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Fault-Tolerant Consensus of Fractional Order Singular Multi-Agent Systems With Uncertainty

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ABSTRACT Consensus of fractional order singular multi-agent systems is considered in this paper. The model of uncertain systems with actuator faults and external disturbances is established. Firstly, augmented transformations are made to systems in order to estimate state vectors and actuator faults simultaneously. Then new admissible consensus criteria of observer-based fractional order singular multi-agent systems (FOSMAS) are proposed. Finally, for the sake of demonstrating the validity of proposed results, some relevant numerical examples are provided.

INDEX TERMS Fractional order systems, multi-agent systems, consensus criteria, actuator fault, Linear matrix inequalities (LMIs).

I. INTRODUCTION

In recent years, with the development of control and computer technology, inspired by the collective behavior of organisms in the ecosystem, distributed coordinated control of multi-agent systems has gradually been paid more and more attention. Compared with the traditional centralized control, the distributed control has obvious advantages, such as strong computing power, convenient operation. Consensus research means that all agents can converge to a unified signal by designing the consensus protocol, which is a basic research issue and becomes a research hotspot and gradually penetrates into various fields, such as biology, intelligent robot, office automation and other fields [1]–[4]. Many scholars have studied the multi-agent systems from different directions, such as robustness, controllability and consistency of multi-agent systems [5]–[7].

However, it is noted that all of the above studies are based on integer order systems, which cannot reflect their own performance when describing materials with memory and viscosity. In fact, the essence of nature is also fractional rather than integer. Therefore, some scientists turn their research objectives into fractional order systems and find that fractional order systems are well reflected the stability of

materials and their own performance [8]–[10]. In recent years, scientists begin to work on study the consistency of FOMAS. For the research on the state consensus of FOMAS, the distributed coordination algorithm of FOMAS is given in the early literature [11] to make the systems achieve the state consensus, on this basis, external disturbance is added in [12], which proposes a pinning control input method such that the state consensus is achieved. Since then, input saturation and other related knowledge are added in [13], so that the state consensus of the system is proved by the Mittag-Leffler stability theory under the condition that input saturation and external disturbance exist at the same time. The issue of unknown nonlinearities is investigated to systems in [14], the consensus analysis of the systems is completed by designing adaptive protocol while unknown nonlinearities and unknown external disturbances exist simultaneously, a sliding mode observer is designed to verify the state consensus of the FOMAS on the basis of nonlinearities in [15]. However, in practical application, state information is difficult to measure directly sometimes, therefore, many scholars focus on the design of output based consensus protocol. Such as the distributed static output feedback protocol proposed in [16], the consensus of the systems with order $0 < \alpha < 2$ is studied by LMI. Based on this, the output consensus of interval FOMAS is studied in [17]. In addition, in practical application, not all agents have

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the same information, and the differences between agents cannot be ignored. Therefore, there are also many literatures that study the consensus of heterogeneous multi-agent systems [18]–[20].

In the 1970s, the concept of singular systems was first put forward, which have a wider form than normal systems. Although singular systems have only developed for more than 20 years, they have become an independent branch of modern control theory and achieved fruitful results in many aspects, such as aircraft model, circuit system [21], [22]. In the past two decades, more and more scholars have extended the general multi-agent system to the singular system in a broader context [23]–[25]. Even some issues cannot be simulated by conventional systems, such as a transistor circuit proposed in [26], a three link planar manipulator network studied in [27]. Therefore, the importance of singular multi-agent systems is obvious. However, there are few papers about FOSMAS. The consensus of FOSMAS is still an open question and there are many unsolved problems in this domain. In practical application, many models have uncertainties, with the increasing complexity of the systems, the issues of faults are inevitable. In engineering practice, the bounds of the faults may be unknown, and agents may be failure in the presence of faults. On this account, the investigation of fault-tolerant control is very necessary in many practical systems, and the estimation of faults is undoubtedly a very important work in fault-tolerant control [28]–[31]. In [30], [31], an unknown input observer is designed to estimate the states and faults, however, the unknown input observer techniques have some limitations, which unknown input decoupling condition is usually not satisfied. In [32], the compensators for agents which cannot access external systems are proposed, however, the distributed observer gain is a parameter, and the ideal performance index cannot be obtained. So in practical application, it is a challenging task to solve the problem of simultaneous fault-tolerant and consensus control of multi-agent systems under distributed architecture. Compared with the decentralized control method, this method can effectively solve the problem of information exchange between adjacent agents. Therefore, we extend the existing results on singular multi-agent systems from FOMAS to FOSMAS, where the dynamic of each agent is described by fractional order singular systems. Combining the fault and state vectors to augmented state vectors, designing a distributed consensus algorithm which can solve the consensus problem and reject the faults, simultaneously. Based on the above motivation, the contributions of this paper can be summarized as follows:

- Compared with the integer multi-agent systems, there are few previous studies on FOSMAS. On this basis, actuator faults and external disturbance are added in the systems and the uncertainties in practical application are considered in this paper.

- The actuator faults are expressed in polynomial form, and the state and actuator faults are estimated at the same time, then design the distributed observer based on relative output

information to estimate the leader and the augmented state vectors.

- Finally, the distributed fault-tolerant consensus protocol is designed and new criteria are proposed to make the systems achieve admissible consensus.

Here are organization structure of the article: In Section 2, some graph theory notions and lemmas of fractional order systems are introduced. Then the corresponding model is established and transformed, some related lemmas are presented in Section 3. New consensus criteria of FOSMAS based on linear matrix inequalities are proposed in section 4. Some examples are given to verify the effectiveness of the approach and conclusions are drawn to end this paper in Section 5.

Notations: I_N represents the $N \times N$ dimensional identity matrix, $\mathbb{R}^{m \times n}$ represents the set of $m \times n$ real matrix, \otimes denotes Kronecker product, $\text{diag}(a_1, \dots, a_n)$ denotes the diagonal matrix with elements a_1, a_2, \dots, a_n , and $\text{blockdiag}(b_1, \dots, b_n)$ denotes a block diagonal matrix, \star represents symmetric part of a matrix, the smallest eigenvalue of matrix A is denoted by $\lambda_{\min}(A)$, the largest eigenvalue of matrix A is denoted by $\lambda_{\max}(A)$.

II. PRELIMINARIES

In this section, some knowledge and lemmas about graph theory and fractional order singular systems are introduced.

A. GRAPH THEORY

Graph theory of directed graph is a tool to express information transfer between agents. Some basic concepts of graph theory are briefly introduced in this paper. Consider a multi-agent system which consists of one leader and N followers. The directed graph is represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ represents the set of all nodes, a set of edges of graph G represented by $\mathcal{E} \subseteq (\mathcal{V}, \mathcal{V})$. Directed pair (v_i, v_j) represents an edge of a graph G , which indicates that node v_i can accept the information of node v_j and there will be $(v_i, v_j) \in \mathcal{E}$, it means that v_i is the neighbor of v_j . \mathcal{A} is defined as an adjacency matrix, this matrix takes a_{ij} as its element, $a_{ij} \geq 0$ when $(v_j, v_i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. For graph \mathcal{G} , the in-degree matrix \mathcal{D} is defined as $\mathcal{D} = \text{diag}(d_1, d_2, \dots, d_N)$ with $d_i = \sum_{j=1}^N a_{ij}$. The Laplacian matrix is $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Let the leader be associated with node 0, set $\bar{\mathcal{V}} = \mathcal{V} \cup \{v_0\}$ and the directed graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$. Moreover, by setting $a_{i0} = 1$ if there is an arc from 0 to i , else $a_{i0} = 0$. $\bar{\mathcal{L}} = \mathcal{L} + \mathcal{A}_0$.

B. CAPUTO FRACTIONAL OPERATOR AND FRACTIONAL ORDER SINGULAR SYSTEMS

The fractional derivative is mainly divided into two classes: Riemann-Liouville derivative and Caputo derivative. Riemann Liouville method leads to the initial condition of useless application, which leads to the conflict between mature mathematical theory and practical needs. In contrast, Caputo method allows the use of the initial value of the classical

integer order derivative with clear physical interpretation. Thus, Caputo definition is used in this paper.

Definition 1 [26]: The Caputo fractional derivative with order α of function $x(t)$ is given by:

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, and n is an integer satisfying $n-1 < \alpha \leq n$.

Definition 2 [33]: For the following linear fractional order system,

$$ED^\alpha x(t) = Ax(t) + Bu(t), \quad (1)$$

when $u(t) = 0$, system (1) is called unforced system, which can be written as triple (E, A, α) , unforced system (1) is admissible if the following three conditions are satisfied simultaneously,

- 1) The triple (E, A, α) is regular, if there exists a constant scalar $s \in \mathbb{C}$ such that $\det(s^\alpha E - A)$ is not identically zero.
- 2) The triple (E, A, α) is impulse free, if system (1) is regular and $\deg(\det(\lambda E - A)) = \text{rank}(E)$.
- 3) The triple (E, A, α) is stable, if $|\arg(\text{spec}(E, A))| > \frac{\pi\alpha}{2}$, where $\text{spec}(E, A) = \{\lambda \mid \det(\lambda E - A) = 0, \forall \lambda \in \mathbb{C}\}$.

Lemma 1 [34]: The next two conditions are equivalent:

1. Unforced system (1) with order $0 < \alpha < 1$ is admissible.
2. There exist matrices $X, Y \in \mathbb{R}^{n \times n}, Q \in \mathbb{R}^{(n-m) \times n}$ such that (2), (3) hold or (2), (4) hold,

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \quad (2)$$

$$\text{sym}\{A(aXE^T - bYE^T + PQ)\} < 0, \quad (3)$$

$$\text{sym}\{A^T(aXE - bYE + \bar{P}Q)\} < 0, \quad (4)$$

where a and b denote $\sin(\frac{\alpha\pi}{2})$ and $\cos(\frac{\alpha\pi}{2})$, respectively, P and \bar{P} are any matrices with full column rank and satisfies $EP = 0, E^T \bar{P} = 0$.

III. PROBLEM STATEMENT

Consider the singular fractional order leader-follower multi-agent systems consisted of one leader and N followers, where the dynamics models of leader can be introduced as follows:

$$\begin{aligned} ED^\alpha x_0(t) &= (A + \Delta A)x_0(t), \\ y_0(t) &= Cx_0(t), \end{aligned} \quad (5)$$

the dynamics of the i th agent can be described as:

$$\begin{aligned} ED^\alpha x_i(t) &= (A + \Delta A)x_i(t) + B(u_i(t) + f_i(t)) + Wx_0(t), \\ y_i(t) &= Cx_i(t), \end{aligned} \quad (6)$$

where $x_i(t) \in \mathbb{R}^n$ represents state vector of node i ($i = 1, 2, \dots, N$), $u_i(t) \in \mathbb{R}^q$ represents the control input vector and $y_i(t) \in \mathbb{R}^r$ denotes the measurement output vector, $x_0(t) \in \mathbb{R}^n$ is the state and $y_0(t) \in \mathbb{R}^r$ is the output of the leader, $f_i(t) \in \mathbb{R}^q$ denotes the unbounded fault signal, $Wx_0(t)$ represents the exosystem disturbances, $\Delta A \in \mathbb{R}^{n \times n}$ denotes uncertainty, which satisfy $\Delta A = D_A F_A(t) E_A$ and

$F_A(t)^T F_A(t) \leq 0, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}$, and $C \in \mathbb{R}^{r \times n}$ are all known constant coefficient matrices, $E \in \mathbb{R}^{n \times n}$ is a singular matrix with $\text{rank}(E) = r < n$.

Assumption 1 [32]: The graph $\bar{\mathcal{G}}$ has a spanning tree and the root node is \mathcal{V}_0 . The pair (A, B) is controllable and the pair (A, C) is detectable.

Remark 1: Actually, errors, disturbances and other factors often occur during the modeling process, uncertainties often exist in systems, which make fault-tolerant control based on fault estimation observer more difficult, therefore, the problem of fault estimation observer and fault tolerant controller for uncertain linear multi-agent system is studied in this paper. Under Assumption 1, all the eigenvalues of the matrix $\bar{\mathcal{L}}$ have positive real parts, which is very helpful to design the distributed consensus algorithm in the case of actuator fault. This condition is important for forming an augmented state vector observer composed of system states and actuator faults.

It is obvious that the actuator fault in system (6) is limited by matrix B , so motivated by literature [6], we study a kind of actuator faults, which can be expressed in the form of polynomial function, by augmenting system (6), the state and the actuator fault can be estimated simultaneously.

The fault signals $f_i(t)$ are represented in the form of the polynomial function as follows,

$$f_i(t) = A_{i,0} + A_{i,1}t + A_{i,2}t^2 + \dots + A_{i,p-1}t^{p-1}. \quad (7)$$

An augmented state vector is defined as the following form,

$$\bar{x}_i = [x_i^T, (f_i^{(p-1)})^T, \dots, \dot{f}_i^T, f_i^T]^T \in \mathbb{R}^{\bar{n}}, \quad (\bar{n} = n + pq). \quad (8)$$

Then, system (6) can be transformed into the following system,

$$\begin{aligned} \bar{E}D^\alpha \bar{x}_i(t) &= (\bar{A} + \Delta \bar{A})\bar{x}_i(t) + \bar{B}u_i(t) + \bar{W}x_0, \\ y_i(t) &= \bar{C}\bar{x}_i(t), \end{aligned} \quad (9)$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & 0_{n \times q} & \dots & 0_{n \times q} & B \\ 0_{q \times n} & 0_{q \times q} & \dots & 0_{q \times q} & 0_{q \times q} \\ 0_{q \times n} & I_q & \dots & 0_{q \times q} & 0_{q \times q} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0_{q \times n} & 0_{q \times q} & \dots & I_q & 0_{q \times q} \end{bmatrix}, \\ \Delta \bar{A} &= \begin{bmatrix} \Delta A & 0_{n \times q} & \dots & 0_{n \times q} & 0_{q \times q} \\ 0_{q \times n} & 0_{q \times q} & \dots & 0_{q \times q} & 0_{q \times q} \\ 0_{q \times n} & 0_{q \times q} & \dots & 0_{q \times q} & 0_{q \times q} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0_{q \times n} & 0_{q \times q} & \dots & 0_{q \times q} & 0_{q \times q} \end{bmatrix}, \\ \bar{E} &= \begin{bmatrix} E & 0_{n \times q} & \dots & 0_{n \times q} & 0_{n \times q} \\ 0_{q \times n} & I_q & \dots & 0_{q \times q} & 0_{q \times q} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0_{q \times n} & 0_{q \times q} & \dots & I_q & 0_{q \times q} \\ 0_{q \times n} & 0_{q \times q} & \dots & 0_{q \times q} & I_q \end{bmatrix}, \end{aligned}$$

$$\bar{B} = \begin{bmatrix} B \\ 0_{q \times q} \\ 0_{q \times q} \\ \vdots \\ 0_{q \times q} \end{bmatrix}, \quad \bar{W} = \begin{bmatrix} W \\ 0_{q \times n} \\ 0_{q \times n} \\ \vdots \\ 0_{q \times n} \end{bmatrix},$$

$$\bar{C} = [C \quad 0_{r \times q} \quad 0_{r \times q} \quad \dots \quad 0_{r \times q}].$$

Remark 2: The faults concerned are assumed to be either incipient faults or abrupt faults in [30], therefore the second-order derivative of the fault $f(t)$ should be zero piecewise, in other words, $\dot{f}_i^T = 0$. On the other hand, one could consider a more general case, i.e., the q th-order derivative of the fault is assumed to be zero as shown in [36]. The case $q = 2$ is considered in [30], but the results can easily be extended to the case when $q \geq 3$. Therefore, a class of actuator faults are considered in this paper, where its p th derivative is assumed to be bounded, particularly, the exponential and ramp faults belong to the unbounded faults. Faults can be expressed in the form of the polynomial function in Eq. (7).

In many systems of real life, in fact, many actuator faults are the special form of the fault (7). In real life, because of the relationship between economic cost and measurement, it is difficult to measure the relevant information of all agents, and the consensus protocol based on output is relatively more practical, so under the relative output information, here, we assume that the perturbations $\Delta \bar{A}$ is measurable [37] and construct the following observer:

$$\bar{E}D^\alpha \hat{x}_i(t) = (\bar{A} + \Delta \bar{A})\hat{x}_i(t) + \bar{B}u_i(t) + \bar{W}\hat{x}_{i0} + v_i(t), \quad (10)$$

where $\hat{x}_i(t) = [\hat{x}_i^T(t), \hat{f}_i^{p-1}, \dots, \hat{f}_i^T(t), \hat{f}_i^T(t)]^T$, $\hat{x}_i(t)$ and $\hat{f}_i(t)$ represent the state and the fault estimation of the i th agent, respectively, then $v_i(t)$ is represented with following pattern,

$$v_i(t) = L \left(\sum_{j=1}^N a_{ij}(\tilde{y}_i(t) - \tilde{y}_j(t)) + a_{i0}\tilde{y}_i(t) \right), \quad (11)$$

with $\tilde{y}_i(t) = y_i(t) - C\hat{x}_i(t)$, $L \in \mathbb{R}^{\bar{n} \times r}$ is the observer gain matrix, which will be designed later. Motivated in [33], the following distributed observer is used in this paper to estimate the leader.

$$ED^\alpha \hat{x}_{i0} = (A + \Delta A)\hat{x}_{i0} - H \left(\sum_{j=1}^N a_{ij}(\hat{x}_{i0} - \hat{x}_{j0} + a_{i0}(\hat{x}_{i0} - x_0)) \right), \quad (12)$$

where $i = 1, 2, \dots, N$, $\hat{x}_{i0} \in \mathbb{R}^n$, and H is a gain matrix. Then the distributed fault-tolerant consensus algorithm is proposed as follows,

$$u_i(t) = K \left(\sum_{j=1}^N a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) + a_{i0}(\hat{x}_i(t) - x_0(t)) - \hat{f}_i(t) - G\hat{x}_{i0}(t) \right), \quad (13)$$

where $K \in \mathbb{R}^{q \times n}$ and $G \in \mathbb{R}^{q \times n}$ are the feedback gain matrices. In this paper, an unknown input observer is designed, which can estimate the state and actuator fault, respectively.

Lemma 2 [6]: For the augmented system (10), there exist an unknown input observer, which is observable, then it can conclude the following condition,

$$\text{rank} \begin{bmatrix} \lambda I_{n+pq} - \bar{A} \\ \bar{C} \end{bmatrix} = n + pq,$$

for all λ with $\lambda \neq 0$.

Letting

$$x(t) = [x_1^T(t), \dots, x_N^T(t)]^T,$$

$$f(t) = [f_1^T(t), \dots, f_N^T(t)]^T.$$

Then, the state and fault estimation errors can be written as follows, $e_x(t) = \hat{x}(t) - x(t)$, $e_f(t) = \hat{f}(t) - f(t)$,

$$e_{i0} = \hat{x}_{i0} - x_0. \quad (14)$$

Combining (9), (10), (12) and (14), the following estimation error equation can be obtained,

$$(I_N \otimes \bar{E})D^\alpha e(t) = (I_N \otimes \bar{A} + \bar{L} \otimes (L\bar{C}))e(t) + (I_N \otimes \bar{W})e_0, \quad (15)$$

with $e(t) = [e_x^T(t), e_f^{T(p-1)}, \dots, \dot{e}_f^T(t), e_f^T(t)]^T$, $e_0 = [e_{10}^T(t), e_{20}^T(t), \dots, e_{N0}^T(t)]^T$, then we can obtain the following equation,

$$(I_N \otimes E)D^\alpha e_0(t) = (I_N \otimes A - \bar{L} \otimes H)e_0(t).$$

Under Assumption 1, according to (10), (13) and (6), the closed-loop dynamics can be written as:

$$ED^\alpha x_i(t) = (A + \Delta A)x_i(t) + BK \left(\sum_{j=1}^N a_{ij}(\hat{x}_i(t) - \hat{x}_j(t)) + a_{i0}(\hat{x}_i(t) - x_0(t)) \right) - B\hat{f}_i(t) + Bf_i(t) - BG\hat{x}_{i0}(t) + Wx_0(t). \quad (16)$$

In order to deal with the different dimensions between the fault and leader, there exists a matrix G such that $W - BG = 0$.

Letting $\varepsilon_i(t) = x_i(t) - x_0(t)$ represents the consensus tracking errors and $e_{xi}(t) = \hat{x}_i(t) - x_i(t)$ represents the state estimation errors, then we have the following equation,

$$ED^\alpha \varepsilon_i(t) = A\varepsilon_i(t) + BK \left(\sum_{j=1}^N a_{ij}(\varepsilon_i(t) - \varepsilon_j(t) + e_{xi}(t) - e_{xj}(t)) + a_{i0}(\varepsilon_i(t) + e_{xi}(t)) \right) - Be_{fi}(t) - We_{i0}. \quad (17)$$

Letting $\varepsilon(t) = [\varepsilon_1(t), \dots, \varepsilon_N(t)]^T$, then the consensus tracking error (17) for leader-follower multi-agent systems (5) can be rewritten as follows,

$$(I_N \otimes E)D^\alpha \varepsilon(t) = (I_N \otimes A + \bar{L} \otimes BK)\varepsilon(t) + (\bar{L} \otimes BK)e_x(t) - (I_N \otimes B)e_f(t) - (I_N \otimes W)e_0. \quad (18)$$

Combining the tracking errors (18) and the estimation error system (15), the closed-loop system can be described in the following compact form,

$$\tilde{E}D^\alpha \xi(t) = \bar{\mathcal{A}}\xi(t) + \mathcal{B}\tilde{e}_0, \quad (19)$$

where the $\xi(t) = [\varepsilon^T(t) \ e^T(t)]^T$, and

$$\bar{\mathcal{A}} = \begin{bmatrix} I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK \\ 0 \\ \bar{\mathcal{L}} \otimes BK 0_{Nn \times N} - I_N \otimes B \\ I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes (\bar{L}\bar{C}) \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} -I_N \otimes W \\ I_N \otimes \bar{W} \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} I_N \otimes E & 0 \\ 0 & I_N \otimes \bar{E} \end{bmatrix} \quad (20)$$

Remark 3: In [38], the consensus algorithm has not fault compensation, which leads non-zero results for consensus tracking error and the algorithm cannot solve the consensus problem. Therefore, the fault compensation controller in Eq. (13) is proposed to reject the fault in each follower. In practice, relative output information is easier to obtain than absolute output information. Compared with [38], the algorithm (13) in this paper uses state estimation instead of state itself, and the fault estimations are determined by the relative output information.

Definition 3: For systems (5) and (6), protocol u_i is said to be asymptotically solved the admissible consensus issue if system (19) is regular, stable, impulse-free and for any initial values, the states of the agents satisfy

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| = 0, \quad (21)$$

for all $i = 1, 2, \dots, N$.

Under Assumption 1, it can obtain all the eigenvalues of $\bar{\mathcal{L}}$ have positive real parts. By Jordan matrix theorem, there exists a nonsingular matrix $U \in R^{N \times N}$ and

$$J(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

such that $\bar{\mathcal{L}} = U^{-1} \tilde{\Delta} U$, where $\tilde{\Delta}$ is the Jordan matrix of $\bar{\mathcal{L}}$, and $\tilde{\Delta} = \text{blockdiag}\{J(\lambda_1), \dots, J(\lambda_k)\}$, $N_1 + N_2 + \dots + N_k = N$. λ_k is the eigenvalue of $\bar{\mathcal{L}}$. Then the matrix $\bar{\mathcal{A}}$ can be transformed into the following form,

$$\bar{\mathcal{A}} = \begin{bmatrix} (A + \Delta A) + \lambda_k BK & (\lambda_k BK 0_{Nn \times N} - B) \\ 0 & (\bar{A} + \Delta \bar{A}) + \lambda_k (\bar{L}\bar{C}) \end{bmatrix}$$

if matrices $(A + \Delta A) + \lambda_k BK$ and $(\bar{A} + \Delta \bar{A}) + \lambda_k (\bar{L}\bar{C})$ are admissible, then the matrix $\bar{\mathcal{A}}$ is admissible.

IV. MAIN RESULTS

In this section, the issue of admissible consensus for system (19) is discussed.

Theorem 1: System (19) can be achieved admissible consensus with protocol (13) if and only if $(I_N \otimes E,$

$I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK, \alpha)$ and $(I_N \otimes \bar{E}, I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes (\bar{L}\bar{C}), \alpha)$ are admissible.

Proof: According to Lemma 1, we can obtain

$$\det(s^\alpha \tilde{E} - \bar{\mathcal{A}}) = \det(s^\alpha (I_N \otimes E) - (I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK)) \times \det(s^\alpha (I_N \otimes \bar{E}) - (I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes \bar{L}\bar{C})), \quad (22)$$

from equation (22), it is obvious that $(\tilde{E}, \bar{\mathcal{A}}, \alpha)$ is regular if and only if $(I_N \otimes E, I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK, \alpha)$ and $(I_N \otimes \bar{E}, I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes (\bar{L}\bar{C}), \alpha)$ are regular simultaneously. The impulse-free condition of $(\tilde{E}, \bar{\mathcal{A}}, \alpha)$ is $\deg(\det(\lambda \tilde{E} - \bar{\mathcal{A}})) = \text{rank}(\tilde{E}) = Nr + N\bar{n}$, from formula (22), we can obtain

$$\deg(\det(\lambda \tilde{E} - \bar{\mathcal{A}})) = \deg(\det(\lambda (I_N \otimes E) - (I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK))) + \deg(\det(\lambda (I_N \otimes \bar{E}) - (I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes \bar{L}\bar{C}))) = \text{rank}(I_N \otimes E) + \text{rank}(I_N \otimes \bar{E}) = Nr + N\bar{n}. \quad (23)$$

The stability of $(\tilde{E}, \bar{\mathcal{A}}, \alpha)$ is equivalent to the stability of $(I_N \otimes E, I_N \otimes (A + \Delta A) + \bar{\mathcal{L}} \otimes BK, \alpha)$ and $(I_N \otimes \bar{E}, I_N \otimes (\bar{A} + \Delta \bar{A}) + \bar{\mathcal{L}} \otimes (\bar{L}\bar{C}), \alpha)$ according to equation (22).

Theorem 2: Assume that Assumption 1 holds, the admissible consensus of system (19) can be achieved if there exist matrices $X_i, Y_i \in R^{n \times n}$, $Q_i \in R^{(n-r) \times n}$, $Z_i \in R^{q \times n}$ $i = 1, 2$, and two scalar constants $\epsilon_i > 0$, $i = 1, 2$, such that

$$\begin{bmatrix} X_i & Y_i \\ -Y_i & X_i \end{bmatrix} > 0, \quad i = 1, 2 \quad (24)$$

$$\begin{bmatrix} \text{sym}(A(aX_1E^T - bY_1E^T + PQ_1) + \lambda_k BZ_1) + \epsilon_1 D_A D_A^T \\ \star \\ (aX_1E^T - bY_1E^T + PQ_1)^T E_A^T \\ -\epsilon_1 I \end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} \text{sym}(\bar{A}^T(aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2) + \lambda_k \bar{C}^T Z_2) + \epsilon_2 \bar{E}_A^T \bar{E}_A \\ \star \\ (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2)^T \bar{D}_A \\ -\epsilon_2 I \end{bmatrix} < 0, \quad (26)$$

where the feedback gain matrices are $K = Z_1(aX_1E^T - bY_1E^T + PQ_1)^{-1}$, $L = (Z_2(aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2)^{-1})^T$.

Proof: It is obvious that $\bar{\mathcal{A}}$ of equation (21) is admissible if $A + \Delta A + \lambda_k BK$ and $\bar{A} + \Delta \bar{A} + \lambda_k (\bar{L}\bar{C})$ are admissible. According to Lemma 2, it can obtain $A + \Delta A + \lambda_k BK$ is admissible if there exist two matrices $X_1, Y_1 \in R^{n \times n}$ and a matrix $Q_1 \in R^{(n-r) \times n}$ such that

$$\text{sym}((A + \Delta A + \lambda_k BK)(aX_1E^T - bY_1E^T + PQ_1)) < 0. \quad (27)$$

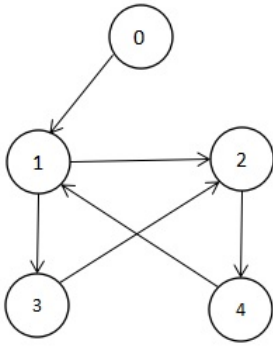
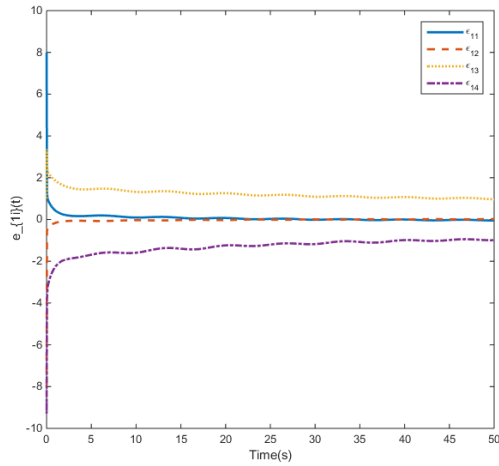
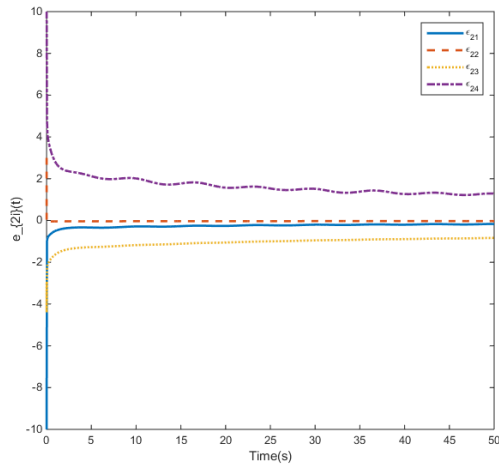


FIGURE 1. Interaction graph for 5 agents. (agent 0 is the leader).



(a)

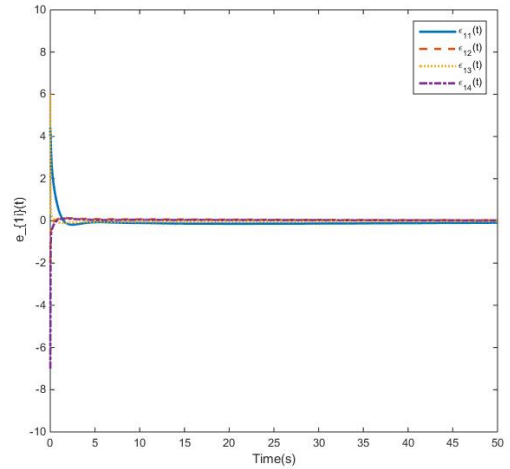


(b)

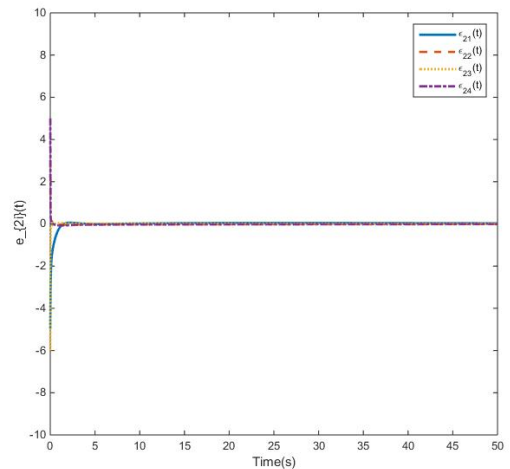
FIGURE 2. Consensus tracking errors of four agents with algorithm in [38].

For the convenience of calculation, inequality (27) can be expanded as follows,

$$\text{sym}((A + \lambda_k BK)(aX_1E^T - bY_1E^T + PQ_1)) + \text{sym}(\Delta A(aX_1E^T - bY_1E^T + PQ_1)) < 0. \quad (28)$$



(a)



(b)

FIGURE 3. Consensus tracking errors of four agents in the present paper.

Letting $Z_1 = K(aX_1E^T - bY_1E^T + PQ_1)$, there exists a scalar $\epsilon_1 > 0$ such that

$$\begin{aligned} & \text{sym}(A(aX_1E^T - bY_1E^T + PQ_1) + \lambda_k BZ_1) + \epsilon_1 D_A D_A^T \\ & + \epsilon_1^{-1} (aX_1E^T - bY_1E^T + PQ_1)^T E_A^T \\ & (aX_1E^T - bY_1E^T + PQ_1) E_A < 0, \end{aligned} \quad (29)$$

using Schur complement in [35], inequality (28) is equivalent to inequality (24).

In the same way, $\bar{A} + \Delta\bar{A} + \lambda_i(L\bar{C})$ is admissible if there exist some matrices $X_2, Y_2 \in \mathbb{R}^{\bar{n} \times \bar{n}}, Q_2 \in \mathbb{R}^{(\bar{n}-r) \times \bar{n}}$ and $Z_2 \in \mathbb{R}^{q \times \bar{n}}$ such that

$$\text{sym}((\bar{A} + \Delta\bar{A} + \lambda_k L\bar{C})^T (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2)) < 0. \quad (30)$$

Letting $Z_2 = L^T (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2)$, there exists a scalar $\epsilon_2 > 0$ such that

$$\begin{aligned} & \text{sym}(\bar{A}^T (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2) + \lambda_k \bar{C}^T Z_2) + \epsilon_2 E_A^T E_A \\ & + \epsilon_2^{-1} (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2)^T D_A D_A^T \\ & (aX_2\bar{E} - bY_2\bar{E} + \bar{P}Q_2) < 0, \end{aligned} \quad (31)$$

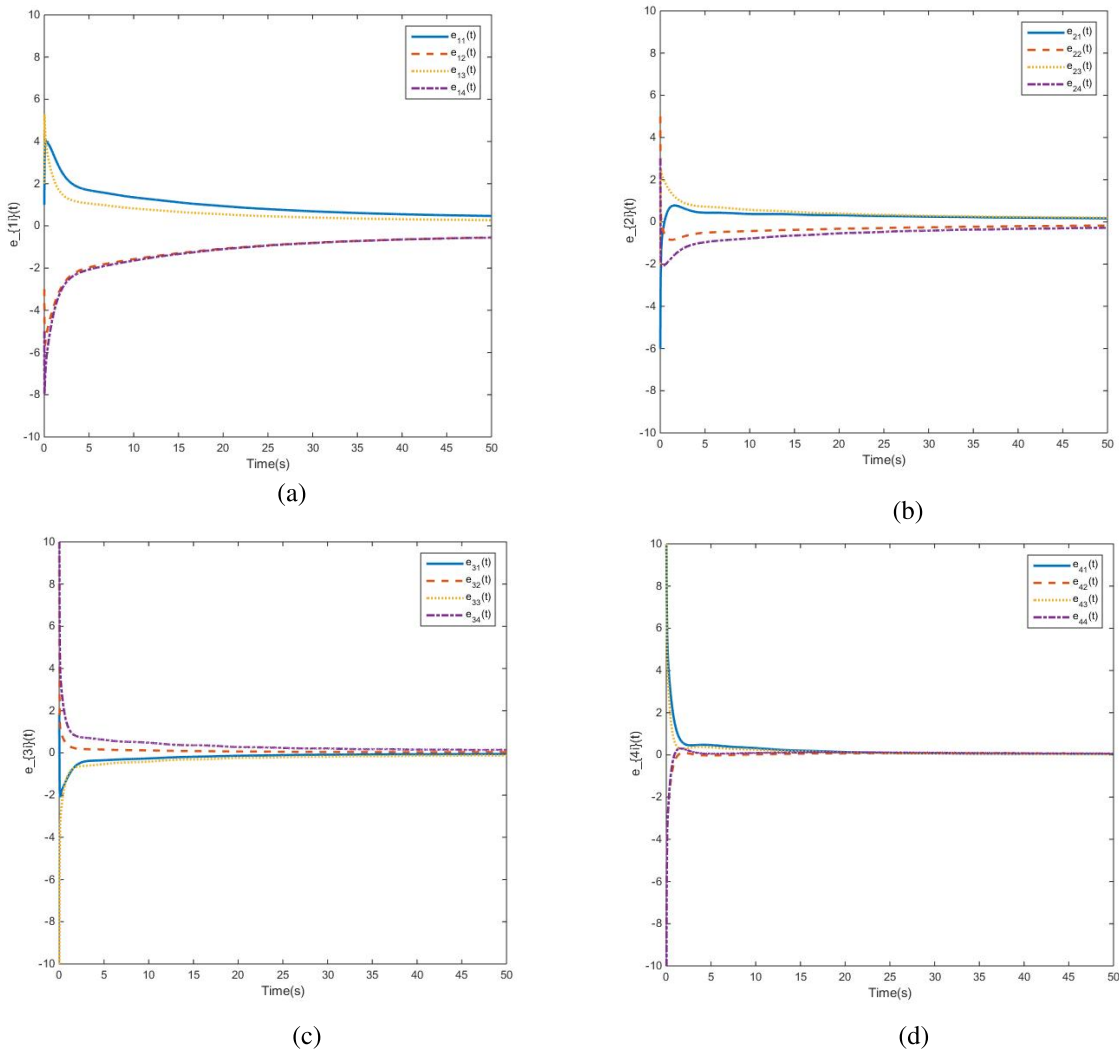


FIGURE 4. State and fault estimate errors of four agents.

using Schur complement in [35], inequality (30) is equivalent to inequality (25).

Remark 4: In this paper, the state and fault estimations are considered in the control algorithm, compared with [39], which state vectors and errors are incorporated into distributed consensus controller, the algorithm has the following characteristics: At first, the effects of external disturbances can be eliminated by state estimation; Then, the influence of faults can be restrained by combining fault estimation. At last, for agents which cannot receive information from the leader, the leader observer can compensate.

V. NUMERICAL SIMULATIONS

In this section, some numerical simulations are presented to expound our theoretical results.

Consider a group of five agents with an interaction graph given by Fig. 1. The fractional order multi-agent systems (4)

and (5) are described by second-order integrators, where

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\
 C &= [1 \ 0], \quad E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\
 D_A &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \quad E_A = \begin{bmatrix} -0.2 & 0.1 \\ -0.2 & 0.3 \end{bmatrix},
 \end{aligned}$$

matrices \mathcal{L} and $\bar{\mathcal{A}}_0$ can be determined as

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad \bar{\mathcal{A}}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the eigenvalues of $\bar{\mathcal{L}}$ are (0.2451, 1.8774+0.7449i, 1.8774-0.7449i, 2.0000). The actuator faults associated with each

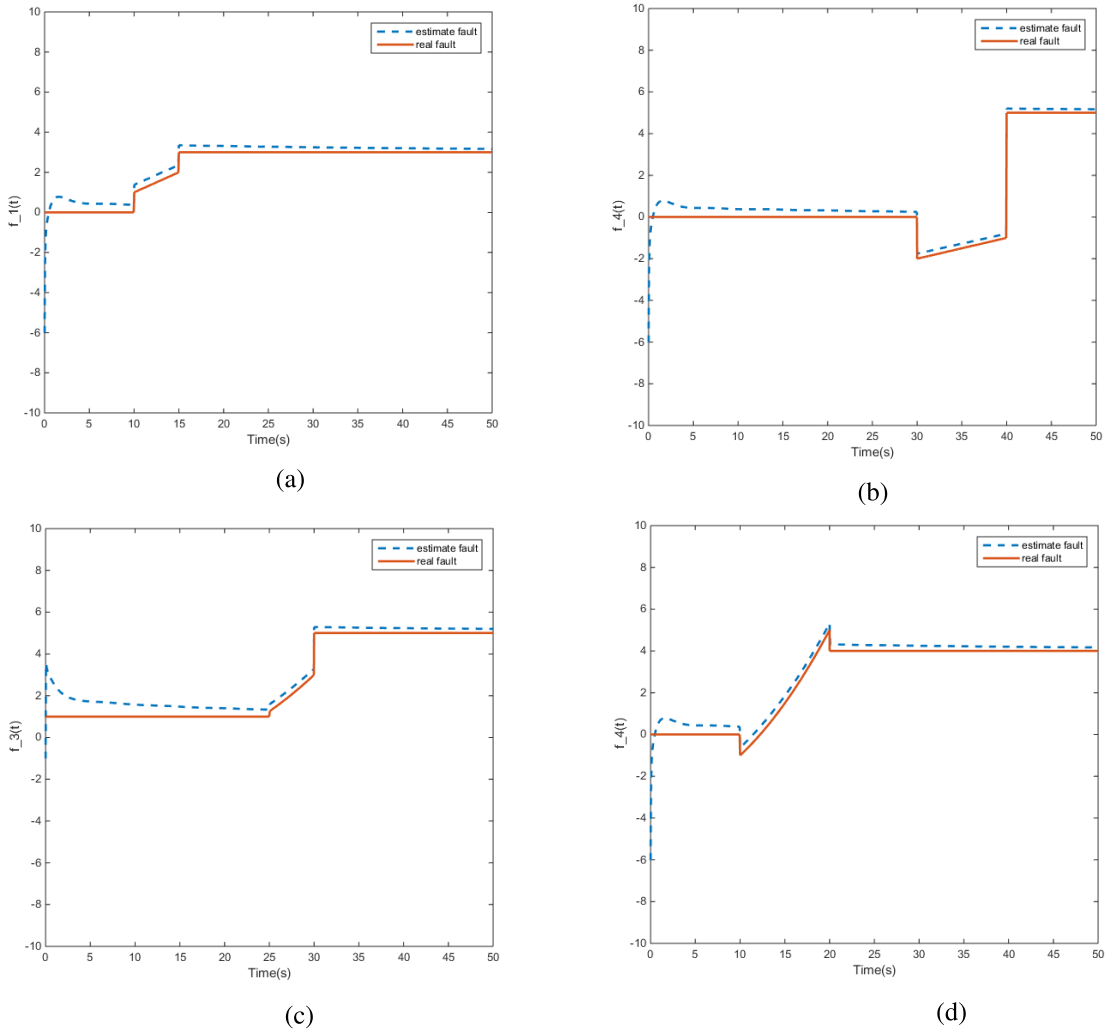


FIGURE 5. Actuator faults and faults estimates.

agent are

$$\begin{aligned}
 f_1(t) &= \begin{cases} 0, & t < 10 \\ 0.2t - 1, & 10 \leq t < 15 \\ 3, & t \geq 15 \end{cases} \\
 f_2(t) &= \begin{cases} 0, & t < 30 \\ 0.1t - 5, & 30 \leq t < 40 \\ 5, & t \geq 40 \end{cases} \\
 f_3(t) &= \begin{cases} 1, & t < 25 \\ 0.01t^2 - 0.2t, & 25 \leq t < 30 \\ 5, & t \geq 30 \end{cases} \\
 f_4(t) &= \begin{cases} 0, & t < 10 \\ 0.02t^2 - 3, & 10 \leq t < 20 \\ 4, & t \geq 20 \end{cases}
 \end{aligned}$$

by solving the inequalities (24), (25) and (26), two scalar constants $\epsilon_1 > 0$, and $\epsilon_2 > 0$ can be obtained as follows,

$$\epsilon_1 = 44.5016, \quad \epsilon_2 = 44.4033,$$

and the feedback gains matrices K and L are obtained

$$\begin{aligned}
 K &= [-1.8874 \quad -0.1808], \\
 L &= [6.6286 \quad -5.8193 \quad -12.7487 \quad -13.8372]^T.
 \end{aligned}$$

Consider the uncertain FOSMAS (6) with the topology, under the directed graph depicted in Fig. 1, the consensus tracking errors of state trajectories of four agents in the present paper and in the [38] are depicted in Fig. 2 and 3, respectively. It is straightforward to see that the state consensus problem can be solved by distributed fault-tolerant consensus algorithm (13). Comparing the algorithm (13) with [38], it is noted that some of consensus errors are not converged to zero, it means that consensus control algorithm in [38] cannot be well reached in the case of actuator faults. The trajectories of state error and fault error are depicted in Fig 4, respectively, faults and their estimations are depicted in Fig 5, which shown that the proposed estimator can effectively estimate the faults, and it is also obvious that the

proposed estimator can effectively estimate the state and fault information.

VI. CONCLUSION

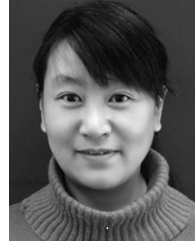
In this paper, the issue of uncertain singular FOMAS with actuator fault and disturbance is studied. An observer based on relative output information is designed for the augmented systems, therefore the states and actuator faults can estimate at the same time. Then distributed fault-tolerant consensus is designed and new criteria are proposed to achieve the admissible consensus. In the end, some numerical examples are provided to testify the availability of the consequences. In the future, FOSMAS with actuator faults and input saturation can be developed similarly according to the approach of theorems in the paper.

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