebraic singularities of $(B(z))^\alpha$ at $z = 2\pi\mathrm{i}k$ for nonzero integers k. (b.) A double-punctured disc, with radius strictly larger than 2π, and punctures at $z = \pm 2\pi\mathrm{i}$.

larger than 2π; the indentations should be at $\pm 2\pi\mathrm{i}$. Such a doubly-punctured disc is also depicted in Figure 1b.

Since we do not want the branch cuts of $(B(z))^\alpha$ to intersect the doubly-punctured disc, *we must instead use a scheme in which the branch cuts are directed away from the origin*, such as the branch cuts depicted in Figure 2a or Figure 2b. Indeed, in Figure 2b, $(B(z))^\alpha$ can be extended beyond the doubly-punctured disc. The region where $(B(z))^\alpha$ is analytic actually extends to the entire complex plane—except for the thick black lines $[2\pi\mathrm{i}, \mathrm{i}\infty)$ and $(-\mathrm{i}\infty, -2\pi\mathrm{i}]$—as shown in Figure 2c.

The singularity analysis theory of [**4**] is very useful for proving Theorem 2. We paraphrase Theorem VI.5 from [**4**], which allows us to determine the asymptotic behavior of $\frac{b_n^{\langle\alpha\rangle}}{n!}$, based on the properties of $(B(z))^\alpha$ near the closest singularities to the origin, at $z = \pm 2\pi\mathrm{i}$. We have the following four conditions:

1. The function $(B(z))^\alpha$ is analytic in $|z| < 2\pi$;
2. The function $(B(z))^\alpha$ has exactly two non-removable singularities on the circle $|z| = 2\pi$, namely, at the points $\zeta_1 = 2\pi\mathrm{i}$ and $\zeta_{-1} = -2\pi\mathrm{i}$;
3. There is a Δ-domain Δ_0 such that $(B(z))^\alpha$ is analytic in the indented disc
$$\mathbf{D} = (\zeta_1 \cdot \Delta_0) \cap (\zeta_{-1} \cdot \Delta_0),$$
with $\zeta \cdot \Delta_0$ the image of Δ_0 by the mapping $z \mapsto \zeta z$;

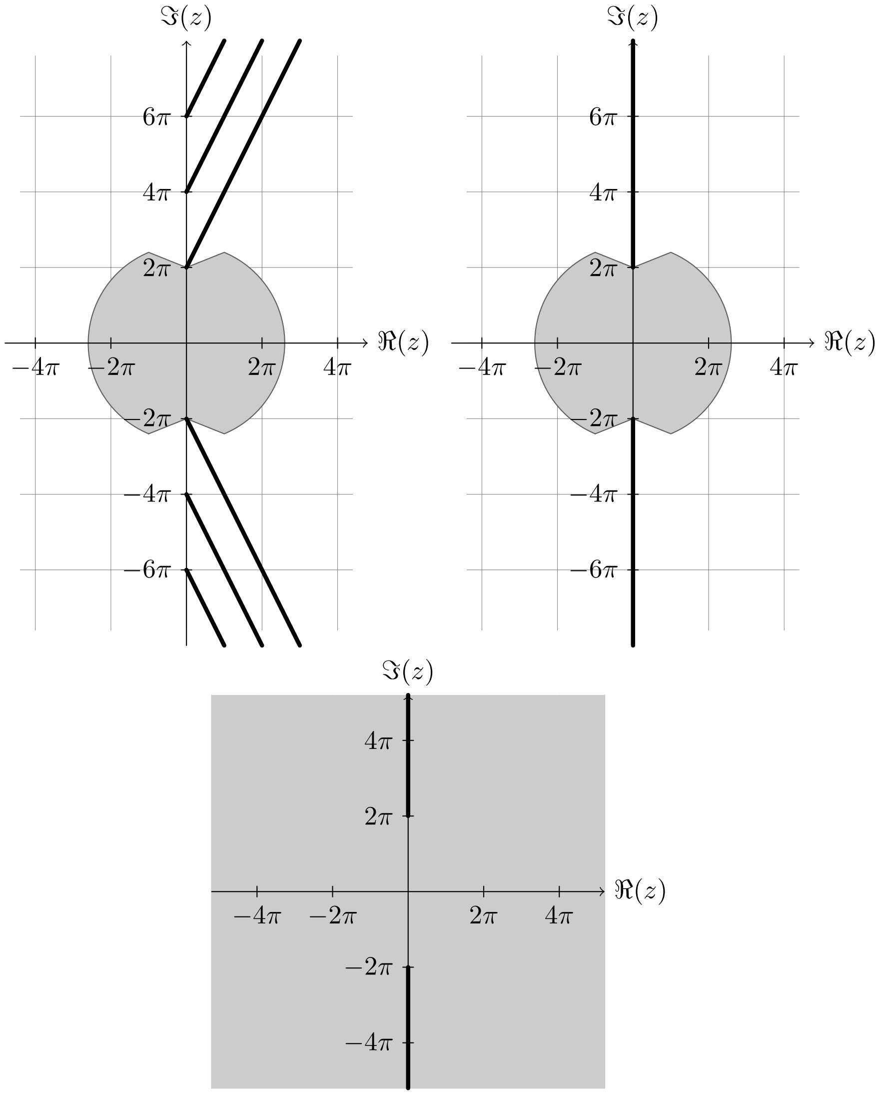

Figure 2. (a.) Branch cuts corresponding to the algebraic singularities of $(B(z))^{\alpha}$ at $z = 2\pi \mathrm{i} k$ for nonzero integers k. These branch cuts are more suitable than those in Figure 1a, because they do not intersect with the doubly-punctured disc from Figure 1b. (b.) Another choice of branch cuts (in which all of the branch cuts in the upper half plane are overlapping, and all of the branch cuts in the lower half plane are overlapping), which is even more suitable. (c.) When all branch cuts in the upper half plane are overlapping, and all of the branch cuts in the lower half plane are overlapping, then $(B(z))^{\alpha}$ is analytic on the entire complex plane except not on the thick black lines, namely $[2\pi\mathrm{i}, \mathrm{i}\infty)$ and $(-\mathrm{i}\infty, -2\pi\mathrm{i}]$.

4. If we define

$$\sigma_1^{\langle\alpha\rangle}(z) = z^\alpha \sum_{j=0}^{T-1} \frac{b_j^{\langle\alpha\rangle}}{j!} \zeta_1^j (1-z)^{j-\alpha} e^{\mathrm{i}\pi(j-\alpha)},$$
$$\sigma_{-1}^{\langle\alpha\rangle}(z) = z^\alpha \sum_{j=0}^{T-1} \frac{b_j^{\langle\alpha\rangle}}{j!} \zeta_{-1}^j (1-z)^{j-\alpha} e^{-\mathrm{i}\pi(j-\alpha)},$$
$$\tau^{\langle\alpha\rangle}(z) = (1-z)^{T-\alpha},$$

then

$$(B(z))^\alpha = \sigma_j^{\langle\alpha\rangle}(z/\zeta_j) + O(\tau^{\langle\alpha\rangle}(z/\zeta_j)) \qquad \text{as } z \to \zeta_j \text{ in } \mathbf{D}, \text{ for } j = -1 \text{ or } 1.$$

Thus, by Theorem VI.5 of [**4**], it follows that

$$[z^n](B(z))^\alpha = \zeta_1^{-n}[z^n]\sigma_1^{\langle\alpha\rangle}(z) + \zeta_{-1}^{-n}[z^n]\sigma_{-1}^{\langle\alpha\rangle}(z) + O\left(\frac{1}{(2\pi)^n n^{T+1-\alpha}}\right).$$

Equivalently,

$$[z^n](B(z))^\alpha = \sum_{j=0}^{T-1} \left(\frac{e^{\mathrm{i}\pi(j-\alpha)}}{\zeta_1^{n-j}} + \frac{e^{-\mathrm{i}\pi(j-\alpha)}}{\zeta_{-1}^{n-j}} \right) \frac{b_j^{\langle\alpha\rangle}}{j!} [z^n] z^\alpha (1-z)^{j-\alpha} + O\left(\frac{1}{(2\pi)^n n^{T+1-\alpha}}\right). \tag{7}$$

To keep our notation compact, we define $c_{n,j}^{\langle\alpha\rangle}$ so that

$$c_{n,j}^{\langle\alpha\rangle} := \begin{cases} 2(-1)^{(n-j)/2}\cos(\pi(j-\alpha)) & \text{for } j \equiv n \bmod 2, \\ 2(-1)^{(n-j-1)/2}\sin(\pi(j-\alpha)) & \text{for } j \not\equiv n \bmod 2, \end{cases}$$

and thus

$$\frac{e^{\mathrm{i}\pi(j-\alpha)}}{\zeta_1^{n-j}} + \frac{e^{-\mathrm{i}\pi(j-\alpha)}}{\zeta_{-1}^{n-j}} = \frac{c_{n,j}^{\langle\alpha\rangle}}{(2\pi)^{n-j}}.$$

Now we can simplify (7) to

$$[z^n](B(z))^\alpha = \sum_{j=0}^{T-1} \frac{c_{n,j}^{\langle\alpha\rangle}}{(2\pi)^{n-j}} \frac{b_j^{\langle\alpha\rangle}}{j!} [z^n] z^\alpha (1-z)^{j-\alpha} + O\left(\frac{1}{(2\pi)^n n^{T+1-\alpha}}\right). \tag{8}$$

Explicit expressions for $[z^n](1-z)^{j-\alpha}$ are well-known (e.g., p. 381–384 of [**4**]). Here, however, an extra z^α is present which fundamentally alters the result, since α is not an integer. The argument is sufficiently intricate that we include the derivation. We follow the technique, and some of the notation, from Theorem VI.1 in [**4**].

To compute $[z^n]z^\alpha(1-z)^{j-\alpha}$, we use Cauchy's coefficient formula:

$$[z^n]z^\alpha(1-z)^{j-\alpha} = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}_0} z^\alpha(1-z)^{j-\alpha}\frac{\mathrm{d}z}{z^{n+1}}, \tag{9}$$

where $\mathcal{C}_0$ is a circle of radius smaller than 1 (so that the circle does not include the singularity at $z = 1$ in the function $z^{\alpha-n-1}(1-z)^{j-\alpha}$), e.g., we could use a circle of radius 1/2, centered at the origin, and oriented in the counterclockwise direction.

Next, we continuously deform $\mathcal{C}_0$ into $\mathcal{C}_R$; such a deformation avoids $z = 1$, so the deformation takes place in the region where the integrand $z^{\alpha-n-1}(1-z)^{j-\alpha}$ of (9) is analytic. The region $\mathcal{C}_R$ is depicted in Figure 3.

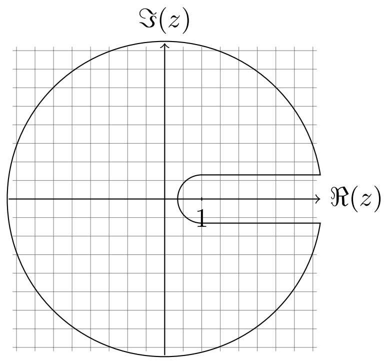

Figure 3. A contour $\mathcal{C}_R$, with counterclockwise orientation, and the outer curve of radius R.

The contribution from the outer circle can be negated as we take $R\to\infty$. The remainder of the contribution can be specified by using the integration path

$$\mathcal{H}(n) = \mathcal{H}^-(n)\cup\mathcal{H}^\circ(n)\cup\mathcal{H}^+(n),$$

(the notation follows [**4**]), where

$$\mathcal{H}^-(n) = \{z = w - \mathrm{i}/n,\ w\ge 1\},$$
$$\mathcal{H}^+(n) = \{z = w + \mathrm{i}/n,\ w\ge 1\},$$
$$\mathcal{H}^\circ(n) = \{z = 1 - e^{\mathrm{i}\theta}/n,\ \theta\in[-\pi/2,\pi/2]\}.$$

Finally, we let $z = 1 + t/n$, so that using the transformation

$$z = 1 + t/n, \quad\text{and}\quad \mathrm{d}z = \mathrm{d}t/n,$$

the integral $\frac{1}{2\pi\mathrm{i}}\int\limits_{\mathcal{C}} z^\alpha(1-z)^{j-\alpha}\frac{1}{z^{n+1}}\,\mathrm{d}z$ simplifies to

$$[z^n]z^\alpha(1-z)^{j-\alpha} = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}}(-t/n)^{j-\alpha}(1+t/n)^{\alpha-n-1}\,\mathrm{d}t/n.$$

We substitute $v = 1/n$ and get

$$[z^n]z^\alpha(1-z)^{j-\alpha} = \frac{n^{\alpha-1-j}}{2\pi\mathrm{i}}\int_{\mathcal{H}}(-t)^{j-\alpha}e^{-t}\sum_k\sum_{\ell=k}^{2k}\lambda_{k,\ell}^{\langle\alpha\rangle}\frac{t^\ell}{n^k}\,\mathrm{d}t,$$

where

$$\lambda_{k,\ell}^{\langle\alpha\rangle} = [v^k t^\ell]e^t(1+vt)^{\alpha-1/v-1}.$$

At this point, our notation and argument diverges from [**4**] since our $\lambda_{k,l}$'s and our e_k's are more general. The analogous terms in [**4**] are the special cases $\lambda_{k,\ell}^{\langle 0\rangle}$ and $e_k(\alpha, 0)$. We obtain

$$[z^n]z^\alpha(1-z)^{j-\alpha} = n^{\alpha-j-1}\sum_k\sum_{\ell=k}^{2k}\lambda_{k,\ell}^{\langle\alpha\rangle}(-1)^\ell\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}}(-t)^{\ell-\alpha+j}e^{-t}\frac{1}{n^k}\,\mathrm{d}t \tag{10}$$

$$= n^{\alpha-j-1} \sum_k \sum_{\ell=k}^{2k} \lambda_{k,\ell}^{\langle\alpha\rangle} (-1)^\ell \frac{1}{\Gamma(\alpha-\ell-j)} \frac{1}{n^k}$$
$$= \frac{n^{\alpha-j-1}}{\Gamma(\alpha-j)} \sum_k e_k(\alpha,j) \frac{1}{n^k}$$

where $e_k(\alpha,j) = \sum_{\ell=k}^{2k} (-1)^\ell \lambda_{k,\ell}^{\langle\alpha\rangle} (\alpha-j-1)^{\underline{\ell}}$.

We only expanded equation (8) to accuracy $O((2\pi)^{-n} n^{-T-1+\alpha})$, and therefore we also only need to use (10) to accuracy $O(n^{-T-1+\alpha})$. Thus, we use this truncated form of (10):

$$[z^n] z^\alpha (1-z)^{j-\alpha} = \frac{n^{\alpha-j-1}}{\Gamma(\alpha-j)} \left(1 + \sum_{k=1}^{T-j-1} \frac{e_k(\alpha,j)}{n^k}\right) + O\left(\frac{1}{n^{T+1-\alpha}}\right).$$

After a substitution into (8), Theorem 2 follows.

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