On Maximum Eigenmode Beamforming and Multi-User Gain

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***Abstract*—This paper is concerned with the advantages of multi-user concurrent transmission in multiple-input multiple-output (MIMO) systems with rate constraints. We first study a maximum eigenmode beamforming (MEB) strategy for fading multiple access channels (MACs). This strategy allows each user to transmit only in its maximum eigenmode direction and applies a suboptimal matched-filter receiver with successive interference cancellation at the base station (BS). We derive a closed-form expression for the average minimum transmitted sum power required by MEB. Based on this, we show that: a) the MEB strategy is asymptotically optimal when the number of simultaneous users is sufficiently large; b) multi-user concurrent transmission has a power advantage, referred to as multi-user gain, over orthogonal transmission approaches such as time-division multiple-access; c) the number of antennas at the BS has a far stronger impact on the system performance than that at each user side. These properties are verified by simulation. Both numerical analyses and simulation results show that a major part of multi-user gain can be achieved in practical environments even with a quite small number of simultaneous users. We also study the MEB strategy for MIMO broadcast channels (BCs). The dirty paper coding (DPC) technique is necessary in this case. It is analytically shown that most observations made for MIMO MACs are extendable to MIMO BCs.**

***Index Terms*—Maximum eigenmode beamforming (MEB), minimum transmitted sum power, multiple-input multiple-output (MIMO), multi-user gain, successive interference cancellation.**

I. Introduction

The multi-user multiple-input multiple-output (MIMO) transmission technique has been extensively studied for future wireless communications [1]–[5]. Currently, most discussions on MIMO systems are on the sum-rate maximization problem with individual/sum power constraints [2]–[12], which is closely relevant to delay-insensitive services such as email and file transfer for data networks. Several user selection criteria [6], [7] and multi-user scheduling algorithms [5], [8], [9] have been proposed to provide fairness among users.

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Power efficiency for wireless systems is a crucial issue that has attracted increasing attention recently. Power efficiency is particularly important for delay-sensitive services such as speech, real-time video and network game. In this case, every user must transmit a certain amount of information within a fixed time period and minimizing the transmitted sum power under certain rate constraints is an appropriate target. This issue is also closely related to throughput maximization in cellular systems where less transmission power implies less interference to other cells and so potentially higher cellular capacity. Several algorithms [13], [14] have been proposed to compute the minimum transmitted sum power (MTSP) of a multi-user MIMO system with rate constraints. However, they involve iterative joint optimization of the transmission covariance matrices and decoding order. They become computationally expensive even when the number of users, denoted by K below, is only moderately large.

In this paper, we study the sum-power minimization problem for multi-user systems with rate constraints. Our focus is on MIMO multiple access channels (MACs). MIMO broadcast channels (BCs) are also briefly discussed. We show that nonorthogonal multi-user concurrent transmission has a significant power advantage over orthogonal ones such as time-division multiple-access (TDMA). This power advantage is referred to as multi-user gain (MUG). It is known that, when the sum-rate maximization problem in MIMO fading channels is considered, the optimal strategy involves multi-user simultaneous transmission [5]. However, to the best of our knowledge, there are only limited efforts to quantify the related gain for the sum-power minimization problem with rate constraints in practical fading channels (e.g., for channels with path loss, lognormal fading and Rayleigh fading). This motivates the work presented in this paper. Using both analytical and numerical results, we show that MUG is very significant in practical environments.

We also investigate near optimal transmission strategies for rate constrained multi-user systems. The low-cost eigenmode beamforming techniques have been studied and shown to be asymptotically optimal for the sum-rate maximization problem [5], [9]–[12]. In this paper, we consider the maximum eigenmode beamforming (MEB) strategy [15] where each user only transmits at a fixed rate in its maximum eigenmode direction. MEB is a simplified case of the dominant eigenmode transmission (DET) technique [16]–[18]. It does not involve the transmission direction optimization of all users that is generally required in DET and thus has a complexity much lower than that of the latter. We show that the MEB strategy is, though simple, still asymptotically optimal for a large K, and its performance is impressive even for a quite small K. The exploration of MUG in practical environments using MEB and low-density parity-check (LDPC) coding is also verified. Simulation results show that the power advantage predicted by theoretical analysis is indeed achievable in practically coded MIMO systems.

We show that, with multi-user concurrent transmission, increasing the number of antennas at the base station has a far stronger impact on the system performance than at the mobile units. Hence an unbalanced MIMO configuration with more antennas at the base station than those at each mobile unit is a good strategy in a multi-user environment. This finding has useful practical implications as it is easier to equip more antennas at the base station than at all mobile units.

If all the mobile units in a multi-user MIMO MAC can cooperate, it results in an equivalent single-user MIMO channel with more transmit antennas than receive antennas when K is large. It is well known that the performance of such an unbalanced single-user MIMO channel is limited by the number of receive antennas (denoted by M) for which the achievable rate R increases linearly with M at fixed average transmission power [19] (when R is large). In this paper, we show that this is also true even when mobile units cannot cooperate.

II. System Model

Consider a K-user system over a single-cell quasi-static fading MIMO MAC with M antennas at the base station and N antennas at each user side (i.e., an $N \times M$ MIMO MAC). Denote by $\boldsymbol{H}_k$ and $\boldsymbol{x}_k$ the channel matrix and transmitted signal for user k, respectively. The received signal at the base station is given by

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{n}$ is a vector of complex additive white Gaussian noise (AWGN) samples with mean zero and variance N_0. $\{\boldsymbol{H}_k\}$ are assumed to be independent and identically distributed (i.i.d.) and perfectly known at both the transmitters and the receiver.

In this paper, we will mainly focus on the sum-power minimization problem given rate constraints. The sum-rate maximization problem given power constraints is briefly discussed in Section IX. For simplicity, we first assume that the instantaneous rate constraint of each user is the same at R/K bits/symbol, where R is referred to as the system *sum rate*. Later we will relax this assumption to the scenario of unequal rate constraints.

Definition 1: For each channel realization $\{\boldsymbol{H}_k\}$, the unconstrained minimum transmitted sum power (MTSP) is the minimum theoretical limit of the aggregate transmitted power (of all users) that supports reliable communication in (1).

Note that the unconstrained MTSP is a function of the channel condition $\{\boldsymbol{H}_k\}$, not the transmission strategy. For different transmission strategies, we have different constrained MTSP, e.g., the MTSP for the maximum eigenmode beamforming strategy to be discussed below. We can also define average MTSP considering the distribution of $\{\boldsymbol{H}_k\}$. In practice, average MTSP is a more useful measure.

Our focus in this paper is on the average constrained MTSP for various transmission strategies and the average unconstrained MTSP of the channel. The problem for finding the unconstrained MTSP of a K-user MIMO system involves joint optimization of the transmission covariance matrices and decoding order for each channel realization. Detailed discussions on this issue can be found in [13] and [14]. The existing methods are highly complicated. Our objective is to develop bounds, asymptotic limits and efficient realization techniques for the MTSP problem.

III. Maximum Eigenmode Beamforming

In this section, we study a simple, low-cost, and suboptimal maximum eigenmode beamforming (MEB) strategy [10], [15]. The corresponding MTSP provides an upper bound for the unconstrained MTSP of the MIMO system in (1). In the next section, we will show that the MEB strategy is asymptotically optimal for a large K.

A. Basic Principles

The basic principle of the MEB strategy is that each mobile unit transmits only in its maximum eigenmode direction. In this paper, we adopt the following low-cost transmitting/receiving operations for MEB.

(i). A simple correlator receiver is used to collect the signals of each user from all receive antennas;
(ii). Successive interference cancellation (SIC) is applied at the receiver. The user with the largest channel gain is decoded first;
(iii). The transmitted power levels of all users are optimized based on the above transmitting/receiving operations.

More specifically, for each channel realization we first perform singular value decomposition (SVD) on the channel matrix $\boldsymbol{H}_k$ of each user k as

$$\boldsymbol{H}_k = \boldsymbol{U}_k \boldsymbol{D}_k \boldsymbol{V}_k^* \tag{2}$$

where $\boldsymbol{U}_k$ and $\boldsymbol{V}_k$ are unitary matrices, $\boldsymbol{D}_k$ an $M \times N$ diagonal matrix consisting of all the singular values of $\boldsymbol{H}_k$ and $\boldsymbol{A}^*$ the conjugate transpose of matrix $\boldsymbol{A}$. Denote by $d_{k,\max}$ the maximum singular value of $\boldsymbol{H}_k$. Let $\boldsymbol{u}_{k,\max}$ and $\boldsymbol{v}_{k,\max}$ be the corresponding left- and right-singular vectors in $\boldsymbol{U}_k$ and $\boldsymbol{V}_k$, respectively. With the MEB strategy, user k transmits only in the direction of $\boldsymbol{v}_{k,\max}$, i.e.

$$\boldsymbol{x}_k = \boldsymbol{v}_{k,\max} \sqrt{p_k} x_k \tag{3}$$

where p_k is the transmitted power of user k and x_k the encoded signal with unit power. Then (1) becomes

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{v}_{k,\max} \sqrt{p_k} x_k + \boldsymbol{n} = \sum_{k=1}^{K} d_{k,\max} \sqrt{p_k} \boldsymbol{u}_{k,\max} x_k + \boldsymbol{n}. \tag{4}$$

Without loss of generality, we assume that

$$d_{1,\max}^2 \le d_{2,\max}^2 \le \cdots \le d_{K,\max}^2 \tag{5}$$

in each channel realization. When decoding x_k at the receiver, we simply correlate the received signal vector $\boldsymbol{y}$ by $\boldsymbol{u}_{k,\max}$. The signal at the correlator output is given by

$$\boldsymbol{u}_{k,\max}^* \boldsymbol{y} = d_{k,\max} \sqrt{p_k} x_k + \sum_{i=1}^{k-1} d_{i,\max} \sqrt{p_i} \boldsymbol{u}_{k,\max}^* \boldsymbol{u}_{i,\max} x_i + \boldsymbol{u}_{k,\max}^* \boldsymbol{n}. \tag{6}$$

Note that in (6), we assume that the interference from users $\{i : d_{i,\max}^2 \geq d_{k,\max}^2\}$ has been successfully decoded and removed by SIC. The signal-to-noise ratio (SNR) for user k (denoted by SNR_k) in (6) is then calculated as

$$SNR_k = \frac{p_k d_{k,\max}^2}{\sum_{i=1}^{k-1} p_i d_{i,\max}^2 |\boldsymbol{u}_{k,\max}^* \boldsymbol{u}_{i,\max}|^2 + N_0} = \frac{p_k d_{k,\max}^2}{\sum_{i=1}^{k-1} p_i d_{i,\max}^2 \phi_{k,i} + N_0} \tag{7}$$

where $\phi_{k,i} \stackrel{\Delta}{=} |\boldsymbol{u}_{k,\max}^* \boldsymbol{u}_{i,\max}|^2$ is a random variable within $[0,1]$. Based on (7) and the Shannon capacity formula $R/K = \log_2(1 + \mathrm{SNR}_k)$, we can compute $\{p_k\}$ recursively in each channel realization as

$$p_k = \frac{(2^{R/K} - 1)\left(\sum_{i=1}^{k-1} p_i d_{i,\max}^2 \phi_{k,i} + N_0\right)}{d_{k,\max}^2}. \tag{8}$$

By averaging the sum power $\sum_k p_k$ calculated using (8) with respect to the fading distribution, we can obtain the average MTSP required by MEB.

The complexity of MEB is much lower than that of the optimal multi-user MIMO scheme. The latter requires more feedback information of the channel matrices from the receiver and involves water-filling and joint optimization of transmission covariance matrices of all users and decoding order [13], [14], while MEB only requires the feedback of a vector to steer beamforming at each transmitter. In what follows, we will show that MEB is also asymptotically optimal when K is large.

B. Average MTSP

We now proceed to derive the average MTSP required by the MEB strategy. In practical environments, each user usually has a power constraint due to its limited battery energy. To avoid an extremely large transmission power level in deep fades, we allow an outage probability $\varepsilon \stackrel{\Delta}{=} \Pr(d_{k,\max}^2 < G_0)$ for each user k. This means that user k doesn't transmit any information if its channel gain is below a given threshold G_0. The subsequent discussions are based on the following assumption.

Assumption 1: All $\{\boldsymbol{u}_{k,\max}\}$ are i.i.d., so are $\{d_{k,\max}^2\}$. Additionally, $\boldsymbol{u}_{k,\max}$ and $d_{k,\max}$ are independent for each user k.

Based on Assumption 1, it can be shown that the mean of $\phi_{k,i}$ defined in (7) is $\mathrm{E}(\phi_{k,i}) = 1/M$. Furthermore, since $d_{k,\max}$ and $d_{i,\max}$ are independent of $\phi_{k,i}$, we also have the following conditional mean:

$$\mathrm{E}\left(\phi_{k,i} \,|\, d_{k,\max}^2, d_{i,\max}^2\right) = 1/M, \quad \forall k \neq i. \tag{9}$$

Equation (9) is the basis of our derivation for Theorem 1 below. This theorem gives a closed-form expression for the average MTSP of MEB in a K-user MIMO system (1) (denoted by $P_{N\times M}^{\mathrm{MEB}}(K,R)$). Its proof can be found in Appendix A.

Theorem 1: Given the target rate R/K and an outage probability ε for each user, we have (10), as shown at the bottom of the page, where $f(\cdot)$ and $F(\cdot)$ are,[1] respectively, the probability density function (pdf) and cumulative distribution function (cdf) of $\{d_{k,\max}^2\}$.

For convenience, we define $t \equiv F(g)$ where t is the probability that user k's channel gain $d_{k,\max}^2$ is no larger than g. Then (10) can be rewritten as (11), shown at the bottom of the page, where $F^{-1}(\cdot)$ is the inverse function of the cdf $F(\cdot)$. This equivalent expression for $P_{N\times M}^{\mathrm{MEB}}(K,R)$ will be frequently used in the rest of this paper.

Since MEB is a particular realization technique, (11) serves as an upper bound for the average unconstrained MTSP (denoted by $P_{N\times M}(K,R)$) of the MIMO system in (1).

Corollary 1:

$$P_{N\times M}(K,R) \leq P_{N\times M}^{\mathrm{MEB}}(K,R). \tag{12}$$

In particular, for a single-input single-output (SISO) system with $M = N = 1$, every user has only one eigenmode and MEB reduces to the conventional SIC scheme. It is shown in [20] that the reduced MEB strategy with (5) is optimal for any K in SISO systems. Hence the average unconstrained MTSP of a SISO system is given by

$$P_{1\times 1}(K,R) = P_{1\times 1}^{\mathrm{MEB}}(K,R) = \int_{\varepsilon}^{1} \frac{N_0 K \left(2^{\frac{R}{K}} - 1\right)\left(1 + \left(2^{\frac{R}{K}} - 1\right)(t - \varepsilon)\right)^{K-1}}{F^{-1}(t)}\, dt. \tag{13}$$

[1]From the i.i.d. property given in Assumption 1, all $\{d_{k,\max}^2\}$ have the same pdf and cdf regardless of k.

$$P_{N\times M}^{\mathrm{MEB}}(K,R) = \int_{G_0}^{\infty} \frac{N_0 K \left(2^{R/K} - 1\right)\left(1 + \left(2^{R/K} - 1\right)(F(g) - \varepsilon)/M\right)^{K-1}}{g} f(g)\, dg \tag{10}$$

$$P_{N\times M}^{\mathrm{MEB}}(K,R) = \int_{\varepsilon}^{1} \frac{N_0 K \left(2^{R/K} - 1\right)\left(1 + \left(2^{R/K} - 1\right)(t - \varepsilon)/M\right)^{K-1}}{F^{-1}(t)}\, dt \tag{11}$$

It can be shown that MEB is also optimal for any K in multiple-input single-output (MISO) MACs. However, for MIMO MACs with finite K, MEB is generally suboptimal.

IV. Asymptotic Optimality of MEB

In this section, we derive the limit of the average unconstrained MTSP of an MIMO MAC when $K \to \infty$ based on the results in the last section, with which we prove the asymptotic optimality of MEB and provide some useful insights into the asymptotic behavior of multi-user MIMO systems.

A. A Lower Bound

We first derive a lower bound for the average unconstrained MTSP with a finite K. Consider the following multiple access system:

$$\tilde{\boldsymbol{y}} = \sum_{k=1}^{K} d_{k,\max} \boldsymbol{I}_M \cdot \tilde{\boldsymbol{x}}_k + \boldsymbol{n} \tag{14}$$

where $\boldsymbol{I}_M$ is an $M \times M$ identity matrix and $d_{k,\max}$ the maximum singular value of $\boldsymbol{H}_k$ in (1). In (14), each user k sees M parallel subchannels with equal gain, one for each receive antenna. Hence (14) can be viewed as a bank of M identical SISO MACs, each with a sum rate R/M. The average unconstrained MTSP of the system in (14), denoted by $P_{N\times M}^{\mathrm{LB}}(K,R)$, can be achieved with every user transmitting at rate R/MK in every subchannel. From (13), we have

$$\begin{aligned} &P_{N\times M}^{\mathrm{LB}}(K,R) = M \cdot P_{1\times 1}(K, R/M) \\ &= \int_{\varepsilon}^{1} \frac{N_0 MK \left(2^{\frac{R}{MK}} - 1\right)\left(1 + \left(2^{\frac{R}{MK}} - 1\right)(t-\varepsilon)\right)^{K-1}}{F^{-1}(t)} dt. \end{aligned} \tag{15}$$

Lemma 1:

$$P_{N\times M}(K,R) \geq P_{N\times M}^{\mathrm{LB}}(K,R). \tag{16}$$

Proof: For each channel realization $\{\boldsymbol{H}_k\}$, assume that $\{\boldsymbol{x}_k\}$ achieve the unconstrained MTSP to support the target rates for (1). Define

$$\tilde{\boldsymbol{x}}_k = \frac{1}{d_{k,\max}} \boldsymbol{H}_k \cdot \boldsymbol{x}_k \tag{17}$$

and substitute (17) into (14). Then the same target rates are supported in the system (14). The related sum power is no larger than the unconstrained MTSP of (1) since

$$\|\tilde{\boldsymbol{x}}_k\|_2 \leq \frac{1}{d_{k,\max}} \|\boldsymbol{H}_k\|_2 \cdot \|\boldsymbol{x}_k\|_2 = \|\boldsymbol{x}_k\|_2, \forall k \tag{18}$$

where $\|\cdot\|_2$ denotes the 2-norm operation. Hence the average unconstrained MTSP of (14) is a lower bound for that of (1). ■

B. Asymptotic Average Unconstrained MTSP

Substituting (15) into (16) and letting $K \to \infty$, we have

$$\begin{aligned} &P_{N\times M}(\infty, R) \\ &\geq \lim_{K\to\infty} \int_{\varepsilon}^{1} \frac{N_0 MK \left(2^{\frac{R}{MK}} - 1\right)\left(1 + \left(2^{\frac{R}{MK}} - 1\right)(t-\varepsilon)\right)^{K-1}}{F^{-1}(t)} dt \\ &= \int_{\varepsilon}^{1} \frac{N_0 R \ln 2 \cdot 2^{R(t-\varepsilon)/M}}{F^{-1}(t)} dt. \end{aligned} \tag{19}$$

On the other hand, when $K \to \infty$, (12) becomes

$$\begin{aligned} &P_{N\times M}(\infty, R) \\ &\leq \lim_{K\to\infty} \int_{\varepsilon}^{1} \frac{N_0 K \left(2^{\frac{R}{K}} - 1\right)\left(1 + \frac{\left(2^{\frac{R}{K}} - 1\right)(t-\varepsilon)}{M}\right)^{K-1}}{F^{-1}(t)} dt \\ &= \int_{\varepsilon}^{1} \frac{N_0 R \ln 2 \cdot 2^{R(t-\varepsilon)/M}}{F^{-1}(t)} dt. \end{aligned} \tag{20}$$

Comparing (19) and (20), we have the following.

Theorem 2: For fixed R and ε, when $K \to \infty$

$$P_{N\times M}(\infty, R) = \int_{\varepsilon}^{1} \frac{N_0 R \ln 2 \cdot 2^{R(t-\varepsilon)/M}}{F^{-1}(t)} dt, \tag{21}$$

which is asymptotically achievable by the MEB strategy.

Theorem 2 indicates that the upper and lower bounds in (12) and (16) are tight for a large K, and the MEB strategy is asymptotically optimal as K increases. Hence (21) gives the average unconstrained MTSP of an MIMO system when K is large. In particular, when the MIMO channel is a set of parallel SISO subchannels, (21) reduces to [21, eq. (34)], where a SISO frequency selective channel is considered.

C. Examples

Fig. 1 compares the MEB performance with the optimal unconstrained system performance over single-cell fading MIMO MACs. The channel matrix of each user involves three multiplicative factors, namely, Rayleigh fading, normalized lognormal fading with $\sigma_s = 8$ and path loss in an edge-length-1 single-hexagonal cell with independent and uniform user distribution and fourth power path-loss law. We assume independent Rayleigh fading for every transmit-receive antenna link and equal lognormal fading and path loss for all the links seen by the same user. The outage probability is set at $\varepsilon = 0.01$. The curves for average unconstrained MTSP shown in the figure are obtained using the iterative algorithm proposed in [14]. We can see from Fig. 1 that the MEB performance converges steadily to the corresponding average unconstrained MTSP as K increases, especially when R is small and less antennas are involved. When $K = 8$, the gaps between them become marginal for both 2×2 and 4×4 systems.

Compared to the optimal strategy that requires joint water-filling of all users and the minimum mean-square-error (MMSE)

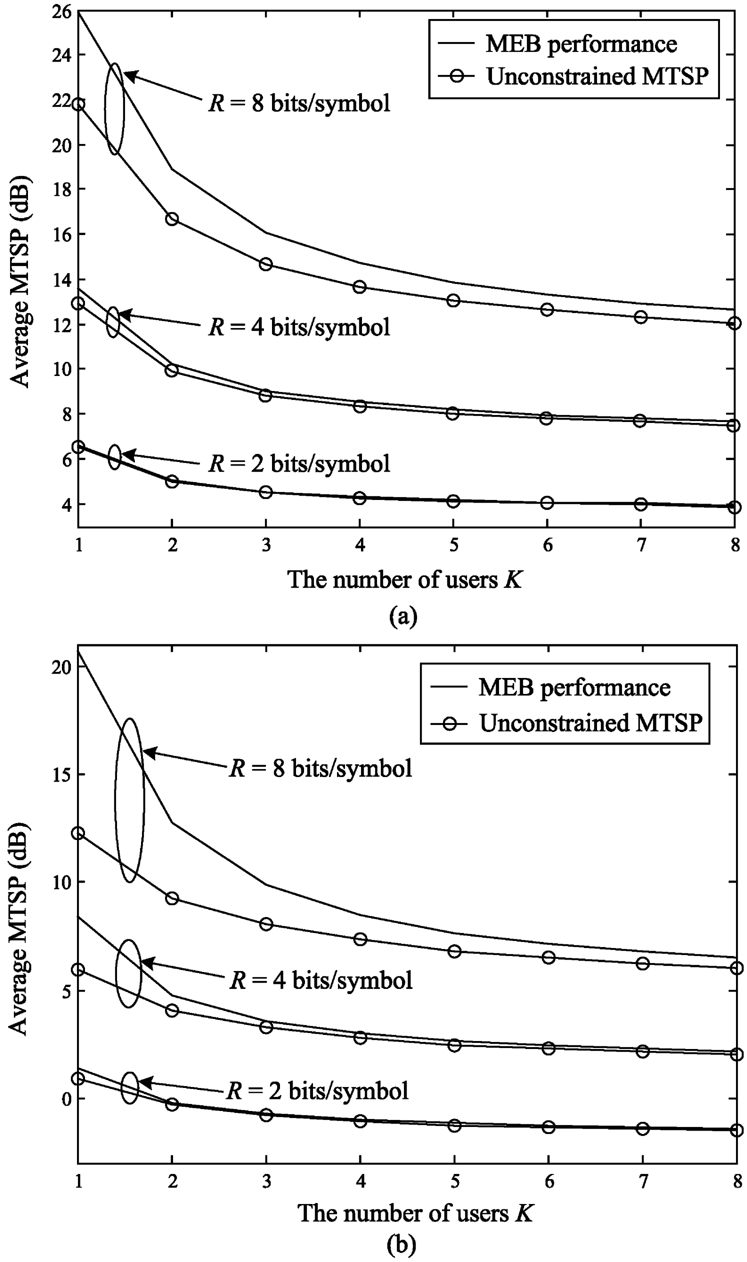

Fig. 1. Comparison between the average constrained (for MEB) and unconstrained MTSP in various MIMO systems with different K over a single-cell fading channel. We set $N_0 = 1$ and $\varepsilon = 0.01$. The antenna settings in (a) and (b) are, respectively, $M = N = 2$ and $M = N = 4$.

receiver, it is surprising that the low-cost MEB strategy can achieve asymptotic optimality as well. Intuitively, this observation attributes to the following reasons. With MEB, although every user only transmits in one direction, the signals of different users arrive at the base station in different directions and can statistically span the whole received signal space when K is large. Hence the MEB strategy with a large K can approximately achieve all the available degrees of freedom (DOF) and is therefore asymptotically optimal.

The asymptotic optimality of the simple correlator receiver used in MEB can be explained using the law of large numbers. When K is large, the interference seen by each user becomes approximately white. Hence the correlator receiver can perform close to the optimal MMSE receiver in this case.

D. The Impacts of M and N

Next we examine the impacts of M and N on (21). Recall that $F(\cdot)$ is the cdf of $\{d^2_{k,\max}\}$. Increasing either M or N leads to reduced $F(g)$ for $\forall g > 0$ (which indicates an increased mean for $\{d^2_{k,\max}\}$) and so reduced average MTSP. Increasing M has an additional benefit since it also reduces the numerator inside the integral in (21). When R is small, this additional benefit is ignorable since the numerator almost remains unchanged. However, when R is sufficiently large, this additional benefit becomes noticeable.

Consider the system DOF with $K \to \infty$ defined by

$$\mathrm{DOF}(M,N) = \lim_{R\to\infty} \frac{dR}{d\left(P_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}. \tag{22}$$

It is more convenient to work on the inverse of DOF as follows:

$$\begin{aligned}&\lim_{R\to\infty}\frac{d\left(P_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}{dR}\\&=\lim_{R\to\infty}\frac{d\left(10\log_{10}\int_{\varepsilon}^{1}\frac{N_0R\ln2\cdot2^{R(t-\varepsilon)/M}}{F^{-1}(t)}dt\right)}{dR}\\&=10\log_{10}e\cdot\lim_{R\to\infty}\frac{d\left(\ln R+\ln\int_{\varepsilon}^{1}\frac{2^{R(t-\varepsilon)/M}}{F^{-1}(t)}dt\right)}{dR}\\&\overset{(a)}{=}10\log_{10}e\cdot\lim_{R\to\infty}\left(\frac{1}{R}+\frac{d(R(1-\varepsilon)/M\cdot\ln2)}{dR}\right)\\&=10\log_{10}2\cdot(1-\varepsilon)/M.\end{aligned}\tag{23}$$

In the above, the equality (a) holds because the integration involved in it is dominated by the point of $t\to1$ when $R\to\infty$. Note that (23) is the asymptotic slope of $P_{N\times M}(\infty,R)$ (in dB form). From (23), we have

$$\begin{aligned}\mathrm{DOF}(M,N)&=\left(\lim_{R\to\infty}\frac{d\left(P_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}{dR}\right)^{-1}\\&=M\cdot\frac{1}{10\log_{10}2\cdot(1-\varepsilon)}.\end{aligned}\tag{24}$$

Equation (24) indicates that, when K is sufficiently large, the DOF for an MIMO MAC is only determined by M, the number of BS antennas. Hence at a high system throughput, R can asymptotically increase linearly with M for a fixed transmission power. Experimentally (using numerical results), we observed that it also approximately holds for quite small R.

This observation has interesting implications in cellular systems. Recall that the capacity of a cellular system is primarily limited by inter-cell interference. The latter is in turn determined by the average transmitted sum power of the users in each cell. Then (24) implies that the cellular capacity increases approximately linearly with M since the average sum power can be maintained unchanged when R and M increase together proportionally. Generally speaking, a cellular system benefits more from increasing the number of antennas at the base station than at each mobile unit.

The above discussion suggests that an unbalanced MIMO configuration with $M>N$ is a good tradeoff between performance and complexity for multi-user systems, since it is difficult to mount multiple antennas at a mobile unit with limited physical size.

Here are some numerical examples. Fig. 2 shows the unconstrained average MTSPs for various multiple access systems over a single-cell fading channel. The channel condition and the outage probability are the same as those used in Fig. 1. We first fix $N=1$ and increase the value of M from 1 to 4 in Fig. 2(a). Then we fix $M=4$ and increase N from 1 to 4 in Fig. 2(b). We can see from Fig. 2 that, in a single-user SISO system with $K=1$, increasing M can only bring about marginal performance improvements [Fig. 2(a)]. However, when we further increase N, significant performance improvements can be achieved [Fig. 2(b)]. This is because, when $K=1$, the system performance is dominated by the available DOF of the system, i.e., the minimum of M and N. This implies that a balanced MIMO configuration with $N=M$ is preferable for single-user systems [19].

On the other hand, for a large K, most power savings can be achieved by increasing M only [Fig. 2(a)] and further increasing N only leads to marginal performance improvements [Fig. 2(b)]. This is because the system DOF now is M and irrelative of N. This agrees with our previous discussion that increasing M has a more noticeable effect than increasing N. Clearly, an unbalanced MIMO configuration with $M>N$ provides an attractive compromise for multi-user systems.

V. Multi-User Gain

We have seen from Figs. 1 and 2 the power advantage of multi-user concurrent transmission. Such an advantage is referred to as multi-user gain (MUG) in this paper and formally defined using the ratio

$$G_{N\times M}(K,R)\triangleq\frac{P_{N\times M}(1,R)}{P_{N\times M}(K,R)}.\tag{25}$$

In (25), the derivation for $P_{N\times M}(1,R)$ is given in Appendix B and $P_{N\times M}(K,R)$ can be obtained using the method proposed in [14]. Similarly, we have the following for MEB:

$$G_{N\times M}^{\mathrm{MEB}}(K,R)\triangleq\frac{P_{N\times M}^{\mathrm{MEB}}(1,R)}{P_{N\times M}^{\mathrm{MEB}}(K,R)}.\tag{26}$$

In this section, we will investigate the MUG in a multi-user system both qualitatively and quantitatively.

A. The Monotonicity of Average Unconstrained MTSP

For a fixed system sum rate R, the average unconstrained MTSP is a monotonic decreasing function of K, which can be seen as follows. For a $(K+1)$-user system, we divide the total transmission time in each channel realization into $K+1$ equal-length slots. We take K users out of $K+1$ each time and form $K+1$ different combinations. Let each combination of K users transmit in a distinguished slot in the optimal way. Provided that all users have the same rate constraint and i.i.d. fading distribution, the resultant average MTSP of such a scheme is clearly equal to the average unconstrained MTSP of a K-user system (after averaging over all possible channel realizations). Thus if the optimal transmission scheme is applied to this $(K+1)$-user system, the corresponding average unconstrained MTSP must be no larger than that of a K-user one. The above discussion is summarized as follows.

Theorem 3:

$$P_{N\times M}(K,R)\geq P_{N\times M}(K+1,R),\quad\forall K,R.\tag{27}$$

Theorem 3 gives an qualitative explanation for the power advantage of multi-user concurrent transmission shown in Fig. 1. Note that the discussion above does not involve any detail of the optimal transmission scheme. Hence Theorem 3 is also applicable to MIMO BCs to be discussed in Section VIII.

B. Near-Far Diversity

To better understand MUG, let us first consider a special case of a K-user fading MAC with $M=N=1$ (i.e., a SISO MAC) and assume SIC with descending decoding order on k. Denote

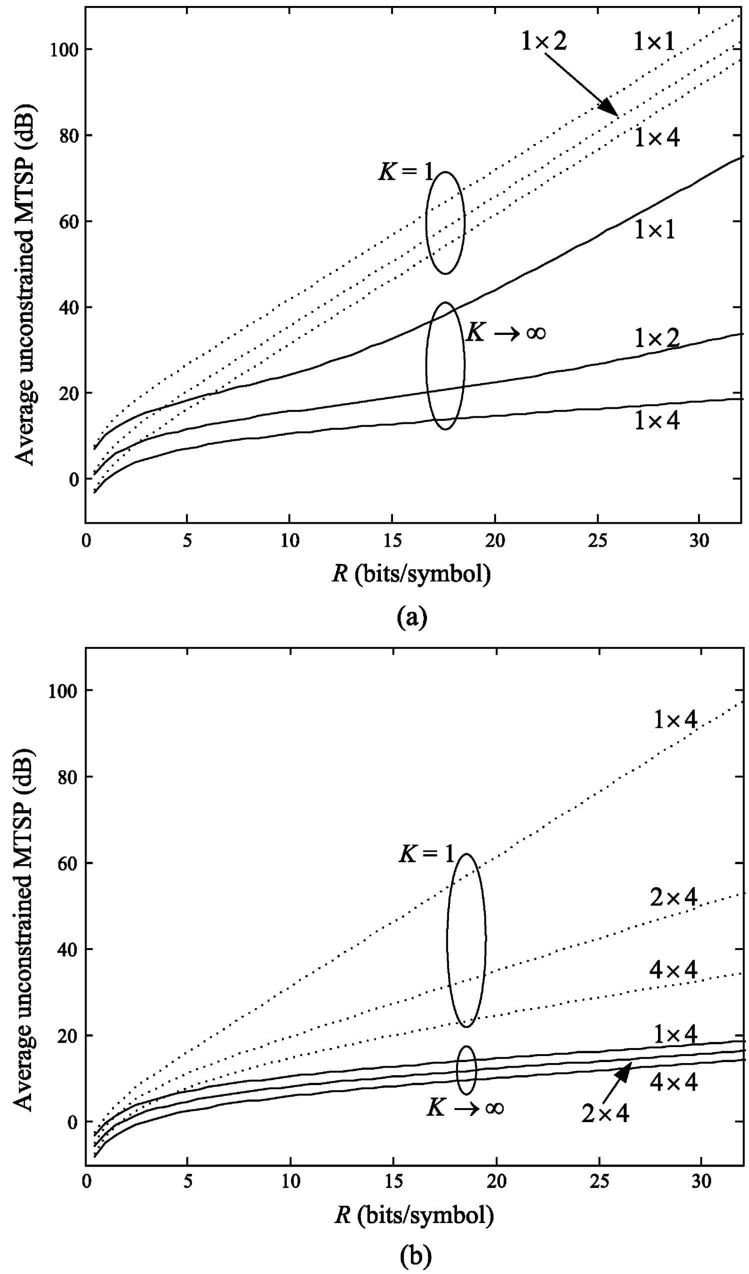

Fig. 2. MTSP versus the sum rate R for various multiple access systems with different M and N over a single-cell fading channel. $N_0 = 1$. $\varepsilon = 0.01$. The antenna settings $N \times M$ are marked on the curves.

by $q_k = p_k d_{k,\max}^2$ the received power of user k. From (8) we have (noting that $\phi_{k,i} \equiv 1, \forall k, i$ in a SISO system)

$$q_k = (2^{R/K} - 1)\left(\sum_{i=1}^{k-1} q_i + N_0\right)$$

or in a nonrecursive form

$$q_k = N_0(2^{R/K} - 1)2^{(k-1)R/K}. \tag{28}$$

For each channel realization, we can adopt power control to obtain the received power level q_k in (28) for each user k. Then the transmitted sum power can be represented as

$$\sum_{k=1}^{K} p_k = \sum_{k=1}^{K} \frac{q_k}{d_{k,\max}^2} = \sum_{k=1}^{K} \frac{N_0(2^{R/K} - 1)2^{(k-1)R/K}}{d_{k,\max}^2}. \tag{29}$$

Given each channel realization $\{d_{k,\max}^2\}$, a permutation of $\{d_{k,\max}^2\}$ can be obtained by re-indexing these elements. For example, $\{d_{1,\max}^2 = 1, d_{2,\max}^2 = 2\}$ is a permutation of $\{d_{1,\max}^2 = 2, d_{2,\max}^2 = 1\}$. The average of (29) over all possible permutations of $\{d_{k,\max}^2\}$ is

$$\sum_{k=1}^{K} \frac{\frac{1}{K}\sum_{i=1}^{K} q_i}{d_{k,\max}^2} = \frac{N_0(2^R - 1)}{K} \sum_{k=1}^{K} \frac{1}{d_{k,\max}^2}. \tag{30}$$

It is interesting to compare (30) with the result of a TDMA-type strategy by dividing the time in each channel realization into K equal-length time slots, each for a unique user transmitting at rate R (so that the average sum rate in each channel re-

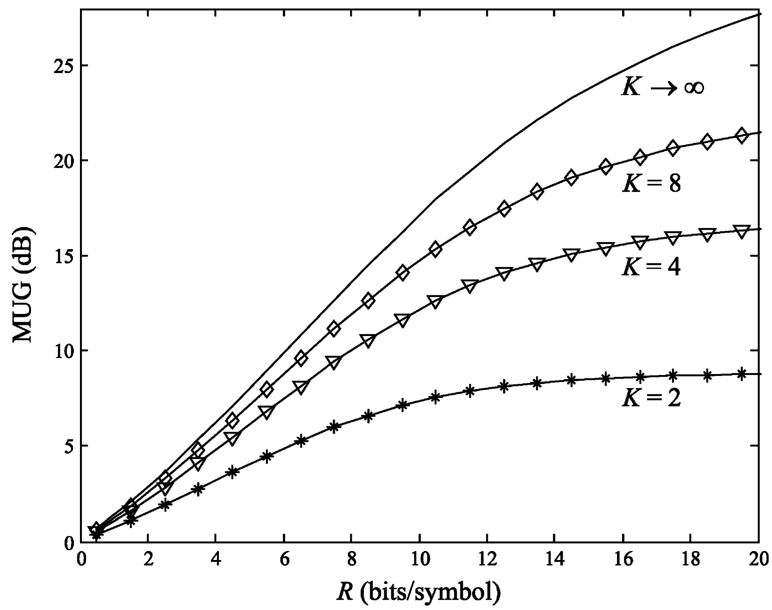

Fig. 3. Achievable MUG for SISO multiple access systems with different number of users K over a single-cell fading channel. $\varepsilon = 0.01$.

alization is still R). The corresponding constrained MTSP (averaged over all time slots) is

$$\frac{1}{K}\sum_{k=1}^{K} p_K^{TDMA} = \frac{N_0(2^R-1)}{K}\sum_{k=1}^{K}\frac{1}{d_{k,\max}^2}. \qquad (31)$$

Clearly, (31) is the same as (30).

We can do better than (30) by choosing a proper permutation of $\{d_{k,\max}^2\}$ for (29). As mentioned in Section III-C, the unconstrained MTSP of a K-user SISO MAC with channel realization $\{d_{k,\max}^2\}$ is given by (29) when the optimal decoding order (5) is applied, which must be less than (30) or (31). Intuitively, this power advantage is achieved by matching the elements of two sets $\{q_k\}$ and $\{d_{k,\max}^2\}$ in a large-to-large/small-to-small manner [20]. Such an ordered matching strategy becomes trivial in a single-user environment or when all users has the same channel gain $\{d_{k,\max}^2\}$. Hence the MUG of a multi-user SISO system comes from the multi-user "near-far diversity" in fading channels together with the matching strategy. (Here the "near-far diversity" is a synonym of "near-far effect" with emphasis on the positive side.)

C. Direction Diversity

Next, we examine another special case of a K-user MIMO MAC with $F^{-1}(t) = 1, \forall t$, i.e., the near-far diversity is excluded. For simplicity, we assume $N = 1$ (the discussion below can be extended to situations with $N > 1$). Thus there is only one nonzero eigenmode for each user and the channel gain is the same for all users. For such a system with $M > 1$, the optimal multi-user concurrent transmission strategy can still achieve power savings when K increases, but the related analysis is complicated due to the lack of concise expressions for the average unconstrained MTSP. However, the problem is much simpler with MEB. When all users have the same unit channel gain, (11) reduces to

$$\begin{aligned} &P_{N\times M}^{\text{MEB}}(K,R) \\ &= \int_0^1 N_0 K\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)t}{M}\right)^{K-1} dt \\ &= N_0 M\left(\left(1+(2^{R/K}-1)/M\right)^K-1\right). \end{aligned} \qquad (32)$$

Here we have set $\varepsilon = 0$ since there is no need to consider outage when the channel gain is a constant.

In the special case of $M = 1$, the system reduces to an AWGN SISO MAC. Then (32) reduces to the standard AWGN relationship

$$P_{1\times 1}^{\text{MEB}}(K,R) = N_0(2^R-1). \qquad (33)$$

Clearly, (33) is independent of K. Thus a SISO system does not provide MUG when all users see the same channel gain, which agrees with our discussion in the previous subsection.

When $M > 1$, it is easy to verify that (32) is a decreasing function of K and so MUG is still available. In this case, the advantage of a large K comes from the diversity of direction of arrival (DOA). (In this paper, we assume that the receive antennas are randomly placed. Thus the meaning of direction here is slightly different from that for a regular beamforming antenna array. Nevertheless, we still borrow this concept for intuitive understanding.) This direction diversity attributes to the fact that signals are more evenly distributed in the whole received signal space with multi-user concurrent transmission and the interference among them is therefore reduced.

In summary, MUG results from two aspects: the near-far diversity discussed in the previous subsection and direction diversity mentioned in this subsection. The latter is not available for an omni-receive-antenna system with $M = 1$.

D. Asymptotic Behavior of MUG

When $R \to 0$ for a finite K, from (11) and (26) we have (34), shown at the bottom of the page.

By applying l'Hôpital's rule to (34) twice, we obtain

$$\lim_{R\to 0} \frac{d}{dR}\left(G_{N\times M}^{\mathrm{MEB}}(K,R)\right)_{\mathrm{dB}} = 10\log_{10} 2 \cdot \frac{K-1}{2K}$$

or equivalently

$$G_{N\times M}^{\mathrm{MEB}}(K,R) = 2^{R(K-1)/2K + o(R)}. \tag{35}$$

Equation (35) provides a simple way to estimate MUG for a small R.

When $R \to \infty$ for a finite K, we have

$$\begin{aligned}
&\lim_{R\to\infty} G_{N\times M}^{\mathrm{MEB}}(K,R)\\
&= \lim_{R\to\infty} \frac{\int_\varepsilon^1 \frac{N_0(2^R-1)}{F^{-1}(t)}\,dt}{\int_\varepsilon^1 \frac{N_0K\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(t-\varepsilon)}{M}\right)^{K-1}}{F^{-1}(t)}\,dt}\\
&= \lim_{R\to\infty} \frac{2^R\int_\varepsilon^1 \frac{N_0(1-2^{-R})}{F^{-1}(t)}\,dt}{2^R\int_\varepsilon^1 \frac{N_0K(1-2^{-\frac{R}{K}})\left(2^{-\frac{R}{K}}+\frac{(1-2^{-\frac{R}{K}})(t-\varepsilon)}{M}\right)^{K-1}}{F^{-1}(t)}\,dt}\\
&= \lim_{R\to\infty} \frac{\int_\varepsilon^1 \frac{1}{F^{-1}(t)}\,dt}{K\int_\varepsilon^1 \frac{((t-\varepsilon)/M)^{K-1}}{F^{-1}(t)}\,dt}
\end{aligned} \tag{36}$$

which is a finite value for a finite K and increases indefinitely with K.

On the other hand when $K \to \infty$, from (23) and (68) in Appendix B, we can obtain the asymptotic slope of MUG in an MIMO system as

$$\begin{aligned}
&\lim_{R\to\infty} \frac{d\left(G_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}{dR}\\
&= \left(\frac{1}{\min(M,N)} - \frac{1-\varepsilon}{M}\right) 10\log_{10} 2.
\end{aligned} \tag{37}$$

We can make the following observations from (37):

(i). When $M \le N$, the asymptotic slope in (37) is almost zero (ignoring the outage probability ε), which means that the achievable MUG increases very slowly with the sum rate R when it is large.

(ii). When $M > N$, the asymptotic MUG (in dB form) increases linearly with R. This indicates that increasing M has a more significant impact than increasing N when R is large.

(iii). The maximum asymptotic slope of MUG is $10\log_{10} 2$, which is approached by allocating only one antenna at each mobile unit side and as many antennas as possible at the base station. This is a single-input-multiple-output (SIMO) situation, indicating that the advantage of multiuser concurrent transmission is most significant for SIMO channels.

We now provide some numerical results. Fig. 3 shows the achievable MUG in a SISO MAC with different K. The channel condition is the same as that in Figs. 1 and 2. We can see that the achievable MUG increases with R and K. The curves for finite K converge to finite values when $R \to \infty$. However, $G(K,R)$ is unlimited when both K and R become infinite. These observations agree with (36).

Fig. 4 shows the asymptotic MUGs for various multiple access systems over a single-cell fading channel. (The MUGs for MISO systems are close to those for MIMO ones and not shown in Fig. 4 for simplicity.) We can see from Fig. 4 that SIMO systems have much higher MUGs than MIMO ones. This is in good agreement with the observation (iii) made above. A SIMO MAC involves multiple antennas at the base station only, which is basically the technique used by most current cellular systems. This indicates the possibility to improve system performance without significantly altering the basic system architecture.

E. Normalized MUG

As we have seen from Theorem 3 and Fig. 3, the available MUG increases with K for a fixed system sum rate R. The corresponding upper limit, i.e., the MUG for $K \to \infty$, is not practically achievable with a finite K. It is therefore worthwhile to examine the relative difference between the MUG values for finite and infinite K. For this purpose, we further define the normalized MUG for a K-user MIMO system as

$$\Psi_{N\times M}(K,R) \triangleq \frac{\left(G_{N\times M}(K,R)\right)_{\mathrm{dB}}}{\left(G_{N\times M}(\infty,R)\right)_{\mathrm{dB}}}. \tag{38}$$

$$\begin{aligned}
\lim_{R\to 0} \frac{d}{dR}\left(G_{N\times M}^{\mathrm{MEB}}(K,R)\right)_{\mathrm{dB}} &= \lim_{R\to 0} \frac{d}{dR} 10\log_{10}\left(\frac{\int_\varepsilon^1 \frac{N_0(2^R-1)}{F^{-1}(t)}dt}{\int_\varepsilon^1 \frac{N_0K\left(2^{R/K}-1\right)\left(1+\left(2^{R/K}-1\right)(t-\varepsilon)/M\right)^{K-1}}{F^{-1}(t)}dt}\right)\\
&= \lim_{R\to 0} \frac{d}{dR} 10\log_{10} \frac{2^R-1}{2^{\frac{R}{K}}-1}\\
&= 10\log_{10} 2 \cdot \lim_{R\to 0} \frac{2^R\left(2^{R/K}-1\right) - \frac{1}{K}2^{R/K}(2^R-1)}{(2^R-1)\left(2^{R/K}-1\right)}
\end{aligned} \tag{34}$$

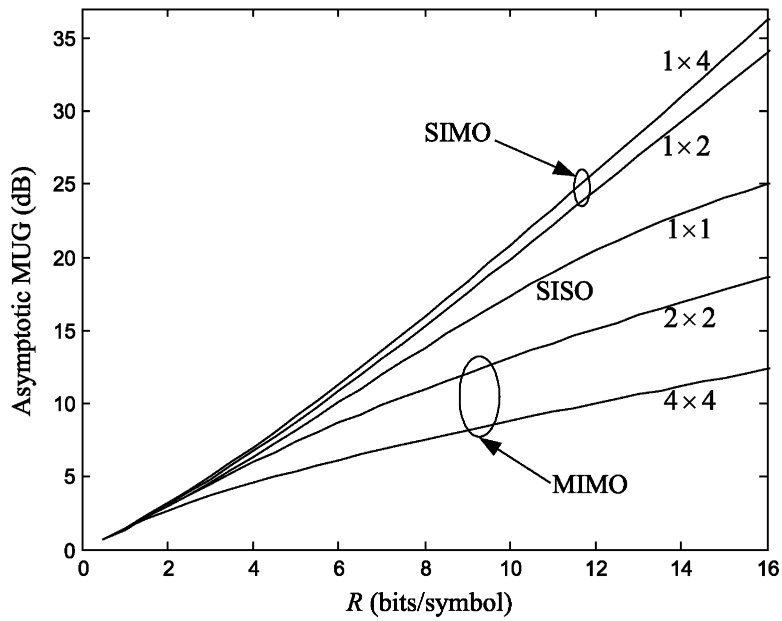

Fig. 4. Asymptotic MUG for various multiple access systems over a single-cell fading channel. $\varepsilon = 0.01$. The antenna settings $N \times M$ are marked on the curves.

Similar to (26), we have the following for MEB:

$$\Psi_{N\times M}^{\mathrm{MEB}}(K,R) \triangleq \frac{\left(G_{N\times M}^{\mathrm{MEB}}(K,R)\right)_{\mathrm{dB}}}{\left(G_{N\times M}^{\mathrm{MEB}}(\infty,R)\right)_{\mathrm{dB}}}. \tag{39}$$

When R is small, the normalized MUG (39) can be estimated using (35) as

$$\lim_{R\to 0}\Psi_{N\times M}^{\mathrm{MEB}}(K,R) = \lim_{R\to 0}\frac{\left(G_{N\times M}^{\mathrm{MEB}}(K,R)\right)_{\mathrm{dB}}}{\lim\limits_{K\to\infty}\left(G_{N\times M}^{\mathrm{MEB}}(K,R)\right)_{\mathrm{dB}}} = \lim_{R\to 0}\frac{R(K-1)/2K}{R/2} = \frac{K-1}{K}. \tag{40}$$

This indicates that a K-user system with a low sum rate can approximately achieve about $(K-1)/K$ of the asymptotic MUG (in dB form), e.g., a 2-user system can achieve about half of the MUG that is achievable by infinite K. Clearly, when R is low, a relatively small K (say, around 4 to 8) is sufficiently good in term of MUG. This observation is useful since a larger K usually implies a higher implementation complexity.

When R is large, we can show based on (36) that

$$\lim_{R\to\infty}\Psi_{N\times M}^{\mathrm{MEB}}(K,R) = 0 \tag{41}$$

for any finite K. This indicates that, when R increases, more simultaneous users are required to provide near optimal performance.

Fig. 5(a) and (b) shows the unconstrained normalized MUGs [i.e., (38)] of various SIMO and MIMO systems with different numbers of users K, respectively. The corresponding normalized MUGs achievable by the MEB strategy [i.e., (39)] are also plotted for reference. From Fig. 5(a) we can see that, for SIMO systems, the gap between the unconstrained normalized MUG and its lower bound,[2] (39), is marginal and vanishes to zero as K increases or R decreases. For MIMO systems, (39) is no longer a lower bound for (38). From Fig. 5(b) we can see that the more antennas are involved, the larger the gap is between the curves for (38) and (39). This is because the single-user MEB performance is inferior to the corresponding optimal single-user water-filling performance in this case. However, this gap will vanish to zero as K increases (referring to Fig. 1).

VI. MEB With Unequal Rate Constraints

So far, we have assumed that every user transmits at the same rate $R_0 = R/K$. The above results can be extended to the unequal rate scenario in a straightforward way. Denote by R_k the designated unequal instantaneous transmission rate for user k. After a similar derivation to that for (10) (the details are omitted here for brevity), we can obtain the following theorem.

Theorem 4: Given the (unequal) rate constraint R_k and an outage probability ε for each user k, the average MTSP of the MEB strategy for a K-user MIMO MAC in (1) (denoted by $P_{N\times M}^{\mathrm{MEB}}(K,\{R_k\})$) is given by

$$P_{N\times M}^{\mathrm{MEB}}(K,\{R_k\}) = \sum_{k=1}^{K}\int_{\varepsilon}^{1}\frac{N_0(2^{R_k}-1)\prod_{i=1,i\neq k}^{K}\left(1+\frac{(2^{R_i}-1)(t-\varepsilon)}{M}\right)}{F^{-1}(t)}\,dt. \tag{42}$$

Equation (42) is the unequal-rate counterpart of the equal-rate average MTSP expression (11) for MEB.

It is also interesting to examine the limiting case of (42) when $K \to \infty$. Let us fix the system sum rate $R = \sum_k R_k$ and

[2]For SIMO systems, it can be seen that $G_{1\times M}(K,R) \geq G_{1\times M}^{\mathrm{MEB}}(K,R)$ for a finite K and $G_{1\times M}(\infty,R) = G_{1\times M}^{\mathrm{MEB}}(\infty,R)$. Hence (39) is a lower bound of (38) in this case, i.e., $\Psi_{1\times M}(K,R) \geq \Psi_{1\times M}^{\mathrm{MEB}}(K,R)$.

assume that $R_k \to 0, \forall k$ when $K \to \infty$. Then from (42) we have

$$\begin{aligned}
&\lim_{\substack{K\to\infty \\ R_k\to 0,\forall k}} P_{N\times M}^{\mathrm{MEB}}(K,\{R_k\}) \\
&= \lim_{\substack{K\to\infty \\ R_k\to 0,\forall k}} \sum_{k=1}^{K} \int_{\varepsilon}^{1} \frac{N_0(2^{R_k}-1)\prod_{\substack{i=1\\ i\neq k}}^{K}\left(1+\frac{(2^{R_i}-1)(t-\varepsilon)}{M}\right)}{F^{-1}(t)}\,dt \\
&= \lim_{K\to\infty} \sum_{k=1}^{K} \int_{\varepsilon}^{1} \frac{N_0 R_k \ln 2 \cdot 2^{R(t-\varepsilon)/M}}{F^{-1}(t)}\,dt \\
&= \int_{\varepsilon}^{1} \frac{N_0 R \ln 2 \cdot 2^{R(t-\varepsilon)/M}}{F^{-1}(t)}\,dt. \qquad (43)
\end{aligned}$$

Similar to Theorem 2, we can adopt the bounding technique used in Section IV to show that (43) gives the unconstrained MTSP for an MIMO system with unequal rate allocation. The detailed proof is omitted here for brevity.

Comparing (43) to (21), we can see that in the limiting case of $K \to \infty$ and under the assumption of $R_k \to 0, \forall k$, the unconstrained MTSP is independent of individual rates $\{R_k\}$ and is a function of the system sum rate R only. Thus most asymptotic properties discussed in Sections V and VI are applicable to the case with unequal rate allocation among users.

VII. Realization Issues

Our discussion above has been focused on capacity aspects. We now consider the realization of MUG using the MEB strategy in practical environments. As mentioned earlier, nonorthogonal multi-user concurrent transmission is essential to achieve MUG. We may use a nonorthogonal code-division multiple-access (CDMA) type technique as a platform to realize MEB. However, the use of spreading codes in a traditional CDMA system reduces the transmission rate of each user, which is a key obstacle for its application in high-rate application environments (e.g., in MIMO systems). To overcome this difficulty, here we adopt the interleave-division multiple-access (IDMA) scheme introduced in [22] that employs different interleavers to distinguish the signals from different users. Very high transmission rates can be supported in this way.

In our simulation example, we fix the system sum rate at $R = 4$ bits/symbol and adopt a length-5000 and rate-1/4 irregular low-density parity-check (LDPC) code for all users. The corresponding edge degree distribution $\lambda(x) = 0.367x^6 + 0.213x^2 + 0.42x$ and $\rho(x) = x^3$ is obtained using the optimization tool in [24]. The basic rate per user is 0.5 bit/symbol with QPSK modulation. If more than 0.5 bit/symbol is requested for each user, we can assign multiple coded streams to each user based on superposition coding principle [23]. A turbo type receiver is applied to perform multi-user linear MMSE detection and single-user *a posteriori* probability (APP) decoding iteratively. (Refer to [22] and [25] for details.) For each channel realization, the transmitted power levels $\{p_k\}$ are optimized using an interior point technique [26].

Fig. 6 shows the BER performance of such LDPC-coded systems with different K over a 4×4 MIMO fading MAC. The channel condition is the same as that used in previous figures. The theoretical limits achieved by the MEB strategy, measured from Fig. 1(b), are also plotted for reference. From Fig. 6, we can see noticeable MUG between the curves for 1 and 8 users at $\mathrm{BER} = 10^{-5}$. Again, a significant portion of the MUG can be achieved with only 2 or 4 users. Further increasing K only leads to marginal improvements. In the above example, no effort has been made to optimize the edge distribution of the LDPC code with respect to the system under consideration. The related optimization technique is an interesting topic for future research.

VIII. MIMO BCs

Consider an MIMO BC. Without confusion, we still assume M antennas at the base station and N antennas at every mobile unit. Let $\boldsymbol{x}_k$ be the transmitted signal of user k at the base station. At the receiver for user k, the received signal $\boldsymbol{y}_k$ can be represented as

$$\boldsymbol{y}_k = \boldsymbol{H}_k^* \sum_{i=1}^{K} \boldsymbol{x}_i + \boldsymbol{n}_k \qquad (44)$$

where $\boldsymbol{H}_k^*$ is the channel matrix seen from the base station to user k and $\boldsymbol{n}_k$ is a vector of complex AWGN samples with zero mean and variance N_0. The MIMO BC in (44) is referred to as the dual of the MIMO MAC in (1). In particular, the dual of a SIMO MAC is a MISO BC.

A. MEB for MIMO BCs

When K is finite, the optimal realization of the MIMO BC in (44) involves complicated optimization of transmission covariance matrices and dirty paper encoding. It is therefore worthwhile to verify the system performance when MEB is directly applied to MIMO BCs. Similar to the MEB strategy proposed for MIMO MACs in Section III, we only allow each user to transmit in its maximum eigenmode direction in an MIMO BC. Without confusion, we still use p_k and x_k to denote the transmitted power and coded signal of user k. Then we have

$$\boldsymbol{x}_k = \sqrt{p_k}\,\boldsymbol{u}_{k,\max} x_k. \qquad (45)$$

At the base station, dirty paper coding is applied, which corresponds to the SIC used in MIMO MACs. Without loss of generality, we still assume that (5) holds and let the user with the largest maximum eigenvalue be encoded first. Then (44) can be rewritten as

$$\boldsymbol{y}_k = \boldsymbol{V}_k \boldsymbol{D}_k \boldsymbol{U}_k^* \sum_{i=k}^{K} \sqrt{p_i}\,\boldsymbol{u}_{i,\max} x_i + \boldsymbol{n}_k. \qquad (46)$$

Note that in (46), the signals for users $\{i, i<k\}$ are assumed to have been pre-cancelled by dirty paper encoding and only those of users $\{i, i>k\}$ are regarded as interference.

At the receiver for each user k, we still use a simple correlator $\boldsymbol{v}_{k,\max}$ to collect its signal. Based on (46), the correlator output is

$$\begin{aligned}
\boldsymbol{v}_{k,\max}^* \boldsymbol{y}_k &= \boldsymbol{v}_{k,\max}^* \boldsymbol{V}_k \boldsymbol{D}_k \boldsymbol{U}_k^* \sum_{i=k}^{K} \sqrt{p_i}\,\boldsymbol{u}_{i,\max} x_i + \boldsymbol{v}_{k,\max}^* \boldsymbol{n}_k \\
&= \sqrt{p_k}\,\boldsymbol{d}_{k,\max} x_k + \sum_{i=k+1}^{K} \sqrt{p_i}\,\boldsymbol{d}_{k,\max} \boldsymbol{u}_{k,\max}^* \boldsymbol{u}_{i,\max} x_i \\
&\quad + \boldsymbol{v}_{k,\max}^* \boldsymbol{n}_k \qquad (47)
\end{aligned}$$

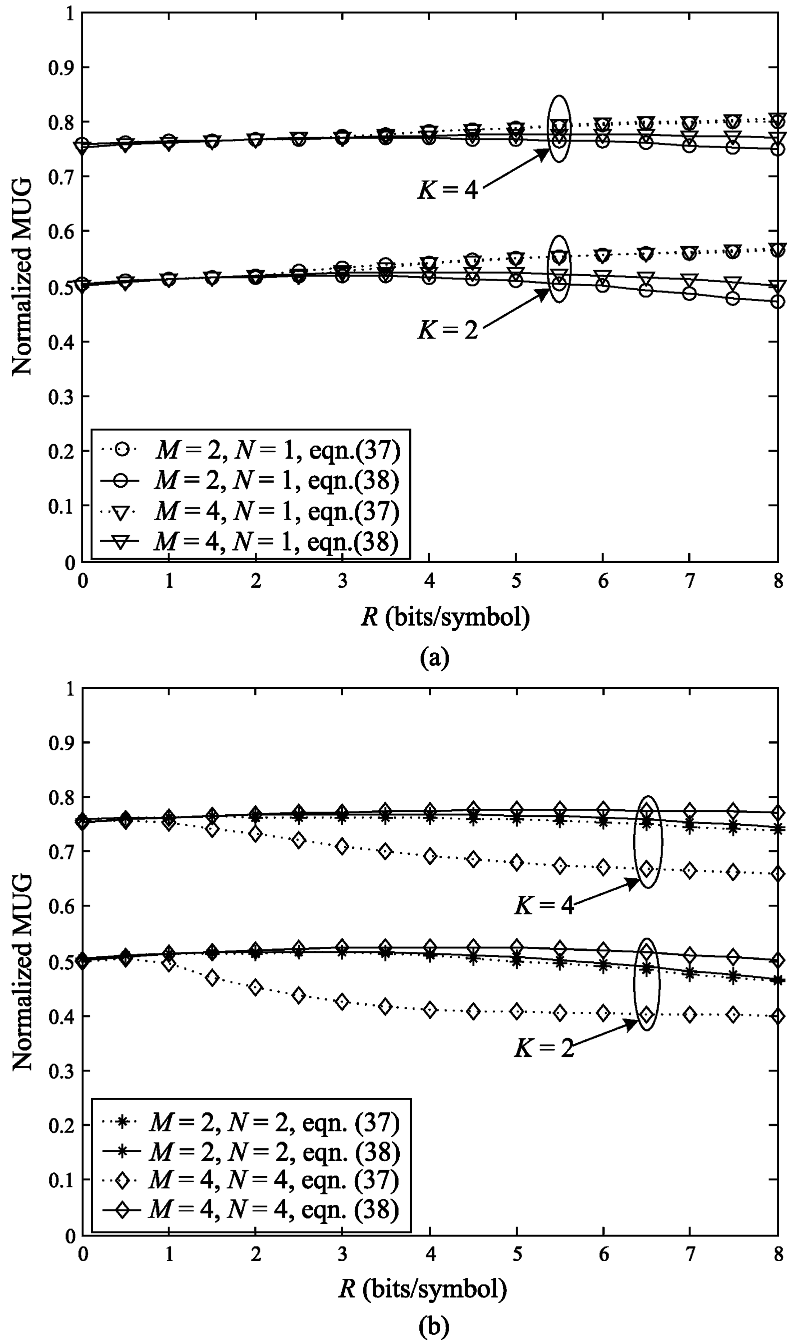

Fig. 5. Normalized MUG achieved by a finite number of users in MIMO multiple access systems over a single-cell fading channel. $\varepsilon = 0.01$.

and the corresponding SNR value for user k can be written as

$$\mathrm{SNR}_k = \frac{p_k d_{k,\max}^2}{\sum_{i=k+1}^{K} p_i d_{k,\max}^2 \phi_{k,i} + N_0}. \tag{48}$$

Assume rate-R/K ideal coding for each user. In each channel realization, the power values $\{p_k\}$ can be evaluated recursively based on (48) and the Shannon capacity formula as

$$p_k = \frac{(2^{R/K} - 1)\left(\sum_{i=k+1}^{K} p_i d_{k,\max}^2 \phi_{k,i} + N_0\right)}{d_{k,\max}^2}. \tag{49}$$

The following lemma provides a connection between the two MEB strategies for the MIMO BC in (44) and the MIMO MAC in (1).

Lemma 2: Given the rate constraint R/K and the channel realization $\boldsymbol{H}_k$ for each user k, the MTSP of the MEB strategy for the MIMO BC in (44) is exactly the same as that of the MEB strategy for the dual MIMO MAC in (1).

The proof of Lemma 2 is given in Appendix C. From Lemma 2, we obtain the following theorem.

Theorem 5: Given the target rate of R/K and an outage probability ε for each user, the average MTSP for the MEB strategy

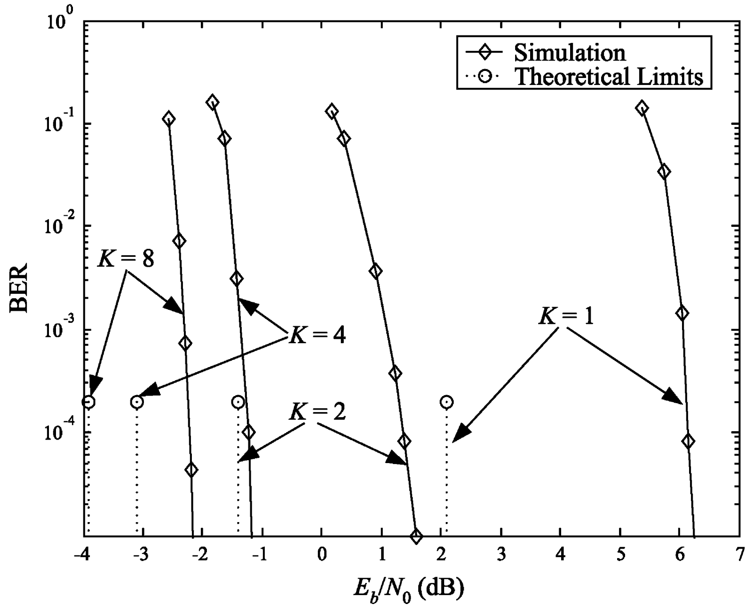

Fig. 6. Simulation results of an MEB-based system with user-specific interleaving and irregular LDPC coding over a 4 × 4 MIMO MAC. The corresponding theoretical limits of MEB are also plotted for reference.

in the MIMO BC (44) (which is still denoted by $P^{\text{MEB}}_{N\times M}(K,R)$ without confusion) is given by

$$P^{\text{MEB}}_{N\times M}(K,R) = \int_{\varepsilon}^{1} \frac{N_0 K\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(t-\varepsilon)}{M}\right)^{K-1}}{F^{-1}(t)}\,dt. \quad (50)$$

Clearly, (50) is the same as (10) for an MIMO MAC. Similar to the discussion for MIMO MACs in previous sections, we can show that the MEB strategy for the MIMO BCs is also asymptotically optimal and the results can be extended to the unequal rate constraint scenario. Hence all the observations related to Theorems 1–4 and Figs. 1–5 can be directly applied to MIMO BCs. Again, increasing the number of antennas at the base station is a more efficient way to enhance performance than increasing that at the mobile units. However, it should be reminded that dirty paper encoding/decoding is still required to guarantee the performance predicted by (50).

B. On the Duality Between MACs and BCs

Reference [27] introduced a duality principle between MIMO MACs and BCs. Here we need clarify some related concepts.

i) *Channel duality*: For example, the MIMO BC in (44) is the dual of the MIMO MAC in (1), and *vice versa*;
ii) *Transmission strategy duality*: Given a transmission strategy for an MIMO MAC in (1), there exists a dual transmission strategy for the dual MIMO BC in (44) such that these two strategies achieve the same rate constraints using the same sum power and *vice versa*;
iii) *Capacity Region duality*: Given the same sum power constraint, the MIMO MAC in (1) and its dual MIMO BC in (44) have the same instantaneous/ergodic capacity region.

From the above discussion, it can be shown that the MIMO MAC in (1) and its dual MIMO BC in (44) have the same average unconstrained MTSP. Hence Theorem 2 for MIMO MACs is also applicable to MIMO BCs. However, note that the BC-MEB defined in (45) is not the dual of the MAC-MEB defined in (3), even though we have proved that they have the same performance.[3] Therefore, we cannot arrive at Theorem 5 via duality directly.

IX. Asymptotic Sum-Rate Optimality of MEB

Before ending this paper, we show that the MEB strategy for MIMO MACs in Section III can also achieve asymptotic optimality when a sum-rate maximization problem given power constraints is considered. Return to the multiple access system in (1). For simplicity, we assume the power constraints of all users are the same at p. The sum capacity of (1) (denoted by $C_{N\times M}(K,p)$) can be written as [1], [5]

$$C_{N\times M}(K,p) = \max_{\{tr(\boldsymbol{Q}_k)\le p\}} \log_2\left|\boldsymbol{I}_M + \frac{1}{N_0}\sum_{k=1}^{K}\boldsymbol{H}_k\boldsymbol{Q}_k\boldsymbol{H}_k^*\right| \quad (51)$$

where $\boldsymbol{Q}_k = \mathrm{E}(\boldsymbol{x}_k\boldsymbol{x}_k^*)$ is the transmitted covariance matrix of user k. The operators $|\cdot|$ and $tr(\cdot)$ denote the determinant and trace of a matrix, respectively.

The proof of the theorem below is given in Appendix D.

Theorem 6: Given the power constraint p for each user, the asymptotic sum capacity of an MIMO MAC in (1) with $K\to\infty$ can be written as

$$C_{N\times M}(\infty,p) = \lim_{K\to\infty} M\cdot\log_2\left(1+\frac{Kp}{MN_0}\int_0^{\infty} g f(g)\,dg\right) \quad (52)$$

[3]Based on the MAC-to-BC duality transformation in Section IV-B of [27], in the BC dual of (3), every user still only involves a single-mode beamforming. However, the beamforming direction of each user depends on the channel matrices of all users and is not its own maximum eigenmode direction.

which is asymptotically achievable by the MEB strategy.

The capacity of a multi-user MIMO system involving resource allocation (in time, frequency and space domains) is a more complicated problem, which is beyond the scope of this paper.

X. Conclusion

In this paper, we have made a comprehensive study on multi-user MIMO systems with rate constraints over MACs and BCs. The main contributions of this paper are summarized here.

- We have derived the closed-form expression for the average MTSP of the low-cost MEB strategy and proved that MEB is asymptotically optimal for a large K. Numerical results show that MEB is also nearly optimal even for a finite K.
- Using MEB, we have proposed upper and lower bounds for the average unconstrained MTSP of MIMO MACs and BCs. These bounds are tight when K is large.
- The limiting form of MEB sum power with $K \to \infty$ facilitates the asymptotic analysis of multi-user MIMO behaviors. In particular, we have shown that: (a) significant power savings, i.e., MUG, can be achieved by multi-user concurrent transmission in fading channels with rate constraints; (b) a large portion of MUG can be achieved by only a few number of users; and (c) an unbalanced MIMO configuration with $M > N$ offers a good tradeoff between performance and complexity for multi-user systems.
- We have provided simulation results to show that significant MUG is achievable in practically coded multi-user systems.

Appendix A
Proof of Theorem 1

For each channel realization, we define an indicator function $I(k,i)$ for any two users k and i as[4]

$$I(k,i) = \begin{cases} 0, & \text{if } d_{i,\max}^2 < G_0 \text{ or } d_{i,\max}^2 > d_{k,\max}^2; \\ 1, & \text{if } G_0 \le d_{i,\max}^2 < d_{k,\max}^2. \end{cases} \tag{53}$$

Then (8) can be rewritten as

$$p_k = \frac{(2^{R/K}-1)\left(\sum_{i=1,i\neq k}^{K} p_i d_{i,\max}^2 \phi_{k,i} I(k,i) + N_0\right)}{d_{k,\max}^2}. \tag{54}$$

[4]The function (53) does not include the event that two users i and k have the same channel gain, i.e., $d_{i,\max}^2 = d_{k,\max}^2$. The probability of such events is zero when $\{d_{i,\max}^2\}$ have continuous distribution.

Our derivation below includes the following four steps.

First, we fix $\{d_{i,\max}^2, \forall i\}$ (noting that $\{I(k,i), \forall i\}$ are also fixed in this case) and take average with respect to $\{\phi_{k,i}\}$ on both sides of (54), i.e.,

$$\begin{aligned} &\mathop{\mathrm{E}}_{\{\phi_{k,i}\}}(p_k) \\ &= \mathop{\mathrm{E}}_{\{\phi_{k,i}\}} \frac{\left(2^{\frac{R}{K}}-1\right)\left(N_0 + \sum_{i=1,i\neq k}^{K} p_i d_{i,\max}^2 \phi_{k,i} I(k,i)\right)}{d_{k,\max}^2} \\ &= \frac{\left(2^{\frac{R}{K}}-1\right)\left(N_0 + \sum_{i=1,i\neq k}^{K} d_{i,\max}^2 I(k,i)\, \mathrm{E}_{\{\phi_{k,i}\}}(p_i\phi_{k,i})\right)}{d_{k,\max}^2}. \end{aligned} \tag{55}$$

From (54), it can be verified that p_i is independent of $\phi_{k,i}$ for $\{k, \forall I(k,i)=1\}$. Then based on (9), we can rewrite (55) as

$$\begin{aligned} &\mathop{\mathrm{E}}_{\{\phi_{k,i}\}}(p_k) \\ &= \frac{\left(2^{\frac{R}{K}}-1\right)\left(N_0 + \sum_{i=1,i\neq k}^{K} \frac{\mathrm{E}_{\{\phi_{k,i}\}}(p_i) d_{i,\max}^2 I(k,i)}{M}\right)}{d_{k,\max}^2} \end{aligned}$$

or in a nonrecursive form

$$\begin{aligned} &\mathop{\mathrm{E}}_{\{\phi_{k,i}\}}(p_k) \\ &= \frac{N_0\left(2^{\frac{R}{K}}-1\right)\prod_{i=1,i\neq k}^{K}\left(1+\left(2^{\frac{R}{K}}-1\right)I(k,i)/M\right)}{d_{k,\max}^2}. \end{aligned} \tag{56}$$

Second, we fix $d_{k,\max}^2$ and take average with respect to $\{I(k,i), \forall i \neq k\}$ on both sides of (56). This is equivalent to taking average with respect to $\{d_{i,\max}^2, \forall i \neq k\}$ based on the definition of $I(k,i)$ in (53). Recall that $\{d_{i,\max}^2, \forall i \neq k\}$ are i.i.d., so are $\{I(k,i)\}$. Then from (56) we have

$$\begin{aligned} &\mathop{\mathrm{E}}_{\{\phi_{k,i}\},\{I(k,i)\}}(p_k) \\ &= \frac{N_0\left(2^{\frac{R}{K}}-1\right)\prod_{\substack{i=1\\ i\neq k}}^{K}\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)\mathrm{E}(I(k,i))}{M}\right)}{d_{k,\max}^2}. \end{aligned} \tag{57}$$

For user k under consideration with channel gain $d_{k,\max}^2$, the probability that another user i's channel gain is smaller than $d_{k,\max}^2$ is simply $F(d_{k,\max}^2)$. So we have

$$\mathrm{E}(I(k,i)) = F(d_{k,\max}^2) - \varepsilon, \quad \forall i \neq k \tag{58}$$

where the term ε corresponds to the situation when user i is in outage. Substituting (58) into (57), we obtain

$$\begin{aligned}&\underset{\{\phi_{k,i}\},\{I(k,i)\}}{\mathrm{E}}(p_k)\\&=\frac{N_0\left(2^{\frac{R}{K}}-1\right)\prod_{i=1,i\neq k}^{K}\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(F(d_{k,\max}^2)-\varepsilon)}{M}\right)}{d_{k,\max}^2}.\\&=\frac{N_0\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(F(d_{k,\max}^2)-\varepsilon)}{M}\right)^{K-1}}{d_{k,\max}^2}.\end{aligned}\tag{59}$$

Third, we take average with respect to the channel gain $d_{k,\max}^2$ on both sides of (59) and obtain

$$\begin{aligned}&\underset{\{\phi_{k,i}\},\{I(k,i)\},d_{k,\max}^2}{\mathrm{E}}(p_k)\\&=\int_{G_0}^{\infty}\frac{N_0\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(F(g)-\varepsilon)}{M}\right)^{K-1}}{g}f(g)\,dg,\end{aligned}\tag{60}$$

which gives the average transmitted power of user k. Note that since user k should be in outage when $d_{k,\max}^2\le G_0$, the integration in (60) is taken within $[G_0,\infty)$, instead of $(0,\infty)$.

Fourth, since we have assumed that all users have the same pdf of channel gain, so are their average transmitted power. Therefore, the corresponding average MTSP for MEB is simply K times of (60), i.e.,

$$\begin{aligned}&P^{\mathrm{MEB}}(K,R)\\&=K\cdot\underset{\{\phi_{k,i}\},\{I(k,i)\},d_{k,\max}^2}{\mathrm{E}}(p_k)\\&=\int_{G_0}^{\infty}\frac{N_0K\left(2^{\frac{R}{K}}-1\right)\left(1+\frac{\left(2^{\frac{R}{K}}-1\right)(F(g)-\varepsilon)}{M}\right)^{K-1}}{g}f(g)\,dg.\end{aligned}\tag{61}$$

Hence Theorem 1 holds.

Appendix B
MTSP for $K=1$

For a single-user MIMO system, the MTSP is achieved by the single-user water-filling algorithm for each channel realization [5]. Suppose one user has its channel matrix with rank r and singular values $\{d_1,d_2,\ldots,d_r\}$. The minimum transmitted power is achieved by

$$p=\sum_{i=1}^{r}\left[p^*-N_0/d_i^2\right]^+\tag{62}$$

where $[x]^+=\max\{x,0\}$ and p^* satisfies

$$\sum_{i=1}^{r}\log_2\left(1+\frac{d_i^2[p^*-N_0/d_i^2]^+}{N_0}\right)=R.\tag{63}$$

Then $P_{N\times M}(1,R)$ can be obtained by taking the average of p over the channel distribution, which also provides the average transmitted MTSP of a TDMA-based multi-user MIMO system. The details are omitted here for brevity.

When R is sufficiently large, p^* in (62) and (63) is also large such that (62) reduces to

$$p=\sum_{i=1}^{r}\left(p^*-N_0/d_i^2\right)\tag{64}$$

where p^* satisfies

$$\sum_{i=1}^{r}\log_2\left(d_i^2p^*/N_0\right)=R.\tag{65}$$

Thus we have

$$\lim_{R\to\infty}\frac{d(p)_{\mathrm{dB}}}{dR}=\lim_{R\to\infty}\frac{d(p)_{\mathrm{dB}}}{dp^*}\cdot\frac{dp^*}{dR}=\frac{10\log_{10}2}{r}\tag{66}$$

or equivalently

$$p=2^{R/r+o(R)}\tag{67}$$

which is the same for all channel realizations.

For most practical fading distributions, the channel matrices have full rank with probability 1. This indicates that $r=\min(M,N)$ for almost all channel realizations. Hence the asymptotic slope of $P_{N\times M}(1,R)$ (in dB form) with an outage probability ε can be calculated as

$$\begin{aligned}&\lim_{R\to\infty}\frac{d\left(P_{N\times M}(1,R)\right)_{\mathrm{dB}}}{dR}\\&=\lim_{R\to\infty}\frac{d}{dR}10\log_{10}\left((1-\varepsilon)2^{\frac{R}{\min(M,N)}+o(R)}\right)\\&=\frac{10\log_{10}2}{\min(M,N)}.\end{aligned}\tag{68}$$

Appendix C
Proof of Lemma 2

We first derive a useful expression for the MTSP of MEB for MIMO MACs. We rewrite (8) as

$$-(2^{R/K}-1)\sum_{i=1}^{k-1}p_id_{i,\max}^2\phi_{k,i}+p_kd_{k,\max}^2=(2^{R/K}-1)N_0$$

or in a matrix form

$$(\boldsymbol{I}_K - \boldsymbol{C}\cdot\boldsymbol{A})\cdot\boldsymbol{G}\cdot\boldsymbol{p}^T = N_0\boldsymbol{C}\cdot\mathbf{1}^T \tag{69}$$

where $\boldsymbol{I}_K$ is a $K\times K$ identity matrix, $\boldsymbol{p} = (p_1, p_2, \ldots, p_K)$ and $\mathbf{1} = (1,1,\ldots,1)$ are two length-K vectors, $\boldsymbol{C} = \{c_{k,k'}\}_{K\times K}$ and $\boldsymbol{G} = \{g_{k,k'}\}_{K\times K}$ are two diagonal matrices with $c_{k,k} = (2^{R/K}-1)$ and $g_{k,k} = d^2_{k,\max}, \forall k$, respectively. $\boldsymbol{A} = \{a_{k,k'}\}_{K\times K}$ is a lower-triangular matrix with

$$a_{k,k'} = \begin{cases} 0, & \text{if } k \le k'; \\ \phi_{k,k'}, & \text{if } k > k'. \end{cases}$$

From (69), we can represent the MEB sum power of an MIMO MAC as

$$\sum_{k=1}^{K} p_k = \mathbf{1}\cdot\boldsymbol{p}^T = N_0\mathbf{1}\cdot\boldsymbol{G}^{-1}\cdot(\boldsymbol{I}_K - \boldsymbol{C}\cdot\boldsymbol{A})^{-1}\cdot\boldsymbol{C}\cdot\mathbf{1}^T. \tag{70}$$

Next, we derive a similar expression for the MEB sum power of MIMO BCs. Equation (49) can be rewritten as

$$p_k d^2_{k,\max} - (2^{R/K}-1)\sum_{i=k+1}^{K} p_i d^2_{k,\max}\phi_{k,i} = (2^{R/K}-1)N_0$$

or equivalently

$$\boldsymbol{G}\cdot\left(\boldsymbol{I}_k - \boldsymbol{C}\cdot\boldsymbol{A}^T\right)\cdot\boldsymbol{p}^T = N_0\boldsymbol{C}\cdot\mathbf{1}^T. \tag{71}$$

Then the MEB sum power of an MIMO BC is given by

$$\sum_{k=1}^{K} p_k = \mathbf{1}\cdot\boldsymbol{p}^T = N_0\mathbf{1}\cdot\left(\boldsymbol{I}_k - \boldsymbol{C}\cdot\boldsymbol{A}^T\right)^{-1}\cdot\boldsymbol{G}^{-1}\cdot\boldsymbol{C}\cdot\mathbf{1}^T. \tag{72}$$

From the Woodbury formula [28], we have

$$(\boldsymbol{I}_K - \boldsymbol{C}\cdot\boldsymbol{A}^T)^{-1} = \boldsymbol{I}_K + \boldsymbol{C}\cdot(\boldsymbol{I}_K - \boldsymbol{A}^T\cdot\boldsymbol{C})^{-1}\cdot\boldsymbol{A}^T. \tag{73}$$

Hence (72) can be rewritten as

$$\begin{aligned}\sum_{k=1}^{K} p_k &= N_0\left(\mathbf{1}\cdot(\boldsymbol{I}_K - \boldsymbol{C}\cdot\boldsymbol{A}^T)^{-1}\cdot\boldsymbol{G}^{-1}\cdot\boldsymbol{C}\cdot\mathbf{1}^T\right)^T \\ &= N_0\left(\mathbf{1}\cdot\left(\boldsymbol{I}_K + \boldsymbol{C}(\boldsymbol{I}_K - \boldsymbol{A}^T\boldsymbol{C})^{-1}\boldsymbol{A}^T\right)\boldsymbol{C}\boldsymbol{G}^{-1}\cdot\mathbf{1}^T\right)^T \\ &= N_0\left(\mathbf{1}\cdot\boldsymbol{C}\left(\boldsymbol{I}_K + (\boldsymbol{I}_K - \boldsymbol{A}^T\boldsymbol{C})^{-1}\boldsymbol{A}^T\boldsymbol{C}\right)\boldsymbol{G}^{-1}\cdot\mathbf{1}^T\right)^T \\ &= N_0\mathbf{1}\cdot\boldsymbol{G}^{-1}\left(\boldsymbol{I}_K + (\boldsymbol{I}_K - \boldsymbol{A}^T\boldsymbol{C})^{-1}\boldsymbol{A}^T\boldsymbol{C}\right)^T\boldsymbol{C}\cdot\mathbf{1}^T \\ &= N_0\mathbf{1}\cdot\boldsymbol{G}^{-1}\cdot\left((\boldsymbol{I}_K - \boldsymbol{A}^T\cdot\boldsymbol{C})^{-1}\right)^T\cdot\boldsymbol{C}\cdot\mathbf{1}^T \\ &= N_0\mathbf{1}\cdot\boldsymbol{G}^{-1}\cdot(\boldsymbol{I}_K - \boldsymbol{C}\cdot\boldsymbol{A})^{-1}\cdot\boldsymbol{C}\cdot\mathbf{1}^T,\end{aligned} \tag{74}$$

which is the same as (70). This ends the proof.

APPENDIX D
PROOF OF THEOREM 6

We still use the bounding technique to prove this theorem. Similar to the proof for Lemma 1, we can readily show that, given the same power constraint p, the system in (14) has a larger sum capacity than (1), i.e.,

$$\begin{aligned} C_{N\times M}(K,p) &\le \max_{\{tr(\tilde{\boldsymbol{Q}}_k)\le p\}} \log_2\left|\boldsymbol{I}_M + \frac{1}{N_0}\sum_{k=1}^{K} d^2_{k,\max}\tilde{\boldsymbol{Q}}_k\right| \\ &= \log_2\left|\boldsymbol{I}_M + \frac{1}{N_0}\sum_{k=1}^{K} d^2_{k,\max}\cdot\frac{p}{M}\boldsymbol{I}_M\right| \\ &= M\cdot\log_2\left(1 + \frac{p}{MN_0}\sum_{k=1}^{K} d^2_{k,\max}\right). \end{aligned} \tag{75}$$

When $K\to\infty$, (75) becomes [29]

$$\begin{aligned} C_{N\times M}(\infty,p) &\le \lim_{K\to\infty} M\cdot\log_2\left(1 + \frac{Kp}{MN_0}\sum_{k=1}^{K}\frac{d^2_{k,\max}}{K}\right) \\ &= \lim_{K\to\infty} M\cdot\log_2\left(1 + \frac{Kp}{MN_0}\int_0^\infty gf(g)\,dg\right). \end{aligned} \tag{76}$$

Next, we derive the sum rate achieved by MEB. Assume (5) and correlator receivers $\{\boldsymbol{u}_{k,\max}\}$ as well as SIC with descending order on k. Based on (7), the rate achieved by user k can be written as

$$R_k = \log_2\left(1 + \frac{pd^2_{k,\max}}{\sum_{i=1}^{k-1} pd^2_{i,\max}\phi_{k,i} + N_0}\right). \tag{77}$$

According to the law of large numbers and (9), when $K\to\infty$, we have

$$\begin{aligned} &\lim_{K\to\infty}\frac{pd^2_{k,\max}}{\sum_{i=1}^{k-1} pd^2_{i,\max}\phi_{k,i} + N_0} \\ &= \lim_{K\to\infty}\frac{pd^2_{k,\max}}{\sum_{i=1}^{k-1} pd^2_{i,\max}/M + N_0} = 0. \end{aligned} \tag{78}$$

Hence (77) can be further rewritten as

$$\begin{aligned} \lim_{K\to\infty} R_k &= \lim_{K\to\infty}\log_2\left(1 + \frac{M\cdot pd^2_{k,\max}/M}{\sum_{i=1}^{k-1} pd^2_{i,\max}/M + N_0}\right) \\ &= \lim_{K\to\infty} M\cdot\log_2\frac{\sum_{i=1}^{k} pd^2_{i,\max}/M + N_0}{\sum_{i=1}^{k-1} pd^2_{i,\max}/M + N_0} \end{aligned} \tag{79}$$

and the corresponding system sum rate achieved by MEB, which is clearly a lower bound of $C_{N\times M}(\infty,p)$, is given by

$$\begin{aligned} C_{N\times M}(\infty,p) &\geq \lim_{K\to\infty}\sum_{k=1}^{K} R_k \\ &= \lim_{K\to\infty} M\cdot\sum_{k=1}^{K}\log_2\frac{\sum_{i=1}^{k} pd_{i,\max}^2/M+N_0}{\sum_{i=1}^{k-1} pd_{i,\max}^2/M+N_0} \\ &= \lim_{K\to\infty} M\cdot\log_2\left(1+\frac{Kp}{MN_0}\sum_{i=1}^{K}\frac{d_{i,\max}^2}{K}\right) \\ &= \lim_{K\to\infty} M\cdot\log_2\left(1+\frac{Kp}{MN_0}\int_0^{\infty} gf(g)dg\right). \end{aligned} \tag{80}$$

Combining (76) and (80), we can see that the upper and lower bound converge to (52) when $K\to\infty$. Hence the theorem is proved.

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