

Randomness Evaluation with the Discrete Fourier Transform Test Based on Exact Analysis of the Reference Distribution

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Abstract

In this paper, we study the problems in the discrete Fourier transform (DFT) test included in NIST SP 800-22 released by the National Institute of Standards and Technology (NIST), which is a collection of tests for evaluating both physical and pseudo-random number generators for cryptographic applications. The most crucial problem in the DFT test is that its reference distribution of the test statistic is not derived mathematically but rather numerically estimated; the DFT test for randomness is based on a pseudo-random number generator (PRNG). Therefore, the present DFT test should not be used unless the reference distribution is mathematically derived. Here, we prove that a power spectrum, which is a component of the test statistic, follows a chi-squared distribution with 2 degrees of freedom. Based on this fact, we propose a test whose reference distribution of the test statistic is mathematically derived. Furthermore, the results of testing non-random sequences and several PRNGs showed that the proposed test is more *reliable* and definitely more *sensitive* than the present DFT test.

Keywords: Computer security, random sequences, statistical analysis

1 Introduction

Random numbers are used in many types of applications, such as cryptography, numerical simulations, and so on. However, it is not easy to generate “truly” random number sequences. Pseudo-random number generators (PRNGs) generate the sequences by iterating some recurrence relation; therefore, the sequences are theoretically not “truly” random. The binary “truly” random sequence is defined as the sequence in which each element has a probability of exactly $\frac{1}{2}$ of being “0” or “1” and in which the elements are statistically independent of

each other. It is also difficult to ascertain if the sequence is truly random; therefore, the randomness of the sequences is evaluated statistically.

NIST SP 800-22 [1, 2] is one of the famous statistical test suites for randomness that was used for selecting the Advanced Encryption Standard (AES) algorithm. NIST SP 800-22 consists of fifteen tests, and every test is hypothesis testing, where the hypothesis is that the input sequence is truly random; if the hypothesis is not rejected in all the tests, it is implied that the input sequences are random. Among the tests included in NIST SP 800-22, the DFT test is of the greatest concern to us. This test detects periodic features of a random number sequence; input sequences are discrete Fourier transformed, and the test statistic is composed of the Fourier coefficients. In 2003, Kim *et al.* [3, 4] reported that the DFT test and the Lempel-Ziv test in the original NIST SP 800-22 [1] have crucial theoretical problems. Regarding the DFT test, it is reported that the test statistic does not follow the expected reference distribution because of the problem that the DFT test regards Fourier coefficients as independent stochastic variables although they are not. Kim *et al.* numerically estimated the distribution of the test statistic with pseudo-random numbers generated with a PRNG and proposed a new DFT test with the estimated distribution. In 2005, Hamano [5] theoretically scrutinized the distribution of the Fourier coefficients in the original DFT test. However, he could not derive the theoretical distribution of the test statistic, but he did make the problems in the DFT test clearer. In 2005, because of these reports, in NIST SP 800-22 version 1.7, the Lempel-Ziv test was deleted, and the DFT test was revised according to the report of Kim *et al.* The DFT test has not subsequently been revised. In 2012, Pareschi *et al.* [6] reviewed three tests included in NIST SP 800-22, and they also numerically estimated the distribution of the test statistic. Consequently, they reported that the distribution estimated by Kim *et al.* is not sufficiently accurate. As stated above, several researchers have attempted to revise the DFT test. However, the distribution of the test statistic has still not been derived theoretically but rather numerically estimated.

In this paper, we review the problems in the DFT test, and we prove three facts, which are important for analyzing the reference distribution of the test statistic: Under the assumption that the input sequence is an ideal random number sequence, when $j \neq 0$,

- The asymptotic distributions of both $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ are the standard normal distribution ($\mathcal{N}(0, 1)$) when $n \rightarrow \infty$.
- When n is sufficiently large, $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ are statistically independent of each other.
- The asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$ is a chi-squared distribution with 2 degrees of freedom (χ_2^2) when $n \rightarrow \infty$.

Here, X is an n -bit binary sequence, $S_j(X)$ is the j -th discrete Fourier coefficient of X , and $c_j(X)$ and $s_j(X)$ are the real and imaginary parts of $S_j(X)$, and they are defined in (1), (2) and (3) in Section 2, respectively. There is no information about these factors in NIST SP800-22, and, to the best of our knowledge, no researchers who have studied the DFT test have ever provided rigorous proofs. These factors are necessary for analyzing the reference

distribution of the test statistic. Furthermore, we propose a new DFT test based on the fact that χ_2^2 is the asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$. By comparing the results of several PRNGs, we show that our test is more *reliable* and definitely more *sensitive* than the present DFT test.

2 Discrete Fourier Transform Test

In this section, we explain the procedure of the original DFT test (DFTT_{original}), released in 2001 [1], before the revision in 2005 [2]. We also explain the problems reported by several researchers [4, 5]. The focus of this test is the peak heights in the discrete Fourier transform of the sequence. The purpose of this test is to detect periodic features in the tested sequence that would indicate a deviation from the assumption of randomness. The intention is to detect whether the number of peaks exceeding the 95 % threshold is significantly different than 5 %.

2.1 The procedure of the original DFT test

- 1) The zeros and ones of the input sequence $E = \{\epsilon_0, \dots, \epsilon_{n-1}\}$ are converted to values of -1 and $+1$ to create the sequence $X = \{x_0, \dots, x_{n-1}\}$, where $x_i = 2\epsilon_i - 1$ ($i \in \{0, \dots, n-1\}$). For simplicity, let n be even.
- 2) Apply a discrete Fourier transform (DFT) to X to produce Fourier coefficients $\{S_j(X)\}_{j=0}^{n-1}$. The Fourier coefficient $S_j(X)$ and its real and imaginary parts $c_j(X)$ and $s_j(X)$ are defined as follows:

$$S_j(X) := \sum_{k=0}^{n-1} x_k \cos \frac{2\pi k j}{n} - \sqrt{-1} \sum_{k=0}^{n-1} x_k \sin \frac{2\pi k j}{n} \quad (1)$$

$$c_j(X) := \sum_{k=0}^{n-1} x_k \cos \frac{2\pi k j}{n} \quad (2)$$

$$s_j(X) := \sum_{k=0}^{n-1} x_k \sin \frac{2\pi k j}{n} \quad (3)$$

- 3) Compute $\{|S_j(X)|\}_{j=0}^{\frac{n}{2}-1}$, where

$$|S_j(X)|^2 = (c_j(X))^2 + (s_j(X))^2.$$

Because $|S_j(X)| = |\overline{S_{n-j}(X)}|$, $\{|S_j(X)|\}_{j=\frac{n}{2}}$ are discarded.

- 4) Compute a threshold value $T_{0.95} = \sqrt{3n}$. The 95% values $\{|S_j(X)|\}_{j=0}^{\frac{n}{2}-1}$ are supposed to be $< T_{0.95}$.

According to SP800-22, $\frac{2}{n}|S_j(X)|^2$ is considered to follow χ_2^2 , and $T_{0.95}$ is defined by the following equation.

$$\begin{aligned} P(|S_j(X)| < T_{0.95}) &= \int_0^{\frac{2}{n}T_{0.95}^2} \frac{1}{2}e^{-\frac{y}{2}} dy \\ &= 1 - e^{-\frac{T_{0.95}^2}{n}} \\ &:= 0.95 \\ \therefore T_{0.95} &= \sqrt{-n \ln(0.05)} \simeq \sqrt{3n} \end{aligned}$$

Several researchers [4, 5] reported that this $T_{0.95} = \sqrt{3n}$ was incorrect, and it was accordingly revised as $T_{0.95} = \sqrt{-n \ln(0.05)}$ in the DFT test in the revised NIST SP800-22 [2].

5) Count

$$N_1 = \# \left\{ |S_j(X)| \mid |S_j(X)| < T_{0.95}, 0 \leq j \leq \frac{n}{2} - 1 \right\}.$$

If $\{|S_j(X)|\}_{j=0}^{\frac{n}{2}-1}$ are mutually independent, then under the assumption of randomness, N_1 can be considered to follow $\mathcal{B}(\frac{n}{2}, 0.95)$, where \mathcal{B} is the binomial distribution.

According to the central limit theorem, when n is sufficiently large, the approximation to $\mathcal{B}(n, p)$ is given by the normal distribution $\mathcal{N}(np, np(1-p))$. Therefore, when n is sufficiently large, under the assumption of randomness,

$$N_1 \sim \mathcal{N} \left(0.95 \frac{n}{2}, (0.95)(0.05) \frac{n}{2} \right).$$

6) Compute a test static

$$d = \frac{N_1 - 0.95 \frac{n}{2}}{\sqrt{(0.95)(0.05) \frac{n}{2}}}.$$

When n is sufficiently large, under the assumption of randomness, the test statistic d can be considered to follow $\mathcal{N}(0, 1)$

7) Compute *P-value*; $p = \text{erfc} \left(\frac{|d|}{\sqrt{2}} \right)$.

If $p < \alpha$, then conclude that the sequence is non-random, where α is a significance level of the DFT test. NIST recommends $\alpha = 0.01$ [2]. Therefore, we also define $\alpha = 0.01$. If $p \geq \alpha$, conclude that the sequence is random.

8) Perform 1) to 7) for m sample sequences $\{X_1, X_2, \dots, X_m\}$; m *P-values* $\{p_1, p_2, \dots, p_m\}$ are computed.

- 9) (Second-level test I: Proportion of sequences passing a test)

Count the number of sample sequences for which $P\text{-value} \geq \alpha$ and define it as m_p . Then, under the assumption of randomness, m_p follows $\mathcal{B}(m, 1-\alpha)$, which approximates $\mathcal{N}(m(1-\alpha), m\alpha(1-\alpha))$ when m is sufficiently large. Therefore, the proportion of sequences passing a test ($= m_p/m$) approximately follows $\mathcal{N}\left((1-\alpha), \frac{\alpha(1-\alpha)}{m}\right)$. The range of acceptable m_p/m is determined using the significance interval defined as

$$1 - \alpha - 3\sqrt{\frac{\alpha(1-\alpha)}{m}} < \frac{m_p}{m} < 1 - \alpha + 3\sqrt{\frac{\alpha(1-\alpha)}{m}}. \quad (4)$$

If the proportion falls outside of this interval, there is evidence that the data are non-random.

- 10) (Second-level test II: Uniform distribution of $P\text{-values}$)

Uniformity may also be determined by applying a χ^2 test and determining a $P\text{-value}$ corresponding to the goodness-of-fit distributional test on the $P\text{-values}$ obtained for an arbitrary statistical test (i.e., the $P\text{-value}$ of the $P\text{-values}$). This is performed by computing

$$\chi^2 = \sum_{i=1}^{10} \frac{(F_i - m/10)^2}{m/10},$$

where F_i is the number of $P\text{-values}$ in sub-interval i . A $P\text{-value}$ P_T is calculated such that

$$P_T = \text{igamc}\left(\frac{9}{2}, \frac{\chi^2}{2}\right),$$

where igamc is the complementary incomplete gamma function. If

$$P_T \geq \alpha_{II} (:= 0.0001), \quad (5)$$

the sequences can be considered to be uniformly distributed, where α_{II} is the significance level for P_T .

- 11) If the set of $P\text{-values}$ $\{p_1, p_2, \dots, p_m\}$ passes both 9) and 10), the physical or pseudo-random number generators that generated the input sequences are concluded to be ideal.

2.2 The fundamental problems of the original and present DFT tests

Kim *et al.* [4] and Hamano [5] reported the following:

- The test statistic $d := \frac{N_1 - 0.95 \frac{n}{2}}{\sqrt{(0.95)(0.05) \frac{n}{2}}}$ does not follow $\mathcal{N}(0, 1)$;

- N_1 does not follow $\mathcal{N}\left(0.95\frac{n}{2}, (0.95)(0.05)\frac{n}{2}\right)$.

Furthermore, Kim *et al.*, using Secure Hash Generator (G-SHA1) [2] as a PRNG, estimated that

$$\begin{aligned} N_1 &\sim \mathcal{N}\left(0.95\frac{n}{2}, (0.95)(0.05)\frac{n}{4}\right); \\ d_{kim} &:= \frac{N_1 - 0.95\frac{n}{2}}{\sqrt{(0.95)(0.05)\frac{n}{4}}} \sim \mathcal{N}(0, 1), \end{aligned}$$

and $\text{DFTT}_{\text{original}}$ was revised according to this report of Kim *et al.* [2]; the present DFT test, denoted as $\text{DFTT}_{\text{present}}$, has not been revised since then. Therefore, the reference distribution of the test statistic of $\text{DFTT}_{\text{present}}$ is not mathematically derived. Furthermore, Pareschi *et al.* reported that the numerical estimation is *not* sufficiently accurate; they numerically estimated that

$$\begin{aligned} N_1 &\sim \mathcal{N}\left(0.95\frac{n}{2}, (0.95)(0.05)\frac{n}{3.8}\right); \\ d_{pareschi} &:= \frac{N_1 - 0.95\frac{n}{2}}{\sqrt{(0.95)(0.05)\frac{n}{3.8}}} \sim \mathcal{N}(0, 1). \end{aligned}$$

Moreover, Pareschi *et al.* proposed that the DFT test with this test statistic ($\text{DFTT}_{\text{pareschi}}$) is more *reliable*. (The definition of the *reliability* of a test is discussed in Section 5.) Therefore, it can be considered that $\text{DFTT}_{\text{present}}$ still has errors. First, $\text{DFTT}_{\text{present}}$ and $\text{DFTT}_{\text{pareschi}}$ are performed based on a PRNG, whose randomness should be evaluated with a randomness test; they cannot be used unless the reference distribution is mathematically derived.

As stated in step 5) in Section 2.1, $\{|S_j(X)|\}_{j=0}^{\frac{n}{2}-1}$ are considered to be mutually independent. However, $\{|S_j(X)|\}_{j=0}^{\frac{n}{2}-1}$ are not mutually independent, and this problem is expected to be the main factor for why N_1 does not follow $\mathcal{N}\left(0.95\frac{n}{2}, (0.95)(0.05)\frac{n}{2}\right)$ [4, 5]. Furthermore, before considering this problem, it is also necessary to ensure that $\frac{2}{n}|S_j(X)|^2$ follows χ_2^2 . Although $\frac{2}{n}|S_j(X)|^2$ is considered to follow χ_2^2 in step 4) in Section 2.1, there is no information about this in SP800-22, and no researchers studying the DFT test have ever provided rigorous proofs to the best of our knowledge. We provide a proof for the DFT test in Section 3.

3 The asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$

In this section, we analyze the asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$. From the definition of $|S_j(X)|$ in (1),

$$\frac{2}{n}|S_j(X)|^2 = \left(\sqrt{\frac{2}{n}}c_j(X)\right)^2 + \left(\sqrt{\frac{2}{n}}s_j(X)\right)^2.$$

When $j = 0$,

$$\frac{2}{n}|S_0(X)|^2 = 2 \left(\frac{\sum_{k=0}^{n-1} x_k}{\sqrt{n}} \right)^2.$$

Under the assumption that X is an ideal random number sequence, $P(x_k = -1) = P(x_k = 1) = \frac{1}{2}$ and $\{x_k\}_{k=0}^{n-1}$ are mutually independent, and $E[x_k] = 0, V[x_k] = 1$. Therefore, as a consequence of the central limit theorem, when n is sufficiently large, $\left(\frac{\sum_{k=0}^{n-1} x_k}{\sqrt{n}}\right)$ follows $\mathcal{N}(0, 1)$, and $\left(\frac{\sum_{k=0}^{n-1} x_k}{\sqrt{n}}\right)^2$ follows a chi-squared distribution with 1 degree of freedom (χ_1^2). Thus, $\frac{2}{n}|S_0(X)|^2$ does not follow χ_2^2 .

In the following, we consider the case when $j \neq 0$. Here, $\frac{2}{n}|S_j(X)|^2$ follows χ_2^2 if the following is true:

- Both $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ follow $\mathcal{N}(0, 1)$.
- $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ are mutually independent.

In the following 2 subsections, we prove the following Theorem 1, Theorem 2 and Theorem 3:

Theorem 1: When n is sufficiently large, both $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ follow $\mathcal{N}(0, 1)$.

Theorem 2: When n is sufficiently large, $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ are mutually independent.

Theorem 3: $\frac{2}{n}|S_j(X)|^2$ follows χ_2^2 when n is sufficiently large.

From the definition of χ_2^2 , Theorem 3 can be proven by combing Theorem 1 and Theorem 2.

3.1 Proof of Theorem 1: The asymptotic distribution of $\sqrt{\frac{2}{n}}c_j(X)$

In this subsection, we prove Theorem 1. Hamano [5] showed that the average, variance, skewness, and kurtosis of $c_j(X)$ and $\mathcal{N}(0, \frac{n}{2})$ are the same. However, it cannot be proven that $\mathcal{N}(0, \frac{n}{2})$ is the asymptotic distribution of $c_j(X)$ based only on these factors.

$\sqrt{\frac{2}{n}}c_j(X)$ is expressed as $\sqrt{\frac{2}{n}}c_j(X) := \sqrt{\frac{2}{n}} \sum_{k=0}^{n-1} x_k a_{k,j}$, where $a_{k,j} = \cos \frac{2\pi k j}{n}$. Under the assumption that X is an ideal random number sequence, the characteristic function of

$\sqrt{\frac{2}{n}}c_j(X)$ denoted by $\phi(t)$ is expressed as follows:

$$\begin{aligned}
\phi(t) &= E_X \left[\exp \left(\sqrt{\frac{2}{n}} \sqrt{-1} t c_j(X) \right) \right] \\
&= E_X \left[\prod_{k=0}^{n-1} \exp \left(\sqrt{\frac{2}{n}} \sqrt{-1} t x_k a_{k,j} \right) \right] \\
&= \prod_{k=0}^{n-1} E_{x_k} \left[\exp \left(\sqrt{\frac{2}{n}} \sqrt{-1} t x_k a_{k,j} \right) \right] \\
&= \prod_{k=0}^{n-1} \cos \left(\sqrt{\frac{2}{n}} t a_{k,j} \right). \\
\therefore \log \phi(t) &= \sum_{k=0}^{n-1} \log \cos \left(\sqrt{\frac{2}{n}} t a_{k,j} \right),
\end{aligned}$$

where

$$\begin{aligned}
E_X(\cdot) &:= \frac{1}{2^n} \sum_{X \in \{-1,1\}^n} (\cdot), \\
E_{x_k}(\cdot) &:= \frac{1}{2} \sum_{x_k \in \{-1,1\}} (\cdot).
\end{aligned}$$

Using the Taylor expansion about a point $t = 0$, we obtain

$$\begin{aligned}
\log \cos \left(\sqrt{\frac{2}{n}} t a_{k,j} \right) &= -\frac{1}{n} a_{k,j}^2 t^2 - \frac{1}{3n^2} a_{k,j}^4 t^4 + O(t^6). \\
\therefore \log \phi(t) &= -\frac{1}{n} \sum_{k=0}^{n-1} a_{k,j}^2 t^2 - \frac{1}{3n^2} \sum_{k=0}^{n-1} a_{k,j}^4 t^4 + O(t^6).
\end{aligned}$$

Since

$$\sum_{k=0}^{n-1} a_{k,j}^2 = \frac{n}{2}, \quad \sum_{k=0}^{n-1} a_{k,j}^{2l} \leq n \quad (l \in \{1, 2, 3, \dots\}),$$

$$\lim_{n \rightarrow \infty} \log \phi(t) = -\frac{1}{2} t^2. \quad \therefore \lim_{n \rightarrow \infty} \phi(t) = e^{-\frac{1}{2} t^2}.$$

Thus, $\mathcal{N}(0, 1)$ is the asymptotic distribution of $\sqrt{\frac{2}{n}}c_j(X)$. Likewise, it can be proven that $\mathcal{N}(0, 1)$ is the asymptotic distribution of $\sqrt{\frac{2}{n}}s_j(X)$.

3.2 Proof of Theorem 2: Statistical independence of $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$

In this subsection, we prove Theorem 2. Let us define a 2-dimensional stochastic variable \mathbf{Y} as the following equation:

$$\mathbf{Y} := (Y_1, Y_2) := \left(\sqrt{\frac{2}{n}}c_j(X), \sqrt{\frac{2}{n}}s_j(X) \right).$$

Under the assumption that X is an ideal random number sequence, the characteristic function of \mathbf{Y} denoted by $\psi(\mathbf{t})$ is expressed as follows:

$$\begin{aligned} \psi(\mathbf{t}) &= E_X[\exp(\sqrt{-1}\mathbf{t}\mathbf{Y}^\top)] \\ &= E_X \left[\exp \left(\sqrt{\frac{2}{n}}\sqrt{-1}(t_1c_j(X) + t_2s_j(X)) \right) \right] \\ &= \prod_{k=0}^{n-1} \cos \left(\sqrt{\frac{2}{n}}(t_1a_{k,j} + t_2b_{k,j}) \right), \end{aligned}$$

where

$$\mathbf{t} = (t_1, t_2), \quad a_{k,j} = \cos \frac{2\pi kj}{n}, \quad b_{k,j} = \sin \frac{2\pi kj}{n}.$$

Therefore,

$$\log \psi(\mathbf{t}) = \sum_{k=0}^{n-1} \log \cos \left(\sqrt{\frac{2}{n}}(a_{k,j}t_1 + b_{k,j}t_2) \right).$$

Using the Taylor expansion about a point $\mathbf{t} = \mathbf{0}$, we obtain

$$\begin{aligned} &\log \cos \left(\sqrt{\frac{2}{n}}(a_k t_1 + b_k t_2) \right) \\ &= -\frac{(a_{k,j}t_1 + b_{k,j}t_2)^2}{n} - \frac{(a_{k,j}t_1 + b_{k,j}t_2)^4}{3n^2} + \dots \end{aligned}$$

Since

$$\sum_{k=0}^{n-1} a_k^2 = \sum_{k=0}^{n-1} b_k^2 = \frac{n}{2}, \quad \sum_{k=0}^{n-1} a_k b_k = 0, \quad \sum_{k=0}^{n-1} a_k^l b_k^m \leq n \quad (l, m \geq 0),$$

we obtain

$$\lim_{n \rightarrow \infty} \log \psi(\mathbf{t}) = -\frac{\mathbf{t}\mathbf{t}^\top}{2}, \quad \therefore \lim_{n \rightarrow \infty} \psi(\mathbf{t}) = \exp \left(-\frac{\mathbf{t}\mathbf{t}^\top}{2} \right).$$

Therefore, when n is sufficiently large, the joint probability distribution function is described as follows:

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi} \exp\left(-\frac{y_1^2 + y_2^2}{2}\right).$$

As we proved before, $\mathcal{N}(0, 1)$ is the asymptotic distribution of both Y_1 and Y_2 . Thus, when n is sufficiently large, the probability distribution functions of Y_1 and Y_2 are $f_{Y_1}(y_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_1^2}{2}\right)$ and $f_{Y_2}(y_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_2^2}{2}\right)$, respectively. Therefore, when n is sufficiently large, the following equation is obtained:

$$f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2).$$

This means that $\sqrt{\frac{2}{n}}c_j(X)$ and $\sqrt{\frac{2}{n}}s_j(X)$ are *mutually independent* when n is sufficiently large.

4 The proposed DFT test

In Section 3, we proved Theorem 3, stating that $\frac{2}{n}|S_j(X)|^2 (j \neq 0)$ follows χ_2^2 when n is sufficiently large. Therefore, if $\{|S_j(X)|\}_{j=1}^{\frac{n}{2}-1}$ are mutually independent, we can consider that N_1 follows $\mathcal{N}\left(0.95\frac{n}{2}, (0.95)(0.05)\frac{n}{2}\right)$. However, $\{|S_j(X)|\}_{j=1}^{\frac{n}{2}-1}$ are not mutually independent. Therefore, it is necessary to mathematically analyze the distribution of the test statistic d under the condition that $\{|S_j(X)|\}_{j=1}^{\frac{n}{2}-1}$ are not mutually independent. Hamano [5] attempted to mathematically derive the distribution of the set $\{|S_j(X)|\}_{j=1}^{\frac{n}{2}-1}$, but he could not do so, and we also could not derive this distribution. However, we rigorously proved that the asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$ is χ_2^2 , and we develop the new DFT test (DFTT_{proposed}) based on this fact. The reference distribution of the test statistic of DFTT_{proposed} is mathematically derived, whereas that of DFTT_{present} is estimated with a PRNG. We explain the test statistic of DFTT_{proposed} in the next subsection.

4.1 The procedure of the proposed DFT test

In the standard approach in NIST SP800-22, each sequence is analyzed; thus, m sequences give m *P-values*. However, DFTT_{proposed} generates $\frac{n}{2} - 1$ (n : length of a sequence) *P-values*. Therefore, more *P-values* are generated since n is generally larger than m . Since the number of *P-values* should not be too large (see Section 5.3), before conducting DFTT_{proposed}, it is necessary to adjust the length of the sequences and make them into more sets of short sequences (see also Table 5), assuming that the set input sequences are continuously generated by an RNG. Therefore, DFTT_{proposed} is theoretically not appropriate for the isolated set of sequences.

The procedure of the proposed DFT test is described as follows:

- 1) The zeros and ones of the m n -length input sequence $\{E_i = \{\epsilon_0^i, \dots, \epsilon_{n-1}^i\}\}_{i=1}^m$ are converted to values of -1 and $+1$ to create the sequence $\{X^i = \{x_0^i, \dots, x_{n-1}^i\}\}_{i=0}^m$, where $x_j^i = 2\epsilon_j - 1$ ($j \in \{0, \dots, n-1\}$). For simplicity, let n be even.
- 2) Apply a discrete Fourier transform (DFT) to each X^i to produce Fourier coefficients $\{S_j(X^i)\}_{j=0}^{n-1}$. The Fourier coefficient $S_j(X^i)$ and its real and imaginary parts $c_j(X^i)$ and $s_j(X^i)$ are defined as follows:

$$\begin{aligned}
S_j(X^i) &:= \sum_{k=0}^{n-1} x_k \cos \frac{2\pi k j}{n} - \sqrt{-1} \sum_{k=0}^{n-1} x_k \sin \frac{2\pi k j}{n}, \\
c_j(X^i) &:= \sum_{k=0}^{n-1} x_k \cos \frac{2\pi k j}{n}, \\
s_j(X^i) &:= \sum_{k=0}^{n-1} x_k \sin \frac{2\pi k j}{n},
\end{aligned}$$

- 3) For all $j \in \{1, \dots, \frac{n}{2} - 1\}$, perform the Kolmogorov-Smirnov (KS) test [8, 9] on the empirical cumulative distribution function of $\{\frac{2}{n}S_j(X_i)\}_{i=1}^m$ defined as $F_m^j(y)$ based on the difference from χ_2^2 and compute the P -value p_j . Here, the KS statistic D_m^j and p_j are defined as follows.

$$\begin{aligned}
D_m^j &= \sqrt{m} \max_{y>0} |F_m^j(y) - F(y)|, \\
p_j &= 1 - H(D_m^j),
\end{aligned}$$

where $H(y)$ is the cumulative distribution function of the Kolmogorov-Smirnov distribution:

$$H(y) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2 y}.$$

Note that $\frac{n}{2} - 1$ P -values $\{p_1, p_2, \dots, p_{\frac{n}{2}-1}\}$ are computed in this step, while the $\text{DFTT}_{\text{present}}$ computes m P -values.

- 4) Perform the second-level tests I and II defined in the original DFT test (see Section 2.1-9, 2.1-10). If the set of P -values $\{p_1, p_2, \dots, p_{\frac{n}{2}-1}\}$ passes both second-level tests I and II, the physical or pseudo-random number generator that generated the input sequences is concluded to be ideal.

5 Experiments

In this section, we explain the experiments that we performed and the conclusions derived from their results. In these experiments, we compare the *reliability* and *sensitivity* of $\text{DFTT}_{\text{present}}$ and $\text{DFTT}_{\text{proposed}}$. The *reliability* of tests means a low probability of *false*

Table 1: Types of error

\mathcal{H}_0 : Null hypothesis = “generator is ideal”		\mathcal{H}_0 is	
		True	False
Judgment of \mathcal{H}_0	Reject	False Positive (Type I error)	True Positive
	Fail to reject	True Negative	False Negative (Type II error)

positives (type I error) (see Table 1), and the *sensitivity* of tests means a low probability of *false negatives* (type II error). Now, the null hypothesis of the tests (\mathcal{H}_0) is that the “generator is ideal”. Therefore, a *false positive* (type I error) means an erroneous identification of an ideal generator as not random, and a *false negative* (type II error) means an erroneous identification of a generator that is not ideal as random. Comparing the probability of type I error and type II error, we can conclude which test is better.

For simplicity, in this experiment, we modify the significance interval of the second-level test I defined in (4) as follows:

$$1 - \alpha - 2.575\sqrt{\frac{\alpha(1 - \alpha)}{m}} < \frac{m_p}{m} < 1 - \alpha + 2.575\sqrt{\frac{\alpha(1 - \alpha)}{m}}. \quad (6)$$

With this modified significance interval, the significance level of the second-level test I ($:= \alpha_I$) is modified to be $\alpha_I = 0.01$.

5.1 Experiment 1: Test results for periodic sequences

In this experiment, we compare the *sensitivity* of $\text{DFTT}_{\text{present}}$ and $\text{DFTT}_{\text{pareschi}}$. *Sensitivity* means a low false negative rate (low probability of type I error), i.e., high true positive rate. Here, we compare the true positive rate of each test result.

$$\begin{aligned} \textit{Sensitivity} &:= \text{low probability of type II error} \\ &= \text{low false negative rate} \\ &= \text{high true positive rate} \end{aligned}$$

Now, we define an nm -length input sequence $\mathcal{X}_{n,m}$ as

$$\begin{aligned} \mathcal{X}_{n,m} &:= \{x_0, x_1, x_2, \dots, x_{mn-1}\} \\ &= \{X_1^n, X_2^n, \dots, X_m^n\}, \end{aligned}$$

where

$$\begin{aligned} X_i^n &= \{x_{(i-1)n}, \dots, x_{in-1}\} \quad (i = 1, 2, \dots, m), \\ x_k &\in \{-1, 1\}. \quad (k = 1, 2, \dots, mn - 1). \end{aligned}$$

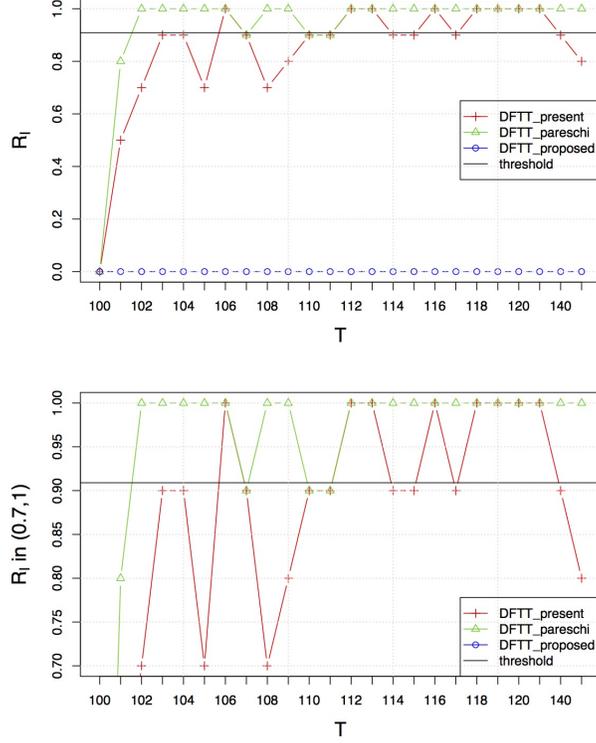


Figure 1: Passing rate R_I in experiment 1. The “threshold” means the lower limit of the significance interval defined in Eq. (7)

We purposely create non-random (periodic) sequences from the mn -length sequence $\mathcal{X}_{n,m}$ using the method described as follows:

$$x_k = \begin{cases} -1 & (k \bmod T = 0 \ \& \ k \bmod 2T = 0) \\ 1 & (k \bmod T = 0 \ \& \ k \bmod 2T \neq 0) \end{cases}.$$

Therefore,

$$\begin{aligned} \mathcal{X}_{n,m}^T &:= \{x_0, \dots, x_T, \dots, x_{2T}, \dots, x_{3T}, \dots, x_{4T}, \dots, x_{mn-1}\} \\ &= \{x_0, \dots, -1, \dots, 1, \dots, -1, \dots, 1, \dots, x_{mn-1}\}. \end{aligned}$$

We can clearly state this sequence is a non-random sequence. Therefore, if the test does not reject the \mathcal{H}_0 (=null hypothesis: “generator is random”), then it is a false negative (type II error).

For each $T \in \{100, 101, 102, \dots, 120, 130, 140, 150\}$, we use 10 sets of an mn -length ($nm = 100,000,000$) input sequence $\mathcal{X}_{n,m}$ generated by the Mersenne Twister algorithm [10] and covert them to non-random mn -length sequences $\mathcal{X}_{n,m}^T$. Table 5 in Section 5.3 shows the parameters n and m for each test. In Section 5.3, we explain why the parameters n and m for $\text{DFTT}_{\text{proposed}}$ are different from the other tests. Note that mn is the same. Table 2, Fig.

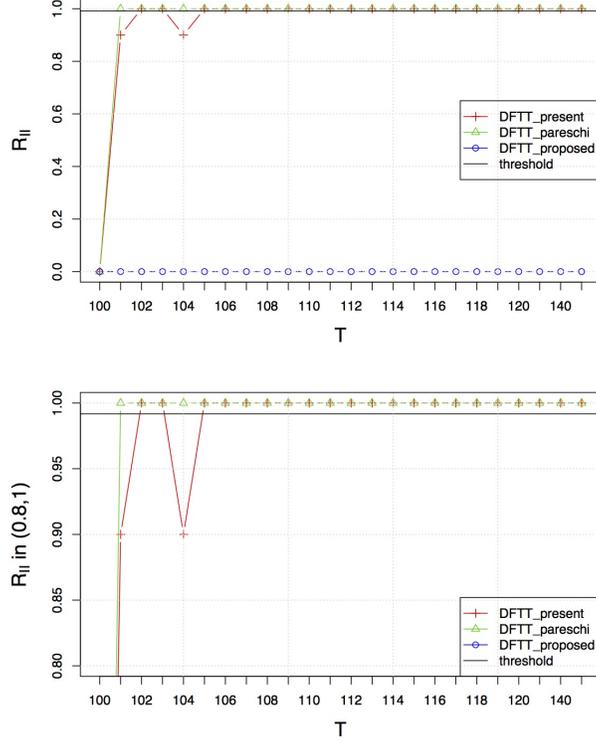


Figure 2: Passing rate R_{II} in experiment 1. The “threshold” means the lower limit of the significance interval defined in Eq. (8)

1 and Fig. 2 show the *passing rate* $R_{I(II)}$, which is defined as follows:

$$\begin{aligned}
 & \text{Passing Rate : } R_{I(II)} \\
 & := \frac{\text{number of } \mathcal{X}_{n,m}^T \text{ passing the second-level test I (II)}}{10}. \\
 & = \begin{cases} \text{True negative rate} & (\text{if } \mathcal{H}_0 = \text{TRUE}) \\ \text{False negative (type II error) rate} & (\text{if } \mathcal{H}_0 = \text{FALSE}) \end{cases}
 \end{aligned}$$

Because we know that $\mathcal{X}_{n,m}^T$ is non-random, we know that $\mathcal{H}_0 = \text{FALSE}$, and the passing rate means a false negative rate in this experiment. Now, the significance levels of second-level tests I and II are $\alpha_I (= 0.01)$ and $\alpha_{II} (= 0.0001)$ (defined in (5)), respectively. Therefore, the significance intervals defined in Eq. (6) of R_I and R_{II} are described as follows:

$$\begin{aligned}
 & \left(1 - \alpha_I - 2.575 \sqrt{\frac{\alpha_I(1 - \alpha_I)}{10}}, 1 - \alpha_I + 2.575 \sqrt{\frac{\alpha_I(1 - \alpha_I)}{10}} \right) \\
 & \simeq (0.991, 1.07)^*, \tag{7}
 \end{aligned}$$

$$\left(1 - \alpha_{II} - 2.575\sqrt{\frac{\alpha_{II}(1 - \alpha_{II})}{10}}, 1 - \alpha_{II} + 2.575\sqrt{\frac{\alpha_{II}(1 - \alpha_{II})}{10}}\right) \simeq (0.9992, 1.008)^*. \quad (8)$$

Therefore, if $R_I < 0.991$ or $R_{II} < 0.9992$, we can conclude that the true positive rate is high, and we can conclude that the test is *sensitive*.

As shown in Table 2, Fig. 1 and Fig. 2, R_I and R_{II} of $\text{DFTT}_{\text{proposed}}$ are all 0.0%, whereas $R_{I(II)}$ of $\text{DFTT}_{\text{present}}$ and $\text{DFTT}_{\text{pareschi}}$ are not as low. From this table and the figures, we can conclude that $\text{DFTT}_{\text{proposed}}$ is more *sensitive* than the other tests.

5.2 Experiment 2: Test results for existing pseudo-random number generators

We use 1000 sets of an mn -length ($mn = 100,000,000$) $\mathcal{X}_{n,m}$ input sequence generated by

- AES Counter Mode (AES-CTR) [11],
- Mersenne Twister [10],
- Xorshift random number generator [12],
- Vector Stream Cipher 2.0 (VSC 2.0) [13],
- Linear congruential generator (LCG) [2],
- Cubic congruential generator (CCG) [2],
- Quadratic congruential generator I (QCG-I) [2],
- Quadratic congruential generator II (QCG-II) [2],
- Micali-Schnorr random bit generator [2].

VSC 2.0 is a stream cipher based on chaos theory, which was proposed by A. Iwasaki and K. Umeno [13]. We test these PRNGs using both the DFT and MS-DFT tests, and we compare the results. The parameter sets of n and m are the same as Table 5 in Section 5.3.

Now, the significance levels of second-level tests I and II are $\alpha_I := 0.01$ and $\alpha_{II} := 0.0001$, respectively, and in this experiment, 1000 mn -length sequences generated by each PRNG are

*These significance intervals range through 1, although R_I and $R_{II} \in [0, 1]$. This is because the number of sets of input sequences in this experiment is 10, and it is too small to provide a good approximation (see Section 2.1-9). Furthermore, $\alpha_{II} (= 0.0001)$ is very small, so the significance interval of R_{II} often ranges through 1.

Table 2: Test results for periodic sequences: passing rate R_I and R_{II} for each T (red cell means that the $R_{I(II)}$ lies outside its significance interval)

Test	DFTT _{present}		DFTT _{pareschi}		DFTT _{proposed}	
	R_I	R_{II}	R_I	R_{II}	R_I	R_{II}
$T = 100$	0.0	0.0	0.0	0.0	0.0	0.0
$T = 101$	0.5	0.9	0.8	1.0	0.0	0.0
$T = 102$	0.7	1.0	1.0	1.0	0.0	0.0
$T = 103$	0.9	1.0	1.0	1.0	0.0	0.0
$T = 104$	0.9	0.9	1.0	1.0	0.0	0.0
$T = 105$	0.7	1.0	1.0	1.0	0.0	0.0
$T = 106$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 107$	0.9	1.0	0.9	1.0	0.0	0.0
$T = 108$	0.7	1.0	1.0	1.0	0.0	0.0
$T = 109$	0.8	1.0	1.0	1.0	0.0	0.0
$T = 110$	0.9	1.0	0.9	1.0	0.0	0.0
$T = 111$	0.9	1.0	0.9	1.0	0.0	0.0
$T = 112$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 113$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 114$	0.9	1.0	1.0	1.0	0.0	0.0
$T = 115$	0.9	1.0	1.0	1.0	0.0	0.0
$T = 116$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 117$	0.9	1.0	1.0	1.0	0.0	0.0
$T = 118$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 119$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 120$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 130$	1.0	1.0	1.0	1.0	0.0	0.0
$T = 140$	0.9	1.0	1.0	1.0	0.0	0.0
$T = 150$	0.8	1.0	1.0	1.0	0.0	0.0

tested. Table 3, Fig. 3 and Fig. 4 show the *passing rate* $R_{I(II)}$, defined as follows:

$$\begin{aligned}
 & \text{Passing Rate : } R_{I(II)} \\
 := & \frac{\text{number of } \mathcal{X}_{n,m} \text{ passing the second-level test I (II)}}{1000} \\
 = & \begin{cases} \text{True negative rate} & (\text{if } \mathcal{H}_0 = \text{TRUE}) \\ \text{False negative (type II error) rate} & (\text{if } \mathcal{H}_0 = \text{FALSE}) \end{cases}
 \end{aligned}$$

Now, the significance intervals (99%) of passing rates R_I and R_{II} are described as,

$$\begin{aligned}
 & \left(1 - \alpha_I - 2.575 \sqrt{\frac{\alpha_I(1 - \alpha_I)}{1000}}, 1 - \alpha_I + 2.575 \sqrt{\frac{\alpha_I(1 - \alpha_I)}{1000}} \right) \\
 \simeq & (0.9819, 0.9982), \tag{9}
 \end{aligned}$$

$$\begin{aligned} & \left(1 - \alpha_{II} - 2.575\sqrt{\frac{\alpha_{II}(1 - \alpha_{II})}{1000}}, 1 - \alpha_{II} + 2.575\sqrt{\frac{\alpha_{II}(1 - \alpha_{II})}{1000}} \right) \\ & \simeq (0.9991, 1.0007)^*, \end{aligned} \tag{10}$$

respectively.

In this experiment, \mathcal{H}_0 for each PRNG is defined as follows:

- \mathcal{H}_0 is TRUE (considered as random): AES-CTR, Mersenne-Twister, Xorshift, VSC 2.0, LCG (Define them as “good PRNGs”).

Because these PRNGs pass all the tests included in NIST SP800-22 [2, 13], we consider them as random in this experiment.

- \mathcal{H}_0 is FALSE (considered as non-random): Micali-Schnorr random bit generator, QCG-I, QCG-II, CCG (Define them as “bad PRNGs”).

Because these PRNGs are rejected by several tests included in NIST SP800-22 [2], we consider them as non-random in this experiment.

Under the assumption that this definition of \mathcal{H}_0 is appropriate, let us consider the *sensitivity* and *reliability* of $\text{DFTT}_{\text{present}}$, $\text{DFTT}_{\text{pareschi}}$ and $\text{DFTT}_{\text{proposed}}$. As shown in Fig. 4, it is difficult to compare the *reliability* from the figure. This is because $R_{II}(= 0.0001)$ is very small, whereas the number of sets of input sequences is 1000. Therefore, in this experiment, we focus on Fig. 3 and derive the conclusion of this experiment as follows.

- *Reliability*; R_I of “good PRNGs” (AES-CTR, Mersenne-Twister, Xorshift, VSC 2.0, and LCG).

If the R_I of “good PRNGs” lies inside its significance interval, we can conclude that the *reliability* of the test is sufficiently high.

As shown in Fig. 3, the R_I of “good PRNGs” of $\text{DFTT}_{\text{proposed}}$ and $\text{DFTT}_{\text{pareschi}}$ lies inside its significance interval, whereas that of $\text{DFTT}_{\text{present}}$ is lower than the threshold. Therefore, we can conclude that the *reliabilities* of $\text{DFTT}_{\text{pareschi}}$ and $\text{DFTT}_{\text{proposed}}$ are sufficiently high. Moreover, we can conclude that the *reliability* of $\text{DFTT}_{\text{present}}$ is low.

- *Sensitivity*; the R_I of “bad PRNGs” (Micali- Schnorr random bit generator, QCG-I, QCG-II, and CCG).

If the R_I of “bad PRNGs” lies lower than the threshold, we can conclude that the *sensitivity* of the test is the highest.

As shown in Fig. 3, except for the Micali-Schnorr random bit generator, the R_I of “bad PRNGs” of $\text{DFTT}_{\text{proposed}}$ are definitely lower than the other tests. The R_I of $\text{DFTT}_{\text{present}}$ are also low, but not as low as $\text{DFTT}_{\text{proposed}}$, and the R_I of $\text{DFTT}_{\text{pareschi}}$ are higher than the R_I of $\text{DFTT}_{\text{present}}$. Therefore, we can conclude that the *reliability* of $\text{DFTT}_{\text{proposed}}$ is definitely high, that of $\text{DFTT}_{\text{present}}$ is high, and that of $\text{DFTT}_{\text{pareschi}}$ is low.

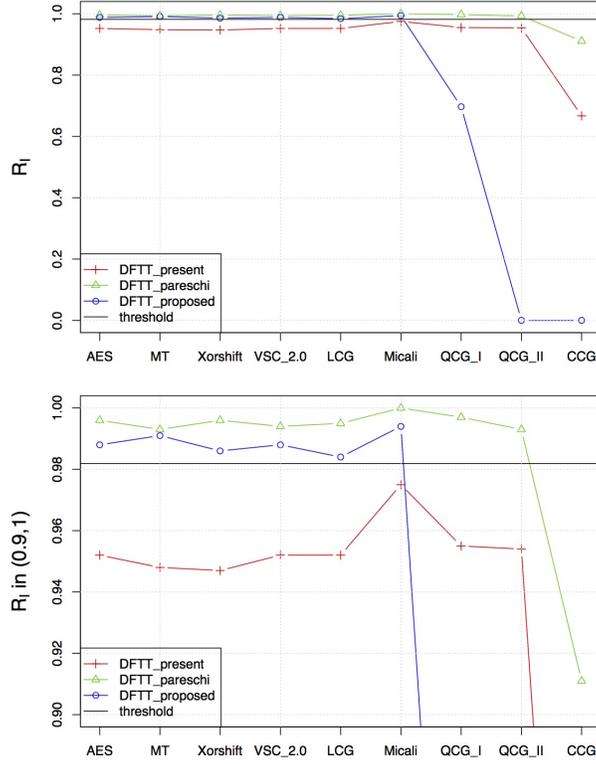


Figure 3: Passing rate R_I in experiment 2. The “threshold” means the lower limit of the significance interval defined in Eq. (9)

Table 3: Test results for existing pseudo-random number generators: Passing rates R_I and R_{II} of each PRNG (red cells mean that the $R_{I(II)}$ lies outside its significance interval)

Test	DFTT _{present}		DFTT _{pareschi}		DFTT _{proposed}	
	R_I	R_{II}	R_I	R_{II}	R_I	R_{II}
AES-CTR	0.95	0.995	0.996	1.000	0.988	1.000
Mersenne-Twister	0.94	0.996	0.993	1.000	0.991	1.000
Xorshift	0.94	0.989	0.996	1.000	0.986	1.000
VSC 2.0	0.95	0.998	0.994	1.000	0.988	1.000
LCG	0.95	0.995	0.99	0.998	0.98	0.999
Micali-Schnorr	0.97	0.993	1.000	1.000	0.994	1.000
QCG-I	0.95	0.994	0.997	1.000	0.69	0.991
QCG-II	0.95	0.993	0.99	0.998	0.00	0.000
CCG	0.66	0.900	0.91	0.995	0.00	0.000

These conclusions from the aforementioned experiment are summarized in Table 4. We can conclude that DFTT_{proposed} is more *reliable* and definitely more *sensitive* than DFTT_{present}.

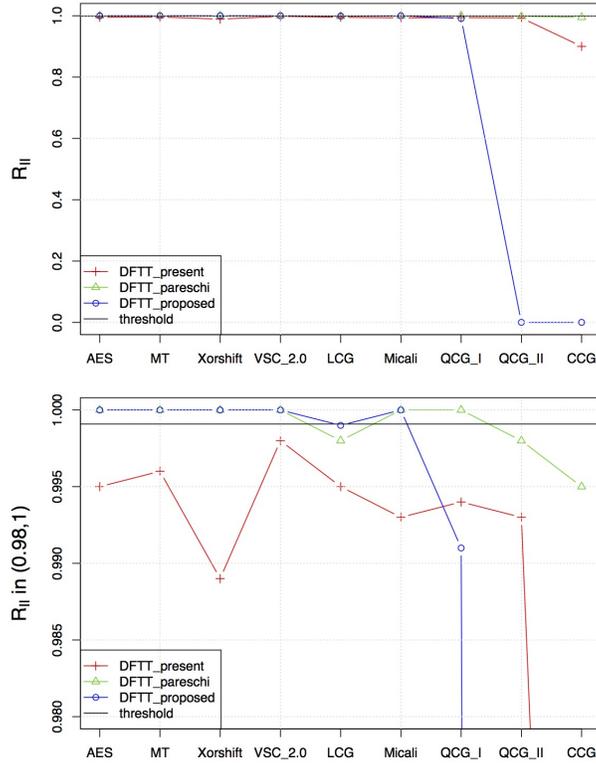


Figure 4: Passing rate R_{II} in experiment 2. The “threshold” means the lower limit of the significance interval defined in Eq. (10)

Table 4: Summary of the conclusions derived from experiments 1 and 2

Test	DFTT _{present}	DFTT _{pareschi}	DFTT _{proposed}
<i>Reliability</i>	low	high enough	high enough
<i>Sensitivity</i>	high	low	definitely high

5.3 Appropriate selection of n and m

As shown in Table 5, the parameters n and m of DFTT_{proposed} are different from the other tests. NIST SP800-22 recommends $n = 1,000,000$ and $m = 1,000$ [2] (in experiments 1 and 2, we defined $n = 100,000$ and $m = 1,000$ for DFTT_{present} and DFTT_{pareschi} to avoid excessive computation because we need 10 and 1000 of mn -length sequences, respectively). However, as we stated in Step 3) in Section 4.1, in the procedure of DFTT_{proposed}, $\frac{n}{2} - 1$ P -values are generated, whereas DFTT_{present} and DFTT_{pareschi} generate m P -values.

Pareschi *et al.* reported that the number of P -values should not be too large because for extremely large numbers of P -values, the second-level tests always fail [15, 16]. Pareschi *et al.* recommended that, in the case that $n = 2^{20} = 1,048,576$, for the frequency test included in NIST SP800-22, the number of P -values should be smaller than 4795. Therefore, in DFTT_{proposed}, n should not be too large (in DFTT_{present}, m should not be too large). However,

Table 5: The parameter sets for each test, and the numbers of P -values generated by each test

Parameter	DFTT _{present}	DFTT _{pareschi}	DFTT _{proposed}
n	100,000	100,000	4,000
m	1,000	1,000	25,000
Number of P -values	$m = 1000$	$m = 1000$	$\frac{n}{2} - 1 = 1999$

Table 6: Trade-off in the selection of n in DFTT_{proposed}

n	small	large
Second-level test	Accurate	Erroneous
Distribution of $\frac{2}{n} S_j(X) ^2$	Erroneous	Accurate

as we proved in Theorem 3, χ_2^2 is the asymptotic distribution of $\frac{2}{n}|S_j(X)|^2$. Therefore, n should be as large as possible. Thus, in DFTT_{proposed}, a selection of the parameter n is a trade-off between the error of the second-level test and the error of the distribution of $\frac{2}{n}|S_j(X)|^2$ (as shown in Table 6). Considering this trade-off, we defined the value of n as shown in Table 5. The appropriate selection of n and m in DFTT_{proposed} still needs to be analyzed more specifically.

6 Conclusion

In this paper, we have considered the DFT test included in the NIST SP800-22 statistical test suite for random number sequences. The most crucial problem in the present DFT test (denoted as DFTT_{present}) is that the reference distribution of its test statistic is *not mathematically* derived but is rather obtained by numerical estimation with a pseudo-random number generator; the basis of the *test for randomness* itself is based on a *pseudo-random number generator*. Therefore, DFTT_{present} cannot be used unless the reference distribution is mathematically derived.

We *proved* that the asymptotic distribution of the power spectrum is χ_2^2 , and based on this fact, we proposed a new DFT test denoted as DFTT_{proposed}, whose distribution of the test statistic is *mathematically* derived.

Furthermore, although appropriate selection of the parameters n and m for DFTT_{proposed} still need to be analyzed more specifically, the results of testing non-random sequences and several pseudo-random number generators showed that DFTT_{proposed} is more *reliable* and definitely more *sensitive* than DFTT_{present}, which is the current standard DFT test.

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