The One-Bit Null Space Learning Algorithm and its Convergence

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Abstract

This paper proposes a new algorithm for MIMO cognitive radio Secondary Users (SU) to learn the null space of the interference channel to the Primary User (PU) without burdening the PU with any knowledge or explicit cooperation with the SU. The knowledge of this null space enables the SU to transmit in the same band simultaneously with the PU by utilizing separate spatial dimensions than the PU. Specifically, the SU transmits in the null space of the interference channel to the PU. We present a new algorithm, called the One-Bit Null Space Learning Algorithm (OBNSLA), in which the SU learns the PU's null space by observing a binary function that indicates whether the interference it inflicts on the PU has increased or decreased in comparison to the SU's previous transmitted signal. This function is obtained by listening to the PU transmitted signal or control channel and extracting information from it about whether the PU's Signal to Interference plus Noise power Ratio (SINR) has increased or decreased.

In addition to introducing the OBNSLA, this paper provides a thorough convergence analysis of this algorithm. The OBNSLA is shown to have a linear convergence rate and an asymptotic quadratic convergence rate. Finally, we derive bounds on the interference that the SU inflicts on the PU as a function of a parameter determined by the SU. This lets the SU control the maximum level of interference, which enables it to protect the PU completely blindly with minimum complexity. The asymptotic analysis and the derived bounds also apply to the recently proposed Blind Null Space Learning Algorithm.

I. Introduction

Multiple Input Multiple Output (MIMO) communication opens new directions and possibilities for Cognitive Radio (CR) networks [1–7]. In particular, in underlay CR networks, MIMO technology enables the SU to transmit a significant amount of power simultaneously in the same band as the Primary User

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(PU) without interfering with it, if the SU utilizes separate spatial dimensions than the PU. This spatial separation requires that the interference channel from the SU to the PU be known to the SU. Thus, acquiring this knowledge, or operating without it, is a major topic of active research in CR [6, 8–15] and in other fields [16]. We consider MIMO primary and secondary systems defined as follows: we assume a flat-fading MIMO channel with one PU and one SU, as depicted in Fig. 1. Let $\mathbf{H}_{ps}$ be the channel matrix between the SU's transmitter and the PU's receiver, hereafter referred to as the SU-Tx and PU-Rx, respectively. In the underlay CR paradigm, SUs are constrained not to inflict "harmful" interference on the PU-Rx. This can be achieved if the SU restricts its signal to lie within the null space of $\mathbf{H}_{ps}$; however, this is only possible if the SU knows $\mathbf{H}_{sp}$. The optimal power allocation in the case where the SU knows the matrix $\mathbf{H}_{ps}$ in addition to its own Channel State Information (CSI) was derived by Zhang and Liang [1]. For the case of multiple SUs, Scutari at al. [3] formulated a competitive game between the secondary users. Assuming that the interference matrix to the PU is known by each SU, they derived conditions for the existence and uniqueness of a Nash Equilibrium point to the game. Zhang et al. [9] were the first to take into consideration the fact that the interference matrix $\mathbf{H}_{ps}$ may not be perfectly known (but is partially known) to the SU. They proposed robust beamforming to assure compliance with the interference constraint of the PU while maximizing the SU's throughput. Another work on the case of an unknown interference channel with known probability distribution is due to Zhang and So [11], who optimized the SU's throughput under a constraint on the maximum probability that the interference to the PU is above a threshold.

The underlay concept of CR in general, and MIMO CR in particular, is that the SU must be able to mitigate the interference to the PU blindly without any cooperation. Zhang [6] was the first to propose a blind solution where the MIMO SU mitigates interference to the PU by null space learning. This work was followed by Yi [13], Chen et al. [12], and Gao et al. [15]. All these works exploit channel reciprocity: specifically, where the SU listens to the PU's transmitted signal and estimates the null space from the signal's second order statistics. Since these works require channel reciprocity, they are restricted to PUs that use Time Division Duplexing (TDD).

Unless there is channel reciprocity, obtaining $\mathbf{H}_{ps}$ by the SU requires cooperation with the PU in the estimation phase; e.g. where the SU transmits a training sequence, from which the PU estimates $\mathbf{H}_{ps}$ and feeds it back to the SU. Cooperation of this nature increases the system complexity overhead, since it requires a handshake between both systems and, in addition, the PU needs to be synchronized with the SU's training sequence. Zhang [10] was the first to propose an interference mitigation mechanism in which a single antenna SU obtains the path-loss of the interference channel to the PU under the condition that the SU can extract the PU's Signal to Interference plus Noise Ratio (SINR) by listening to its transmitted signal or control channel. By transmitting an interfering signal, and measuring the effect of this signal on the PU SINR, the SU obtains the path-loss of the interference channel. This enables the SU to set its power low enough to maintain its interference below some predefined level. However, it does not enable the SU to exploit other spatial degrees of freedom than those used by the PU. In [14], we proposed the Blind Null Space Learning Algorithm (BNSLA), which enables a MIMO underlay CR to learn the null space of $\mathbf{H}_{ps}$ by observing some unknown monotone continuous function of the PU's SINR. For example, if the PU is using continuous power control, the PU's signal power is a monotone function of its SINR. During this learning, the PU does not cooperate at all with the SU and operates as though there were no other systems in the medium (the way current PUs operate today).

This paper makes two contributions. The first contribution is a new algorithm, called the One-Bit Null Space Learning Algorithm (OBNSLA), which requires much less information than the BNSLA; namely, the SU can infer whether the interference it inflicts on the PU has increased or decreased compared to a previous time interval with a one-bit function. In other words, in the OBNSLA the SU measures a one-bit function of the PU's SINR, rather than a continuous-valued function as in the BNSLA. Using this single bit of information, the SU learns $\mathbf{H}_{ps}$'s null space by iteratively modifying the spatial orientation of its transmitted signal and measuring the effect of this modification on the PU's SINR. The second contribution of the paper is to provide a thorough convergence analysis of the OBNSLA. We show that the algorithm converges linearly and has an asymptotically quadratic convergence rate. In addition, we derive upper bounds on the interference that the SU inflicts on the PU; these results enable the SU to control the interference to the PU without any cooperation on its part. Furthermore, all the bounds and the convergence results apply equally to the BNSLA.

II. The One-Bit Null Space Learning Problem

Consider a flat fading MIMO interference channel with a single PU and a single SU without interference cancellation; i.e., each system treats the other system's signal as noise. The PU's received signal is

$$\mathbf{y}_p(t) = \mathbf{H}_{pp}\mathbf{x}_p(t) + \mathbf{H}_{ps}\mathbf{x}_s(t) + \mathbf{v}_p(t),\ t \in \mathbb{N} \tag{1}$$

where $\mathbf{x}_p$, $\mathbf{x}_s$ is the PU's and SU's transmitted signal, respectively, $\mathbf{H}_{pp}$ is the PU's direct channel; $\mathbf{H}_{ps}$ is the interference channel between the PU Rx and the SU Tx, and $\mathbf{v}_p(t)$ is a zero mean stationary noise. In the underlay CR paradigm, the SU is constrained not to exceed a maximum interference level at the

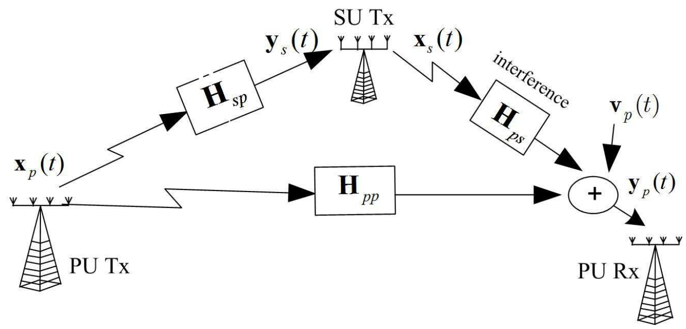


Fig. 1. Our cognitive radio scheme. $\mathbf{H}_{ps}$ is unknown to the secondary transmitter and $\mathbf{v}_p$ (t) is a stationary noise (which may include stationary interference). The interference from the SU, $\mathbf{H}_{ps}\mathbf{x}_s(t)$, is treated as noise; i.e., there is no interference cancellation.

PU Rx; i.e.,

$$\|\mathbf{H}_{ps}\mathbf{x}_s(t)\|^2 \leq \eta_{\max}, \tag{2}$$

where $\eta_{\max} > 0$ is the maximum interference constraint. In this paper, all vectors are column vectors. Let $\mathbf{A}$ be an $l \times m$ complex matrix; then, its null space is defined as $\mathcal{N}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{C}^m : \mathbf{A}\mathbf{y} = \mathbf{0}\}$ where $\mathbf{0} = [0, ..., 0]^{\mathrm{T}} \in \mathbb{C}^l$.

Since our focus is on constraining the interference caused by the SU to the PU, we only consider the term $\mathbf{H}_{ps}\mathbf{x}_s(t)$ in (1). Hence, $\mathbf{H}_{ps}$ and $\mathbf{x}_s$ will be denoted by $\mathbf{H}$ and $\mathbf{x}$, respectively. We also define the Hermitian matrix $\mathbf{G}$ as

$$\mathbf{G} = \mathbf{H}^*\mathbf{H} \tag{3}$$

The time line $\mathbb{N}$ is divided into N-length intervals, each referred to as a transmission cycle (TC), as depicted in Figure 2. For each TC, the SU's signal is constant; i.e.,

$$\mathbf{x}_s((n-1)N + N') = \mathbf{x}_s((n-1)N + 1) = \cdots = \mathbf{x}_s(Nn + N' - 1) \triangleq \tilde{\mathbf{x}}(n), \tag{4}$$

where the time interval $nN < t \leq nN + N' - 1$ is the snapshot in which the SU measures a sequence $q(n)$, where each $q(n)$ is a function of the interference that the SU inflicts on the PU. We assume that the SU can extract one-bit of information from the sequence $q(n)$, which indicates whether the interference it inflicts on the PU, at the nth TC, has increased or decreased with respect to the previous TCs. This assumption is described in the following.

Observation Constraint (OC) on the function $q(n)$: Let $q(n),\ n = 1, 2, ...$ be a sequence observed by the SU where n is the index of a TC, and let $\mathbf{H}\tilde{\mathbf{x}}(n)$ be the interference that the SU inflicts on the

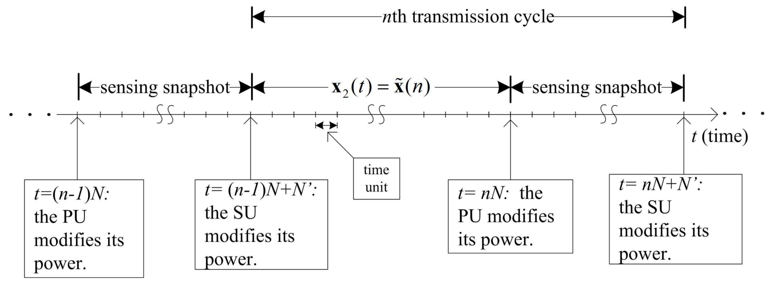


Fig. 2. The time indexing used in this paper. t indexes the basic time unit (pulse time) where N time units constitute a TC that is indexed by n. Furthermore, K transmission cycles constitute a learning phase (not shown in this figure).

PU at the nth TC. Then, $q(n)$ is a function of the interference that the SU inflicts on the PU as follows: There exists some integer $M \geq 1$, such that from $q(n-m), ..., q(n)$, the SU can extract the following function

$$\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-m)) = \begin{cases} 1, \text{ if } \|\mathbf{H}\tilde{\mathbf{x}}(n)\| \geq \|\mathbf{H}\tilde{\mathbf{x}}(n-m)\| \\ -1, \text{ otherwise} \end{cases} \tag{5}$$

for every $m \leq M$.

The SU's objective is to learn $\mathcal{N}(\mathbf{H})$ from $\{\tilde{\mathbf{x}}(n), q(n)\}_{n\in\mathbb{N}}$. This problem, referred to as the One-bit Blind Null Space Learning (OBNSL) problem, is illustrated in Figure 3 for $M = 1$. The OBNSL problem is similar to the Blind Null Space Learning (BNSL) problem [14] except for one important difference. In the latter, the SU observes a continuous-valued function of the PU's SINR whereas in the OBNSL problem, it observes a one-bit function. In both problems, the SU obtains $q(n)$ by measuring the PU's transmit energy, or any other parameter that indicates the PU's SINR (see Sec. II-B in [14] for examples). However, in the OBNSL problem, the SU is more flexible since it can obtain $q(n)$ from, for example, incremental power control[1] or other quantized functions of the PU's SINR such as modulation size. Another way for the SU to extract information about the interference to the PU is by decoding the PU's control signal to obtain parameters such as channel quality indicator feedback or ACK/NAK feedback [8]. From a system point of view, the OC means that between m consecutive transmission cycles, the PU's SINR is mostly affected by variations in the SU's signal. Note that the OC is less restrictive for smaller values of m. The TC length is the minimum time it takes the SU to modify its learning signal. This length must be equal or greater to the PUs interference adaptation interval for the learning to be accurate. In addition, variations in other sources of interference and in the PU's direct channel should occur on

[1]This is power control that is carried out using one-bit command which indicates weather to increase or decrease the power by a certain amount.

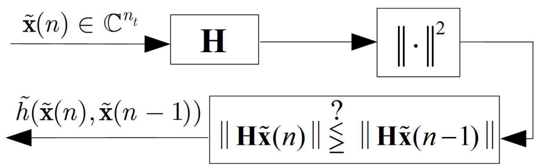


Fig. 3. Block Diagram of the One-bit Blind Null Space Learning Problem. The SU's objective is to learn the null space of $\mathbf{H}$ by inserting a series of $\{\tilde{\mathbf{x}}(n)\}_{n\in\mathbb{N}}$ and measuring $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-1))$ as output. In practice, the SU does not measure $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-1))$ directly, but rather it measures a sequence $q(n)$, for each $\tilde{\mathbf{x}}(n)$, and from $q(n), q(n-1)$, the SU extracts $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-1))$.

a much slower timescale than the TC length or else the learning may not converge. It is important to stress that the latter constraint applies only to the TC time and not to the entire time it takes the SU to learn the null space of $\mathbf{H}$. This is because the OBNSL problem is based only on the variation in the interference with respect to the previous TC, and if $M > 1$ it is with respect to the variations in the Mth previous TCs. It is therefore possible that the environment and the PU's direct channel vary faster than $\mathbf{H}$ as long as these variations are slow with respect to the TC. In Sec. V we study the effect of a time-varying environment on the proposed learning scheme, via simulation, and show that it is possible to learn the null space even when the PU direct channel varies faster then $\mathbf{H}$. Note that in the case where $q(n)$ is not extracted from the PU's SINR, the PU's path-loss may not affect the OC. Consider the case where the PU constantly measures the interference power, or the interference spatial covariance matrix at the Rx and feeds it back to its Tx. For example, such a mechanism is necessary if the PU has full CSI at its Tx. In this case, if the SU can decode the PU's control signal, it can extract $q(n)$ from it without being affected by variations in the PU direct channel; i.e., $\mathbf{H}_{pp}(t)$.

The learning process unfolds as follows. In the first TC ($n = 1$), the SU transmits $\tilde{\mathbf{x}}(1)$, and measures $q(1)$. In the next TC, the SU transmits $\tilde{\mathbf{x}}(2)$ and measures $q(2)$ from which it extracts $\tilde{h}(\tilde{\mathbf{x}}(1), \tilde{\mathbf{x}}(2))$. This process is repeated until the null space is approximated. Note that while $\tilde{h}(\tilde{\mathbf{x}}(1), \tilde{\mathbf{x}}(2))$ requires two TCs, $\tilde{h}(\tilde{\mathbf{x}}(n-1), \tilde{\mathbf{x}}(n))$ for $n > 2$ requires a single TC. Note that the OC does not provide an explicit relation between $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-m))$ and $q(n)$. This is because the way $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-m))$ is extracted from $q(n)$ depends on the PU's communication protocol. For example, if the PU is using incremental power control, and the SU observes these power control commands; i.e., $q(n)$ is equal to the PU's power command at the nth TC, then $h(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-1))$ will be equal to $q(n)$, and M will be equal to one. On the other hand, if the PU is using a continuous power control and the $q(n)$ that the SU observes is some monotone function of the PUs power[2], then $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-m))$ will be the difference between $q(n)$ and $q(n-m)$. The value of M in this case will be the number of TCs in which the PU's direct channel and the interference from the rest of the environment remains constant.

III. The One-Bit Blind Null Space Learning Algorithm (OBNSLA)

We now present the OBNSLA by which the SU approximates $\mathcal{N}(\mathbf{H})$ from $\{\tilde{\mathbf{x}}(n), q(n)\}_{n=1}^{T}$ under the OC, where the approximation error can be made arbitrarily small for sufficiently large T. Once the SU learns $\mathcal{N}(\mathbf{H})$, it can optimize its transmitted signal, regardless of the optimization criterion, under the constraint that its signal lies in $\mathcal{N}(\mathbf{H})$. Let $\mathbf{U\Sigma V}^*$ be $\mathbf{H}$'s Singular Value Decomposition (SVD), where $\mathbf{V}$ and $\mathbf{U}$ are $n_t \times n_t$ and $n_r \times n_r$ unitary matrices, respectively, and assume that $n_t > n_r$. The matrix $\mathbf{\Sigma}$ is an $n_r \times n_t$ diagonal matrix with real nonnegative diagonal entries $\sigma_1, ..., \sigma_d$ arranged as $\sigma_1 \geq \ \sigma_2, \geq \cdots \geq \sigma_d > 0$. We assume without loss of generality that $n_r = d (= \mathrm{Rank}(\mathbf{H}))$. In this case $\mathcal{N}(\mathbf{H}) = \mathrm{span}(\mathbf{v}_{n_r+1}, ..., \mathbf{v}_{n_t})$, where $\mathbf{v}_i$ denotes $\mathbf{V}$'s ith column. From the SU's point of view, it is sufficient to learn $\mathcal{N}(\mathbf{G})$ (recall, $\mathbf{G} = \mathbf{H}^*\mathbf{H}$), which is equal to $\mathcal{N}(\mathbf{H})$ since

$$\mathbf{G} = \mathbf{V\Lambda V}^*, \tag{6}$$

where $\mathbf{\Lambda} = \mathbf{\Sigma}^{\mathrm{T}}\mathbf{\Sigma}$. The decomposition in (6) is known as the Eigenvalue Decomposition (EVD) of $\mathbf{G}$. In order to obtain $\mathcal{N}(\mathbf{H})$ it is sufficient to obtain $\mathbf{G}$'s EVD. However, in the OBNSL problem, $\mathbf{G}$ is not observed, so the SU needs to obtain the EVD using only one-bit information.

To illustrate that $\mathcal{N}(\mathbf{H})$ can be obtained via only one bit we consider a simple example in which $\mathbf{H} = \alpha[\sqrt{3}, -1]^{\mathrm{T}}$ where $\alpha > 0$. In this case

$$\mathbf{G} = \frac{\alpha^2}{4}\begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{V} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} \tag{7}$$

Note that the null space is spanned by $\mathbf{r}(-\pi/6)$ where $\mathbf{r}(\theta) = [\sin(\theta), -\cos(\theta)]$. Thus, the null space can be obtained by minimizing $\mathbf{r}(\theta)^{\mathrm{T}}\mathbf{G}\mathbf{r}(\theta)$ over $\theta \in [-\pi, \pi]$. The latter is true due to the fact that $\mathbf{r}(-\pi/6)^{\mathrm{T}}\mathbf{G}\mathbf{r}(-\pi/6) = 0$ is the global minimum of the function $\mathbf{r}(\theta)^{\mathrm{T}}\mathbf{G}\mathbf{r}(\theta)$. Because $\mathbf{r}(\theta)^{\mathrm{T}}\mathbf{G}\mathbf{r}(\theta) = \left(\sqrt{3}\sin(\theta) + \cos(\theta)\right)^2/4$ is a sinusoidal function with a period of π, it is possible to search for the null space by transmitting $\mathbf{r}(\theta)$ for different values of θ and receiving the one bit information given in (5), with linear complexity; i.e., a complexity that grows with $1/\epsilon$, where ϵ is the desired accuracy. The extension of

[2]See [14, Sec. II-B] for examples and conditions under which this is possible.

the above idea to practical MIMO channels, which are complex and might have more than two antennas, poses some challenges. The first challenge is that because each search point is obtained via a TC, it is highly desirable to reduce the search complexity. The second problem is that even for the same dimensions, i.e., a one dimensional null space in $\mathbb{C}^2$, the null space cannot be parameterized by a single parameter θ via $\mathbf{r}(\theta)$ but must be parameterized by two parameters, e.g. (θ, ϕ) via $\mathbf{r}(\theta, \phi) = [\cos(\theta), e^{-i\phi}\sin(\theta)]$. Thus, it is necessary to perform an efficient two dimensional search based on the one bit of information in (5). This problem is even more complicated when the dimension of the null space is greater than one. In this section we address these issues and present the OBNSLA. In the case of $n_t > 2$, to avoid searching over more than a two dimensional parameter space, we will utilize the well-known Cyclic Jacobi Technique (CJT) for Hermitian matrix diagonalization, which is based on two dimensional rotations. Then, we show that for two dimensional rotations it is possible to reduce the complexity of the search from a linear complexity to a logarithmic complexity. The proposed algorithm is a blind realization of the CJT. Because of the restriction to two dimensional rotations, except for the case of $n_t = 2$, the OBNSLA does not obtain the null space after a finite amount of rotations, but rather converges to the null space as the number of rotations increases. Nevertheless, we will show (Sec IV) that the OBNSLA converges to the null space very fast. We begin with a review the CJT.

A. *Review of the Cyclic Jacobi Technique*

The CJT [see e.g. 17] obtains the EVD of the $n_t \times n_t$ Hermitian matrix $\mathbf{G}$ via a series of 2-dimensional rotations that eliminates two off-diagonal elements at each step (indexed by k). It begins by setting $\mathbf{A}_0 = \mathbf{G}$ and then performs the following rotation operations $\mathbf{A}_{k+1} = \mathbf{V}_k \mathbf{A}_k \mathbf{V}_k^*,\ k = 1, 2, ...,$ where

$$\mathbf{V}_k = \mathbf{R}_{l,m}(\theta, \phi) \tag{8}$$

is an $n_t \times n_t$ unitary rotation matrix that is equal to $\mathbf{I}_{n_t}$ except for its mth and lth diagonal entries that are equal to $\cos(\theta)$, and its (m, l)th and (l, m)th entries that are equal to $e^{-i\phi}\sin(\theta)$ and $-e^{i\phi}\sin(\theta)$, respectively. For each k, the values of θ, ϕ are chosen such that $[\mathbf{A}_k]_{l,m} = 0$, or stated differently, θ and ϕ are chosen to zero the l, m and m, l off diagonal entries of $\mathbf{A}_k$ (which are conjugate to each other). Note that in an $n_t \times n_t$ Hermitian matrix, there are $(n_t - 1)n_t/2$ such pairs. The values of l, m are chosen in step k according to a function $J : \mathbb{N} \longrightarrow \{1, ..., n_t\} \times \{1, ..., n_t\}$, i.e $J_k = (l_k, m_k)$. It is the choice of J_k that differs between different Jacobi techniques. In the cyclic Jacobi technique, l_k, m_k satisfy $1 < l_k < n_t - 1$ and $l_k < m_k \leq n_t$ such that each pair (l, m) is chosen once in every $(n_t - 1)n_t/2$ rotations. Such $(n_t - 1)n_t/2$ rotations are referred to as a Jacobi sweep. An example of a single sweep of the CJT for $n_t = 3$ is the following series of rotations: $J_1 = (1, 2),\ J_2 = (1, 3), J_3 = (2, 3)$. The next sweep is $J_4 = (1, 2), J_5 = (1, 3), J_6 = (2, 3)$ and so forth.

The convergence of the CJT has been studied extensively over the last sixty years. The first proof of convergence of the CJT for complex Hermitian matrices was given in [18]. However, this result did not determine the convergence rate. The convergence rate problem was addressed in [19], which proved that the CJT for real symmetric matrices has a global linear convergence rate[3] if $\theta_k \in [-\pi/4, \pi/4]$ for every k. This result was extended to complex Hermitian matrices in [20]. It was later shown in [21, 22] that for a matrix with well separated eigenvalues, the CJT has a quadratic convergence rate[4]. This result was extended in [23] to a more general case which includes identical eigenvalues and clusters of eigenvalues (that is, very close eigenvalues). Studies have shown that in practice the number of iterations that is required for the CJT to reach its asymptotic quadratic convergence rate is a small number, but this has not been proven rigorously. In [24] it is argued heuristically that this number is $O(\log_2(n_t))$ cycles for $n_t \times n_t$ matrices. Extensive numerical results show that quadratic convergence is obtained after three to four cycles (see e.g. [17, page 429], [25, page 197]). Thus, since each Jacobi sweep has $n_t(n_t - 1)/2$ rotations, the overall number of rotations in the CJT roughly grows as n_t^2. For further details about the CJT and its convergence, the reader is referred to [17, 25].

B. *The One-Bit Line Search*

The learning in the OBNSLA is carried out in learning stages, indexed by k, where each stage performers one Jacobi rotation. The SU approximates the matrix $\mathbf{V}$ by $\mathbf{W}_{k_s}$, where

$$\mathbf{W}_k = \mathbf{W}_{k-1}\mathbf{R}_{l,m}(\theta_k, \phi_k),\ k = 1, ..., k_s, \tag{9}$$

and $\mathbf{W}_0 = \mathbf{I}$. Recall that in the CJT, one observes the matrix

$$\mathbf{A}_{k-1} = \mathbf{W}_{k-1}^*\mathbf{G}\mathbf{W}_{k-1} \tag{10}$$

and chooses $\theta_k = \theta_k^{\mathrm{J}}$, $\phi_k = \phi_k^{\mathrm{J}}$ such that

$$[\mathbf{R}_{l,m}^*(\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}})\mathbf{A}_{k-1}\mathbf{R}_{l,m}(\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}})]_{l,m} = 0 \tag{11}$$

[3] A sequence a_n is said to have a linear convergence rate of $0 < \beta < 1$ if there exists $n_0 \in \mathbb{N}$ such that $|a_{n+1}| < \beta|a_n|$ for every $n > n_0$. If $n_0 = 1$, a_n has a global linear convergence rate.

[4] A sequence is said to have a quadratic convergence rate if there exists $\beta > 0, n_0 \in \mathbb{N}$ such that $|a_{n+1}| < \beta|a_n|^2, \forall n > n_0$.

In the OBNSL problem, the SU needs to perform this step using only $\{\tilde{\mathbf{x}}(n), q(n)\}_{n=1}$, without observing the matrix $\mathbf{A}_{k-1}$. The following theorem, proved in [14], is the first step towards such a blind implementation of the Jacobi technique. The theorem converts the problem of obtaining the optimal Jacobi rotation angles into two one-dimensional optimizations of the function $S(\mathbf{A}_{k-1}, \mathbf{r}_{l,m}(\theta,\phi))$ (which is continuous, as shown in [14]), where $S(\mathbf{A},\mathbf{x}) = \mathbf{x}^*\mathbf{A}\mathbf{x}$ and $\mathbf{r}_{l,m}(\theta,\phi)$ is $\mathbf{R}_{l,m}(\theta,\phi)$'s lth column.

Theorem 1: [14, Theorem 2] Consider the $n_t \times n_t$ Hermitian matrix $\mathbf{A}_{k-1}$ in (10), and let $S(\mathbf{A},\mathbf{x}) = \mathbf{x}^*\mathbf{A}\mathbf{x}$ and $\mathbf{r}_{l,m}(\theta,\phi)$ be $\mathbf{R}_{l,m}(\theta,\phi)$'s lth column. The optimal Jacobi parameters θ_k^{J} and ϕ_k^{J}, which zero out the (l,m)th entry of $\mathbf{R}^*_{l,m}(\theta_k^{\mathrm{J}},\phi_k^{\mathrm{J}})\mathbf{A}_{k-1}\mathbf{R}_{l,m}(\theta_k^{J},\phi_k^{\mathrm{J}})$, are given by

$$\begin{aligned} \phi_k^{\mathrm{J}} &= \arg\min_{\phi\in[-\pi,\pi]} S\left(\mathbf{A}_{k-1}, \mathbf{r}_{l,m}(\pi/4,\phi)\right) \quad (12) \\ \theta_k^{\mathrm{J}} &= T_k(\phi_k^{\mathrm{J}}) \quad (13) \end{aligned}$$

where

$$T_k(\phi) = \begin{cases} \tilde{\theta}_k(\phi) & \text{if } -\frac{\pi}{4} \le \tilde{\theta}_k(\phi) \le \frac{\pi}{4} \\ \tilde{\theta}_k(\phi) - \mathrm{sign}(\tilde{\theta}_k(\phi))\pi/2 & \text{otherwise} \end{cases} \quad (14)$$

where $\mathrm{sign}(x) = 1$ if $x > 0$ and -1 otherwise, and

$$\tilde{\theta}_k(\phi) = \arg\min_{\theta\in[-\pi/2,\pi/2]} S\left(\mathbf{A}_{k-1}, \mathbf{r}_{l,m}(\theta,\phi)\right) \quad (15)$$

The theorem enables the SU to solve the optimization problems in (12) and (15) via line searches based on $\{\tilde{\mathbf{x}}(n), q(n)\}_{n=1}^{T}$. This is because under the OC, the SU can extract $\tilde{h}(\tilde{\mathbf{x}}(n), \tilde{\mathbf{x}}(n-m))$, which indicates whether $S(\mathbf{G},\tilde{\mathbf{x}}(n)) > S(\mathbf{G},\tilde{\mathbf{x}}(n-m))$ is true or false. It is possible, however, to further reduce the complexity of the line search, which is important, since each search point requires a TC. To see this, consider the line search in (12) and denote $w(\phi) = S(\mathbf{A}_k, \mathbf{r}_{l,m}(\pi/4,\phi)) = \|\mathbf{H}\mathbf{W}_{k-1}\mathbf{r}_{l,m}(\pi/4,\phi)\|^2$. According to the OC, for each ϕ_1,ϕ_2 the SU only knows whether $w(\phi_1) \ge w(\phi_2)$ or not. Assume that the SU tries to approximate ϕ_k^{J} by searching over a linear grid, with a spacing of η, on the interval $[-\pi,\pi]$. The complexity of such a search is at least $O(1/\eta)$ since each point in the grid must be compared to a different point at least once. The two line searches in (12) and (15) would be carried out much more efficiently if binary searches could be invoked. However, a binary search is feasible only if the objective function has a unique local minimum point, which is not the case in (12) and (15) because

$$S(\mathbf{G}, \mathbf{r}_{l,m}(\theta,\phi)) = \cos^2(\theta)\,|g_{l,l}| + \sin^2(\theta)\,|g_{m,m}| - |g_{l,m}|\sin(2\theta)\cos(\phi + \angle g_{l,m}) \quad (16)$$

Thus, before invoking the binary search, a single-minimum interval (SMI) must be determined; i.e, an interval in which the target function in (12) or (15) has a single local minimum. This is possible via the following proposition:

Proposition 2: Let $w(\phi) = \|\mathbf{H}\mathbf{r}_{l,m}(\pi/4, \phi)\|^2$ where $\mathbf{r}_{l,m}$ is defined in Theorem 1. Let $\check{\phi} \in [-\pi, \pi]$ be a minimum[5] point of $w(\phi)$, then

(a) $\check{\phi} \in [-3\pi/4, -\pi/4]$ if $w(-\pi), w(0) \geq w(-\pi/2)$.

(b) $\check{\phi} \in [-\pi/4, \pi/4]$ if $w(-\pi) \geq w(-\pi/2) \geq w(0)$.

(c) $\check{\phi} \in [\pi/4, 3\pi/4]$ if $w(-\pi), w(0) \leq w(-\pi/2)$.

(d) $\check{\phi} \in [3\pi/4, \pi] \cup [-\pi, -3\pi/4]$ if $w(-\pi) \leq w(-\pi/2) \leq w(0)$.

Proof: From (16), $w(\phi)$ can be expressed as $w(\phi) = B - A\cos(\phi - \check{\phi}),\ A, B \geq 0$. If $B = 0$, every $\phi \in [-\pi, \pi]$ is a minimum point and the proposition is true. We now assume that $B > 0$. By substituting $w(0) = B - A\cos(\check{\phi}), w(\pi/2) = B - A\sin(\check{\phi})$ and $w(\pi) = B + A\cos(\check{\phi})$ into $w(-\pi), w(0) \geq w(-\pi/2)$, one obtains that the latter is equivalent to $\pm\cos(\check{\phi}) > \sin(\check{\phi})$ which is equivalent to $(-3\pi/4 < \check{\phi} < \pi/4) \cap ((-\pi < \check{\phi} < -\pi/4) \cup (3\pi/4 < \check{\phi} < \pi))$. The last set can be written as $\check{\phi} \in (-3\pi/4, -\pi/4)$, which establishes (a). The proof of (b)-(d) is similar. $\square$

Note that unless w is a horizontal line, it is a 2π periodical sinusoid. In the latter case, there cannot be more than a single local (and therefor global) minimum within an interval of $\pi/2$. If w is a horizontal line, every point is a minimum point. In both cases the SU can efficiently approximate $\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}}$ by $\hat{\theta}_k^{\mathrm{J}}$ and $\hat{\phi}_k^{\mathrm{J}}$, respectively, using a binary search, such that

$$|\hat{\theta}_k^{\mathrm{J}} - T_k(\hat{\phi}_k^{\mathrm{J}})| \leq \eta, |\hat{\phi}_k^{\mathrm{J}} - \phi_k^{\mathrm{J}}| \leq \eta, \tag{17}$$

where $\eta > 0$ determines the approximation accuracy. The SU uses Proposition 2 to determine an SMI via $u_n(\pi, \pi/2)$ and $u_n(\pi/2, 0)$, where $u_n(\phi_n, \phi_{n-1}) = \tilde{h}_n(\mathbf{r}_{l,m}(\pi/4, \phi_n), \mathbf{r}_{l,m}(\pi/4, \phi_{n-1}))$, and $\tilde{h}$ is defined in (5). The one-bit line search is given in Algorithm 1. In determining the SMI, the one-bit line search requires 3 TCs: two TCs for $u_n(\pi, \pi/2)$, and one more for $u_n(\pi, 0)$. Given an SMI of length a and an accuracy of $\eta > 0$, it takes $\lfloor -\log_2(\eta/a) \rfloor + 1$ search points to obtain the minimum to within that accuracy. In the search for ϕ_k^{J}, $a = \pi/2$, thus $\hat{\phi}_k^{\mathrm{J}} \in [\phi_k^{\mathrm{J}} - \eta, \phi_k^{\mathrm{J}} + \eta]$ is obtained using $\lceil -\log(2\eta/\pi) \rceil + 1$ TCs, plus the 3 TCs required for determining the SMI and 2 more TCs to compare the initial boundaries of the SMI. Then, θ_k^{J} can be approximated to within η in the same way.

[5]Since w is a 2π periodical sinusoid, such a point always exists, though might not be unique.

We conclude with a discussion of parameter M in the OC. The proposed line search can obtain $w(\phi)$'s minimum even for $M = 1$. However, the number of TCs is lower if M is larger. Assume that the SU has obtained an SMI $[\phi^{\min}, \phi^{\max}]$. It takes the SU two TCs to determine whether $w(\phi_{\max}) > w(\phi_{\min})$, where, in the first TC, it transmits $r_{l,m}(\pi/4, \phi_{\max})$ and measures $q(1)$, and in the second TC, it transmits $r_{l,m}(\pi/4, \phi_{\min})$, and measures $q(2)$. If the latter inequality is false, $\phi_{\max}$ is set as $\phi_{\max} = (\phi_{\min}+\phi_{\max})/2$. In the next phase of the binary search the SU cannot use $q(1)$ and it needs an additional TC to do so. In general, $M > \lfloor -\log(\eta/\pi) + 1 \rfloor$ guarantees that each search point requires one TC.

C. *The OBNSLA*

Now that we have established the one-bit line search, we can present the OBNSLA. In the OBNSLA, the SU performs two line searches for each k. The first search is carried out to find $\hat{\phi}_k$ that minimizes $\|\mathbf{H}\mathbf{W}_k\mathbf{r}_{l_k,m_k}(\pi/4, \phi))\|^2$, where each search point, ϕ_n, is obtained by one TC in which the SU transmits

$$\mathbf{x}_s(t) = \tilde{\mathbf{x}}(n) = \mathbf{W}_k\mathbf{r}_{l_k,m_k}(\pi/4, \phi_n) \in \mathbb{C}^{n_t} \qquad , \forall (n-1)N \leq t \leq nN, \tag{18}$$

and measures $q(n)$. In the first line search, the SU obtains $\hat{\phi}_k^{\mathrm{J}}$ which is then used in the second line search to obtain $\tilde{\theta}_k(\hat{\phi}_k^{\mathrm{J}})$ according to (15), and then to obtain $\hat{\theta}_k^{\mathrm{J}}$ according to (13). The indices (l_k, m_k) are chosen as in the CJT. After performing k_s iterations the SU approximates the matrix $\mathbf{V}$ (see (6)) by $\mathbf{W}_{k_s}$. It then chooses its pre-coding matrix $\mathbf{T}_{k_s}$ as

$$\mathbf{T}_{k_s} = [\mathbf{w}_{i_1}^{k_s}, ..., \mathbf{w}_{i_{n_t-n_r}}^{k_s}], \tag{19}$$

where $\mathbf{w}_i^k$ is $\mathbf{W}_k$'s ith column, and $i_1, i_2, ..., i_{n_t}$ is an indexing such that $(\mathbf{w}_{i_q}^{k_s})^*\mathbf{G}\mathbf{w}_{i_q}^s \leq (\mathbf{w}_{i_v}^{k_s})^*\mathbf{G}\mathbf{w}_{i_v}^s$ for every $q \leq v$. Thus, the interference power that the SU inflicts on the PU is bounded as $\|\mathbf{H}\mathbf{x}\|^2 \leq p_s\|\mathbf{H}\mathbf{w}_{i_{n_t-n_r}}^{k_s}\|^2, \forall \|\mathbf{x}\|^2 = p_s$, where p_s is the SU's transmit power. A high level description of the OBNSLA algorithm is given in Fig. 4 and the exact algorithm is given in Algorithm 2.

Although the SU becomes "invisible" to the PU after it learns $\mathcal{N}(\mathbf{H}_{ps})$, it interferes with the PU during this learning process. Furthermore, this interference is an important ingredient in the learning since it provides the SU with the means to learn $\mathcal{N}(\mathbf{H}_{ps})$, i.e. $q(n)$. Nevertheless, the SU must also protect the PU during the learning process. Hence we assume that there exists an additional mechanism enabling the SU to choose $\tilde{\mathbf{x}}(n)$'s power to be high enough to be able to extract $q(n)$, but not too high, so as to meet the interference constraint (2). We give examples for such mechanisms in [14, Sec. II-C].

Algorithm 1 $[z, n] = \textbf{OneBitLineSearch}(\{\tilde{h}_l\}_{l\in\mathbb{N}}, z_{\max}, n, \eta, \mathbf{x}(z))$

Initialize: $L \leftarrow z_{\max}$,
$u_n(z_1, z_2) \leftarrow \tilde{h}_n(\mathbf{x}(z_1), \mathbf{x}(z_2))$
$a \leftarrow u_n(-L, -L/2), n++.$
$b \leftarrow u_n(0, -L/2),\ n++$
$z_{\max} \leftarrow (3 + 2b - 2a(1 + 2b))L/4;$
$z_{\min} \leftarrow z_{max} - L/2.$
while $|z_{\max} - z_{\min}| \geq \eta$ **do**
  $z \leftarrow (z_{\max} + z_{\min})/2$
  $a \leftarrow u(z_{\max}, z_{\min}), n++$
  if a=1 **then**
    $z_{\max} \leftarrow z$
  else
    $z_{\min} \leftarrow z$
  end if
end while

Algorithm 2 The OBNSL Algorithm

Input: $\{\tilde{h}_v\}_{v\in\mathbb{N}}$, defined in (5).
Output:$\mathbf{W}$
initialize: $n = 1$
$[\mathbf{W}, n] = \textbf{OBNSLF}(\{\tilde{h}_v\}_{v\in\mathbb{N}}, n_t, n)$
End

Function: $[\mathbf{W}, n]$= OBNSLF($\{\tilde{h}_v\}_{v\in\mathbb{N}}, n_t, n$)

Initialize: $k = 1$, $\mathbf{W} = \mathbf{I}_{n_t}$, $\Delta_j = 2\eta, \forall j \leq 0$
while $\left(\max_{j\in\{k-n_t(n_t-1)/2,\ldots,k\}} \Delta_j \geq \eta\right)$ **do**
  $\mathbf{x}(\phi) \leftarrow \mathbf{W}\mathbf{r}_{l_k,m_k}(\pi/4, \phi)$
  $[\hat{\phi}_k, n] \leftarrow \textbf{OneBitLineSearch}\left(\{\tilde{h}_l\}_{l\in\mathbb{N}}, \pi, n, \eta, \mathbf{x}(\phi)\right)$
  $\mathbf{x}(\theta) \leftarrow \mathbf{W}\mathbf{r}_{l_k,m_k}(\theta, \hat{\phi}_k)$
  $[\tilde{\theta}_k, n] \leftarrow \textbf{OneBitLineSearch}\left(\{\tilde{h}_l\}_{l\in\mathbb{N}}, \pi/2, n, \eta, \mathbf{x}(\theta)\right)$
  $\hat{\theta}_k \leftarrow \tilde{\theta}_k$ if $\tilde{\theta}_k \leq |\pi/4|$, otherwise $\hat{\theta}_k \leftarrow \tilde{\theta}_k - \pi\text{sign}(\tilde{\theta}_k)/2$.
  $\Delta_k \leftarrow |\theta_k|$
  $\mathbf{W} \leftarrow \mathbf{W}\mathbf{R}_{l_k,m_k}(\hat{\theta}_k, \hat{\phi}_k)$
  $k \leftarrow k + 1.$
end while

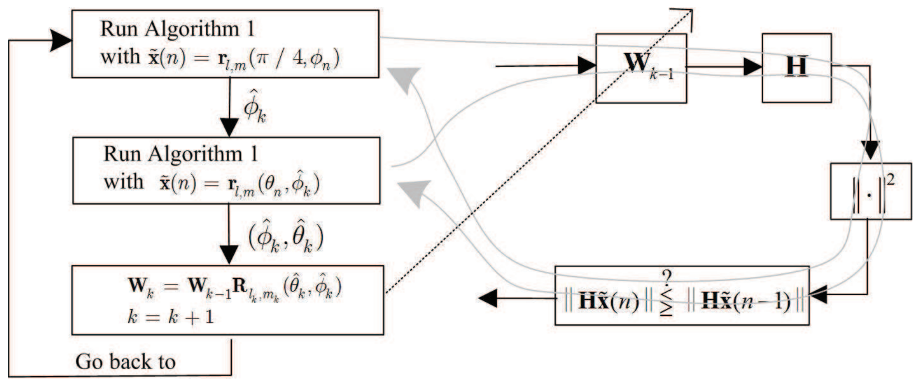


Fig. 4. High-level description of the One-bit Blind Null Space Learning Algorithm. The gray arrows represent the path $\tilde{\mathbf{x}}(n)$ undergoes before $q(n)$ is obtained by the SU. The dashed line represents the action of updating of the precoding matrix $\mathbf{W}_{k-1}$.

IV. Algorithm Convergence

The OBNSLA is, in fact, a blind implementation of the CJT whose convergence properties have been extensively studied over the last 60 years. However, the convergence results of the CJT do not apply directly to the OBNSLA. This is because of the approximation in (17); i.e., due to the fact that for every k, the rotation angles $\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}}$ are obtained by a binary search of accuracy η. Thus the off diagonal entries are not completely annihilated; i.e., $[\mathbf{A}_{k+1}]_{l_k,m_k} \approx 0$ instead of $[\mathbf{A}_{k+1}]_{l_k,m_k} = 0$. Moreover, we would like to make this line search accuracy as low as possible (that is, to make η as large as possible) in order to reduce the number of TCs. It is therefore crucial to understand how η affects the performance of the OBNSLA algorithm, in terms of convergence rate and the interference reduction to the PU. In this section, we extend the classic convergence results of the CJT to the OBNSLA and indicate the required accuracy in the binary search that assures convergence and bounds the maximum reduction level of the interference inflicted by the SU on the PU. It will also be shown that the same convergence analysis applies to the BNSLA proposed in [14].

The following theorem shows that for a sufficiently good line search accuracy, the OBNSLA has a global linear convergence rate.

Theorem 3: Let $\mathbf{G}$ be a finite dimensional $n_t \times n_t$ complex Hermitian matrix and P_k denotes the Frobenius norm of the off diagonal upper triangular (or lower triangular) part of $\mathbf{A}_k = \mathbf{W}_{k-1}^*\mathbf{G}\mathbf{W}_{k-1}$ where $\mathbf{W}_k$ is defined in (9) and let $m = n_t(n_t-1)/2$. Let η be the accuracy of the binary search (see

(17)), then the OBNSLA satisfies

$$P_{k+m}^2 \leq P_k^2 \left(1 - 2^{-(n_t-2)(n_t-1)/2}\right) + (n_t^2 - n_t)(7 + 2\sqrt{2})\eta^2 \|\mathbf{G}\|^2 \tag{20}$$

Proof: See Appendix A.

In what follows, it will be shown that for sufficiently small η, the OBNSLA has an asymptotic quadratic convergence rate, but in order to obtain this, we modify the algorithm slightly as follows. Let $I : \{1, ..., n_t\} \to \{1, ..., n_t\}$ be the identity operator, i.e. $I(x) = x$. At the beginning of each sweep, i.e. for every $k = q(n_t^2 - n_t)/2$ were $q \in \mathbb{N}$, the SU sets $I_q = I$ and for each $k \in \{q(n_t^2 - n_t)/2 + 1, ..., (q + 1)(n_t^2 - n_t)/2\}$, the SU modifies I_q as follows

$$I_q(l_k) = \begin{cases} l_k \text{ if } a_{l_k l_k} \geq a_{m_k, m_k} \\ m_k \text{ otherwise} \end{cases}, \tag{21}$$

where $a_{l,m}$ is the (l, m)th entry of $\mathbf{A}_k$ (defined in (10)), and (l_k, m_k) are determined by the rotation function $J_k = (l_k, m_k)$ of the CJT, as discussed in Section III-A. At the end of each sweep, i.e. for $k = (q + 1)(n_t^2 - n_t)/2$, the SU permutes the columns of $\mathbf{W}_k$ such that $\mathbf{W}_k$'s lth column becomes its $I_q(l)$'s column. Note that this modification does not require extra TCs and that the convergence result in Theorem 3 is still valid. We refer to the OBNSLA after this modification as the modified OBNSLA.

Besides the fact that this modification is necessary for guaranteeing the quadratic convergence rate, as will be shown in the following theorem, it will also be shown that it helps the SU to identify the null space (the last n_r columns of $\mathbf{W}_k$) blindly without taking extra measurements.

Theorem 4: Let η be the accuracy of the binary search, $\{\lambda_l\}_{l=1}^{n_t}$ be $\mathbf{G}$'s eigenvalues and let

$$\delta = \min_{\lambda_l \neq \lambda_r} |\lambda_l - \lambda_r|/3 \tag{22}$$

Let P_k be the Frobenius norm of the off diagonal upper triangular part of $\mathbf{A}_k = \mathbf{W}_{k-1}^* \mathbf{G} \mathbf{W}_{k-1}$, where $\mathbf{W}_k$ is defined in (9) and let $m = (n_t^2 - n_t)/2$. Assume that the modified OBNSLA has reached a stage k, such that $P_k^2 < \delta^2/8$, then

$$P_{k+2m}^2 \leq O\left(\left(\frac{P_{k+m}^2}{\delta}\right)^2\right) + O\left(\frac{\eta P_{k+m}^{3/2}}{\delta}\right) + O\left(\frac{\eta^2 P_{k+m}^{1/2}}{\delta}\right) + 2\left(n_t^2 - n_t\right)\eta^2 \|\mathbf{G}\|^2 \tag{23}$$

Furthermore, the last $n_t - n_r$ columns of $\mathbf{W}_{k+m}$ inflict minimum interference to the PU; i.e., $\|\mathbf{H}_{ps}\mathbf{w}_i^{k+m}\| \leq \|\mathbf{H}_{ps}\mathbf{w}_j^{k+m}\|,\ \forall 1 \leq j \leq n_r < i \leq n_t$.

Proof: See Appendix B.

Theorem 4 shows that to guarantee the quadratic convergence rate, the accuracy, η, should be much smaller than P_k^2; that is, let k_0 be an integer such that $P_{k_0}^2 < \delta^2/8$, then

$$P_{k_0+2m} \leq O\left(\left(\frac{P_{k_0+m}}{\sqrt{\delta}}\right)^2\right) \tag{24}$$

if $\eta << P_{k_0}^2$. This implies that once P_k becomes very small such that $P_k = O(\eta)$, one cannot guarantee that P_{k+2m} will be smaller than P_{k+m}^2 since at $k+1$ it will be $O(\eta)$.

The asymptomatic quadratic convergence rate of Theorem 4 is determined by $1/\delta$ where 3δ is the minimal gap between $\mathbf{G}$'s eigenvalues. In addition, the quadratic convergence rate takes effect only after $P_k^2 < \delta/8$. Such a condition implies that if δ is very small, it will take the modified OBNSLA many cycles to reach its quadratic convergence rate. This is problematic since MIMO wireless channels may have very close singular values (recall that $\mathbf{H}_{12}$'s square singular values are equal to $\mathbf{G}$'s first n_r eigenvalues). If we were using the optimal Cyclic Jacobi technique (i.e. no errors because of finite line search accuracy) this would not have practical implications since a quadratic decrease in P_k, which is independent of δ, occurs prior to the phase where $P_k^2 < \delta/8$ [23]. In the following theorem, we extend this result to the modified OBNSLA.

Theorem 5: Let η be the accuracy of the line search, $\{\lambda_l\}_{l=1}^{n_t}$ be $\mathbf{G}$'s eigenvalues such that there exists a cluster of eigenvalues; i.e., there exists a subset $\{\lambda_{i_l}\}_{l=1}^{v} \subset \{\lambda_l\}_{l}^{n_t}$ such that $\lambda_{i_l} = \lambda + \xi_l$, for $l \in L_2 = \{i_1, ..., i_v\}$, where $\sum_{l=1}^{v} \xi_l = 0$ and the rest of the non-equal eigenvalues satisfy $\delta_c > 16\sqrt{\sum \xi_l^2}$, where

$$3\delta_c = \min(\Lambda_1 \cup \Lambda_2) \tag{25}$$

$$\begin{aligned} \Lambda_1 &= \{|\lambda_l - \lambda_r| : l \in L \setminus L_2,\ \lambda_l \neq \lambda_r\} \\ \Lambda_2 &= \{|\lambda_l - \lambda| : l \in L \setminus L_2\} \\ L &= \{1, \ldots, n_t\} \end{aligned} \tag{26}$$

Then, once the modified OBNSLA reaches a k such that

$$2\delta_c \sqrt{\sum_{l \in L_2} \xi_l^2} \leq P_k^2 \leq \delta_c^2/8 \tag{27}$$

it satisfies

$$P_{k+2m}^2 \leq O\left(\left(\frac{P_{k+m}}{\delta_c}\right)^4\right) + O\left(\left(\frac{\eta P_{k+m}^3}{\delta_c}\right)\right) + O\left(\left(\frac{\eta^2 P_{k+m}}{\delta_c}\right)\right) + 2\left(n_t^2 - n_t\right)\eta^2 \|\mathbf{G}\|^2 \tag{28}$$

Proof: See Appendix C.

Theorem 5 states that in the presence of a single eigenvalue cluster; i.e. $\sqrt{\sum_l \xi_l^2} << \delta_c$, and if

$\eta_k = o(P_k)$, the modified OBNSLA has four convergence regions: The first region is $P_k^2 \geq \delta_c^2/8$, the second is $2\delta_c\sqrt{\sum_l \xi_l^2} \leq P_k^2 \leq \delta_c^2/8$, the third is $\delta/8 \leq P_k^2 \leq 2\delta_c\sqrt{\sum_l \xi_l^2}$ and the fourth is $P_k^2 \leq \min_l \xi_l^2/8$. In the first and the third regions, the modified OBNSLA has at least a linear convergence rate while in the second and fourth regions, it has a quadratic convergence rate. This means that from a practical point of view, a close cluster of eigenvalues; i.e. $\sqrt{\sum_l \xi_l^2}/\delta_c << 1$, is not a problem. This is because once the algorithm enters the second convergence region; i.e., it reaches the stage $k = k_2$ such that $2\delta_c\sqrt{\sum_l \xi_l^2} << P_{k_2}^2 \leq \delta_c/8$, P_k will decrease quadratically until $k = k_3$ such that $P_{k_3}^2 \leq 2\delta_c\sqrt{\sum_l \xi_l^2}$. But the latter inequality implies that $P_{k_3} << P_{k_2}$, a fact that guarantees a significant reduction P_k; i.e., from P_{k_2} to P_{k_3} with a quadratic rate. Nevertheless, P_k will eventually decrease quadratically as P_k^2 becomes smaller than $\delta/8$ as required by Theorem 4. This phenomenon is also a characteristic of the Cyclic Jacobi technique [23].

We now consider the maximum level of interference that the SU inflicts on the PU. Our aim here is to relate the asymptotic behavior of the maximum interference to that of P_k, and to obtain bounds on the maximum interference as a function of η. We begin with the following proposition:

Proposition 6: Let $\mathbf{T}_k$ be the SU's pre-coding matrix defined in (19), $\boldsymbol{t}_i^k$ be its ith column, $Q = \{1, ..., n_t - n_r\}$, and P_k be the norm of the off diagonal upper triangular (or lower triangular) part of $\mathbf{A}_k$ (where $\mathbf{A}_k$ is defined in (10)). Then

$$\max_{q \in Q} \|\mathbf{H}_{12}\boldsymbol{t}_q^k\|^2 \leq 2P_k^2 \tag{29}$$

Proof: This is an immediate result of [26, Corollary 6.3.4] which states that for every eigenvalue $\hat{\lambda}$ of $\mathbf{B} + \mathbf{E}$, where $\mathbf{B}$ is an $n_t \times n_t$ Hermitian matrix with eigenvalues $\lambda_i, i = 1, ..., n_t$, there exists λ_i such that $|\hat{\lambda} - \lambda_i|^2 \leq \|\mathbf{E}\|^2$, where $\|\cdot\|$ is the Forbinus norm. Thus, if one expresses $\mathbf{A}_k$ as $\mathbf{A}_k = \mathbf{B} + \mathbf{E}$, where $\mathbf{B} = \mathrm{diag}(\mathbf{A}_k), \mathbf{E} = \mathrm{offdiag}(\mathbf{A}_k)$, (29) follows. □

Since the maximum interference to the PU, for a single column of $\mathbf{T}_k$, is bounded by $2P_k^2$ (from Proposition 6), it follows that the maximum interference satisfies $\|\mathbf{H}_{12}\mathbf{T}_k\|^2 \leq 2(n_t - n_r)P_k^2$. Thus, it is possible to apply the results of Theorems 4 and 5 and to bound the maximum interference. These bounds are valuable since they relate the asymptotic level of interference to the accuracy of the line search η (which is determined by the SU), thus enabling the SU to control the interference reduction to the PU.

Before obtaining the first bound on the interference, we need the following corollary of Theorem 3:

Corollary 7:

$$\limsup_k P_k^2 \leq \frac{(n_t^2 - n_t)(7 + 2\sqrt{2})\eta^2\|\mathbf{G}\|^2}{2^{-(n_t-2)(n_t-1)/2}}. \tag{30}$$

Proof: See Appendix D.

From Corollaries 6 and 7 we obtain the following bound:

$$\limsup_{k} \max_{q\in Q} \|\mathbf{H}_{12}\boldsymbol{t}_q^k\|^2 \leq \frac{2(n_t^2-n_t)(7+2\sqrt{2})\eta^2\|\mathbf{G}\|^2}{2^{-(n_t-2)(n_t-1)/2}}. \tag{31}$$

We now derive a tighter bound than (31) which is valid only if the conditions of Theorem 4 are satisfied; i.e., that the OBNSLA is replaced by the modified OBNSLA and that there exists k such that $P_k^2 < \delta^2/8$. In this case, by combining Proposition 6 and Theorem 4, one obtains

$$\max_{q\in Q} \|\mathbf{H}_{12}\boldsymbol{t}_q^k\|^2 \leq O\left(\frac{P_k^2}{\delta}\right)^2 + O\left(\frac{\eta P_k^{3/2}}{\delta}\right) + O\left(\frac{\eta^2 P_k^{1/2}}{\delta}\right) + 2\left(n_t^2-n_t\right)\eta^2\|\mathbf{G}\|^2 \tag{32}$$

Furthermore, if P_k becomes sufficiently small such that $\eta > P_k$, the dominant term in the RHS of (32) will be $O(\eta^2)$; i.e., we effectively have:

$$\max_{q\in Q} \|\mathbf{H}_{12}\boldsymbol{t}_q^k\|^2 \leq 2\left(n_t^2-n_t\right)\eta^2\|\mathbf{G}\|^2 + O(\eta^{2.5}) \tag{33}$$

Thus, the parameter η gives the SU autonomous control on the maximum interference to the PU.

We conclude with the following corollary, which extends the convergence analysis presented in this section to the BNSLA.

Corollary 8: Theorems 3, 4, 5, Proposition 6 and Corollary 7 apply to the BNSLA presented in [14].

Proof: The proofs of Theorems 3, 4, 5, Proposition 6 and Corollary 7 rely on the fact that the only difference between the CJT and the OBNSLA is in the rotation angles. In the OBNSLA, the CJT's rotation angles , $\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}}$, are approximated according to (17). Furthermore, note that the BNSLA and the OBNSLA are identical except for the way in which each algorithm determines its SMI (which are not identical SMIs) before invoking the binary search. However, (17) is satisfied as long as each SMI contains the desired minimum point. Because the latter is satisfied by both algorithms, as indicated by Proposition 2 for the OBNSLA and by Proposition 3 in [14] for the BNSLA, Theorems 3, 4, 5, Proposition 6 and Corollary 7 apply to the BNSLA. □

We conclude this section with a discussion of the effective interference channel between the SU-Tx and the PU-Rx [6]. In many MIMO communication systems the PU-Rx applies a spatial decoding matrix $\mathbf{B}$ to its received signal, e.g, $\mathbf{B}$ might be a projection matrix into the column space of the PU's direct channel $\mathbf{H}_{11}$. In this case, the equivalent received signal is $\tilde{\mathbf{y}}_p(t) = \mathbf{B}\mathbf{y}_s(t)$. Thus, the SU's effective interference to the PU-Rx is $\mathbf{B}\mathbf{H}\tilde{\mathbf{x}}(n)$, rather than $\mathbf{H}\tilde{\mathbf{x}}(n)$. We now discuss the effect of $\mathbf{B}$ on the OBNSLA algorithm and on the bounds in (31) and (33), in different cases. The first is the case where the SU extracts $q(n)$

from the PU's SINR and the PU calculates its SINR based on the effective interference; i.e., $\mathbf{B}\mathbf{H}_{ps}\tilde{\mathbf{x}}(n)$. In this case, $q(n)$ will be a one-bit function of $\|\mathbf{B}\mathbf{H}\tilde{\mathbf{x}}(n)\|$, rather than $\|\mathbf{H}\tilde{\mathbf{x}}(n)\|$. Thus, the OBNSLA will converge to the null space of the effective interference channel[6]; i.e., $\mathcal{N}(\mathbf{B}\mathbf{H})$ and the bounds in (31) and (33) will be valid by replacing $\mathbf{H}$ with $\mathbf{B}\mathbf{H}$ and setting $\mathbf{G} = \mathbf{H}^*\mathbf{B}^*\mathbf{B}\mathbf{H}$. Note that for the OBNSLA to work, the matrix $\mathbf{B}$ must be constant during the learning process. If the PU modifies $\mathbf{B}$ due to the modifications in the SU's learning signal, the OBNSLA will not converge to the null space. The second case is where the SU observes $q(n)$ which carries information only on $\|\mathbf{H}\tilde{\mathbf{x}}(n)\|$, while the effective interference is $\|\mathbf{B}\mathbf{H}\tilde{\mathbf{x}}(n)\|$. In this case, the OBNSLA will converge to $\mathcal{N}(\mathbf{H})$, but the bounds will not necessarily hold. If $\mathbf{B}$ is a projection matrix; i.e., it projects to some subspace of $\mathbb{C}^{n_r}$ and it does not amplify the signal, the bounds in (31) and (33) will be satisfied. The SU can lose degrees of freedom by restricting its signal to $\mathcal{N}(\mathbf{H})$ rather than $\mathcal{N}(\mathbf{B}\mathbf{H})$ because $\mathcal{N}(\mathbf{H}) \subseteq \mathcal{N}(\mathbf{B}\mathbf{H})$.

V. SIMULATIONS

In this section we study the performance of the OBNSLA via simulations. We first compare the OBNSLA to the OBNSLA [14] in the non-asymptotic regime. We examine the effect of important practical aspects, such as time-varying environments and quantization. In the second part of this section, we compare the asymptotic properties of the two algorithms with respect to the asymptotic analysis in Sec. IV.

A. Non-Asymptotic Comparison of the BNSLA and the OBNSLA

We now examine the OBNSLA and BNSLA in time-varying environments where the SU measures the PU's transmitted signal and extracts $q(n)$ from it. Recall that it takes the SU some time to learn the null space of the interference channel. Thus, if the channel varies faster than the SU's learning period, it will not be able to effectively mitigate interference to the PU. Another problem that we address in this section is the effect of the variations in the PU direct link; i.e., $\mathbf{H}_{pp}$, on the performance of the OBNSLA and the BNSLA. We assume that the primary user performs a power adaptation every 1 msec and that the SU sets this as the TC's length; the considerations for this choice are the following. Let T_p be the time cycle in which the PU performs power control, $T_{\mathbf{H}_{pp}}$ and $T_{\mathbf{H}_{ps}}$ be the coherence times of $\mathbf{H}_{pp}(t)$ and $\mathbf{H}_{ps}(t)$, respectively, and T_{TC} be the length of the TC (which is equal to N time units, as depicted in Fig. 2). Note that $T_{\mathrm{TC}} \geq T_p$. Because the OBNSLA is based on the OC, the SU must choose T_{TC}

[6] The channel $\mathbf{B}\mathbf{H}$ is a special case of the effective interference channel which is defined in [6] for the case for a MIMO TDD PU, where the SU interferes with both the PU-Rx and the PU-Tx; i.e., uplink and downlink.

such that $T_{\mathrm{TC}} << T_{\mathbf{H}_{ps}}$, where the variations in the SU's signal must be the dominant factor affecting the PU's SINR. If the PU adapts its transmission scheme fast, such that T_p is smaller than typical values of $T_{\mathbf{H}_{pp}}$ and $T_{\mathbf{H}_{sp}}$, the SU can set $T_{\mathrm{TC}} = T_p$. In LTE, for example, the PU can adapt its signal's power, modulation and coding every 1 msec[7]. Other examples are all the third generation cellular systems which perform power control every 2/3 msec (W-CDMA) or 5/4 msec (CDMA2000)[8] (see e.g. [28], Appendix D).

Figure 5 presents simulation results where the PU performs a power adaptation to maintain a target 10 dB SINR at the receiver, and the SU inflicts interference on the PU and measures $q(n)$ by listening to the PU signal's power at the SU-Rx[9]. Fig. 5(a). presents the interference reduction of the BNSLA and the OBNSLA as a function of $\mathbf{H}_{ps}$'s Doppler spread. The results show that both algorithms perform similarly but the BNSLA shows a slightly better interference reduction than the OBNSLA for higher values of Doppler spread and vice-versa. The fact that the BNSLA is slightly better than the OBNSLA in low Doppler spread can be explained by the fact that the BNSLA is a little faster than the OBNSLA, as we discuss below. In [14, Proposition 3] it was shown that the BNSLA determines a $\pi/4$-length SMI using $w_k(0), w_k(\pi/2), w_k(-\pi/2)$ if

$$|w_k(0) - w_k(\pi/2)| \geq |w_k(\pi/2) - w_k(\pi)|, \tag{34}$$

where $w_k(\phi)$ is some monotone function of the PU SINR such that obtaining its value for a given ϕ requires a TC. Thus, if (34) is true, the BNSLA is faster than the OBNSLA by one TC. This is due to the fact that although both algorithms require three TCs to determine an SMI, the BNSLA's SMI is half the length of the OBNSLA SMI. If (34) is false, the SU will determine a $\pi/4$-length SMI by testing a similar condition to (34) using $w_k(0), w_k(-\pi/2), w_k(-\pi)$. This requires one more TC to obtain $w(-\pi)$. The same phenomenon occurs in the search for the θ_k^{J}. On the other hand, the inequality in (34) also suggests a possible explanation for the slightly better performance of the OBNSLA for low Doppler spread. The fact that the BNSLA, in order to determine an SMI, must check a condition that involves

[7]The power control in LTE is typically no more than a few hundred Hz. [see e.g. 27, Section. 20.3]. However, in the future it may be possible to have power control every 1 msec, since this is the length of a Transmission Time Interval (TTI), the smallest interval in which a base station can schedule any user for transmission (uplink or downlink).

[8]Note that CDMA channels are not necessarily narrow-band. However, the SU can still learn the null space of a narrow-band portion since the PU will perform power adaptation in the presence of narrow-band interference as long as it is not too narrow. For instance the bandwidths of most CDMA2000 systems is 1.25 MHz, but the coherence bandwidth typically varies between 50 KHz and 3 MHz.

[9]The SU sets $q(n) = \frac{1}{N}\sum_{t=Nn}^{nN+N'-1} \|\mathbf{y}_s(t) - \bar{\mathbf{y}}_s\|^2$, where $\bar{\mathbf{y}}_s$ is the average of $\bar{\mathbf{y}}_s(t)$ over $t = N(n-1)+N', \ldots Nn+N'-1$. The consideration for choosing such q are described in [14, Sec II-B1].

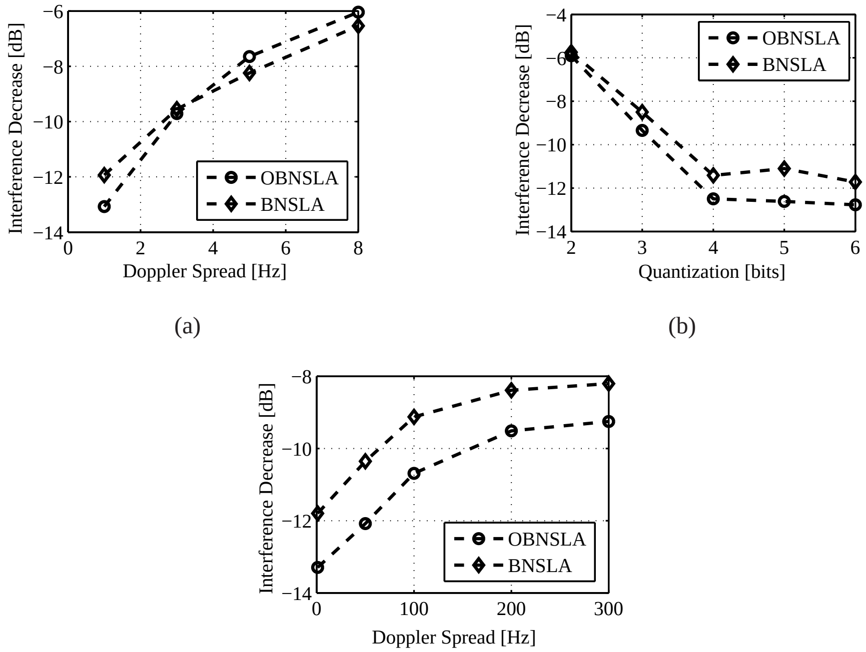
three noisy search points, while the OBNSLA does so via two noisy search points, which makes the OBNSLA more robust to noise. In low Doppler spread, this advantage compensates for the fact that the OBNSLA is slightly slower than the BNSLA. However, as the Doppler spread increases, the BNLSA's interference reduction becomes equal to that of the OBNSLA and eventually becomes better.

An important practical issue in the implementation the OBNSLA and the BNSLA is the granularity in the PU's SINR, which prevents the SU from detecting small variations in the PU SINR. A full theoretical convergence analysis of this problem is an important topic for future research. In this paper, we test this problem using simulation. Fig. 5(b) presents simulation results for a scenario where the PU's power control process is based on a quantized measurement of the PU's SINR in the range -5 dB to 20 dB. It is shown that the interference reduction is not improved for more than 4 bits of quantization. This means that small granularity does have a practical affect on the performance of the OBNSLA and BNSLA.

In the last simulation we investigate the effect of variations in the PU direct channel on the performance of the OBNSLA and BNSLA. Recall that the OC requires that the PU's SINR be mostly affected by the variations in the interference inflicted by the SU. However, if the PU direct channel varies, it will be impossible for the SU to distinguish whether the variations in the PU SINR are due to the SU interference signal or to the PU's direct channel path loss, and will lead to errors. To study this phenomenon we run a simulation for the case where the PU experiences fast fading. Fig. 5(c) presents the interference reduction of the two algorithms as a function of $\mathbf{H}_{pp}$'s Doppler spread, where $\mathbf{H}_{pp}$ is Rayleigh fading. The result shows that OBNSLA outperforms the BNSLA at all frequencies. Even for a 150 Hz doppler spread, the OBNSLA achieves a 10 dB interference reduction.

B. Asymptotic analysis

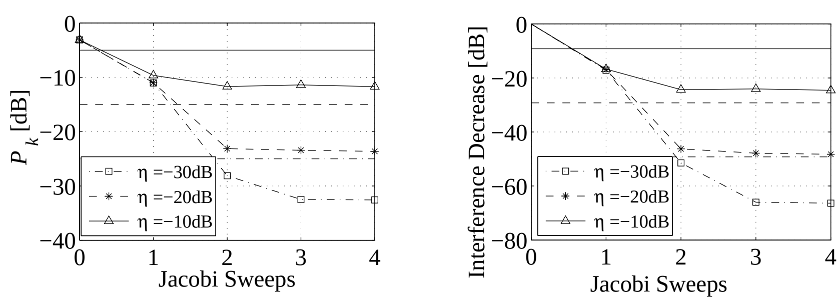
We now compare the asymptotic properties of the OBNSLA to the bounds derived in Section IV under optimal conditions; i.e, $q(n)$ is perfectly observed. Figure 6(a) depicts P_k and the bound on it as given in (31), versus complete OBNSLA sweeps; i.e. $(n_t^2 - n_t)/2$ learning phases. It shows that for sufficiently small η the OBNSLA converges quadratically. The quadratic decrease breaks down when the value of P_k becomes as small as the order of magnitude of η. This result is consistent with Theorem 4. Figure 6(b) depicts the interference decrease and the bound on it as given in (33) versus the number of transmission cycles. It shows that the asymptotic level of the interference to the PU is $O(\eta^2)$. Note that because the simulations in Fig. 6 are under optimal conditions; i.e., $q(n)$ is not noisy and the channel is not time-varying, we would obtain the same results if we applied the BNSLA instead of the OBNSLA. In what follows, we study asymptotic performance under non-optimal conditions.



(a) (b)

(c)

Fig. 5. Interference reduction after a single Jacobi sweep as a function of (a) $\mathbf{H}_{ps}$'s Doppler spread, (b) SINR quatization bits, (c) $\mathbf{H}_{pp}$'s Doppler spread. The maximum power constraint of each Tx is 23 dBm. The channels' path-losses are calculated as $128.1 + 37.6\log_{10}(R)$, where R is the distance between the Rx and the TX in meters as used by the 3GPP (see page 61 3GPP Technical Report 36.814). The locations of the PU-TX is randomly chosen from a uniform distribution over a 300 m disk, and the locations of the SU-Tx and the SU-Rx are randomly chosen from a uniform distribution over a 400 m disk. Both disks are centered at the location of the PU-Rx. The minimum distance between the PU-Rx to the PU-Tx, and the PU-Rx to the SU-Tx is 20 m and 100 m, respectively. For each t the entries of the channel matrices $\mathbf{H}_{pp}(t), \mathbf{H}_{sp}(t)$ and $\mathbf{H}_{ps}(t)$ are i.i.d. where each entry is 15 KHz flat fading Rayleigh channel. In all plots, the Doppler spread of $\mathbf{H}_{sp}(t)$ is 15 Hz, which is also $\mathbf{H}_{pp}$'s Doppler spread in (a) and (b). The Doppler spread of $\mathbf{H}_{ps}$ in (b) and (c) is 1 Hz. All channels are generated using the Improved Rayleigh Fading Channel Simulator [29]. The noise level at the receivers is -121 dBm and the SU transmit power during the learning process is 5 dBm. The numbers of antennas are $n_{t_s} = 3$, $n_{t_p} = 2$, $n_{r_p} = 1$ $n_{r_s} = 2$.



(a) (b)

Fig. 6. Simulation results for different values η of the OBNSL algorithm to obtain the null space of $\mathbf{H}$ with $n_t = 3$ transmitting antennas and $n_r = 2$ antennas at the PU receiver. The matrix $\mathbf{G} = \mathbf{H}^*\mathbf{H}$ was normalized such that $\|\mathbf{G}\|^2 = 1$. The unmarked lines in (a) and (b) represent the asymptotic upper bound of (31) and (33) respectively on the corresponding marked line; e.g. the solid unmarked line is a bound on the solid line with squares. We used 200 Monte-Carlo trials where the entries of $\mathbf{H}$ are i.i.d. complex Gaussian random variables.

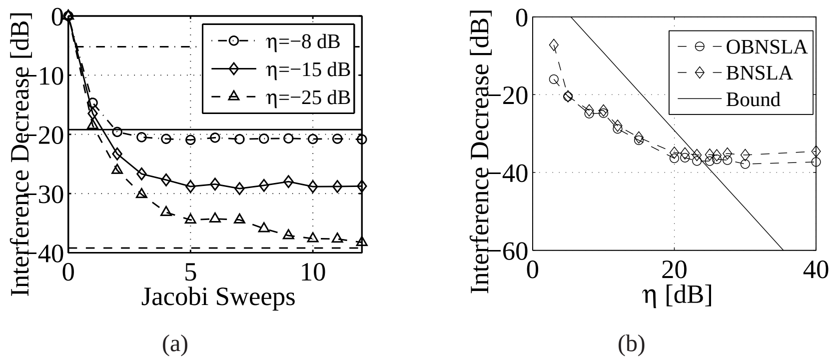


Fig. 7. Interference reduction (marked lines) of the OBNSLA, as a function of Jacobi Sweeps (a), and of the OBNSLA and BNSLA as a function of η (b). $\mathbf{H}_{ps}$ was generated according to [30], which represents a fixed Rx-Tx channel where one antenna is 1.75 m in height and the second antenna is 25 m in height. The Rician factor and the Doppler spread of $\mathbf{H}_{ps}$ were determined according to Equations (13) and (14) in [30]. The unmarked lines represent the bound in (33) for the corresponding marked lines with the same pattern, e.g., in Subfigure (a) the dotted-dashed unmarked line represent the bound on the OBNSLA's interference reduction with $\eta = -8$ dB, which is represented by the dotted-dashed line that is marked with circles. The numbers of antennas are $n_{t_s} = 2$ $n_{t_p} = 2$, $n_{r_p} = 1$ $n_{r_s} = 2$. The results were averaged over 1000 Monte-Carlo trials.

The bounds in this paper are derived under the assumption that the OC holds perfectly. In practice, however, $q(n)$ is affected by measurement error such as noise. Furthermore, the channel matrices $\mathbf{H}_{pp}$, $\mathbf{H}_{sp}$ vary with time, a fact that may also affect the function $q(n)$. For example, variations in $\mathbf{H}_{pp}$ affect the PU's SINR. In addition, $\mathbf{H}_{ps}$ is also time-varying, which leads to some discrepancy between the estimated null space and the true null space. In what follows, we show in simulations that the derived bounds are still useful in practice. A full theoretical convergence analysis of the O/BNSLA in practical conditions, which extends the bounds derived in this paper to account for measurement noise and time variations in the channel is a topic for future research, beyond the scope of this paper.

Figure 7 presents simulation results for an identical scenario as in Figure 5 except for $\mathbf{H}_{ps}$ which is generated assuming that both the SU-Tx and PU-Rx are fixed. The result shows that for an interference reduction smaller or equal to 37 dB, the bound in (33) predicts the behavior of the interference reduction (i.e., it decreases as η^2) of both the OBNSLA and the BNSLA. Furthermore, it is shown that the BNSLA and the OBNSLA have the same asymptotic properties; i.e. convergence rate and asymptotic interference reduction.

VI. Summary and Future Research

This paper proposed the OBNSLA, which enables a MIMO CR SU to learn the null space of the interference channel to the PU by observing a binary function that indicates the variations (increase or decrease) in the PU's SINR. Such information can be extracted, for instance, from the quantized version of the variation in the PU's SINR, or in the PU's modulation. We also provided a convergence analysis of the OBNSLA, which also applies to the BNSLA [14]. It was shown that the two algorithms have a global linear and an asymptotically quadratic convergence rate. It was also shown in simulations that just like in the Cyclic Jacobi technique, the OBNSLA and the BNSLA reach their quadratic convergence rates in only three to four cycles. In addition, we derived asymptotic bounds on the maximum level of interference that the SU inflicts on the PU. The derived bounds have important practical implications. Due to the fact that these bounds are functions of a parameter determined by the SU, it enables the SU to control the maximum level of interference caused to the PU. This gives the OBNSLA (or the BNSLA) a useful stopping criterion which guarantees the protection of the PU. The analytical convergence rates and interference bounds were validated by extensive simulations.

We consider the theoretical analysis of the OBNSLA and BNSLA under measurement noise as an important topic for future research. Note that in the presence of noise, the analysis of the two algorithms is not identical since the BNSLA relies on a continuous-valued function of the PU's SINR, whereas the OBNSLA relies on a binary function. Noise, which is continuous-valued, will thus affect these functions and hence the performance and convergence of the two algorithms quite differently.

APPENDIX A

Consider the first sweep of the BNSL algorithm; i.e. $k = 1, 2, ..., n_t(n_t - 1)/2$. Denote the number of rotated elements in the lth row by $b_l = n_t - l$ and let

$$c_l = \sum_{j=1}^{l} b_j = (2n_t - 1 - l)l/2;\ \ Z(l,k) = \sum_{j=1}^{n_t-l} |[\mathbf{A}_k]_{j+l,l}|^2;\ \ W(l,k) = \sum_{j=l+1}^{n_t-1} Z(j,k) \tag{35}$$

Note that $W(0,k) = P_k^2$. In every sweep, each entry is eliminated once; we therefore denote $\mathbf{A}_k$'s p,q entry before its annihilation as $g_{q,p}(t)$ where t denotes the number of changes since $k = 0$. After $g_{q,p}(t)$ is annihilated once, it will be denoted by $\tilde{g}_{q,p}(\tilde{t})$ where $\tilde{t}$ is the number of changes after the annihilation. The diagonal entries of $\mathbf{A}_k$ will be denoted by x since we are not interested in their values in the course of the proof. This is illustrated in the following example of a 4×4 matrix

$$\mathbf{A}_0 = \begin{pmatrix} g_{1,1}(0) & g_{1,2}(0) & g_{1,3}(0) & g_{1,4}(0) \\ g_{2,1}(0) & g_{2,2}(0) & g_{2,3}(0) & g_{2,4}(0) \\ g_{3,1}(0) & g_{3,2}(0) & g_{3,3}(0) & g_{3,4}(0) \\ g_{4,1}(0) & g_{4,2}(0) & g_{4,3}(0) & g_{4,4}(0) \end{pmatrix}, \qquad \mathbf{A}_1 = \begin{pmatrix} x & \epsilon & g_{1,3}(1) & g_{1,4}(1) \\ \epsilon & x & g_{2,3}(1) & g_{2,4}(1) \\ g_{3,1}(1) & g_{3,2}(1) & x & g_{3,4}(0) \\ g_{4,1}(1) & g_{4,2}(1) & g_{4,3}(0) & x \end{pmatrix},$$
$$\mathbf{A}_2 = \begin{pmatrix} x & \tilde{g}_{1,2}(0) & \epsilon & g_{1,4}(2) \\ \tilde{g}_{2,1}(0) & x & g_{2,3}(2) & g_{2,4}(1) \\ \epsilon & g_{3,2}(2) & x & g_{3,4}(1) \\ g_{4,1}(2) & g_{4,2}(1) & g_{4,3}(1) & x \end{pmatrix}, \qquad \mathbf{A}_3 = \begin{pmatrix} x & \tilde{g}_{1,2}(1) & \tilde{g}_{1,3}(0) & \epsilon \\ \tilde{g}_{2,1}(1) & x & g_{2,3}(2) & g_{2,4}(2) \\ \tilde{g}_{3,1}(0) & g_{3,2}(2) & x & g_{3,4}(2) \\ \epsilon & g_{4,2}(2) & g_{4,3}(2) & x \end{pmatrix} \tag{36}$$

For arbitrary n_t, after the first c_1 sweeps $\mathbf{A}_{c_1}$'s first column is equal to the following vector:

$$[x, \tilde{g}_{2,1}(n_t - 3), ..., \tilde{g}_{n_t-1,1}(0), \epsilon_{c_1}]^{\mathrm{T}} \tag{37}$$

and

$$Z(1, c_1) \leq |\tilde{g}_{2,1}(n_t - 3)|^2 + ... + |\tilde{g}_{n_t-1,1}(0)|^2 + |\epsilon_{c_1}|^2 \tag{38}$$

From (10) it follows that for $q = 2, ..., n_t$,

$$\begin{aligned} \tilde{g}_{q,1}(n_t - q - 1) &= \cos(\theta_{n_t-1})\, \tilde{g}_{q,1}(n - q - 2) - e^{i\phi_{n_t-1}} g_{q,n_t}(1) \sin(\theta_{n_t-1}) \\ &\vdots \\ \tilde{g}_{q,1}(1) &= \cos(\theta_{q+1})\, \tilde{g}_{q,1}(0) - e^{i\phi_3} g_{q,q+2}(1) \sin(\theta_3) \\ \tilde{g}_{q,1}(0) &= \epsilon_{q-1} \cos(\theta_q) - e^{i\phi_q} g_{q,q+1}(1) \sin(\theta q) \end{aligned} \tag{39}$$

where $\tilde{g}_{q,1}(-1) = \epsilon_1$. The following bounds on $\{\tilde{g}_{q,1}(l)\}_{l=0}^{n_t-q-1}$ are obtained recursively (i.e., by obtaining a bound on $\tilde{g}_{q,1}(0)$, substituting and obtaining a bound on $\tilde{g}_{q,1}(1)$ and so on)

$$\begin{aligned} \tilde{g}_{q,1}(n_t - q - 1) &\leq |\epsilon_{q-1} \textstyle\prod_{v=q}^{n_t-1} \cos(\theta_v) - \sum_{j=q}^{n_t-1} e^{i\phi_j} \sin(\theta_j)\, g_{q,j+1}(1) \prod_{v=j+1}^{n_t-1} \cos(\theta_v)\, | \\ &\leq |\mathbf{v}(q)^{\mathrm{T}} \mathbf{y}(q)| + \epsilon \textstyle\prod_{v=q}^{n_t-1} \cos(\theta_v) \end{aligned} \tag{40}$$

where $\mathbf{v}, \mathbf{y} \in \mathbb{C}^{n_t-q}$ such that $[\mathbf{v}(q)]_j = e^{i\phi_{j+q-1}} g_{q,j+q}(1)$, $[\mathbf{y}(q)]_j = \sin(\theta_{j+q-1}) \prod_{v=j+q}^{n_t-1} \cos(\theta_v)$, $j = 1, ..., n_t - q$, and $\epsilon = \max_q |\epsilon_q|$. It follows that

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \quad \leq |\mathbf{y}^{\mathrm{T}}(q)\mathbf{v}(q)|^2 + |\epsilon|^2 \textstyle\prod_{v=q}^{n_t-1} \cos^2(\theta_v) \leq \|\mathbf{y}(q)\|^2 \|\mathbf{v}(q)\|^2 + |\epsilon|^2 \prod_{v=q}^{n_t-1} \cos^2(\theta_v) \tag{41}$$

Proposition 9:

$$\|\mathbf{y}(q)\|^2 = 1 - \textstyle\prod_{i=q}^{n-1} \cos(\theta_i) \tag{42}$$

Proof: This is shown by induction. By definition

$$\|\mathbf{y}(q)\|^2 = \textstyle\sum_{i=q}^{n-1} \sin^2(\theta_i) \prod_{v=i+1}^{n-1} \cos^2(\theta_v) \tag{43}$$

where $\prod_{i=l}^{m} v_i \triangleq 1, \text{if } \mathrm{l} > \mathrm{m}$. Assume that (42) is true for $n = m \in \mathbb{N}$, then, for $m+1$ (42) and (43) yields

$$\begin{aligned}&\textstyle\sum_{i=q}^{m} \sin^2(\theta_i) \prod_{v=i+1}^{m} \cos^2(\theta_v) = \sum_{i=q}^{m-1} \sin^2(\theta_i) \prod_{v=i+1}^{m} \cos^2(\theta_v) + \sin^2(\theta_m) \prod_{v=m+1}^{m} \cos^2(\theta_v)\\ &\quad = \cos^2(\theta_m) \textstyle\sum_{i=q}^{m-1} \sin^2(\theta_i) \prod_{v=i+1}^{m-1} \cos^2(\theta_v) + \sin^2(\theta_m),\end{aligned}$$

According to the supposition (42)

$$\cos^2(\theta_m) \left(1 - \textstyle\prod_{i=q}^{m-1} \cos^2(\theta_i)\right) + \sin^2(\theta_m) = 1 - \prod_{i=q}^{m} \cos^2(\theta_i), \tag{44}$$

which establishes the desired result. □

By substituting Proposition 9 into (41) one obtains

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \le \underbrace{\left(\textstyle\sum_{i=1}^{n_t - q} |g_{q,i+q}(1)|^2 \right)}_{=Z(q,c_1)} \left(1 - \textstyle\prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i)\right) + |\epsilon|^2 \underbrace{\prod_{v=q}^{n_t-1} \cos^2(\theta_v)}_{\le 1} \tag{45}$$

thus,

$$|\tilde{g}_{q,1}(n_t - q - 1)|^2 \le \left(1 - \textstyle\prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i)\right) Z(q, c_1) + |\epsilon|^2 \tag{46}$$

and by summing both sides of (46) over $q = 2, ..., n_t$

$$\begin{aligned}&Z(1, c_1) \le \textstyle\sum_{q=2}^{n_t} \left(1 - \prod_{i=c_0+q}^{n_t-1} \cos^2(\theta_i)\right) Z(q, c_1) + (n_t - 1)|\epsilon|^2\\ &\le (1 - \textstyle\prod_{i=c_0+2}^{c_1} \cos^2(\theta_i)) \underbrace{\sum_{q=2}^{n} Z(q, c_1)}_{W(1,c_1)} + (n_t - 1)|\epsilon|^2 \le (1 - \prod_{i=c_0+2}^{c_1} \cos^2(\theta_i)) W(0, 0) + (n_t - 1)|\epsilon|^2\end{aligned} \tag{47}$$

where the last inequality is due to $P_{c_1} = W(1, c_1) + Z(1, c_1)$, $W(0, 0) = P_0$, and because P_k is a monotonically decreasing sequence[10]. It follows that

$$Z(1, c_1) = \sin^2(\Psi_{c_0+2,c_1}) W(0, 0) + (n_t - 1)|\epsilon|^2 \tag{48}$$

where

$$\sin^2\left(\Psi_{c_{l-1}+2,c_l}\right) = 1 - \textstyle\prod_{i=c_{l-1}+2}^{c_l} \cos^2(\tilde{\theta}_i) \tag{49}$$

[10]Forsythe and Henrici [18] showed that the sequence P_k is a monotonically decreasing sequence.

and $\tilde{\theta}_i$ is an angle that satisfies $|\tilde{\theta}_i| \leq |\theta_i|$. Thus,

$$P_{c_1} = W(1, c_1) + Z(1, c_1) \leq W(0, 0) = P_0 \tag{50}$$

substituting (48) we obtain

$$W(1, c_1) \leq W(0, 0)\cos^2\left(\Psi_{2,c_1}\right) - (n_t - 1)|\epsilon|^2 \tag{51}$$

Now that this relation is established, it can be applied to $\mathbf{A}_{c_l}$'s lower $(n_t - l) \times (n_t - l)$ block-diagonal, thus

$$W(l, c_l) \leq W(l-1, c_{l-1})\cos^2\left(\Psi_{c_{l-1}+2,c_l}\right) - (n_t - l)|\epsilon|^2 \tag{52}$$

By substituting (52) recursively into itself, one obtains

$$W(l, c_l) \leq W(0, 0)\prod_{j=1}^{l}\cos^2\left(\Psi_{c_{j-1}+2,c_j}\right) - \epsilon^2\sum_{j=1}^{l} b_j \prod_{v=j+1}^{l}\cos^2\left(\Psi_{c_{v-1}+2,c_v}\right) \tag{53}$$

Thus

$$\begin{aligned} Z(l, c_l) = \sin^2\left(\Psi_{c_{l-1}+2,c_l}\right) W(l-1, c_{l-1}) + (n_t - l)|\epsilon^2| \leq W(0,0)\sin^2\left(\Psi_{c_{l-1}+2,c_l}\right)\textstyle\prod_{j=1}^{l-1}\cos^2\left(\Psi_{c_{j-1}+2,c_j}\right) \\ - |\epsilon|^2 \textstyle\sum_{j=1}^{l-1} b_j \prod_{v=j+1}^{l-1}\cos^2\left(\Psi_{c_{v-1}+2,c_v}\right) + (n_t - l)|\epsilon^2| \end{aligned} \tag{54}$$

After a complete sweep

$$\begin{aligned} P^2_{c_{n_t-1}} = \; & \textstyle\sum_{l=1}^{n_t-2} Z(l, c_{n_t-1}) + |\epsilon|^2 = \sum_{l=1}^{n_t-2} Z(l, c_l) \leq W(0,0)\sum_{l=1}^{n_t-2}\sin^2\left(\Psi_{c_{l-1}+2,c_l}\right)\prod_{j=1}^{l-1}\cos^2\left(\Psi_{c_{j-1}+2,c_j}\right) \\ & - \textstyle\sum_{l=1}^{n_t-2}|\epsilon|^2\sum_{j=1}^{l-1} b_j \prod_{v=j+1}^{l-1}\cos^2\left(\Psi_{c_{v-1}+2,c_v}\right) + |\epsilon^2|\sum_{l=1}^{n_t-1}(n_t - l) \end{aligned} \tag{55}$$

where the first equality is due to the fact that for $k = c_l + 1, ..., c_{n_t}$, the sum of squares of the lth column remains unchanged; thus, $Z(l, k) = Z(l, c_l), \forall k > c_l$. Similar to proposition 9, it can be shown that $\sum_{l=1}^{n}\sin^2(\tau_l)\prod_{j=1}^{l-1}\cos^2(\tau_j) = 1 - \prod_{j=1}^{n}\cos^2(\tau_j)$. Thus

$$P^2_{c_{n-1}} \leq \; W(0,0)\left(1 - \textstyle\prod_{j=1}^{n-2}\cos^2\left(\Psi_{c_{j-1}+2,c_j}\right)\right) - \sum_{l=1}^{n-2}\epsilon^2\sum_{j=1}^{l-1} b_j\prod_{v=j+1}^{l-1}\cos^2\left(\Psi_{c_{v-1}+2,c_v}\right) + |\epsilon^2|\sum_{l=1}^{n-1} b_l \tag{56}$$

From (49) we have $\cos^2\left(\Psi_{c_{l-1}+2,c_l}\right) \geq \prod_{v=c_{l-1}+2}^{ck_l}\cos^2(\theta_v)$, therefore

$$\begin{aligned} P^2_{c_{n-1}} \leq \; & W(0,0)\left(1 - \textstyle\prod_{j=1}^{n-2}\prod_{v=c_{j-1}+2}^{c_j}\cos^2(\theta_v)\right) \\ & - \textstyle\sum_{l=1}^{n-2}|\epsilon|^2\sum_{j=1}^{l-1}(n-j)\prod_{v=j+1}^{l-1}\prod_{r=c_{v-1}+2}^{c_v}\cos^2(\theta_r) + \frac{|\epsilon^2|(n^2-n)}{2} \end{aligned} \tag{57}$$

Recall that $|\theta_i| < \pi/4$, therefore

$$\begin{aligned} P^2_{c_{n-1}} &\leq W(0,0)\left(1-2^{-(n-2)(n-1)/2}\right) - |\epsilon^2|\left(\sum_{l=1}^{n-2}\sum_{j=1}^{l-1}(n-j)2^{\frac{l^2}{2}-ln+\frac{l}{2}+9n-45} - \frac{(n^2-n)}{2}\right) \\ &\leq W(0)\left(1-2^{-(n-2)(n-1)/2}\right) + |\epsilon^2|\frac{(n^2-n)}{2} \end{aligned} \tag{58}$$

It remains to relate ϵ to the accuracy of the line search η. Note that the error ϵ in (58) is due to (17) which is a result of the two finite-accuracy (of η accuracy) line-searches in (12), and (13). If η were zero, $\mathbf{A}_k$'s l, m off diagonal entry would be zero after the kth sweep, i.e.

$$u(\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}}) = 0 \tag{59}$$

where

$$u(\theta,\phi) \stackrel{\triangle}{=} |[\mathbf{R}_{l,m}(\theta,\phi)\mathbf{A}_k\mathbf{R}^*_{l,m}(\theta,\phi)]_{l,m}|^2 = u_1(\theta,\phi) + u_2(\theta,\phi), \tag{60}$$

$$\begin{aligned} u_1(\theta,\phi) &= 4(a^k_{l,m})^2\sin^2\left(\gamma_{l,m}+\phi\right) \\ u_2(\theta,\phi) &= \left(2\cos(2\theta)a^k_{l,m}\cos\left(\angle a^k_{l,m}+\phi\right) + \sin(2\theta)\left(a^k_{l,l}-a^k_{m,m}\right)\right)^2 \end{aligned} \tag{61}$$

and $(\theta_k^{\mathrm{J}}, \phi_k^{\mathrm{J}})$ is the value given in Theorem 1 when substituting $\mathbf{G} = \mathbf{A}_k$. Recall that $(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}})$ (see (17)) is the non optimal value that is obtained by the two line searches, then

$$|\epsilon|^2 = \max_k u(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}}) \tag{62}$$

The error $u(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}})$ can be bounded because $\phi_k^{\mathrm{J}} = \angle a^k_{l,m}$, thus $\hat{\phi}_k^{\mathrm{J}} = -\angle a^k_{l,m} + \eta_\phi$ where $|\eta_\phi| < \eta$, and

$$u_1(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}}) = 4(a^k_{l,m})^2\sin^2\left(\angle a^k_{l,m} + \hat{\phi}_k^{\mathrm{J}}\right) \leq 4(a^k_{l,m})^2\eta^2 \leq 2\|\mathbf{G}\|\eta^2 \tag{63}$$

$$u_2(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}}) = \left(2a^k_{l,m}\cos\left(\eta_\phi\right)\cos^2(2\hat{\theta}_k^{\mathrm{J}}) + \sin(2\hat{\theta}_k^{\mathrm{J}})\left(a^k_{l,l}-a^k_{m,m}\right)\right)^2 \tag{64}$$

To bound $u_2(\hat{\theta}_k^{\mathrm{J}}, \hat{\phi}_k^{\mathrm{J}})$, note that if $a^k_{ll} = a^k_{mm}$, then $\hat{\theta}_k^{\mathrm{J}} = \theta_k^{\mathrm{J}} \in \{0, \pi/4\}$ since the line search will not miss these points. Now for the case where $a^k_{ll} \neq a^k_{mm}$ we have $\hat{\theta}_k^{\mathrm{J}} = \theta_k^s + \eta_\theta$ where

$$\theta_k^s = \frac{1}{2}\tan^{-1}\left(x_k\right) \tag{65}$$

and

$$x_k = \frac{2|a^k_{l,m}|\cos(\eta_\phi)}{a^k_{m,m} - a^k_{l,l}} \tag{66}$$

Note that

$$u_2(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}}) = \left(2\cos\left(\eta_\phi\right)a_{l,m}^k\left(\cos\left(2\theta_k^s\right) - 2\eta_\theta\sin\left(2\theta^*\right)\right) + \left(a_{l,l}^k - a_{m,m}^k\right)\left(\sin\left(2\theta_k^s\right) + 2\cos\left(2\theta^*\right)\eta_\theta\right)\right)^2$$

where (θ^*,ϕ^*) is a point on the line that connects the points $(\theta_k^{\mathrm{J}},\phi_k^{\mathrm{J}})$, $(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}})$. By substituting (65) we obtain

$$u_2(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}}) = \left(\frac{2\cos(\eta_\phi)a_{l,m}^k + x_k a_{l,l}^k - x_k a_{m,m}^k}{\sqrt{x_k^2+1}} - 4\eta_\theta\sin\left(2\theta^*\right)\cos\left(\eta_\phi\right)a_{l,m}^k + 2\eta_\theta\cos\left(2\theta^*\right)\left(a_{l,l}^k - a_{m,m}^k\right)\right)^2 \tag{67}$$

Using (66) and the fact that the sinusoidal is bounded by one, and because $|\eta_\theta| \leq \eta$, it follows that

$$\begin{aligned} &u_2(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}}) \leq 4\eta^2\left(2|\sin(2\theta^*)|a_{l,m}^k + \cos(2\theta^*)\left|a_{l,l}^k - a_{m,m}^k\right|\right)^2 \\ &\leq 4\eta^2\left(4\sin^2(2\theta^*)|a_{l,m}^k|^2 + 2\sin(4\theta^*)|a_{l,m}^k||a_{ll}^k - a_{mm}^k| + \cos^2(2\theta^*)|a_{ll}^k - a_{mm}^k|^2\right) \\ &\leq 4\eta^2\left(2|a_{l,m}^k|^2 + 2\sin(4\theta^*)|a_{l,m}^k||a_{ll}^k - a_{mm}^k| + |a_{ll}^k - a_{mm}^k|^2 + 2|a_{l,m}^k|^2\right) \end{aligned} \tag{68}$$

$$u_2(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}}) \leq 4\eta^2\left(2\|\mathbf{G}\|^2 + \sqrt{2}\|\mathbf{G}\|\|\mathbf{G}\| + \|\mathbf{G}\|^2\right) \tag{69}$$

Thus

$$|\epsilon|^2 = \max_k u(\hat{\theta}_k^{\mathrm{J}},\hat{\phi}_k^{\mathrm{J}}) \leq 2(7+2\sqrt{2})\eta^2\|\mathbf{G}\|^2 \tag{70}$$

This expression is substituted into (58) and the desired result follows. □

Appendix B

Without loss of generality, we assume that $W(0,0) \leq \delta^2/8$ where $W(k,l)$ is defined in (35)[11]. We first prove the theorem assuming that $\mathbf{G}$'s eigenvalues are all distinct. From (40) it follows that

$$|\tilde{g}_{q,1}(n_t-q-1)|^2 \leq \sum_{j=q}^{n_t-1}\sin^2\left(\theta_j\right)|g_{q,j+1}(1)|^2 + \epsilon^2\prod_{v=q}^{n_t-1}\cos^2\left(\theta_v\right) \tag{71}$$

Similar to the derivation of (46), but without applying Proposition 9, one obtains

$$|\tilde{g}_{q,1}(n_t-q-1)|^2 \leq Z(q,c_1)\sum_{j=q}^{n_t-1}\sin^2\left(\theta_j\right) + |\epsilon|^2 \leq Z(q,c_1)\sum_{j=2}^{n_t-1}\sin^2\left(\theta_j\right) + |\epsilon|^2 \tag{72}$$

[11] I.e. let k_0 be the smallest integer such that $P_{k_0} < \delta^2/8$ and $(l_{k_0}, m_{k_0}) = (1,2)$, we set $\mathbf{A}_{k_0} = \mathbf{G}$).

and by summing both sides of (72) (similar to the derivation of (47)) over $q = 2, ..., n_t$ it follows that

$$Z(1,c_1) \le \left(\sum_{j=2}^{n_t-1} \sin^2(\theta_j)\right) \underbrace{\sum_{q=2}^{n_t} Z(q,c_1)}_{W(1,c_1)} + (n_t-1)|\epsilon|^2 \le \left(\sum_{j=2}^{n_t-1} \sin^2(\theta_j)\right) W(0,0) + (n_t-1)|\epsilon|^2$$

Now that we have established this relation we can apply it to the reduced $n_t - l + 1$ lower block diagonal matrix and obtain $Z(l,c_l) \le \left(\sum_{j=c_{l-1}+1}^{c_l} \sin^2(\theta_j)\right) W(0,0) + (n_t - l)|\epsilon|^2$. After a complete sweep we have

$$\begin{aligned} P^2_{c_{n_t-1}} \le \quad & \sum_{l=1}^{n_t-2} Z(l, c_{n_t-1}) + |\epsilon|^2 = \sum_{l=1}^{n_t-2} Z(l,c_l) + |\epsilon|^2 \\ & \le W(0,0) \sum_{j=1}^{n_t(n_t-1)/2} \sin^2(\theta_j) + |\epsilon^2| \sum_{l=1}^{n_t-1} (n_t - l) \end{aligned} \tag{73}$$

We now relate $\sum_{j=1}^{n_t(n_t-1)/2} \sin^2(\theta_j)$ to $W(0,0)$ (recall that $P_0^2 = W(0,0)$). Note that $|a^k_{ll} - a^k_{mm}|^2 = |a^k_{ll} - \lambda_l - a^k_{mm} + \lambda_m + \lambda_l - \lambda_m|^2 \ge |\lambda_l - \lambda_m|^2 - |a^k_{ll} - \lambda_l|^2 - |a^k_{mm} - \lambda_m|^2$, furthermore, by [21, Theorem 1], there exists a permutation to $\{\lambda_i\}_{i=1}^{n_t}$ such that

$$|a^k_{ii} - \lambda_i| \le \sqrt{2} P_k, \tag{74}$$

thus,

$$|a^k_{ii} - \lambda_i| \le \delta/2, \tag{75}$$

and

$$|a^k_{ll} - a^k_{mm}| \ge 2\delta - \delta/2 - \delta/2 = \delta. \tag{76}$$

Recall that the optimal rotation angle satisfies $\tan(2\theta^{\mathrm{J}}_k) = 2|a^k_{l_k m_k}|/|a^k_{l_k l_k} - a_{m_k m_k}|$ while the actual the rotation angel is

$$\hat{\theta}^{\mathrm{J}}_k = \theta^{\mathrm{J}}_k + \eta_\theta \tag{77}$$

It follows that

$$\begin{aligned} |\sin^2(\hat{\theta}^{\mathrm{J}}_k)| \le |\sin^2(\theta^{\mathrm{J}}_k)| + |\eta_\theta \sin(2\theta^{\mathrm{J}}_k)| \le \tfrac{1}{4}|2\theta_k|^2 + |\eta_\theta| \tan(2\theta^{\mathrm{J}}_k)| \le \tfrac{1}{2^2}\tan^2(2\theta^{\mathrm{J}}_k) + |\eta_\theta| \tan(2\theta^{\mathrm{J}}_k)| \\ \le \tfrac{|a^k_{l_k,m_k}|^2}{\delta^2} + 2|\eta_\theta| \tfrac{|a^k_{l_k,m_k}|}{\delta} \le \tfrac{|a^k_{l_k,m_k}|^2}{\delta^2} + \tfrac{2|\eta_\theta|\sqrt{W(0,k)|}}{\delta} \end{aligned} \tag{78}$$

Therefore

$$\sum_{k=1}^{n_t(n_t-1)/2} \sin^2(\hat{\theta}^{\mathrm{J}}_k) \le \quad \sum_{k=1}^{n_t(n_t-1)/2} \left(\tfrac{|a^k_{l_k,m_k}|^2}{\delta^2} + \tfrac{2|\eta_\theta|\sqrt{W(0,k)|}}{\delta} \right) = \tfrac{1}{\delta^2} W(0,k) + \tfrac{\eta_\theta (n_t^2 - n_t)}{\delta} \sqrt{W(0,k)} \tag{79}$$

By substituting (79) into (73) one obtains

$$P^2_{c_{n_t-1}} \leq W(0,0)\left(\frac{1}{\delta^2}W(0,0) + \frac{(n_t^2-n_t)|\eta_\theta|}{\delta}\sqrt{W(0,k)}\right) + \frac{|\epsilon^2|}{2}\left(n_t^2-n_t\right), \tag{80}$$

It remains to relate η_θ in (77) to η. Recall that the calculation of $\hat{\theta}^{\mathrm{J}}_k$ relies on the calculation of $\hat{\phi}^{\mathrm{J}}_k$. Thus, η_θ depends on η_ϕ as well, as we now show. Form the proof of [14, Theorem 2] we know that if an accurate line search were invoked, it would produce $\hat{\phi}^{\mathrm{J}}_k = -\angle a^k_{l,m}$. However, the actual line search yields $\hat{\phi}^{\mathrm{J}}_k = -\angle a^k_{lm} + \eta_\phi$, where $|\eta_\phi| \leq \eta$. Thus, θ_k is obtained by searching the minimum of a perturbed version of $S(\mathbf{A}_k, \mathbf{r}_{l,m}(\theta, \phi^{\mathrm{J}}_k))$, i.e.

$$\tilde{S}(\mathbf{A}_k, \mathbf{r}_{l,m}(\theta, \hat{\phi}^{\mathrm{J}}_k)) = h_k(\cos^2(\theta)a^k_{l,l} - \cos(\eta_\phi)\sin(2\theta)a^k_{l,m} + \sin^2(\theta)a^k_{m,m}) \tag{81}$$

We first assume that $a^k_{ll} \neq a^k_{mm}$. From the proof of [14, Theorem 2], the optimal value of θ is $\theta^{\mathrm{J}}_k = \frac{1}{2}\tan^{-1}(p_k)$, where $x_k = \frac{2|a^k_{l,m}|}{a^k_{m,m}-a^k_{l,l}}$. If one takes into consideration the non-optimality of the line-search which obtains $\hat{\phi}^{\mathrm{J}}_k$ and ignores the non-optimality of the line search that obtains $\hat{\theta}^{\mathrm{J}}_k$, then the minimizer of (81) would be $\theta^s_k = \frac{1}{2}\tan^{-1}(x_k\cos(\eta_\phi))$ and

$$|\theta^{\mathrm{J}}_k - \theta^s_k| = \left|\frac{1}{2}\tan^{-1}(x_k\cos(\eta_\phi)) - \frac{1}{2}\tan^{-1}(x_k)\right| \leq \frac{|\eta_\phi\sin\left(\eta^*_\phi\right)p_k|}{\cos^2\left(\eta^*_\phi\right)p_k^2+1} \leq \eta^2_\phi\frac{|x_k|}{\cos^2(\eta_\phi)x_k^2+1} \tag{82}$$

where $|\eta^*_\phi| \leq \eta_\phi$. It can be shown that $\frac{|x_k|}{\cos^2(\eta_\phi)x_k^2+1} \leq \frac{1}{|\cos(\eta_\phi)|}$, and because $\hat{\theta}^{\mathrm{J}}_k = \theta^{\mathrm{J}}_k + \theta^s_k - \theta^{\mathrm{J}}_k + \eta_\phi$ and $|\eta_\phi| < \eta$, the accumulated effect of the finite accuracy of both line searches is bounded by $\eta_\theta \leq \eta + \frac{\eta^2}{|\cos(\eta)|}$. For sufficiently small η (e.g. $\eta \leq \pi/20$) we obtain

$$\eta_\theta \leq 6\eta/5 \tag{83}$$

By substituting (83) and (70) into (80) it follows that

$$P^2_{c_{n_t-1}} \leq W(0,0)\left(\frac{1}{\delta^2}W(0,0) + \eta\frac{6(n_t^2-n_t)}{5\delta}\sqrt{W(0,0)}\right) + (10+2\sqrt{2})(n_t^2-n_t)\eta^2\|\mathbf{G}\|^2 \tag{84}$$

Thus, as long as η is negligible with respect to $W(0,0)$, the BNSLA will have a quadratic convergence rate for $\mathbf{G}$ whose all eigenvalues are distinct. This is not sufficient since we are interested in a matrix $\mathbf{G}$ $n_t - n_r$ with zero eigenvalues.

To extend the proof to the case where the matrix $\mathbf{G}$ has $n_t - n_r$ zero eigenvalues and n_r distinct eigenvalues we use the following theorem:

Theorem 10 ([25] Theorem 9.5.1): Let $\mathbf{A}$ be an $n_t \times n_t$ Hermitian matrix with eigenvalues $\{\lambda_l\}_{l=1}^{n_t}$

that satisfy

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_{n_r} \neq \lambda_{n_r+1} = \lambda_{n_r+2} = \cdots = \lambda_{n_t} = \lambda \tag{85}$$

Consider the following partition:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B} \\ \mathbf{B} & \mathbf{A}_2 \end{bmatrix} \tag{86}$$

where $\mathbf{A}_1$ is $n_r \times n_r$ and $\mathbf{A}_2$ is $(n_t - n_r) \times (n_t - n_r)$ and let $\delta' > 0$. If $\|(\mathbf{A}_1 - \lambda \mathbf{I})^{-1}\| < 1/\delta'$, then

$$\|\mathbf{A}_2 - \lambda \mathbf{I}\| \leq \|\mathbf{B}\|^2/\delta' \tag{87}$$

To apply Theorem 10 to the modified OBNSLA, we need to show that $\mathbf{A}_k$ satisfies its conditions. This however is only satisfied by $\mathbf{A}_k$ with $k \geq m+1$. To show this, note that (75) and (76) are satisfied by $\mathbf{A}_k, k \leq m$ for some permutations of the eigenvalues. Thus, due to the permutation in (21), $\mathbf{A}_k, k > m$ satisfies (75) and (76), for the ordering of (85). For the rest of the proof, it is assumed that $k > m$. Let $\mathbf{A}_1^k, \mathbf{A}_2^k, \mathbf{B}^k$ be $\mathbf{A}_k$'s submatrices that correspond to the partition in (86). Recall that in our case, $\lambda = 0$, thus, (75) implies that $\|\mathbf{A}_1^k\| > \delta$, and also implies that $a_{ll}^k \geq 5\delta/2, \forall 0 \leq l \leq n_r$. Furthermore, by [26, Corollary 6.3.4] $|\lambda_l(\mathbf{A}_1^k) - a_{ll}^k| \leq \|\mathbf{A}_k\|_{\text{off}} \leq \delta/2$, thus

$$\lambda_l(\mathbf{A}_1^k) > 0, \ \forall 0 \leq l \leq n_r \tag{88}$$

and therefore, the matrix $\mathbf{A}_1^k$ is invertible, and because $\|\mathbf{A}_1^k\| > \delta$, it follows that $\|(\mathbf{A}_1^k)^{-1}\| \leq 1/\delta$, and by Theorem 10 we obtain

$$\|\mathbf{A}_2^k\| \leq \|B_k\|^2/\delta \tag{89}$$

To show that (89) leads to quadratic convergence, one must show that the affiliation of the diagonal entries in the upper $n_r \times n_r$ -block of $\mathbf{A}_k$ remains unchanged and that the eigenvalue that correspond are arranged in decreasing order, i.e.

$$l = \arg \min_{1 \leq m \leq n_t} |\lambda_l - a_{mm}^k| = \arg \min_{1 \leq m \leq n_t} |\lambda_l - a_{mm}^{k+1}|, \ \forall l \in \{1, ..., n_r\} \tag{90}$$

and

$$\lambda_l \geq \lambda_m, \ \forall l \leq m \tag{91}$$

To show (90), note that

$$\left|a^k_{l_k,l_k} - a^k_{m_k,m_k}\right|^2 \le \sin^2(\theta_k)\left(2\cos(\theta_k)a^k_{l_k,m_k}\cos\left(\phi_k - \angle a^k_{l_k,m_k}\right) + \sin(\theta_k)\left(a^k_{l_k,l_k} - a^k_{m_k,m_k}\right)\right)^2 \tag{92}$$

and that for every θ_k such that $l_k \le n_r$, (78) is satisfied. Thus

$$\begin{aligned}
&\left|a^k_{l_k,l_k} - a^{k+1}_{l'_k,l_k}\right|^2 \le \sin^2(\theta_k)\left(a^k_{l_k,m_k} + \sin(\theta_k)\left(a^k_{l_k,l_k} - a^k_{m_k,m_k}\right)\right)^2 \\
&\quad \le \sin^2(\theta_k)\left(a^k_{l_k,m_k} + \left(\left(a^k_{l_k,m_k}\right)^2/\delta^2 + 2\eta_\theta\, a^k_{l_k,m_k}\big/\delta\right)^{1/2}\delta\right)^2 \\
&\quad \le \sin^2(\theta_k)\left(a^k_{l_k,m_k} + \left(\left(a^k_{l_k,m_k}\right)^2 + 2\delta\eta_\theta a^k_{l_k,m_k}\right)^{1/2}\right)^2 \le \sin^2(\theta_k)\left(a^k_{l_k,m_k} + \left(\delta^2/4 + 2\delta\eta_\theta\delta/2\right)^{1/2}\right)^2 \\
&\quad \le \sin^2(\theta_k)\left(a^k_{l_k,m_k} + \delta\left(1/4 + \eta_\theta\right)^{1/2}\right)^2 \le \tfrac{\delta^2}{4}\left(1 + 4\eta_\theta\right)\left(1/2 + \sqrt{1/2 + \eta_\theta}\right)^2
\end{aligned} \tag{93}$$

By restricting $\eta \le 1/100$ and considering (83) it follows that

$$|a^k_{l_k,l_k} - a^{k+1}_{l_k,l_k}| \le 0.65\delta \tag{94}$$

which establishes (90).

Now that (90) is established, (91) immediately follows and for every l, m such that $l \neq m$ and $1 \le l \le n_r$, we have $|a^k_{ll} - a^k_{mm}| \ge \delta$. And (73) can be written as

$$\begin{aligned}
P_{c_{n_r}+m} \le\ & \textstyle\sum_{l=1}^{n_t-1} Z(l, c_{n_r}+m) + |\epsilon^2| \sum_{l=1}^{n_r}(n_t - l) \\
& \le W(0,m)\textstyle\sum_{j=1}^{c_{n_r}} \sin^2(\theta_j) + \sum_{l=n_r+1}^{n_t-1} Z(l, c_{n_r}+m) + |\epsilon^2| \sum_{l=1}^{n_r}(n_t - l)
\end{aligned} \tag{95}$$

Recall that $|\epsilon|^2 \le max_k u(\theta_k, \phi_k)$ where $u_1(\theta_k, \phi_k)$ and $u_2(\theta_k, \phi_k)$ are defined in (63) and (64). From (63) and (68) $u(\theta_k, \phi_k) \le 4\eta^2(5P_k^2 + 4P_k\|\mathbf{G}\| + \|\mathbf{G}\|^2)$. Because $|a^k_{l_k l_k} - a^k_{m_k m_k}| \ge \delta$ for $m < k \le m + c_{n_r}$, (79) is satisfied and similar to (84) we obtain

$$\begin{aligned}
&P_{c_{n_r}+m} \le W(0,m)\left(\tfrac{1}{\delta^2}W(0,m) + \eta\tfrac{(n_t^2 - n_t)}{\delta}\sqrt{W(0,k)}\right) \\
&+2\left(2n_t n_r - n_r^2 - n_r\right)\eta^2(5W(0,m) + 4\sqrt{W(0,m)}\|\mathbf{G}\| + \|\mathbf{G}\|^2) + \textstyle\sum_{l=n_r+1}^{n_t-1} Z(l, c_{n_r}+m)
\end{aligned} \tag{96}$$

It remains to bound the term $\sum_{l=c_{n_r+1}}^{n_t-1} Z(l, c_{n_r}+m)$. Note that for every θ_k such that $m < k \le c_{n_r}+m$, (78) is satisfied. Let $Q = \{(l,m) : 1 \le l \le n_r < m \le n_t\}$. Note that for every k such that $(l_k, m_k) \in Q$, $a^k_{l_k l_k}$ and $a^k_{m_k m_k}$ are located in $\mathbf{A}^k_1$ and $\mathbf{A}^k_2$ respectively. Thus,

$$\begin{aligned}
|a^{k+1}_{q,m_k}|^2 &\le |a^k_{q,m_k}|^2 + \sin^2(\theta_k)|a^k_{l_k,q}|^2, \text{ for } n_r < q < m_k \\
|a^{k+1}_{m_k,q}|^2 &\le |a^k_{m_k,q}|^2 + \sin^2(\theta_k)|a^k_{l_k,q}|^2, \text{ for } m_k < q \le n_t
\end{aligned} \tag{97}$$

and from (78)

$$\begin{aligned} |a_{m_k,q}^{k+1}|^2 &\le |a_{m_k,q}^{k}|^2 + \left(\tfrac{|a_{l_k,m_k}^k|^2}{\delta^2} + 2\eta_\theta \tfrac{|a_{l_k,m_k}^k|}{\delta}\right) |a_{l_k,q}^{k}|^2 \text{ for } m_k < q \le n_t \\ |a_{q,m_k}^{k+1}|^2 &\le |a_{q,m_k}^{k}|^2 + \left(\tfrac{|a_{l_k,m_k}^k|^2}{\delta^2} + 2\eta_\theta \tfrac{|a_{l_k,m_k}^k|}{\delta}\right) |a_{l_k,q}^{k}|^2, \text{ for } n_r < q < m_k \end{aligned} \tag{98}$$

These can be bounded by

$$|a_{m_k,q}^{k+1}|^2, |a_{q,m_k}^{k+1}|^2 \le W^2(0,m)\left(1+\tfrac{1}{\delta^2}\right) + \tfrac{2\eta_\theta}{\delta} W^{3/2}(0,m) \tag{99}$$

Thus, for every $k \in \{m, ..., m + c_{n_r}\}$,

$$\textstyle\sum_{l=n_{r+1}}^{n_t-1} Z(l, c_{n_r} + m) = \displaystyle\sum_{q=n_r+1}^{n_t-1} \sum_{t=q+1}^{n_t} |a_{q,t}^k|^2 \le O\left(\left(\tfrac{W(0,0+m)}{\delta}\right)^2\right) + O\left(\left(\tfrac{\eta_\theta W^{3/2}(0,0+m)}{\delta}\right)\right) \tag{100}$$

This, together with (96) and (83) show

$$P_{c_{n_r}}^2 \le O\left(\left(\tfrac{W(0,0+m)}{\delta}\right)^2\right) + O\left(\left(\tfrac{\eta W^{3/2}(0,0+m)}{\delta}\right)\right) + O\left(\left(\tfrac{\eta^2 W^{1/2}(0,0+m)}{\delta}\right)\right) + 2\left(n_t^2 - n_t\right)\eta^2 \|\mathbf{G}\|^2 \tag{101}$$

Since P_k is a decreasing sequence, the desired result follows. $\square$

Appendix C

We first prove the theorem for the case where the non-clustered eigenvalues are the largest; i.e., $\lambda_i \ge \lambda_{i+1} + \delta_c$ and $\lambda_i - \lambda \ge \delta_c$ for $i = 1, ..., n_r - v$. Note that $\lambda_i = \lambda + \xi_{i-n_r-v}$ for $i \in L_2 = \{n_r - v + 1, ..., n_r\}$ and $\lambda_i = 0$ for $i = n_r + 1, ..., n_t$. Without loss of generality, we assume that $W(0,0) \le \delta_c^2/8$ where $W(k,l)$ is defined in (35). Let $\mathbf{V}_k \Lambda \mathbf{V}_k^\star = \mathbf{A}_k$ be $\mathbf{A}_k$'s EVD, and let $\tilde{\mathbf{A}}^k = \mathbf{V}_k \tilde{\mathbf{\Lambda}} \mathbf{V}_k^\star$, $\hat{\mathbf{A}}^k = \mathbf{V}_k \hat{\mathbf{\Lambda}} \mathbf{V}_k^\star$ where

$$\begin{aligned} \tilde{\mathbf{\Lambda}} &= \operatorname{diag}(\lambda_1, \cdots, \lambda_{n_r-v}, \underbrace{\lambda \cdots \lambda}_{v}, \underbrace{0 \cdots 0}_{n_t-v-n_r}) \\ \hat{\mathbf{\Lambda}} &= \operatorname{diag}(\underbrace{0 \cdots 0}_{n_r-v}, \xi_1, \cdots, \xi_v, \underbrace{0 \cdots 0}_{n_t-n_r-v},) \end{aligned} \tag{102}$$

Let $L_1 = \{1, ..., n_r - v\}$, $L_3 = L \backslash (L_1 \cup L_2)$ and $L_s = (L_1 \times L) \cup (L_2 \times L_3)$, $L_c = L_s \cap \{(l,m) : l < m\}$. By combining (74) and the condition $P_k^2 < \delta_c^2/8$, it follows that (75) and (76) hold for $\delta = \delta_c$. Thus, due to the permutation in (21), the inequalities (75) and (76) are satisfied for $\mathbf{A}_k, k > m$, $\forall (l,m) \in L_c$ and $\delta = \delta_c$. In the rest of the proof, we assume that $k > m$. Because $|a_{ll}^k - a_{mm}^k| < \delta_c, \forall (l,m) \in L_c$, $\mathbf{A}_k$ can be partitioned to blocks $\{\mathbf{A}_{ij}^k\}_{i,j=1}^3$, where $\mathbf{A}_{22}^k \in \mathbb{C}^{(n_r-v)\times(n_r-v)}$, $\mathbf{A}_{33}^k \in \mathbb{C}^{v\times v}$, and $\mathbf{A}_{ij}$ and $\mathbf{A}_{qt}$ has the same number of rows if $i = q$ and same number columns if $j = t$. In this partition, the diagonal entries of $\mathbf{A}_{11}^k$ are separated by more than δ_c, and in addition, two diagonal entries where each belongs to a different diagonal block (i.e. $\mathbf{A}_{11}, \mathbf{A}_{22}, \mathbf{A}_{33}$) are also separated by more than δ_c. Now it is possible to apply [23, Lemma 2.3] which asserts that

$$\|\mathbf{A}_{ll}^k\|_{\text{off}} \leq \frac{P_k^2}{2\delta_c}, \text{for } l = 2, 3, \tag{103}$$

where $\|\cdot\|_{\text{off}}$ is the sum of squares of the off-diagonal entries.

To show that (103) establishes (28) we first show that the affiliation of the diagonal entries in the upper $\mathbf{A}_{11}^k$-block remains unchanged and that no diagonal entry leaves the $\mathbf{A}_{22}^k$ and $\mathbf{A}_{33}^k$ blocks; i.e., for $i = 1, 2, 3$

$$R_{k+1}(v) \in L_i \text{ if } v \in L_i, \text{and } R_k(v) = v \text{ if } v \in L_1, \tag{104}$$

$$\text{s.t. } R_k(v) = \arg\min_{l \in L} |\lambda_v - a_{ll}^k| \tag{105}$$

This follows from (92) and because for every k such that $(l_k, m_k) \in L_c$, (78) is satisfied with replacing δ by δ_c. Thus, similar to (93), for every k such that $(l_k, m_k) \in L_s$ $\left|a_{l_k,l_k}^k - a_{l_k,l_k}^{k+1}\right|^2 \leq \frac{\delta_c^2}{4}\left(1 + 4\eta_\theta\right)\left(1/2 + \sqrt{1/2 + \eta_\theta}\right)^2$. By taking $\eta \leq 1/100$ and considering (83) it follows that $|a_{l_k,l_k}^k - a_{l_k,l_k}^{k+1}| \leq 0.65\delta_c$, which establishes (105); and therefore, for every $(l, m) \in L_s$, $|a_{ll}^k - a_{mm}^k| \geq \delta_c$.

Similar to the derivation of (96),

$$\begin{aligned} P_{c_{n_t-v-r}+m} &\leq W(0,m)\left(\tfrac{1}{\delta^2}W(0,m) + \eta\tfrac{n_t^2-n_t}{\delta}{}_c\sqrt{W(0,k)}\right) \\ &+ 4c_{n_t-v-r}\eta^2(5W(0,m) + 4\sqrt{W(0,m)}\|\mathbf{G}\| + \|\mathbf{G}\|^2) + \textstyle\sum_{l=n_t+v+r+1}^{n_t-1} Z(l, c_{n_t-v-r} + m) \end{aligned} \tag{106}$$

and similar to the derivation of (101), we obtain

$$P_{c_{n_t-v-r}+m}^2 \leq O\left(\left(\tfrac{W(0,m)}{\delta_c}\right)^2\right) + O\left(\left(\tfrac{\eta W^{3/2}(0,m)}{\delta_c}\right)\right) + O\left(\left(\tfrac{\eta^2 W^{1/2}(0,m)}{\delta_c}\right)\right) + 2\left(n_t^2 - n_t\right)\eta^2\|\mathbf{G}\|^2$$

Since P_k is a decreasing sequence, the desired result follows.

Appendix D

The corollary follows from Theorem 3 and from the following proposition:

Proposition 11: Let $b > 0, 0 < \rho < 1$ and let a_n be a non-negative sequence that satisfies $a_{n+1} \leq \rho a_n + b,\ \forall n \in \mathbb{N}$, then,$\limsup_n a_n \leq \frac{b}{1-\rho}$.

Proof: We first assume that for some $n \in \mathbb{N}$ $a_n \geq \frac{b}{1-\rho}$. In this case we have $a_{n+1} \leq a_n$ which means that a_n is a monotonic decreasing sequence as long as $a_n \geq \frac{b}{1-\rho}$. In the case where $a_n < \frac{b}{1-\rho}$ we have $a_{n+1} < \frac{b}{1-\rho}$. These mean that either a_n converges to a limit $\xi > \frac{b}{1-\rho}$, or that it satisfies $\limsup_n a_n \leq \frac{b}{1-\rho}$. Assume that the previous statement is true, then for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$

such that $\xi - \epsilon \leq a_n \leq \xi + \epsilon,\ \forall n > n_\epsilon$. By substituting it into $a_{n+1} \leq \rho a_n + b$, i.e., substituting $\xi - \epsilon$ for a_{n+1} and $\xi + \epsilon$ for a_n it follows that for every $\epsilon > 0$, $\xi(1-\rho) \leq b + \epsilon(1+\rho)$. This is equivalent to $\xi \leq \frac{b}{1-\rho} + \frac{\epsilon(1+\rho)}{(1-\rho)},\ \forall \epsilon > 0$ which is a contradiction. □

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