Splines and Geometric Modeling

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INTRODUCTION

Piecewise polynomials of some fixed degree and continuously differentiable upto some order are known as splines or finite elements. Splines are used in applications ranging from image processing, computer aided design, to the solution of partial differential equations via finite element analysis. The spline fitting problem of constructing a mesh of finite elements that interpolate or approximate multivariate data is by far the primary research problem in geometric modeling. Parametric splines are vectors of multivariate polynomial (or rational) functions while implicit splines are zero contours of multivariate polynomials. This survey shall dwell mainly on spline surface fitting methods in $\mathbb{R}^3$ Tensor product splines in (Section xx.1,...), triangular basis splines (Section xx.7,...). The following criteria may be used in evaluating these spline methods:

- Implicit or parametric representations

- Algebraic and geometric degree of the spline basis

- Number of surface patches required

- Computation and memory required

- Stability of fitting algorithms

- Local or non-local interpolation

- Splitting or non–splitting of surface patches

- Convexity or non-convexity of the input and solutio

27.1 TENSOR PRODUCT SURFACES

GLOSSARY

B–spline surface: A deformation of a planar domain, traditionally tessellated into a rectilinear grid. Any B–spline can be written in piecewise Bézier form. Surface splines may be treated as a collection of tensor product polynomial patches defined over rectangles.

C^k ***continuity:*** Smoothness is defined in terms of matching derivatives along patch boundaries.

TABLE 27.1.1   Tensor Product Patches

TYPE	INPUT	PROPERTIES
Piecewise Bi-Cubic Bezier	rectangular grid of points, corner twists	C^1, initial global survey of the data to determine the tangent and cross-derivative vectors at the patch corners
Piecewise Bi-Cubic Hermite	rectangular grid of points, partials, mixed partials	C^1, initial global survey of the data to determine the tangent and cross-derivative vectors at the patch corners
Bi-Cubic B-spline	rectangular grid of points	C^2, overlap control polyhedra of adjacent patches
Coons patches	4 boundary curves	C^1, Gregory Square
Gordon surfaces	rectangular network of curves	C^1
Bi-Quadratic A-patches	rectilinear 3D grid points	C^1, local calculation of first-order cross derivatives

Bi-linear interpolation: The "simplest" surface defined by values at four points.

Ruled (lofted) surface: A surface that interpolates two given curves using linear interpolation.

Transfinite interpolation: Interpolating entire curves as opposed to values at discrete points.

Blending functions: The basis functions used by interpolation schemes such as Gordons surfaces.

Parametric Bézier and B-SPLINES

Parametric representations possess good properties which include easy to order, easy to generate points on, simple patches, compact storage, and irreducibility. B–splines have emerged as the polynomial basis of choice for working with parametric surfaces. The theory of tensor product patches requires that data have a rectangular geometry and that the parametrizations of opposite boundary curves be similar. It is based on the concept of bilinear interpolation. Tensor product Bézier surfaces are obtained by repeated applications of bilinear interpolation. Properties of tensor product Bézier patches include affine invariance, convex hull property, and the variation diminishing property. The boundary curves of a patch are polynomial curves which have their Bézier polygon given by the boundary polygons of the control net of the patch. Hence the four corners of the control net lie on the patch.

Piecewise bicubic Bézier patches may be used to fit a C^1 surface through a rectangular grid of points. After the rectangular network of curves has been created there are four coefficients left to determine the corner twists of each patch. These four corner twists cannot be specified independently and must satisfy a "compatibility constraint". Common twist estimation methods include zero twists, Adini's

twist, Bessel twist, and Brunet's twist. To obtain C^1 continuity between two patches the directions and lengths of the polyhedron edges must be matched across the common polyhedron boundary defining the common boundary curve. Piecewise bicubic Hermite patches are similar to the piecewise bicubic Bézier patches but take points, partials, and mixed partials as input. The mixed partials have effect only on the interior shape of the patch and are also called twist vectors.

It is not possible to model a general closed surface or a surface with handles as a single non-degenerate B–spline. To represent free-form surfaces a significant amount of recent work has been done in the areas of geometric continuity, non-tensor product patches, and generalizing B–splines. Common schemes include splitting, convex combinations of blending functions, subdivision, and local interpolation by construction.

COONS PATCHES AND GORDON SURFACES

Instead of being described by control points, Coons patches and Gordon surfaces work by generating a surface from a network of curves. Coons patches are based on a generalization of ruled, or lofted, surfaces. Coons patches interpolate four boundary curves. They are constructed by composing two lofted surfaces and one bilinear surface, and hence are called bilinearly blended surfaces. A network of curves may be filled in with a C^1 surface using bicubically blended Coons patches. For this the four twists at the data points and the four cross boundary derivatives must be computed. Compatibility problems may arise in computing the twists. If $\mathbf{x}(u, v)$ is twice differentiable, we have $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, but this simplification does not apply here. One approach is to adjust the given data so that the incompatibilities disappear. Or if the data can not be changed one can use a method known as *Gregory's square* that replaces the constant twist terms by variable twists that are computed from the cross boundary derivatives. The resulting surface does not have continuous twists at the corners and is rational parametric, which may not be acceptable in certain geometric modeling environments.

Gordon surfaces are a generalization of Coons patches used to construct a surface $\mathbf{g}$ that interpolates a rectangular network of curves. The idea is to take a univariate interpolation scheme, apply it to all curves, add the resulting surfaces, and subtract the tensor product interpolant that is defined by the univariate scheme. Polynomial interpolation or spline interpolation schemes may be used. Methods for Coons patches and Gordon surfaces can be formulated in terms of boolean sums and projectors. This has also been generalized to create triangular Coons patches.

27.2 GENERALIZED B–SPLINE SURFACES USING MULTI–SIDED PATCHES

GLOSSARY

Constrained Domain Mapping: Representing a domain point for an n–sided

TABLE 27.2.1   Multi-sided Schemes

TYPE	LIMITATIONS	PROPERTIES	DOMAIN POINTS	SOURCE
Sabin	n=3,5	C^1	constrained domain mapping, symmetric system of parameters	[Sab83]
Gregory/Charrot	n=3,5	C^1	barycentric coordinates	[Gre86]
Hosaka/Kimura	$n <= 6$	C^1	constrained domain mapping, symmetric system of parameters	[HF84]
Varady		VC^1	$2n$ variables constrained along polygon sides	[Var91]

patch by n dependent parameters. If the rest of the parameters can be computed when any two parameters are independently chosen it is called a symmetric system of parameters.

B–spline surfaces have been generalized to include multi–sided patches by using convex combinations of blending functions. Multi–sided patches can be generated in basically two ways. Either the polygonal domain which is to be mapped into $I\!\!R^3$ is subdivided in the parametric plane, or one uniform equation is used as a combination of equations. In the first case triangular or rectangular elements are put together or recursive subdivision is applied. And in the later case either the known control point based methods are generalized or a weighted sum of interpolants is used.

S–Patches

Loop and DeRose [LD89, LD90] present generalizations of biquadratic and bicubic B-spline surfaces that are capable representing surfaces of arbitrary topology by placing restrictions on the connectivity of the control mesh, relaxing C^1 continuity to G^1 continuity, and allowing n-sided finite elements. This generalized view considers the spline surface to be a collection of possibly rational polynomial maps from independent n-sided polygonal domains, whose union possesses continuity of some number of geometric invariants, such as tangent planes. In this view patches are required to meed with geometric continuity (denoted G^k continuity). This more general view allows patches to be sewn together to describe free form surfaces in more complex ways.

GENERALIZED TRIANGULAR SURFACES USING SPLITTING

With splitting schemes every triangle in the triangulation of the data points (also called a "macro–triangle") is split into several "mini–triangles". Split-triangle interpolants do not require derivative information of higher order than the continuity of the desired interpolant. The simplest of the split–triangle interpolants in the C^1 Clough–Tocher interpolant. Each vertex is joined to the centroid, and the macro–triangle is split into three mini–triangles. The first order data that this interpolant

requires are position and gradient value at the vertices macro–triangle plus some cross–boundary derivative at the midpoint of each edge. There are twelve data per macro–triangle and cubic polynomials are used over each mini–triangle. The C^1 Powell–Sabin interpolants produce C^1 piecewise quadratic interpolants to C^1 data at the vertices of a triangulated data set. Each macro–triangle is split into six or twelve mini triangles.

GENERALIZED TRIANGULAR SURFACES USING BLOSSOMING

The B–patches developed by Seidel [Sei89] are based on the study of symmetric recursive evaluation algorithms and are defined by generalizing the de Boor algorithm for the evaluation of a B–spline segment from curves to surfaces. A polynomial surface that has a symmetric recursive evaluation algorithm is called a *B–Patch*. B–patches generalize Bézier patches over triangles and are characterized by control points and a three parameter family of knots. Every bivariate polynomial $F : R^2 \rightarrow R^d$ of degree n has a unique representation

$$F(U) = \sum_{|\vec{i}|=n} N_{\vec{i}}^n(U) P_{\vec{i}}, \qquad P_{\vec{i}} \in R^d$$

as a B–patch with parameters $R_0, \ldots, R^{n-1}, S_0, \ldots, S^{n-1}, T_0, \ldots, T^{n-1}$ in R^2 if the parameters (R_i, S_j, T_k) are affinely independent for $0 \leq |\vec{i}| \leq n-1$. The real-valued polynomials $N_{\vec{i}}^n(U)$ are called the *normalized B–weights* of degree n over $\mathcal{K}$.

GENERALIZED TRIANGULAR SURFACES USING MULTI– SIDED PATCHES

One approach to create multi–sided patches has been by introducing base points into rational parametric functions. Base points are parameter values for which the homogeneous coordinates (x,y,z,w) are mapped to (0,0,0,0) by the rational parameterization.

Gregory's patch [Gre83] is defined using a special collection of rational basis functions that evaluate to 0/0 at vertices of the parametric domain and thus introduce base points in the resulting parameterization. It is possible to describe Gregory's patch solely in terms of control points and weights. Warren [War92] uses base points to create parameterizations of four-, five-, and six–sided surface patches using rational Bézier surfaces defined over triangular domains. Setting a triangle of weights to zero at one corner of the domain triangle produces a four–sided patch that is the image of the domain triangle. This technique can be generalized to create five- and six–sided patches by treating each vertex of the triangular domain independently.

27.3 SPLINE SURFACES OVER MESHES WITH ARBITRARY TOPOLOGY

The representation of free–form surfaces is one of the major issues in geometric modeling. These surfaces are generally defined in a piecewise manner by smoothing joining several mostly four–sided patches. The patches are given in vector valued parametric form, mapping a rectangular parametric domain into $\mathbb{R}^3$.

Local construction, blending polynomial pieces, and splitting are common approaches to constructing surfaces over irregular meshes. Because of its nonlinear nature and the advantages of a local construction, different approaches have been centered around selecting geometrically meaningful variables that can be fixed (as input or derived from data) so as to arrive at a sufficient and consistent set of linear constraints on the remaining variables. Blending polynomial pieces means constructing k pieces for a k-sided mesh facet such that each piece matches a part of the facet data and a convex combination of the pieces matches the whole. Blending approaches prescribe a mesh of boundary curves and their normal derivatives. However for this approach the existence of a well–defined tangent plane at the data points is not sufficient to guarantee the existence of a C^1 mesh interpolant since the mixed derivatives p_{uv} and p_{vu} are given independently at any point p. Splitting approaches on the other hand expect at least tangent vectors at the data points and sometimes the complete boundary to be given. In Mann et al. [MLL$^+$92] it is concluded that local polynomial interpolants generally produce unsatisfactory shapes.

Peters [Pet91] considers the interpolation of a mesh of curves by a smooth regularly parameterized surface with one polynomial piece per facet. Not every mesh with a well–defined tangent plane at the mesh points has such an interpolant. Necessary and sufficient *vertex enclosure* constraints on a mesh of polynomial curves that guarantees the existence of a regular smooth interpolant are given. The curvature of mesh curves emanating from mesh points with an even number of neighbors must satisfy an additional "vertex enclosure constraint". The vertex enclosure constraint is automatically satisfied by the splitting construction and can be satisfied by singularly parameterizing one of the boundary curves. An algorithm for the local interpolation of a cubic curve mesh by a piecewise [bi]quartic C^1 surface is described. The scheme is based on a sufficient constraint that forces the mesh curves to interpolate second–order data at the mesh points. Rational patches, singular parameterizations, and the splitting of patches are interpreted as techniques to enforce the vertex enclosure constraint.

Reif [Rei93] constructs a piecewise G^1 spline surface based on biquadratic rectangular Bézier patches from a set of control points on meshes with arbitrary topology. Geometrical smoothness conditions are used only near the singular vertices of a mesh. He constructs an additional ring of "G–edges" around singular vertices and expresses the smoothness conditions in a system of linear equations.

Peters [Pet93] gives an algorithm for refining an irregular mesh of points into a bivariate C^1 surface. The algorithm generalizes the construction of quadratic splines from a mesh of control points, and an explicit parameterization of the surface with quadratic and cubic pieces is given. When the mesh is regular then a quadratic spline surface is generated. Irregular input meshes with nonquadrilateral mesh cells more or fewer than four cells meeting at a point are allowed and generate spline spaces that generalize the space of quadratic splines. The main idea is to refine the irregular input mesh by the averaging process of Doo–Sabin and generate strips of regular mesh points that isolate regions of irregular points. The algorithm can

model bivariate open or closed surfaces of general topological structure. However the algorithm generates a large number of patches relative to the number of faces in the control mesh.

Loop [Loo94] constructs a piecewise G^1 spline surface composed of sextic triangular Bézier patches in one-to-one correspondence with the faces of a triangular control mesh. Surfaces of arbitrary topological type are created by approximating any mesh that represents a triangulated 2–manifold. In the case of a regular parameterization the surface generated by this method is equivalent to a quartic C^2 triangular B–spline. The question of whether there exists values in the space of the shape parameters so that the affine combinations of mesh vertices are strictly convex has not been resolved.

27.4 SUBDIVISION SURFACES

Subdivision techniques can be used to produce generally pleasing surfaces from arbitrary control meshes. Subdivision consists of splitting and averaging. Each edge or face is split and each new vertex introduced by the splitting is positioned at a fixed affine combination of its neighbor's weights.

The algorithms start with a polyhedral configuration of points, edges, and faces. The control mesh will in general consist of large regular regions and isolated singular regions. Subdivision enlarges the regular regions of the control net and shrinks the singular regions. Each application of the subdivision algorithm constructs a refined polyhedron, consisting of more points and smaller faces, tending in the limit to a smooth surface. In general the new control points are computed as a linear combination of old control points. The associated matrix is called the *subdivision matrix*.

The earliest of these approaches are the recursive subdivision schemes of Doo and Sabin [DS78] and Catmull and Clark [CC78]. These algorithms generate C^1 surfaces that interpolate the centroids of all faces at every step of subdivision. Nasri [Nas91] describes a recursive subdivision surface scheme that is capable of interpolating points on irregular networks as well as normal vectors given at these points. The subdivision scheme developed by Loop [Loo87] splits each triangle of a triangular mesh into four triangles. Each new vertex is positioned using a fixed convex combination of the vertices of the original mesh. The final limit surface has a continuous tangent plane. Hoppe et al. [H. 94] extends Loop's method to incorporate sharp edges into the final limit surface. The vertices of the initial polyhedron are tagged as belonging on a face, edge, or vertex of the final limit surface. Based on this tag different averaging masks are used to produce new polyhedra.

Storry and Ball [SB89] demonstrate that a B–spline subdivision patch can be fitted into a general n–sided area of a bicubic surface with at least tangent plane continuity on the boundary. A Hermite formulation is used for the surface patches, and after the control points have been determined the subdivision algorithm is applied to produce an n–sided patch of optimal continuity properties. One degree of freedom is identified and related to shape control. Reif [Rei92] presents a unified approach to subdivision algorithms for meshes with arbitrary topology and gives

a sufficient condition for the regularity of the surface. The existence of a smooth regular parametrization for the generated surface near the point is determined from the leading eigenvalues of the subdivision matrix and an associated characteristic map.

27.5 MULTIVARIATE BOX SPLINES AND SIMPLEX SPLINES

Multivariate splines are a generalization of univariate B–splines to a multivariate setting. Multivariate splines have applications in data fitting, computer–aided design, the finite element method, and image analysis. Work on splines has traditionally been for a given planar triangulation using a polynomial function basis. Box–splines are multivariate generalizations of B-splines with uniform knots. Many of the basis functions used in finite element calculations on uniform triangles occur as special instances of box splines. In general a box spline is a locally supported piecewise polynomial. One can define translates of box splines which form a negative partition of unity.

In the bivariate case box splines correspond to surfaces defined over a regular tessellation of the plane. If the tessellation is composed of triangles, it is possible to represent the surface as a collection of Bernstein-Bézier patches. The two most commonly used special tesselations arise from a rectangular grid by drawing in lines in north–easterly diagonals in each subrectangle or by drawing in both diagonals for each subrectangle. For these special triangulations there is an elegant way to construct locally supported splines.

Splines over arbitrary triangulations of the parameter plane were first considered by Dahmen and Michelli [DM82] and Höllig [Hol82]. These multivariate splines are defined as projections of simplices and are therefore called simplex splines.

Auerbach [AMNS91] constructs approximations with simplex splines over irregular triangles. Bivariate quadratic simplicial B–splines defined by their corresponding sets of knots derived from a (suboptimal) constrained Delaunay triangulation of the domain are employed to obtain a C^1 surface. This approach is well suited for scattered data. Each vertex of a given triangle is associated with two additional points which give rise to six configurations of five knots defining six linearly independent bivariate quadratic B–splines supported over the convex hull of the corresponding five knots. The coefficients of the linear combinations of normalized simplicial B–splines are visualized as geometric control points satisfying the convex hull property.

Fong and Seidel [FS86, FS92] construct multivariate B–splines for quadratics and cubics by matching B–patches with simplex splines. The surface scheme is an approximation scheme based on blending functions and control points and allows the modeling of C^{k-1} continuous piecewise polynomial surfaces of degree k over arbitrary triangulations of the parameter plane. The resulting surfaces are defined as linear combinations of the blending functions and are parametric piecewise polynomials over a triangulation of the parameter plane whose shape is determined by their control points.

27.6 IMPLICT POLYNOMIAL SPLINES

A–splines are a suitable polynomial basis for working with piecewise implicit polynomial curves and surfaces. While it is possible to model a general closed surface of arbitrary genus as a single A–spline the geometry of impliict surfaces has proven to be more difficult to specify, interactively control, and polygonize than parametrics. The main shortcoming held against the use of implicit representations is that zeros of polynomials being multivalued may cause the zero contour to have multiple real sheets, self–intersections and several other undesirable singularities. On the positive end, using polynomials of the same degree, implicit polynomial splines have more degrees of freedom compared with parametric and hence potentially are more flexible to approximate a complicated surface with fewer number of pieces and to achieve a higher order of smoothness. The potential of implicits remains largely latent and virtually all commercial and many research modeling systems are based on the parametric representation. An exception is SHASTRA which allows modeling with both implicit and parametric splines [Baj93].

There are two main advantages of using implicit surfaces instead of parametrics. First, the set of algebraic surfaces are closed under basic modeling operations such as offset and intersection that are often required in a solid modeling system. Second, for the same polynomial of degree n, implicit algebraic surfaces have more degrees of freedom ($\frac{(n+3)(n+2)(n+1)}{6} - 1$) compared with $2(n+2)(n+1) - 1$ degrees of freedom for rational parametric surfaces.

27.7 SURFACES OVER MESHES WITH ARBITRARY TOPOLOGY

Sederberg [Sed85] showed how various smooth implicit algebraic surfaces in trivariate Bernstein basis can be manipulated as functions in Bézier control tetrahedra with finite weights. He showed that if the coefficients of the Bernstein-Bézier form of the trivariate polynomial on the lines that parallel one edge, say L, of the tetrahedron all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the zero contour algebraic surface patch at most once. Patrikalakis and Kriezis [PK89] extended this by considering implicit algebraic surfaces in a tensor product B–spline basis. However the problem of selecting weights or specifying knot sequences for C^1 meshes of implicit algebraic surface patches which fit given spatial data was left open.

The problem of constructing a C^1 mesh of implicit algebraic patches based on an input polyhedron $\mathcal{P}$ has been considered by many. Dahmen [Dah89] presented a scheme for constructing C^1 continuous piecewise quadric surface patches over a data triangulation in space. In his construction each triangular face is split and replaced by six micro quadric triangular patches, similar to the splitting scheme of Powell–Sabin [PS77]. Dahmen's technique however works only if the original triangulation of the data set allows a transversal system of planes, and hence is quite restricted. Moore and Warren [MW91] extended the marching cubes scheme to compute a C^1

TABLE 27.7.1   Point Sampling

INPUT	EXAMPLES
rectangular grid	results of physical simulations
slices	contour data extracted from a CT scan
scattered	laser range scanners, 3D scanners to measure RGB components of object color

piecewise quadratic approximation to scattered data using a Powell–Sabin like split over subcubes. Guo [Guo91] used cubics to create free–form geometric models and enforced monotonicity conditions on a cubic polynomial along the direction from one vertex to a point of the opposite face of the vertex. A Clough-Tocher split is used to subdivide each tetrahedron of the simplicial hull. He derived a condition $a_{\lambda - e_1 + e_4} - a_\lambda \geq 0$ for all λ with $\lambda_1 \geq 1$, where a_λ are the coefficients of the cubic in Bernstein-Bézier form. [DTS93] utilize a single cubic patch per tetrahedron.

Lodha [Lod92] constructed low degree surfaces with both parametric and implicit representations and investigated their properties. A method is described for creating quadratic triangular Bézier surface patches which lie on implicit quadric surfaces. And another method is described for creating biquadratic tensor product Bézier surface patches which lie on implicit cubic surfaces. The resulting patches satisfy all the standard properties of parametric Bézier surfaces, including interpolation of the corners of the control polyhedron and the convex hull property.

Bajaj and Ihm [BI91] construct low–degree algebraic surfaces that approximate or contain with C^1 continuity any collection of points and algebraic space curves with derivative information. Their Hermite interpolation algorithm solves a homogeneous linear system of equations to compute the coefficients of the polynomial defining the algebraic surface. Bajaj, Ihm and Warren [BIW93] extend this idea to C^k (rescaling continuity) interpolate or least squares approximate implicit or parametric curves in space. They show this problem can be formulated as a constrained quadratic minimization problem, where the algebraic distance is minimized instead of the geometric distance.

Bajaj, Ihm, Guo, and Dahmen [Dah89, Guo91, Guo93, BI91] provide heuristics based on monotonicity and least square approximation to circumvent the multiple sheeted and singularity problems of implicit patches. Bajaj [Baj92, BI92, BIW93] constructed implicit surfaces to solve the scattered data fitting problem. Bajaj and Ihm [BI91] considered an arbitrary spatial triangulation $\mathcal{T}$ consisting of vertices in $I\!R^3$ (or more generally a simplicial polyhedron $\mathcal{P}$ when the triangulation in closed) with possibly normal vectors at the vertex points. Their algorithm constructs a C^1 continuous mesh of real implicit algebraic surface patches over $\mathcal{T}$ or $\mathcal{P}$. The scheme is local (each patch has independent free parameters) and there is no local splitting. The algorithm first converts the given triangulation or polyhedron into a curvilinear wireframe with at most cubic parametric curves which C^1 interpolate all the vertices. The curvilinear wireframe is then fleshed to produce a single implicit surface patch of degree at most 7 for each triangular face $\mathcal{T}$ of $\mathcal{P}$. If the triangulation is convex then the degree is at most 5. Similar techniques exist for parametrics [Pet91, Far86, Sar87] however the geometric degree of the solution surfaces tend to be prohibitively high.

Bajaj, Chen and Xu [BCX95] construct 3- and 4-sided A–patches that are implicit surfaces in Bernstein–Bézier(BB) form that are smooth and single–sheeted. They give sufficiency conditions for the BB form of a trivariate polynomials within a tetrahedron such that the zero contour of the polynomial is a single sheeted non–singular surface within the tetrahedron and its cubic-mesh complex for the polyhedron $\mathcal{P}$ is guaranteed to be both nonsingular and single sheeted. They distinguish between convex and non-convex facets and edges of the triangulation. For non-convex facets and edges a double-sided tetrahedra is built and for convex facets and edges single-sided tetrahedra are built. A generalization of Sederberg's condition is given for a three–sided j–patch where any line segment passing through the j-th vertex of the tetrahedron and its opposite face intersects the patch only once. Instead of having coefficients be monotonically increasing or decreasing there is a single sign change condition. There are also free parameters for both local and global shape control. [BCX94] uses these conditions for C^1 cubic and C^2 quintic schemes to approximate a polyhedron $\mathcal{P}$ using a simplicial hull construction. Reconstructing surfaces and scalar fields defined over the surface from scattered data using implicit Bézier splines are given in [BX94, BBX95].

ENERGY BASED SPLINES

Terzopoulos, Platt, Barr, Barzel, Fleischer, and Witkin [TF88, PB88, PTB87, WFB87] have presented discrete models which are based extensively on the theory of elasticity and plasticity and use energy fields to define and enforce constraints. Haumann [Hau87] used the same approach but used a triangularized model and a simpler physical model based on points, springs, and hinges. Thingvold and Cohen [TC90] defined a model of elastic and plastic B–spline surfaces which supports both animation and design operations. The basis for the physical model is a generalized point mass-spring-hinge model which has been adapted into simultaneous refinement of the geometric/physical model. Always having a sculptured surface representation as well as the physical hinge/spring/mesh model allows the user to intertwine physical based operations, such as force application, with geometrical modeling. Refinement operations for spring and hinge B–spline models are compatible with the physics and mathematics of B–spline models. The models of elasticity and plasticity are written in terms of springs and hinges, and can be implemented with standard integration techniques to model realistic motions of elastic and plastic surfaces. These motions are controlled by the physical properties assigned and by kinematic constraints on various portions of the surface.

Terzopoulos and Qin [TQ94] develop a dynamic generalization of the nonuniform rational B–spline (NURBS) model. They present a physics–based model that incorporates mass distributions, internal deformation energies, and other physical quantities into the NURBS geometric substrate. These dynamic NURBS can be used in applications such as rounding of solids, optimal surface fitting to unstructured data, surface design from cross–sections, and free–form deformations. Qin and Terzopoulos [QT95] develop a dynamic freefrom surface model based on swung NURBS surfaces which is useful for representing objects with symmetries and topological variability.

HIERARCHIAL SPLINES

Hierarchical splines are a multi–resolution approach to the representation and manipulation of free–form surfaces. A hierarchal B–spline is constructed from a base surface (level 0) and a series of overlays are derived from the immediate parent in the hierarchy. Forsey [FB88] presents a refinement scheme that uses a hierarchy of rectangular B–spline overlays to produce C^2 surfaces. Overlays can be added manually to add detail to the surface, and local or global changes to the surface can be made by manipulating control points at different levels.

Forsey and Wang [FW93] create hierarchical bicubic B–spline approximations to scanned cylindrical data. The resulting hierarchical spline surface is interactively modifiable using editing capabilities of the hierarchical surface representation allowing either local or global changes to surface shape while retaining the details of the scanned data. However oscillations occur when the data has high–amplitude or high–frequency regions.

27.8 SOURCES AND RELATED MATERIAL

SURVEYS

All results not given an explicit reference above may be traced in these surveys.

- [Baj92]: Summary of data fitting with implicit algebraic splines.

- [Far86, Far93]: Summary of the history of triangular Bernstein-Bézier patches.

- [Alf89]: Scattered data fitting and multivariate splines.

- [DM83]: Scattered data fitting and multivariate splines.

- [Hol82]: Scattered data fitting and multivariate splines.

- [Sch94]: Scattered data fitting and multivariate splines.

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