Quantum Speed-Up for Multidimensional (Zero Correlation) Linear Distinguishers*

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Abstract. This paper shows how to achieve a quantum speed-up for multidimensional (zero correlation) linear distinguishers. A previous work by Kaplan et al. has already shown a quantum quadratic speed-up for one-dimensional linear distinguishers. However, classical linear cryptanalysis often exploits multidimensional approximations to achieve more efficient attacks, and in fact it is highly non-trivial whether Kaplan et al.'s technique can be extended into the multidimensional case. To remedy this, we investigate a new quantum technique to speed-up multidimensional linear distinguishers. Firstly, we observe that there is a close relationship between the subroutine of Simon's algorithm and linear correlations via Fourier transform. Specifically, a slightly modified version of Simon's subroutine, which we call Correlation Extraction Algorithm (CEA), can be used to speed-up multidimensional linear distinguishers. CEA also leads to a speed-up for multidimensional zero correlation distinguishers, as well as some integral distinguishers through the correspondence of zero correlation and integral properties shown by Bogdanov et al. and Sun et al. Furthermore, we observe possibility of a more than quadratic speed-ups for some special types of integral distinguishers when *multiple* integral properties exist. Especially, we show a single-query distinguisher on a 4-bit cell SPN cipher with the same integral property as 2.5-round AES. Our attacks are the first to observe such a speed-up for classical cryptanalytic techniques without relying on hidden periods or shifts. By replacing the Hadamard transform in CEA with the general quantum Fourier transform, our technique also speeds-up generalized linear distinguishers on an arbitrary finite abelian group.

Keywords: symmetric-key cryptography · quantum cryptanalysis · linear cryptanalysis · integral cryptanalysis · more-than-quadratic speed-up

1 Introduction

Research in the past decade has revealed possible quantum attacks on symmetric cryptosystems are not limited to the exhaustive key search with Grover's algorithm [31] or the collision search by the BHT algorithm [19]. A notable line of

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research is the one initiated by Kuwakado and Morii showing that Simon's algorithm breaks lots of classically secure schemes in polynomial time [46, 47, 42, 13]. Other previous works show how to speed-up classical cryptanalytic techniques such as differential and linear cryptanalysis, MITM, and integral attacks [43, 37, 15]. Some recent papers study dedicated quantum collision attacks on concrete hash functions such as SHA-2 and SHA-3 [38, 29, 39, 32].

In this paper, we investigate the possibility to achieve more quantum speed-up for major classical cryptanalytic techniques than previous works.

Q1 and Q2. For quantum cryptanalysis on symmetric cryptosystems, there are two attack models called Q1 and Q2 [43]. The Q1 model assumes the existence of a quantum computer but oracles of keyed functions are classical[1], whereas Q2 assumes that oracles are also quantum[2]. For instance, a Q2 attack on a cipher is allowed to query quantum superposition of messages to the encryption oracle. Such an attack is called a Quantum Chosen-Plaintext Attack (QCPA). This paper studies attacks in the Q2 model.

Significance of Studying Q2 Attacks. The Q1 model is more realistic than Q2 in that oracles in Q1 are the same as classical ones, and thus Q1 attacks become real threats as soon as a large-scale fault-tolerant quantum computer is available. Still, studying Q2 attacks is important for the following two reasons. First, a new non-trivial Q1 attack may be found based on Q2 attacks. For instance, the so-called offline Simon's algorithm by Bonnetain et al. [12], which is a Q1 attack, is developed by modifying the Q2 attack by Leander and May [48]. Second, Q2 attacks can be converted into Q1 attacks when the key length is sufficiently long: Let E_K be an n-bit block cipher with k-bit keys. Suppose that $k > 2n$, and that there is a Q2 attack on E_K with time complexity $T < 2^{k/2}$. Now, assume we are in the Q1 model and run the following attack. First, query all the (classical) inputs to E_K, storing the results in a qRAM. Second, simulate the quantum oracle of E_K by accessing the qRAM, and execute the Q2 attack with the simulated oracle. This is a valid Q1 attack since the resulting complexity $T' = \max\{T, 2^n\}$ is less than $2^{k/2}$, the complexity of the exhaustive key-search by Grover's algorithm. Even if $k \leq 2n$, some Q2 attacks may similarly be converted into Q1 if quantum queries are required only on some small portion of inputs.

Quantum Speed-Up for Linear Cryptanalysis. Linear cryptanalysis [50] is one of the most fundamental techniques in symmetric cryptanalysis. Kaplan et al. [43] has already shown a quadratic quantum speed-up for linear attacks. However, their distinguisher uses only one-dimensional linear approximations, while classical attacks often exploit multidimensional linear approximations to reduce

[1] Note that attacks that gather encrypted data now and execute quantum algorithms later (after the realization of a large-scale quantum computer) are also in Q1.

[2] When attack targets are primitives without secret keys, e.g. hash functions, it is reasonable to assume attackers can compute all functions in quantum superposition. Namely, attacks are always Q2, or there is no distinction between Q1 and Q2.

complexity [33–35]. In fact, it is unclear whether Kaplan et al.'s distinguisher can be sped-up further even if multiple linear approximations are available, due to the following reason.

Kaplan et al.'s distinguisher relies on the quantum counting algorithm [18], which (approximately) counts the number of x satisfying $F(x) = 1$ for an (efficiently computable) Boolean function F. Since classical one-dimensional linear distinguishers work just by counting the number of messages satisfying a linear approximation, such F is naturally defined in the one-dimensional case, and the quantum counting algorithm can be applied.

Meanwhile, classical multidimensional linear distinguishers are based on sophisticated statistical tests exploiting a relationship between capacity and a sum of squared correlations in a clever way. It is highly unclear whether there exists an efficiently computable Boolean function F such that just counting the number of x satisfying $F(x) = 1$ corresponds to performing such statistical tests.

Thus it is natural to ask whether there exists another quantum technique for linear distinguishers running faster than Kaplan et al.'s when a multidimensional linear approximation is available.

Multidimensional linear cryptanalysis has many variants including (multidimensional) zero correlation linear cryptanalysis [9] and generalized linear cryptanalysis on an arbitrary finite abelian group [4]. However, no previous work has shown quantum speed-up for them. A technique speeding-up multidimensional linear distinguishers may lead to a speed-up for such variants, which is of another interest. It may also lead to a speed-up for some integral distinguishers, because a class of multidimensional zero correlation linear distinguishers corresponds to integral distinguishers based on balanced functions, as shown by Bogdanov et al. and Sun et al [8, 60].

Quadratic Barrier. Due to Grover's generic quadratic speed-up for exhaustive search, the only way to break more rounds in the quantum setting, especially for key-recovery and distinguishing attacks, is to achieve a super-quadratic speed-up[3]. Hence, such a speed-up is one of the main goals in quantum cryptanalysis on symmetric-key cryptosystems.

Some previous works have already achieved such a speed-up, even in the Q1 model [16], but the types of techniques are limited. All of them exploit algebraic

[3] The reason of this is as follows: Consider attacks on a k-bit-key block cipher. Assume that, in the classical setting, we know a valid dedicated attack (i.e., an attack faster than 2^k) on r rounds of the cipher, but know only an *invalid* attack (i.e., attack requiring time more than 2^k) for $(r+1)$-rounds. Especially, r rounds of the cipher are classically broken but $(r+1)$ rounds are not. Let T_c be the classical complexity of the invalid $(r+1)$-round attack. Since this attack is classically invalid, $T_c > 2^k$ holds. Suppose we achieve some quantum speed-up for the $(r+1)$-round attack and the resulting complexity is T_q. Then, since the generic complexity of key-recovery is $\sqrt{2^k}$ in the quantum setting (by the Grover search), the attack after quantum speed-up is valid (i.e., $T_q < \sqrt{2^k}$ and $(r+1)$ rounds are broken) only if the speed-up is more-than-quadratic and $T_q < \sqrt{T_c}$ holds.

structures such as *hidden periods* or *shifts* of target ciphers by using Simon's algorithm or a related algorithm solving an algebraic problem.

Moreover, few previous works have succeeded to achieve a more than quadratic speed-up on classical cryptanalytic techniques such as differential, linear, or integral cryptanalysis. The only one exception is the quantum versions of (advanced) slide attacks [42, 14], but they also rely on special algebraic structures like hidden periods. Whether a more than quadratic speed-up is possible for other major classical techniques (without relying on periods or shifts) has been an important open problem for years.

1.1 Our Contributions

This paper shows that quantum speed-up for multidimensional (zero correlation) linear and integral distinguisher can be achieved by using a modified version of the subroutine of Simon's algorithm, without exploiting hidden periods or shifts. Especially, we show that some special versions of integral distinguishers achieve a more-than-quadratic speed-up.

First, we observe that Simon's algorithm has a close relationship with linear correlations of functions via Fourier transform. Simon's algorithm iterates a subroutine, which is composed of the Hadamard transform and an oracle query to the target function. We find that, with a slight modification made, the subroutine outputs a pair of linear masks of the target function with probability proportional to the squared linear correlation. Since it extracts linear correlations of a function into quantum amplitude, we call the subroutine after the modification the *correlation extraction algorithm*, or CEA for short.

Second, we show that multidimensional linear distinguishers can be sped-up by combining CEA and the Quantum Amplitude Amplification (QAA) technique. As an application example, we show that the multidimensional distinguishers on FEA-1 and FEA-2 by Beyne [6] can be sped-up from $O(2^{(r/4-3/4)n})$ and $O(2^{(r/6-3/4)n})$ to $O(2^{(r/8-1/4)n})$ and $O(2^{(r/12-1/4)n})$, respectively, when messages are n bits and the number of rounds is r.

Then we show that CEA also leads to a speed-up for multidimensional zero correlation linear distinguishers. Our technique leads to quantum distinguishers on 5-round balanced Feistel running in time $O(2^{n/2})$ when round functions are bijections and the entire width of the cipher is n, and distinguishers on Type-I/II generalized Feistel structures. (See Table 2 in the appendix for details.)

Finally, we show a speed-up for integral distinguishers. The speed-up is obtained via the correspondence of integral and zero correlation properties observed by Bogdanov et al. and Sun et al. [8, 60], and applicable when integral properties are based on balanced functions. Especially, we observe the possibility of a more than quadratic speed-up when there are *multiple* integral properties on mutually orthogonal subspaces, which appear in some SPN ciphers such as the 3.5-round AES. As a notable example, we show that a toy 4-bit-cell SPN cipher having the same integral property as the 2.5-round AES is distinguished only by a *single* quantum query. Such a single-query attack is almost impossible in the classical setting (unless another weakness exists), and the example illustrates a new type

of qualitative difference between classical and quantum computation that has not been observed before[4].

Note that all of our attacks do not require the target cipher to have algebraic structures such as hidden periods or shifts. It is somewhat surprising that (a modified) Simon's algorithm, which was primarily developed to solve an algebraic problem of hidden periods, leads to non-trivial speed-ups for various classical attacks not relying on hidden periods nor shifts.

Our technique extends to generalized linear distinguishers on arbitrary finite abelian groups [4] by replacing the Hadamard transform in CEA with the general Quantum Fourier Transform (QFT). As an application, we show a speed-up for the distinguisher by Beyne [6] on the FF3-1 structure. The amount of speed-up is the same as that for FEA-1.

A drawback of our technique is that it cannot be applied to integral distinguishers based on zero-sum properties, although zero-sum properties are usually used to extend distinguishers into key-recovery attacks on more rounds. Especially, it does not directly lead to breaking more rounds than classical attacks. Still, we believe that our techniques are novel and general, and will inspire other new types of quantum attacks in both of the Q1 and Q2 models.

1.2 Related Works

Quantum speed-up for integral attacks has already been studied in, e.g., [15], but zero-sum properties are used and the distinguisher part itself is not sped-up. A recent work by Shi et al. [58] also studies zero correlation linear attacks in the quantum setting, but it mainly focuses on how to find zero correlation linear approximations by using quantum computers, and does not have much overlap with our results.

Schrottenloher's Key-Recovery Attack. Another recent concurrent and independent work by Schrottenloher [57] showed how to obtain quantum speed-up for linear *key-recovery* attacks.

Classical linear distinguishers are often combined with efficient key-recovery attacks using the FFT [23, 61, 30]. What Schrottenloher did is to combine such classical techniques with the QFT. Computing convolution of some Boolean functions related to linear approximations in quantum superposition, Schrottenloher's algorithm produces some quantum superposition of *subkey candidates* in such a way that the quantum amplitude are proportional to their experimental correlations. Then the amplitude of the right key is amplified by QAA.

Note that our main interest is to achieve a speed-up for *multidimensional* (zero correlation) linear distinguishers. Schrottenloher's work [57] also deals with multiple linear approximations, but the existence of multiple approximations improves only precision of attack by a constant factor, and essentially does not

[4] Bonnetain showed a single query attack on the one-time pad encryption scheme by making quantum registers for messages and ciphertexts disentangled [11], but the attack target and technique are quite different from ours.

contribute much to reducing the time complexity. Additionally, zero correlation linear or integral attacks are not studied in [57].

One would expect that a more speed-up for key-recovery is obtained by combining our technique and Schrottenloher's. Still, the mechanism of the two techniques is quite different (Schrottenloher uses the QFT to compute convolution in superposition to obtain a superposition of key candidates, while we use it to extract correlations of multidimensional linear approximations), and so far we do not have any idea on how to combine them. Studying theoretical connection between them and reducing the time complexity of key-recovery exploiting (zero correlation) multidimensional linear approximations is definitely an important and interesting future work.

1.3 Organization

Section 2 introduces basic notions and facts. Section 3 reviews classical (multidimensional) linear distinguishers and Kaplan et al.'s quantum one-dimensional linear distinguisher. Section 4 studies relationships between the Simon's subroutine and linear correlations, and introduces CEA. Sections 5, 6, and 7 show how to achieve a quantum speed-up with CEA for multidimensional linear, zero correlation multidimensional linear, and integral distinguishers, respectively. Section 8 shows the extension to generalized linear distinguishers on an arbitrary finite abelian group. Section 9 concludes the paper.

2 Preliminaries

$\mathbb{F}_2$ denotes the Galois field of order two. We identify the set of n-bit strings $\{0,1\}^n$ and the n-dimensional $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$. Especially, by "bit string" we denote an element of $\mathbb{F}_2^n$ for some n. By $\mathbf{e}_i$ we denote the n-bit string (for some n) of which the i-th bit is 1 and other bits are 0. $x \oplus y$ denotes the addition of x and y in $\mathbb{F}_2^n$, and $x||y$ denotes the concatenation as bit strings. For a bit string $x \in \mathbb{F}_2^n$, we denote the i-th bit (from the left) by x_i. Namely we represent x as $x = x_1||\cdots||x_n$. For $x, y \in \mathbb{F}_2^n$, the dot product of x and y is defined by $x \cdot y := (x_1 \cdot y_1) \oplus \cdots \oplus (x_n \cdot y_n)$. For a vector space $V \subset \mathbb{F}_2^n$ (resp., vector x), $V^\perp$ (resp., $x^\perp$) denotes the subspace that is composed of y satisfying $y \cdot x = 0$ for all $x \in V$ (resp., y satisfying $y \cdot x = 0$). For two vector spaces $V_1, V_2 \subset \mathbb{F}_2^n$, we write $V_1 \perp V_2$ if $v_1 \cdot v_2 = 0$ for all $v_1 \in V_1$ and $v_2 \in V_2$. The event that a (classical or quantum) algorithm $\mathcal{A}$ outputs a classical bit string x is denoted by $x \leftarrow \mathcal{A}$. For a bit string $x \in \mathbb{F}_2^n$ (resp., function $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$), by $\mathsf{msb}_u[x]$ (resp., $\mathsf{msb}_u[f]$) we denote the most significant u bits of x (resp., the function that returns $\mathsf{msb}_u[f(x)]$ for each input x). The notations $\mathsf{lsb}_u[x]$ and $\mathsf{lsb}_u[f]$ are similarly defined for least significant u bits. For a distribution D and a real value X_w depending on a parameter w, $\mathbb{E}_{w\sim D}[X_w]$ denotes the expected value of X_w when w is sampled according to D. It is also denoted by $\mathbb{E}_w[X_w]$ or just $\mathbb{E}[X_w]$ if the distribution is clear from the context. Similar notations are used for variance and the probability of an event. For a unitary operator U, its adjoint is

denoted by U^*. In cryptanalysis of a block cipher E, we regard the unit of time as the time to encrypt a message by E. We assume that readers are familiar with Pearson's chi-squared test of goodness-of-fit. For those who are not, we provide a brief overview about the relationship between the test and distinguishers in Section A in the appendix.

2.1 Linear Approximations and Correlations

The (one-dimensional) linear approximation of a function $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$ for an input mask $\alpha \in \mathbb{F}_2^m$ and output mask $\beta \in \mathbb{F}_2^n$ is the Boolean function defined by $x \mapsto (\alpha \cdot x) \oplus (\beta \cdot f(x))$. The correlation $\mathrm{Cor}(f; \alpha, \beta)$ of this linear approximation is defined by $\mathrm{Cor}(f; \alpha, \beta) := \Pr_x [\alpha \cdot x = \beta \cdot f(x)] - \Pr_x [\alpha \cdot x \neq \beta \cdot f(x)]$. It is well-known that the linear correlation satisfies

$$\mathrm{Cor}(f; \alpha, \beta) = \sum_{x \in \mathbb{F}_2^m} \frac{(-1)^{\alpha \cdot x \oplus \beta \cdot f(x)}}{2^m}. \tag{1}$$

In addition, we need the following claim for analysis of attacks.

Claim 1 (Distribution of capacity of a random permutation.) *Let $V \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$ be a vector space and S be an arbitrary basis of V. Then, for a randomly chosen permutation P, the value $2^n \cdot \sum_{(\alpha,\beta) \in V - \{\mathbf{0}\}} \mathrm{Cor}(P; \alpha, \beta)^2$ approximately follows the χ^2 distribution with $2^v - 2^u - 2^w + 1$ degrees of freedom. Here, $v := \dim(V)$, $u := \dim(V \cap \mathbb{F}_2^n \times \{0^n\})$ and $w := \dim(V \cap \{0^n\} \times \mathbb{F}_2^n)$.*

This claim is conjectured in [2]. We do not have a formal proof, but explain why the claim is plausible in Section D in the appendix.

2.2 Balanced Function and Zero-Sum Property

Integral cryptanalysis [45], which was initially proposed as a dedicated attack on the block cipher SQUARE [25], exploits the *zero-sum property* of (a part of) ciphers. Here, we say that a function $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$ has the zero-sum property if $\sum_x f(x) = 0$. Moreover, we say that a function f is balanced if $|f^{-1}(y)| = |f^{-1}(y')|$ holds for any y, y' in the range of f. A balanced function has the zero-sum property but the converse does not necessarily hold. In some previous works, the zero-sum property is called "balanced property", but this paper uses the term "balanced" only when referring to a balanced function in the above sense.

2.3 Quantum Computation

We assume that the readers are familiar with quantum computation and linear algebra (see, e.g., [54] for basics of quantum computation). We adopt the standard quantum circuit model and do not take the cost of quantum error correction into account. I_m denotes the identity operator on an m-qubit system and H denotes the (1-qubit) Hadamard transform. For a function $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$, U_f

denotes the unitary operator defined by $U_f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$. Namely, U_f is the quantum oracle of f. All quantum attacks in this paper are Quantum Chosen-Plaintext Attacks (QCPAs, in the Q2 model), and the quantum encryption oracle U_{E_K} of a target cipher E_K is assumed to be available. If E_K is a tweakable block cipher, adversaries query tweaks also in quantum superposition.

Quantum Amplitude Amplification. Here we recall the Quantum Amplitude Amplification (QAA) technique [18], which is a generalization of Grover's algorithm [31]. Let $f : \mathbb{F}_2^m \to \mathbb{F}_2$ be a Boolean function, U be a unitary operator acting on an m-qubit system, and p denote the probability that we observe a bit string x satisfying $f(x) = 1$ when the state $U|0^m\rangle$ is measured by the computational basis. In addition, let $\mathcal{S}_f$ and $\mathcal{S}_0$ be the unitary operators acting on an m-qubit quantum system defined by $\mathcal{S}_f |x\rangle = (-1)^{f(x)} |x\rangle$ and $\mathcal{S}_0 |x\rangle = (-1)^{\delta_{0^m,x}} |x\rangle$, where $\delta_{0^m,x}$ is Kronecker's delta.

Proposition 1 (Quantum amplitude amplification). *In the above setting, let $Q(U,f) := -U\mathcal{S}_0 U^* \mathcal{S}_f$. When the state $Q(U,f)^i U|0^m\rangle$ is measured by the computational basis for some $i > 0$, an outcome x satisfying $f(x) = 1$ is obtained with probability $\sin^2((2i+1)\cdot \arcsin(\sqrt{p}))$. Especially, such an x is obtained with probability at least $\max\{p, 1-p\}$ by setting $i = \lceil \pi/4 \arcsin(\sqrt{p}) \rceil$.*

Grover's algorithm is obtained when $U = H^{\otimes m}$. Here, $p = |f^{-1}(1)|/2^m$ and an $x \in f^{-1}(1)$ is found by applying $Q(H^{\otimes m}, f)$ at most $\sqrt{2^m/|f^{-1}(1)|}$ times.

Applications to Distinguishers. A typical task in cryptanalysis is to distinguish two distributions of functions. That is, under the assumption that a function f is chosen from a distribution D_1 or D_2, an adversary tries to judge which distribution f is chosen from. For linear distinguishers, D_1 (resp., D_2) corresponds to a linear approximation of a real block cipher (resp., a random permutation).

A counterpart of such a task in the quantum setting is to distinguish two distributions of unitary operators. That is, under the assumption that a unitary operator U is chosen according to a distribution D_1 or D_2, an adversary tries to judge which distribution U is chosen from[5].

QAA can be used to solve such a task. Assume that an adversary has access to not only U but U^*, and that U acts on an n-qubit system[6]. Moreover, suppose that we know a Boolean function $F : \mathbb{F}_2^n \to \mathbb{F}_2$ satisfying the following conditions.

(1) If U is chosen from D_1, then the probability $p_U := \Pr\left[x \xleftarrow{\text{measure}} U|0^n\rangle : F(x) = 1\right]$ is relatively high on average.
(2) If U is chosen from D_2, then p_U is relatively low on average.

[5] A typical example is that D_1 corresponds to the quantum encryption oracle U_{E_K} of a block cipher E_K while D_2 to the oracle U_P of a random permutation P (choosing K or P randomly corresponds to sampling according to D_1 or D_2).

[6] If U is the quantum oracle U_f of a function f, then $U_f^* = U_f$ holds. Especially, an access to $U = U_f$ automatically means an access to U^*.

Specifically, assume we know a value t satisfying $\mathbb{E}_{U\sim D_1}[p_U] \geq t \gg \mathbb{E}_{U\sim D_2}[p_U]$. Then we can distinguish D_1 and D_2 by using QAA on U and F: If U is chosen from D_1, then QAA with $O(\sqrt{t^{-1}})$ applications of U, U^*, and $\mathcal{S}_F$ will find x satisfying $F(x) = 1$ because $p_U \geq t$. If U is chosen from D_2, such QAA will not find x because $t \gg p_U$ and the number of iterations is not large enough.

More precisely, since we know only the lower bound of $\mathbb{E}_{U\sim D_1}[p_U]$, we run multiple instances of QAAs with the number of iteration randomized as follows.

QAA for Distinguisher ($\mathcal{QD}$).

1. For $j = 1, \ldots, s$, do:
   (a) Choose i from the set of integers from 0 to $\left\lfloor \frac{1}{\sin\left(2\cdot\arcsin\left(\sqrt{t}\right)\right)} \right\rfloor$ uniformly at random.
   (b) Apply $Q(U,F)^i U$ to $|0^n\rangle$ and measure the entire state by the computational basis, and let x be the outcome.
   (c) Compute $F(x)$. If $F(x) = 1$, return 1 and abort.
2. Return 0.

Here, s is a positive integer constant chosen depending on applications. We denote the above algorithm by $\mathcal{QD}$.

The idea of randomly choosing the number of iteration is just a straightforward adaptation of previous works on Grover's algorithm and QAA without knowing initial success probability [18, 17].

Proposition 2. *With the above setting and notions, suppose* $1/4 > t > 0$. *Then,* $\mathcal{QD}$ *applies* U, U^*, *and* $\mathcal{S}_F$ *at most* $s(\frac{1}{\sqrt{t}} + 1)$ *times and (1) returns* 1 *with probability at least* $(1 - (\frac{3}{4})^s) \cdot \Pr_{U\sim D_1}[1/4 > p_U \geq t]$ *if* U *is chosen according to* D_1 *and (2) returns* 1 *with probability at most* $s \cdot (\frac{16t'}{t} + \frac{20t'}{\sqrt{t}}) + \Pr_{U\sim D_2}[t' < p_U]$ *for any* $t' > 0$ *satisfying* $4\sqrt{t'/t} + 2\sqrt{t'} < \pi/2$ *if* U *is chosen according to* D_2.

The interpretation of the proposition is as follows. Suppose that p_U is distributed around t (resp., t') if U is chosen according to D_1 (resp., D_2), and $1/4 > t \gg t'$ holds. For a sufficiently large constant s (e.g., $s = 3$), the proposition guarantees that $\mathcal{QD}$ returns 1 with probability $\geq 1/2$ (resp., only with a negligibly small probability) when U is chosen according to D_1 (resp., D_2). Hence D_1 is distinguished from D_2. The proof of Proposition 2 is a straightforward application of some lemmas in previous works [17, 18], though, we provide a proof in Section B in the appendix for completeness.

Simon's algorithm. Simon's quantum algorithm [59] finds a period of a periodic function. More precisely, it solves the following problem.

Problem 1. Let $s \in \mathbb{F}_2^m$ be a (secret) constant, and $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$ be a function satisfying the following properties C1 and C2.

C1. $f(x \oplus s) = f(x)$ for all x. Namely, f is a periodic function with period s.
C2. $f(x) \neq f(y)$ if $x \neq y$ and $x \oplus s \neq y$.

Given the (quantum) oracle of f, find s.

The classical complexity to solve the problem is $\Theta(2^{m/2})$ but Simon's algorithm, which runs as follows, solves it in polynomial time with high probability.

1. For $i = 1, 2, \ldots, 2m$, execute the following subroutine (a)-(e).
   (a) Prepare the initial state $|0^m\rangle |0^n\rangle$.
   (b) Apply the m-qubit Hadamard transform $H^{\otimes m}$ on the first m qubits.
   (c) Apply U_f on the state (i.e., make a quantum query to f).
   (d) Apply the $H^{\otimes m} \otimes I_n$ on the state.
   (e) Measure the first m qubits by the computational basis, discard the remaining n-qubits, and return the observed m-bit string (denoted by α_i).
2. If $\dim(\mathrm{Span}_{\mathbb{F}_2}(\alpha_1, \ldots, \alpha_{2m})) = m - 1$, compute and output the unique $s' \in \mathbb{F}_2^m \setminus \{0^m\}$ such that $s' \cdot \alpha_i = 0$ for $i = 1, \ldots, 2m$. If $\dim(\mathrm{Span}_{\mathbb{F}_2}(\alpha_1, \ldots, \alpha_{2m})) \neq m - 1$, output $\perp$.

Simon showed that each α_i is uniformly distributed over the subspace $\{v \in \mathbb{F}_2^m | v \cdot s = 0\}$, and thus the algorithm returns the period s with high probability. We refer to the subroutine (a)-(e) as Simon's subroutine.

Many papers (e.g., [46, 47, 42]) showed polynomial-time quantum attacks on symmetric cryptosystems by using Simon's algorithm. In fact only C1 is satisfied in those applications and C2 is not necessarily satisfied. Still, C1 guarantees that the subroutine (a)-(e) always returns an α_i satisfying $\alpha_i \cdot s = 0$ [42].

3 Classical and Kaplan et al.'s Linear Distinguishers

Here we review classical (multidimensional) linear distinguishers and Kaplan et al.'s quantum one-dimensional linear distinguisher [43].

3.1 Classical One-Dimensional Linear Distinguisher

The linear correlation $\mathrm{Cor}(P; \alpha, \beta)$ of an n-bit random permutation P approximately follows the normal distribution $\mathcal{N}(0, 2^{-n})$ for an arbitrary mask (α, β) with $\alpha, \beta \neq 0^n$ [27]. Thus, if the correlation $\mathrm{Cor}(E_K; \alpha, \beta)$ for a block cipher E_K with $\alpha, \beta \neq 0^n$ significantly deviates from the segment $[-2^{-n/2}, 2^{-n/2}]$, then E_K can be distinguished by collecting a list $L = \{(P_1, C_1), \ldots, (P_N, C_N)\}$ for random $P_1, \ldots, P_N$, and checking if the estimated empirical correlation

$$\widehat{\mathrm{Cor}}(E_K; \alpha, \beta) = \frac{\#\{(P, C) \in L | \alpha \cdot P = \beta \cdot C\} - \#\{(P, C) \in L | \alpha \cdot P \neq \beta \cdot C\}}{N}$$

is far from $[-2^{-n/2}, 2^{-n/2}]$. The attack succeeds with a high probability if $N \gtrapprox \mathrm{Cor}(E_K; \alpha, \beta)^{-2}$.

3.2 Classical Multidimensional Linear Distinguishers

A natural idea to enhance the power of linear distinguishers is to utilize multiple linear approximations. Some early works indeed show such attacks, assuming the existence of statistically independent multiple approximations [41, 7]. However, the assumption does not necessarily hold in general [51]. Instead, Hermelin et al. [36] proposed to use multidimensional linear approximations, i.e., sets of linear approximations of which the input-output masks form a vector space.

Specifically, let $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$ be a function, $V \subset \mathbb{F}_2^m \times \mathbb{F}_2^n$ be a set of input-output masks for f that is a vector space, and $S := \{(\alpha_1, \beta_1), \ldots, (\alpha_\ell, \beta_\ell)\}$ be a basis of V. Then the multidimensional linear approximation of f (w.r.t. (V, S)) is defined as the function $\mathsf{Lin}_S^f : \mathbb{F}_2^m \to \mathbb{F}_2^\ell$ such that

$$\mathsf{Lin}_S^f(x) = (\alpha_1 \cdot x \oplus \beta_1 \cdot f(x), \ldots, \alpha_\ell \cdot x \oplus \beta_\ell \cdot f(x)).$$

Define a distribution p_S^f on $\mathbb{F}_2^\ell$ by $p_S^f(z) := \Pr\left[x \xleftarrow{\$} \mathbb{F}_2^m : \mathsf{Lin}_S^f(x) = z\right]$.

Below we denote the zero vector $(0^m, 0^n)$ by $\mathbf{0}$. We say that the input and output masks are linearly independent if $V = V_1 \times V_2$ holds for some $V_1 \subset \mathbb{F}_2^m$ and $V_2 \subset \mathbb{F}_2^n$. Moreover, we say that the input and output masks are linearly completely dependent if there exists a basis $\{(\alpha_i, \beta_i)\}_{1 \le i \le \dim(V)}$ of V such that both of $\{\alpha_i\}_{1 \le i \le \dim(V)}$ and $\{\beta_i\}_{1 \le i \le \dim(V)}$ are linearly independent in $\mathbb{F}_2^n$.

The advantage of considering a set of masks forming a vector space is that we can utilize a link of the sum of the squared correlations to the *capacity* of p_S^f and Pearson's chi-squared test: Here, the capacity of a probability function (distribution) p over $\mathbb{F}_2^\ell$ is the value defined[7] by

$$\mathrm{Cap}(p) := 2^\ell \sum_{z \in \mathbb{F}_2^\ell} (p(z) - 2^{-\ell})^2.$$

The important well-known fact is that

$$\mathrm{Cap}(p_S^f) = \sum_{(\alpha,\beta) \in V - \{\mathbf{0}\}} \mathrm{Cor}(f; \alpha, \beta)^2 \tag{2}$$

holds for the multidimensional approximation of f. Moreover, suppose a list of random input-output pairs $L = \{(P_1, C_1), \ldots, (P_N, C_N)\}$ is given. Then the capacity $\mathrm{Cap}(\hat{p}_S^f)$ of the estimated empirical distribution $\hat{p}_S^f$ (defined by $\hat{p}_S^f(z) := \frac{\#\{(P,C) \in L | \mathsf{Lin}_S^f(P) = z\}}{N}$) multiplied by N is equal to the test statistic of the Pearson's chi-squared goodness-of-fit test (for testing the goodness-of-fit of p_S^f and the uniform distribution on $\mathbb{F}_2^\ell$).

The idea of multidimensional linear distinguishers for a block cipher E_K is that the distribution $p_S^{E_K}$ is far from uniform if the right hand side of Eq. (2) with $f = E_K$ is sufficiently large for random K, and thus E_K can be distinguished

[7] In fact this is the χ^2-divergence between p and the uniform distribution over $\mathbb{F}_2^\ell$. We use the term *capacity* following previous works on linear cryptanalysis.

from random by checking whether the test statistic of the Pearson's chi-squared test is larger than a certain threshold. Specifically, given a list of (real) random plaintext-ciphertext pairs $L = \{(P_1, C_1), \ldots, (P_N, C_N)\}$, we count $\mathsf{num}(z) := \{(P_i, C_i) \in L | \mathsf{Lin}_S^{E_K}(P_i) = z\}$ for each z, and compute the test statistic $\chi^2_{\text{real}} := N2^\ell \sum_z (\mathsf{num}(z)/N - 2^{-\ell})^2 = N \cdot \text{Cap}(\hat{p}_S^{E_K})$. Then χ^2_{real} is approximately distributed around $(2^\ell - 1) + N \sum_{(\alpha,\beta) \in V - \{\mathbf{0}\}} \text{Cor}(E_K; \alpha, \beta)^2$. If the plaintext-ciphertext pairs are generated from a random permutation, then $\mathsf{num}(z)$ approximately follows the uniform distribution. Thus, the similarly computed statistic χ^2_{ideal} approximately follows the χ^2 distribution with $(2^\ell - 1)$ degrees of freedom (denoted by $\chi^2_{2^\ell - 1}$), of which the standard deviation is $\sqrt{2(2^\ell - 1)}$. Hence E_K can be distinguished from a random permutation with a constant advantage when[8] $N \gg \sqrt{2^\ell} / \sum_{(\alpha,\beta) \in V - \{\mathbf{0}\}} \text{Cor}(E_K; \alpha, \beta)^2 = \sqrt{2^\ell}/\text{Cap}(p_S^{E_K})$, by checking whether the test statistic is larger than $(2^\ell - 1) + \sqrt{2(2^\ell - 1)}$ or not.

Some Remarks. The arguments in the above paragraph are mainly based on [6, Section 4.3]. Strictly speaking, the statistic in the ideal world χ^2_{ideal} does not follow $\chi^2_{2^\ell - 1}$ actually because the squared correlation $\text{Cor}(P; \alpha, \beta)^2$ is not zero on average even for a random permutation P for $\alpha, \beta \neq 0^n$. Still, the difference of χ^2_{ideal} and $\chi^2_{2^\ell - 1}$ is very small compared to the difference of χ^2_{real} and $\chi^2_{2^\ell - 1}$, and it is usually (and implicitly) assumed that the above arguments heuristically work in practice. Meanwhile, zero-correlation linear cryptanalysis *does* exploit such small difference, which we will explain later.

Some early works showed that distinguishers based on the Log Likelihood Ratio (LLR) test [3, 33, 34] requires only $O(1/\text{Cap}(p_S^{E_K}))$ data instead of $O(\sqrt{2^\ell}/\text{Cap}(p_S^{E_K}))$ of the χ^2-test-based distinguishers, and the LLR-test-based distinguishers perform better. However, the LLR test requires accurate knowledge on key-dependent distributions of multidimensional linear approximations, which is not often the case as pointed out by Cho [22].

3.3 Kaplan et al.'s Quantum One-Dimensional Linear Distinguisher

Kaplan et al. [43] observed that a quadratic quantum speed-up can be obtained for linear distinguishers by using the quantum counting algorithm [18].

Roughly speaking, the quantum counting algorithm achieves a quadratic speed-up to solve the problem of estimating $M := \#\{x | F(x) = 1\}$ for a Boolean function F. Making $O(q)$ quantum queries to F, it returns an approximation $\tilde{M}$ of M satisfying $|\tilde{M} - M| \leq O\left(\frac{\sqrt{M(2^n - M)}}{q} + \frac{2^n}{q^2}\right)$.

Now, suppose that there exists a linear approximation of an n-bit block cipher E_K satisfying $c := |\text{Cor}(E_K; \alpha, \beta)| \gg 2^{-n/2}$ for a random key K, and let F be

[8] The squared correlation and capacity can significantly change depending on keys in general, but they are often estimated by their averages under the assumption that they concentrate around the mean. As the first step of achieving quantum speed-up for multidimensional linear attacks, we also assume this. An in-depth study about the key-dependence of complexity is an important future work.

the Boolean function such that $F(x) = 1$ iff $\alpha \cdot x \oplus \beta \cdot E_K(x) = 0$. Then, E_K can be distinguished by estimating $M = \#\{x|F(x) = 1\}$ and checking whether $|M - \frac{2^n}{2}| \gg 2^{n/2}$. Using the quantum counting algorithm, one can obtain an estimation of $\tilde{M}$ with sufficient precision for distinguisher ($|\tilde{M} - M| \leq \frac{M}{a}$ for a small integer $a > 0$) in time $O(1/c)$. Compared to the classical complexity of $O(1/c^2)$, a quadratic speed-up is achieved.

Extension to Multidimensional Linear Distinguishers? After seeing Kaplan et al.'s work, it is natural to ask whether their technique can be extended to multidimensional linear distinguishers. However, to apply the quantum counting algorithm to solve a problem, one has to construct an efficiently computable Boolean function F in such a way that counting the number of x satisfying $F(x) = 1$ solves the problem. In the one-dimensional case F is obtained in a quite natural way as explained above, but in the multidimensional case essentially we have to construct F in such a way to achieve a quadratic speed-up for Pearson's chi-squared test applied to the distribution $p_S^{E_K}$. It seems highly unclear whether such F exists, and thus we seek for another technique.

4 New Observation on Simon's Algorithm

As explained in Section 2, the subroutine of Simon's algorithm uses only the quantum oracle of a target function and the Hadamard transform, which is the Fourier transform over the group $(\mathbb{Z}/2\mathbb{Z})^n$. Meanwhile, a well-know fact is that linear correlations have strong relationships with Fourier transform. This section shows that a slightly modified version of Simon's subroutine, which we call CEA, returns input and output masks of a function with a probability proportional to the linear correlations. Later we show that CEA can be utilized to obtain speed-up for various techniques including multidimensional linear distinguishers. Since $\mathbb{F}_2^n$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ as Abelian groups, in what follows we identify $\mathbb{F}_2^n$ with $(\mathbb{Z}/2\mathbb{Z})^n$.

4.1 Fourier Transform

First, we recall the Fourier transform (over $(\mathbb{Z}/2\mathbb{Z})^n$) and its relationship with linear cryptanalysis and quantum computation. The Fourier transform of a function $F : \mathbb{F}_2^n \to \mathbb{C}$ is the function $\mathcal{F}F : \mathbb{F}_2^n \to \mathbb{C}$ defined by $\mathcal{F}F(x) := \sum_{y \in \mathbb{F}_2^n} \frac{(-1)^{x \cdot y} F(y)}{\sqrt{2^n}}$.

Relationship with Linear Correlations. It is well-known that the linear correlation of an arbitrary function f is obtained by applying the Fourier transform on a function naturally defined from f [24, 63].

For arbitrary function $f : \mathbb{F}_2^m \to \mathbb{F}_2^n$, let $f_{\mathrm{emb}} : \mathbb{F}_2^m \times \mathbb{F}_2^n \to \mathbb{C}$ be the function defined by $f_{\mathrm{emb}}(x, y) = 1$ if $f(x) = y$ and $f_{\mathrm{emb}}(x, y) = 0$ otherwise[9]. Then some

[9] "emb" is an abbreviation of "embedding".

straightforward calculation shows

$$\mathcal{F}f_{\mathrm{emb}}(\alpha,\beta) = \sqrt{2^{m-n}} \cdot \mathrm{Cor}(f;\alpha,\beta). \tag{3}$$

Relationship with Quantum Computation. The relationship with quantum computation is quite clear. The Fourier transform on $\mathbb{F}_2^n$ exactly corresponds to the Hadamard operator $H^{\otimes n}$. For instance, let $\psi : \mathbb{F}_2^n \to \mathbb{C}$ and $|\psi\rangle := \sum_{x\in\mathbb{F}_2^n} \psi(x)\,|x\rangle$. Then

$$H^{\otimes n}\,|\psi\rangle = \sum_{y\in\mathbb{F}_2^n} \mathcal{F}\psi(y)\,|y\rangle \tag{4}$$

holds. (Note that this property holds regardless of the norm of $|\psi\rangle$.) In fact this is one of the most important sources of quantum speed-up: While the classical FFT requires time $O(n2^n)$ to compute the Fourier transform of a function, an application of the Hadamard transform to a quantum state requires time $O(1)$.

4.2 Extracting Correlations by (Modified) Simon's Subroutine

Here we show that Simon's subroutine with a slight modification returns input and output masks for linear approximations with high correlation. We call the resulting algorithm Correlation Extraction Algorithm (CEA) because it extracts linear correlations into the quantum amplitude of a state, and we denote it by CEA^f when applied to a function f. Specifically, the algorithm runs as follows.

Algorithm CEA^f.

(a) Prepare the initial state $|0^m\rangle\,|0^n\rangle$.
(b) Apply the m-qubit Hadamard transform $H^{\otimes m}$ on the first m qubits.
(c) Apply U_f on the state (i.e., make a quantum query to f).
(d) Apply the $(m+n)$-qubit Hadamard transform $H^{\otimes(m+n)}$ on the state.
(e) Measure the entire $(m+n)$ qubits by the computational basis and return the observed $(m+n)$-bit string $\alpha\|\beta$ ($\alpha \in \mathbb{F}_2^m$ and $\beta \in \mathbb{F}_2^n$).

The underlines indicate the parts modified from the original Simon's subroutine on p.9. CEA^f is different from the original Simon's subroutine only in that CEA^f does not discard the last n qubits and measure them after applying $H^{\otimes n}$.

Note that this change does not affect the distribution of α in Step (e). Especially, α is just uniformly distributed over the subspace $\{v \in \mathbb{F}_2^m | v \cdot s = 0\}$ if f satisfies the conditions of Problem 1. Thus there is nothing new if we focus only on α. However, we observe that CEA^f shows an interesting link to linear correlations when β is taken into account, as in the following proposition.

Proposition 3. *The quantum state of* CEA^f *before the final measurement is*

$$\sum_{\alpha\in\mathbb{F}_2^m,\beta\in\mathbb{F}_2^n} \frac{\mathrm{Cor}(f;\alpha,\beta)}{\sqrt{2^n}}\,|\alpha\rangle\,|\beta\rangle\,. \tag{5}$$

In particular, for any subset $S \subset \{0,1\}^m \times \{0,1\}^n$,

$$\Pr\left[(\alpha,\beta) \leftarrow \mathsf{CEA}^f : (\alpha,\beta) \in S\right] = \sum_{(\alpha,\beta)\in S} \frac{\mathrm{Cor}(f;\alpha,\beta)^2}{2^n} \tag{6}$$

holds.

Proof. The quantum state of CEA^f before the final measurement is

$$\begin{aligned}
H^{\otimes(m+n)} U_f \left(H^{\otimes m} \otimes I_n\right) |0^m\rangle |0^n\rangle &= H^{\otimes(m+n)} U_f \sum_{x\in\mathbb{F}_2^m} \frac{1}{\sqrt{2^m}} |x\rangle |0^n\rangle \\
&= H^{\otimes(m+n)} \sum_{x\in\mathbb{F}_2^m} \frac{1}{\sqrt{2^m}} |x\rangle |f(x)\rangle \\
&\overset{\text{Def. of } f_{\mathrm{emb}}}{=} H^{\otimes(m+n)} \sum_{x\in\mathbb{F}_2^m, y\in\mathbb{F}_2^n} \frac{f_{\mathrm{emb}}(x,y)}{\sqrt{2^m}} |x\rangle |y\rangle \\
&\overset{\text{Eq. (4)}}{=} \sum_{\alpha\in\mathbb{F}_2^m, \beta\in\mathbb{F}_2^n} \frac{\mathcal{F} f_{\mathrm{emb}}(\alpha,\beta)}{\sqrt{2^m}} |\alpha\rangle |\beta\rangle \\
&\overset{\text{Eq. (3)}}{=} \sum_{\alpha\in\mathbb{F}_2^m, \beta\in\mathbb{F}_2^n} \frac{\mathrm{Cor}(f;\alpha,\beta)}{\sqrt{2^n}} |\alpha\rangle |\beta\rangle .
\end{aligned}$$

Hence we have Eq. (5). Eq. (6) immediately follows from Eq. (5). □

Some Remarks. CEA is quite close to the Bernstein-Vazirani algorithm [5] when $n = 1$ and some previous works [20, 64] already observes similar relationships between linear correlations and the Bernstein-Vazirani algorithm. Still, analysis in the previous works is done only in the case of $n = 1$. To obtain speed-up for multidimensional (zero correlation) linear and integral distinguishers, our analysis for general n involving both input and output masks is essential. Furthermore, we observe that a similar relationship holds for generalized linear correlations over an arbitrary finite abelian group and the general quantum Fourier transformation. See Section 8 for details.

5 Speed-Up for Multidimensional Linear Distinguishers

By using the CEA in the previous section, here we show quantum linear distinguishers achieving a bigger speed-up than Kaplan et al.'s when a multidimensional linear approximation with high correlations exists. Recall that what the algorithm CEA^f does is to apply the unitary operator $H^{\otimes(m+n)} U_f (H^{\otimes m} \otimes I_n)$ on $|0^m\rangle |0^n\rangle$ and measure the entire state by the computational basis. By abuse of notation, let CEA^f also denote the operator $H^{\otimes(m+n)} U_f (H^{\otimes m} \otimes I_n)$ itself.

We show three distinguishers[10] $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. $\mathcal{A}_1$ is a general distinguisher applicable to arbitrary multidimensional linear approximations. $\mathcal{A}_2$ (resp., $\mathcal{A}_3$) is applicable only when the input and output masks are linearly independent (resp., completely dependent). Here are some remarks on notations and assumptions.

- We assume that $\sum_{(\alpha,\beta)\in V-\{\mathbf{0}\}} \mathrm{Cor}(E_K;\alpha,\beta)^2 \geq c$ holds with a high probability for some $c > 0$, and that we know the value of c. $\mathcal{O}$ denotes the given oracle, which is either E_K for a random K or a random permutation P.
- (Notations for $\mathcal{A}_2$) When the input and output masks are linearly independent, i.e., $V = V_1 \times V_2$ holds for some subspaces $V_1, V_2 \subset \mathbb{F}_2^n$, we denote $\dim(V_1)$ and $\dim(V_2)$ by u and w, respectively. In addition, $S_1 := \{\alpha_1, \ldots, \alpha_u\}$ and $S_2 := \{\beta_1, \ldots, \beta_w\}$ denotes basis of V_1 and V_2. Without loss of generality, we assume $V_2 = \{\beta || 0^{n-w} | \beta \in \mathbb{F}_2^w\}$ and $\beta_i = \mathbf{e}_i$[11]. Especially, we regard V as a subspace of $\mathbb{F}_2^n \times \mathbb{F}_2^w$.
- (Notations for $\mathcal{A}_3$) When the input and output masks are linearly completely dependent, we fix a basis $S := \{(\alpha_i, \beta_i)\}_{1\leq i\leq \dim(V)}$ of V such that both of $\{\alpha_i\}_{1\leq i\leq \dim(V)}$ and $\{\beta_i\}_{1\leq i\leq \dim(V)}$ are linearly independent in $\mathbb{F}_2^n$. W.l.o.g., we assume $\beta_i = \mathbf{e}_i$[12]. Especially, we regard V as a subspace of $\mathbb{F}_2^n \times \mathbb{F}_2^{\dim(V)}$.

Distinguishers $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. All the three distinguishers are obtained by applying the algorithm $\mathcal{QD}$ of Proposition 2. The difference between the distinguishers is the choice of the parameters s and t, the unitary operator U, and the Boolean function[13] F, which is as follows.

$\mathcal{A}_1$ **(general case):** $(s,t) := (3, c/2^n)$ and $U := \mathsf{CEA}^{\mathcal{O}}$. F is the Boolean function of which the domain is $\mathbb{F}_2^n \times \mathbb{F}_2^n$ and $F(\alpha,\beta) = 1$ iff $(\alpha,\beta) \in V - \{\mathbf{0}\}$.

$\mathcal{A}_2$ **(linearly independent masks):** $(s,t) := (3, c/2^w)$ and $U := \mathsf{CEA}^{\mathsf{msb}_w[\mathcal{O}]}$. F is the Boolean function of which the domain is $\mathbb{F}_2^n \times \mathbb{F}_2^w$ and $F(\alpha,\beta) = 1$ iff $(\alpha,\beta) \in V - \{\mathbf{0}\}$.

$\mathcal{A}_3$ **(linearly completely dependent masks):** $(s,t) := (3, c/2^{\dim(V)})$ and $U := \mathsf{CEA}^{\mathsf{msb}_{\dim(V)}[\mathcal{O}]}$. F is the Boolean function of which the domain is $\mathbb{F}_2^n \times \mathbb{F}_2^{\dim(V)}$ and $F(\alpha,\beta) = 1$ iff $(\alpha,\beta) \in V - \{\mathbf{0}\}$.

Here, sampling a unitary U according to D_1 (resp., D_2) in Proposition 2 corresponds to sampling a random key K for a real cipher (resp., choosing an ideally random permutation P).

[10] The three distinguishers might be unified into a single one by restricting inputs and outputs appropriately. Still we focus on these cases because the three distinguishers are enough for the examples of interest shown later, and to avoid unnecessarily complex analysis.

[11] Let M be an arbitrary full-rank $n \times n$ matrix over $\mathbb{F}_2$ satisfying $M^T \mathbf{e}_i = \beta_i$. Then we have $\beta_i \cdot E_K(x) = (M^T \mathbf{e}_i) \cdot E_K(x) = \mathbf{e}_i \cdot M(E_K(x))$, and thus distinguishing E_K by using output mask β_i is equivalent to distinguishing $M \circ E_K$ by using output mask $\mathbf{e}_i$. Since E_K can be distinguished from a random permutation P iff $M \circ E_K$ can be distinguished, we can assume them without loss of generality.

[12] The reasoning is almost the same as before.

[13] See Section F in the appendix for details on how to efficiently compute F on a quantum circuit.

Analysis. If input and output masks are linearly independent, $\mathcal{A}_2$ distinguishes E_K and P in time $O(\sqrt{2^w/c})$ roughly due to the following reasoning. If the oracle given to $\mathcal{A}_2$ is E_K, the probability that we observe $(\alpha,\beta) \in F^{-1}(1)$ when measuring $\mathsf{CEA}^{\mathsf{msb}_w[E_K]}\left|0^n\right\rangle\left|0^w\right\rangle$ is approximately lower bounded by $c/2^w$. Hence, QAA on $\mathsf{CEA}^{\mathsf{msb}_w[E_K]}$ and F with $O(\sqrt{2^w/c})$ iterations returns $(\alpha,\beta) \in F^{-1}(1)$ (i.e., $\mathcal{A}_2$ returns 1) with high probability. On the other hand, if the oracle given to $\mathcal{A}_2$ is a random permutation P, from Claim 1 it follows that the probability that we observe $(\alpha,\beta) \in F^{-1}(1)$ when measuring $\mathsf{CEA}^{\mathsf{msb}_w[P]}\left|0^n\right\rangle\left|0^w\right\rangle$ is approximately upper bounded by $2^{\dim(V)}/2^{n+w}$. Especially, the probability that QAA on $\mathsf{CEA}^{\mathsf{msb}_w[P]}$ and F with $O(\sqrt{2^w/c})$ iterations returns $(\alpha,\beta) \in F^{-1}(1)$ (i.e., $\mathcal{A}_2$ returns 1) is negligibly small. For similar reasons, $\mathcal{A}_1$ and $\mathcal{A}_3$ distinguish E_K and P in time $O(\sqrt{2^n/c})$ and $O(\sqrt{2^{\dim(V)}/c})$, respectively.

More precisely, define parameters c_i, pb_i, and T_i for $i = 1, 2, 3$ as follows.

- $\mathsf{c}_1 := \mathsf{c}_3 = 2^{-n}$, $\mathsf{c}_2 := 2^{-n-w+\dim(V)}$.
- $\mathsf{pb}_1 := \frac{2^{\dim(V)+7}(n+1)}{2^{2n}\cdot c} + 2^{-\dim(V)+1}\cdot n^{-2}$, $\mathsf{pb}_2 := \frac{2^{\dim(V)+7}(n+1)}{2^{n+w}\cdot c} + 2^{-\dim(V)+1}\cdot n^{-2}$, and $\mathsf{pb}_3 := \frac{2^7(n+1)}{2^n\cdot c} + 2^{-\dim(V)+1}\cdot n^{-2}$.
- $\mathsf{T}_1 := 6\sqrt{2^n/c}$, $\mathsf{T}_2 := 6\sqrt{2^w/c}$, and $\mathsf{T}_3 := 6\sqrt{2^{\dim(V)}/c}$.

Then the following proposition holds.

Proposition 4. *Let $i = 1, 2$, or 3. Suppose that $c \gg \mathsf{c}_i$, and that $1/4 > \sum_{(\alpha,\beta)\in V-\{\mathbf{0}\}} \mathrm{Cor}(E_K;\alpha,\beta)^2 \geq c$ holds with a constant probability p when K is randomly chosen. If $\mathcal{A}_i$ runs relative to the real cipher E_K, then the probability that $\mathcal{A}_i$ outputs 1 is at least $p/2$. If $\mathcal{A}_i$ runs relative to a random permutation P, then the probability that $\mathcal{A}_i$ outputs 1 is approximately upper bounded by pb_i. In addition, the time complexity of $\mathcal{A}_i$ is at most T_i. (The probabilities are taken not only over the randomness of $\mathcal{A}_i$ but also over the randomness of choices of K or P.)*

This proposition can be proven by applying Proposition 2 and Claim 1 in a straightforward manner. Still, we provide a proof in Section C in the appendix for completeness.

Some Remarks. The speed-ups in this section are not always quadratic. Still, a quadratic speed-up is obtained in the specific case when input-output masks are linearly independent and $u = w$ (by applying $\mathcal{A}_2$). In this case, the classical complexity is about $2^{\ell/2}/(\text{capacity}) = 2^w/(\text{capacity})$ because $\ell = \dim(V) = \dim(V_1) + \dim(V_2) = u + w = 2w$. Meanwhile, if $\mathcal{A}_2$ is applied[14], the complexity drops to about $\sqrt{2^w/(\text{capacity})}$. For other cases, the speed-up is not quadratic in general, except for the one-dimensional case.

In the one-dimensional case, the asymptotic complexity of our technique is the same as Kaplan et al.'s, but the non-asymptotic complexity become smaller in a specific situation. For example, suppose that we have a one-dimensional

[14] $\mathcal{A}_1$ is also applicable here but performs worse than $\mathcal{A}_2$.

linear approximation of a cipher, and the absolute value of the linear correlation is concentrated in a very narrow range around a known value $c > 0$. Then, some analysis shows that the combination of CEA and QAA distinguishes the cipher by making $(\sqrt{2}\pi) \cdot (1/c)$ queries to the oracle. (This is faster than $\mathcal{A}_3$ in Proposition 4. Here, we consider to run QAA only once, whereas $\mathcal{A}_3$ runs QAA multiple times. A single run of QAA is sufficient here since the variance of the correlation is assumed to be small.) Meanwhile, Kaplan et al.'s distinguisher requires about $(2\sqrt{2}\pi) \cdot (1/c)$ queries. (See Appendix E for details.) Thus our attack is about 2 times faster in this situation. For general cases where the variance of the correlation may be large, we do not observe evident difference between ours and Kaplan et al.'s because we have to run QAA multiple times (as $\mathcal{A}_3$ does).

So far we have discussed how to distinguish block ciphers from random permutations, but we expect the above distinguishers are also applicable to distinguish keyed functions from random functions of n-bit inputs, without changing the asymptotic complexity (in the same way as classical linear distinguishers work not only for permutations but Below we give some application examples, but they are essentially distinguishers on keyed functions from random functions, rather than block ciphers from random permutations.

5.1 Application Example: FEA-1 and FEA-2 Structures

FEA is a Korean standard (TTAK.KO-12.0275) for format preserving encryption [49], which has two variants named FEA-1 and FEA-2. Both variants adopt *tweakable* Feistel structures. Here we study linear distinguishers on these structures when round functions are ideally random.

The FEA-1 and FEA-2 structures look like Fig. 1. As in usual Feistel structures, plaintexts are divided into two parts. We focus on the case when the widths of the two branches are equal. A tweak T is also divided into two parts, denoted by T_L and T_R, and processed in an alternate manner. In FEA-1, the i-th round function takes T_L (resp., T_R) when $i \equiv 1$ (resp., $i \equiv 0$) mod 2. In FEA-2, the i-th round function takes T_L (resp., T_R) when $i \equiv 2$ (resp., $i \equiv 0$) mod 3. The $(3j+1)$-th round function of FEA-1 does not take any tweak (or equivalently, take a constant value instead). For simplicity, we assume the tweak length is sufficiently large.

At CRYPTO 2021, Beyne showed multidimensional linear distinguishers on these structures [6]. The multidimensional linear approximation[15] for FEA-1 is a vector space V of completely linearly dependent input-output masks with $\dim(V) = n/2$ (when n is the block size of Feistel), and the sum of the squared correlations $\sum_{(\alpha,\beta)\in V} \mathrm{Cor}(\alpha,\beta)^2$ is equal to $2^{n(1-r/4)}$. Meanwhile, the approximation for FEA-2 is a vector space V' of linearly independent input-output masks with $\dim(V) = \dim(V'_2) = n/2$ (here, we assume V' is decomposed as $V' = V'_1 \times V'_2$), and the sum of the squared correlation is equal to $2^{n(1-r/6)}$.

[15] See the original paper [6] on details of linear approximations. What is significant here is only the dimensions of the approximations and the sum of the squared correlations.

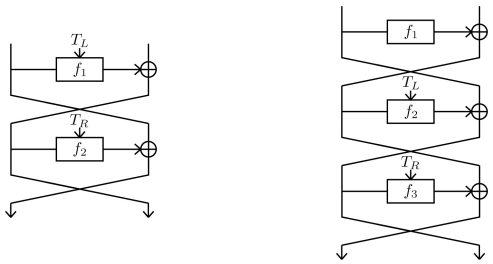


Fig. 1: The FEA-1 structure (left) and FEA-2 structure (right).

The classical distinguishing complexity is $O(2^{(r/4-3/4)n})$ for FEA-1 (resp., $O(2^{(r/6-3/4)n})$ for FEA-2). By applying our quantum distinguishers above, the complexity is reduced to $O(2^{(r/8-1/4)n})$ (resp., $O(2^{(r/12-1/4)n})$).

Remark 1. In [6], linear distinguishers are extended to message recovery attacks and key recovery attacks. Our distinguishers could also be extended to message or key recovery attacks in the quantum setting by just guessing the secret information with the Grover search, though, non-trivial extension of interest (beyond just applying Grover) would require another new idea and not be straightforward.

6 Speed-Up for Zero Correlation Linear Distinguishers

This section shows how CEA can be used to speed-up (multidimensional) zero correlation linear distinguishers [9]. We first recall the basic ideas of attacks in the classical setting.

6.1 Classical Zero Correlation Linear Distinguishers

Unlike linear cryptanalysis, zero correlation linear cryptanalysis exploits linear approximations of which the correlation is exactly zero.

For instance, let E_K be an n-bit block cipher and suppose $\mathrm{Cor}(E_K; \alpha, \beta) = 0$ holds for some input and output masks $\alpha, \beta \neq 0^n$. Then, $(\mathrm{Cor}(P; \alpha, \beta))^2$ for a random permutation P is distributed around 2^{-n} and non-zero with high probability. Hence we can distinguish E_K from P if we have sufficiently many ($\approx 2^n$) plaintext-ciphertext pairs by checking whether the estimated empirical correlation is zero or not.

This idea naturally extends to attacks exploiting multidimensional linear approximations of correlation zero (below we follow the notations of Section 3.2). Again, let E_K be an n-bit block cipher and $V \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$ be a vector space such that $\mathrm{Cor}(E_K; \alpha, \beta) = 0$ for all $(\alpha, \beta) \in V$. Moreover, let S be an arbitrary basis of V. Then the distribution $p_S^{E_K}$ over $\mathbb{F}_2^{\dim(V)}$ defined by $p_S^{E_K}(z) :=$

$\Pr_x\left[\mathsf{Lin}_S^f(x) = z\right]$ exactly matches the uniform distribution. On the other hand, the distribution p_S^P similarly defined for a random permutation P is slightly different from the uniform distribution. Hence E_K and P can be distinguished by using suitable statistical tests. Indeed, Bogdanov et al. [8] showed that E_K can be distinguished in time $O(2^n/\sqrt{2^{\dim(V)}})$ in such a setting[16].

Remark 2. In the special case where the input-output masks are independent and $V = V_1 \times V_2$ holds, we can achieve the time complexity $O(2^n/2^{\dim(V_1)})$ instead of $O(2^n/\sqrt{2^{\dim(V)}})$ by using the link between zero correlation linear cryptanalysis and integral cryptanalysis, which we will elaborate in Section 7.

6.2 Quantum Speed-Up by CEA

Next, we study how to speed-up (multidimensional) zero correlation linear distinguishers by using CEA and QAA.

As well as linear distinguishers in Section 3.2, we introduce three distinguishers which we denote by $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. $\mathcal{B}_1$ is a general distinguisher applicable to arbitrary multidimensional linear approximations. $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) is applicable when the input and output masks are linearly independent (resp., completely dependent).

In what follows, we assume that $\mathrm{Cor}(E_K; \alpha, \beta)^2 = 0$ holds for all $(\alpha, \beta) \in V - \{\mathbf{0}\}$. $\mathcal{O}$ denotes the oracle, which is either E_K for a random K or a random permutation P. For notations related to $\mathcal{B}_2$ and $\mathcal{B}_3$, we use the same ones as those for $\mathcal{A}_2$ and $\mathcal{A}_3$ introduced on p.16.

Distinguisher $\mathcal{B}_1$ (General Case). When nothing can be assumed on linear dependence of masks, a natural way to mount a distinguisher by using QAA and CEA is to run the following procedure.

1. Let $F : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{F}_2$ be the Boolean function such that $F(\alpha, \beta) = 1$ iff $(\alpha, \beta) \in V - \{\mathbf{0}\}$.
2. Apply QAA on $\mathsf{CEA}^{\mathcal{O}}$ and F with the number of iterations $\lfloor \frac{\pi}{4}\sqrt{2^{2n-\dim(V)}} \rfloor$. Namely, let the unitary operator $Q(\mathsf{CEA}^{\mathcal{O}}, F)^i \mathsf{CEA}^{\mathcal{O}}$ act on $|0^n\rangle\,|0^n\rangle$ with $i = \lfloor \frac{\pi}{4}\sqrt{2^{2n-\dim(V)}} \rfloor$. Then, measure the resulting state by the computational basis and let (α, β) be the observed bit string.
3. If $F(\alpha, \beta) = 0$, return 1. Otherwise, return 0.

Some analysis shows that this algorithm distinguishes E_K and P with high probability. However, the running time of $\mathcal{B}_1$ is $O(\sqrt{2^{2n-\dim(V)}}) = O(2^n/\sqrt{2^{\dim(V)}})$, which is the same as the complexity of the classical distinguisher. Namely, $\mathcal{B}_1$ does not obtain any speed-up from classical attacks. Meanwhile, we can obtain some quantum speed-up when input-output masks are linearly independent or linearly completely dependent, which we explain below.

[16] Bogdanov and Wang showed a similar result assuming that many statistically independent linear approximations exist [10], but the assumption often does not hold.

Remark 3. $\mathcal{B}_1$ runs QAA only once, unlike $\mathcal{QD}$ of Proposition 2 (or its applications $\mathcal{A}_1$-$\mathcal{A}_3$) running QAA multiple times. This is because the probability $\Pr[F(\alpha,\beta)=1]$ is exactly zero when (α,β) is obtained by measuring the state $\mathsf{CEA}^{E_K}|0^n\rangle|0^n\rangle$, and thus we can achieve a sufficiently high advantage with a single run of QAA.

Distinguishers $\mathcal{B}_2$ and $\mathcal{B}_3$. Here we show distinguishers $\mathcal{B}_2$ and $\mathcal{B}_3$ for linearly independent and completely dependent masks, respectively[17].

$\mathcal{B}_2$ is obtained by modifying the unitary operators and the number of iterations for QAA in $\mathcal{B}_1$. Specifically, we change

1. the unitary operator for QAA of $\mathcal{B}_1$ from $\mathsf{CEA}^{\mathcal{O}}$ to $\mathsf{CEA}^{\mathsf{msb}_w[\mathcal{O}]}$, and
2. the number of iterations from $\lfloor\frac{\pi}{4}\sqrt{2^{2n-\dim(V)}}\rfloor$ to $\lfloor\frac{\pi}{4}\sqrt{2^{n+w-\dim(V)}}\rfloor = \lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor$.

$\mathcal{B}_3$ is obtained just by changing the parameter w appeared in $\mathcal{B}_2$ to $\dim(V)$.

$\mathcal{B}_2$ distinguishes E_K and P with high probability, roughly for the following reason: If the oracle given to $\mathcal{B}_2$ is E_K, $\mathcal{B}_2$ always returns 1. If the oracle given to $\mathcal{B}_2$ is a random permutation P, Claim 1 guarantees[18] that the probability that we observe $(\alpha,\beta)\in F^{-1}(1)$ when measuring $\mathsf{CEA}^{\mathsf{msb}_w[P]}|0^n\rangle|0^w\rangle$ is approximately equal to $2^{\dim(V)}/2^{n+w} = 2^{u-n}$. Hence the QAA with $O(\sqrt{2^{n-u}})$ iterations in Step 2 of $\mathcal{B}_2$ returns $(\alpha,\beta)\in F^{-1}(1)$ with high probability, and $\mathcal{B}_2$ returns 0. Thus $\mathcal{B}_2$ distinguishes E_K and P. Especially, $\mathcal{B}_2$ achieves a quadratic speed-up in the special case where $w=1$ (see Remark 2). Similarly, $\mathcal{B}_3$ distinguishes E_K in time $O(\sqrt{2^n})$. More precisely, the following proposition holds.

Proposition 5. *If $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) runs relative to E_K, then $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) always outputs 1. If $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) runs relative to a random permutation P, then the probability that $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) outputs 0 is approximately lower bounded by $\frac{1}{2}\cdot\left(1-2^{-\dim(V)+1}\right)$. In addition, the running time of $\mathcal{B}_2$ (resp., $\mathcal{B}_3$) is at most $2\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor+1$ (resp., $2\lfloor\frac{\pi}{4}\sqrt{2^n}\rfloor+1$) encryptions by E_K. (The probabilities are taken not only over the randomness of $\mathcal{B}_2$ or $\mathcal{B}_3$ but also over the randomness of choices of K or P.)*

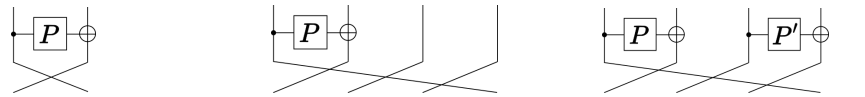
A proof of the proposition is given in Section G in the appendix.

6.3 Applications

Both of $\mathcal{B}_2$ and $\mathcal{B}_3$ have various immediate applications. For instance, Bogdanov and Rijmen showed multidimensional zero correlation linear approximations on the 5-round balanced Feistel structure, 18-round 4-branch Type-I generalized Feistel structure, and 9-round 4-branch Type-II generalized Feistel structure (see Fig. 2 and Table 1 in the appendix) when round functions are bijections. The input-output masks of the linear approximations are linearly completely

[17] Recall that we use the same notations as those for $\mathcal{A}_2$ and $\mathcal{A}_3$. See p.16 for details.
[18] Note that $2^n\cdot\sum_{(\alpha,\beta)\in V-\{\mathbf{0}\}}\mathrm{Cor}(P;\alpha,\beta)^2$ in Claim 1 is equal to $2^n\cdot\mathrm{Cap}(p_S^P)$.

dependent. Thus $\mathcal{B}_3$ distinguishes these constructions in time $O(2^{n/2})$ (when inputs and outputs are n bits). In fact the linear approximations on the 4-branch Type-I/II generalized Feistel structures can be extended to k-branch structures for general[19] k in a straightforward manner, and $\mathcal{B}_3$ distinguishes (k^2+k-2)-round (resp., $(2k+1)$-round) k-branch Type-I (resp., Type-II) generalized Feistel structure in time $O(2^{n/2})$.



(a) Balanced Feistel (b) Type-I Generalized Feistel (c) Type-II Generalized Feistel

Fig. 2: One round of Balanced and (4-branch) generalized Feistel structures. What we assume is only that P and P' are bijections. Our attacks work regardless of whether P (and P') for different rounds are independent or not.

Balanced	k-branch Type-I	k-branch Type-II
$(\alpha\|\|0^{n/2}, 0^{n/2}\|\|\alpha)$	$(\beta\|\|0\|\|\cdots\|\|0, 0\|\|\beta\|\|0\|\|\cdots\|\|0$	$(\alpha\|\|0\|\|\cdots\|\|0, 0\|\|\cdots\|\|0\|\|\beta)$

Table 1: Input-output mask patterns for balanced and generalized Feistel structures. $\alpha \in \mathbb{F}_2^{n/2}$ and $\beta \in \mathbb{F}^{n-\frac{n}{k}}$ are non-zero values. "0" for generalized Feistel structures denotes $0^{n/k} \in \mathbb{F}_2^{n/k}$.

Note that the numbers of rounds we attack here are larger than those broken by previous polynomial time attacks using Simon's algorithm in a black-box way. The number of rounds of balanced (resp., k-branch Type-I and Type-II) Feistel broken by previous polynomial time attacks is 4 [40] (resp., k^2-k+1 [53] and $k+1$ [28]). See also Table 2 in the appendix.

In fact, the complexity of our distinguishers may also be achieved just by speeding-up a one-dimensional zero-correlation linear distinguisher with simpler techniques. Still, to the authors' best knowledge, we are the first to point out the existence of attacks with such complexity.

There also exist lots of other previous works showing zero correlation approximations [10, 9, 8, 60, 1] and our $\mathcal{B}_2$ or $\mathcal{B}_3$ can be applied to all of them in principle. The amount of quantum speed-up compared to classical distinguishers depends on linear approximations, and we can achieve at most quadratic speed-up.

[19] k must be even for Type-II structures.

7 Speed-Up for Integral Distinguishers

This section shows applications of CEA to integral cryptanalysis. As shown by Bogdanov et al. [8] and Sun et al. [60], balanced property of a cipher is equivalent to multidimensional zero correlation linear properties of which the input-output masks are linearly independent. Specifically, the following proposition holds[20].

Proposition 6 ([8, 60]). *Let $F : \mathbb{F}_2^m \to \mathbb{F}_2^n$ be a function. Let $V_1 \subset \mathbb{F}_2^m, V_2 \subset \mathbb{F}_2^n$ be sub-vector spaces, and $V := V_1 \times V_2$. Then the following conditions are equivalent.*

1. *V is the set of input-output masks of a multidimensional zero correlation linear approximation of F, i.e., $\mathrm{Cor}(F; \alpha, \beta) = 0$ for all $(\alpha, \beta) \in V - \{\mathbf{0}\}$.*
2. *The function $G : x \mapsto \beta \cdot F(x \oplus \lambda)$ is balanced over $V_1^\perp$ for all $\lambda \in \mathbb{F}_2^m$ and $\beta \in V_2 - \{\mathbf{0}\}$.*

Remark 4. Note that this equivalence holds only for balanced property but not for zero-sum property. Our quantum attacks below also rely on the above equivalence. Especially, the attacks are applicable only if a balanced property exists.

Recall that the distinguisher $\mathcal{B}_2$ (Proposition 5) is applicable when a multidimensional zero correlation linear approximation exists and the input-output masks are linearly independent. Together with Proposition 6, this implies the following proposition.

Proposition 7. *Let E_K be an n-bit block cipher. Suppose some output bits of E_K are balanced over a vector space $V \subset \mathbb{F}_2^n$. (W.l.o.g., we assume the most significant w bits are balanced, and let $V' := \{x || 0^{n-w} | x \in \mathbb{F}_2^w\}$.) Then, by applying $\mathcal{B}_2$ on the zero correlation multidimensional linear approximations of $V^\perp \times V'$, we can distinguish E_K from P with time and query complexity at most $2\lfloor \frac{\pi}{4}\sqrt{2^{\dim(V)}} \rfloor + 1$. $\mathcal{B}_2$ always outputs 1 if the given encryption oracle is the real cipher E_K. If the oracle is a random permutation P, then $\mathcal{B}_2$ outputs 0 with probability at least $\frac{1}{2}\left(1 - 2^{-\dim(V)+1}\right)$.*

This proposition shows that we can obtain (almost) quadratic speed-up for integral distinguisher because the complexity of $\mathcal{B}_2$ is $\approx 1.6\sqrt{2^{\dim(V)}}$ while the complexity of the classical integral distinguisher is $2^{\dim(V)}$.

Still, this is at most quadratic speed-up. At first glance, achieving a more than quadratic speed-up seems impossible for integral distinguishers. However, we see possibility of a more than quadratic speed-up in some situations.

7.1 Possibility of More than Quadratic Speed-Up

Roughly speaking, if a part of the outputs of a cipher (e.g., a specific byte of ciphertexts) is balanced on *multiple mutually orthogonal vector spaces* included

[20] This equivalence was first shown by [8] and later refined by [60]. [60] proves the equivalence only in the special case $\dim(V_2) = 1$ but it immediately implies the equivalence for $\dim(V_2) > 1$.

in the input space, there exists possibility to achieve a more than quadratic quantum speed-up by using CEA.

Specifically, let $E_K : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a block cipher, and suppose there exist sub-vector spaces $V_1, \ldots, V_s \subset \mathbb{F}_2^m$ satisfying the following conditions.

1. $V_1, \ldots, V_s$ are mutually orthogonal, i.e., $V_i \perp V_j$ for $i \neq j$.
2. There exists some $d \leq n/2$ and $\dim(V_i) = d$ holds for all i.
3. A part of the outputs of E_K is balanced on $V_i \oplus \lambda$ for all $1 \leq i \leq s$ and arbitrary $\lambda \in \mathbb{F}_2^m$. (For ease of explanation, below we assume the most significant w bits of outputs of E_K are balanced.)

Then, by Proposition 6 we have $\mathrm{Cor}(\mathsf{msb}_w[E_K]; \alpha, \beta) = 0$ if $\alpha \in (V_1)^\perp \cup \cdots \cup (V_s)^\perp - \{\mathbf{0}\}$ and $(\alpha, \beta) \neq (0, 0)$. This means

$$\Pr_K \left[(\alpha, \beta) \leftarrow \mathsf{CEA}^{\mathsf{msb}_w[E_K]} : \alpha \perp V_i \text{ for some } i \text{ and } \alpha \neq 0 \text{ and } \beta \neq 0 \right] = 0.$$

Meanwhile, for a random permutation P we have

$$\begin{aligned}
&\Pr_P \left[(\alpha, \beta) \leftarrow \mathsf{CEA}^{\mathsf{msb}_w[P]} : \alpha \perp V_i \text{ for some } i \text{ and } \alpha \neq 0 \text{ and } \beta \neq 0 \right] \\
&\quad \overset{(*)}{=} \sum_{\substack{\alpha \neq 0, \beta \neq 0 \\ \alpha \perp V_i \text{ for some } i}} \mathbb{E}_P \left[\frac{\mathrm{Cor}(\mathsf{msb}_w[P]; \alpha, \beta)^2}{2^w} \right] \\
&\quad = \sum_{\substack{\alpha \neq 0, \beta \neq 0 \\ \alpha \perp V_i \text{ for some } i \\ \mathsf{lsb}_{n-w}[\beta] = 0}} \mathbb{E}_P \left[\frac{\mathrm{Cor}(P; \alpha, \beta)^2}{2^w} \right] \overset{(**)}{=} \sum_{\substack{\alpha \neq 0, \beta \neq 0 \\ \alpha \perp V_i \text{ for some } i \\ \mathsf{lsb}_{n-w}[\beta] = 0}} \frac{1}{2^w (2^n - 1)} \\
&\quad = \# \{ \alpha \in \mathbb{F}_2^n - \{\mathbf{0}\} | \alpha \perp V_i \text{ for some } i \} \cdot \frac{\# \{ \beta \in \mathbb{F}_2^n - \{\mathbf{0}\} | \mathsf{lsb}_{n-w}[\beta] = 0 \}}{2^w (2^n - 1)} \\
&\quad \geq \left(\sum_{1 \leq i \leq s} |V_i^\perp| - \sum_{1 \leq i < j \leq s} |V_i^\perp \cap V_j^\perp| - 1 \right) \cdot \frac{2^w - 1}{2^w (2^n - 1)} \\
&\quad \overset{V_i \perp V_j \text{ for } i \neq j}{\geq} \left(s 2^{n-d} - s^2 2^{n-2d} - 1 \right) \cdot \frac{2^w - 1}{2^w (2^n - 1)} \quad \approx \quad \frac{s}{2^d}.
\end{aligned}$$

Here, $(*)$ and $(**)$ follow from Proposition 3 and Proposition 11, respectively.

Therefore, E_K can be distinguished from P in time about $\frac{\pi}{2}\sqrt{2^d/s}$ by applying QAA on $\mathsf{CEA}^{\mathsf{msb}_w[E_K]}$ (or $\mathsf{CEA}^{\mathsf{msb}_w[P]}$) and the Boolean function $F : \mathbb{F}_2^n \times \mathbb{F}_2^w \to \mathbb{F}_2$ such that $F(\alpha, \beta) = 1$ iff $\alpha \perp V_i$ for some $i = 1, \ldots, s$ and $\alpha \neq 0$ and $\beta \neq 0$.

This can lead to a more than quadratic speed-up compared to the corresponding classical integral distinguisher (when $s \geq 4$) because the classical complexity is 2^d: Even if we have such multiple spaces $V_1, \ldots, V_s$, what we can do in the classical setting is just to choose a single space V_i and check whether (a part of) E_K is balanced on that space, unless some additional properties can be assumed.

Application Examples. To see how the above distinguisher can be applied to concrete ciphers, let us recall the 3.5-round integral property of AES for an example [26]. If a tuple of certain four cells of inputs take all values while others being constant, we can make a single column after the first round take all values while others remain constant, and each cell after 3.5 rounds balanced (see Fig. 3). Since there are four choices on which tuple of four cells to activate (i.e., which column after the first round to activate), we are in the situation of the distinguisher explained above with $d = 32$ and $s = 4$ (V_i corresponds to a tuple of four active cells of inputs).

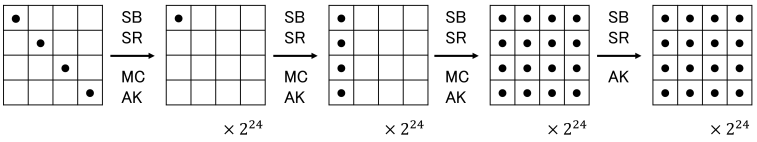


Fig. 3: The integral property of the 2.5-round AES. Cells with filled circles are those taking all values and others are constants. "$\times 2^{24}$" indicates that 2^{24} sets of the same active cell pattern shown in the figure is observed.

In fact this example itself is not so significant because more efficient distinguishers exist: If a tuple of four cells of inputs take all values like Fig. 3, actually a certain tuple of four cells of outputs take all values. This means that the integral property specifies a 32-bit permutation between part of inputs and outputs. Hence the 3.5-round AES is distinguished by checking if this part contains a collision in time $\approx \sqrt[3]{2^{32}}$ with the BHT algorithm [19].

Still, we observe an interesting attack when s is relatively large. For instance, suppose $s = 2^d$ holds. This situation happens if, e.g., E_K is a 4-bit cell SPN cipher with the same integral property as the 2.5-round AES (the latter 2.5 round of Fig. 3). Then $\Pr[F(\alpha, \beta) = 1] \approx s/2^d = 1$ holds when (α, β) is obtained by measuring $\mathsf{CEA}^{\mathsf{msb}_w[P]}$, while the probability is always zero for the real cipher E_K. Thus we can distinguish E_K only with a *single* quantum query, which apparently exhibits a more than quadratic speed-up.

The margin compared to the square-root of the classical complexity is not large, but this example is important in that a new-type example illustrating a qualitative difference between classical and quantum computation is achieved by using a classical cryptanalytic technique.

8 Extension to Generalized Linear Distinguishers

Linear cryptanalysis is useful when group operations are done in $\mathbb{Z}_2^n$, but some ciphers use other group operations such as modular additions (i.e., additions in $\mathbb{Z}/2^n\mathbb{Z}$), where generalized linear cryptanalysis on arbitrary finite groups [4] is

more useful. Generalized linear cryptanalysis uses group characters instead of bit masks, but we observe again there exists a close relationship between (generalized) correlations and quantum computation via Fourier transform. This section shows how the technique of Section 5 extends to generalized linear distinguishers. In this section, the symbol "$\oplus$" denotes the direct sum of groups.

8.1 Fourier Transform on Arbitrary Finite abelian Group

Let G be an arbitrary finite abelian group. Then, by the Chinese remainder theorem, there is a group isomorphism from G to $\mathbb{Z}/N_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/N_m\mathbb{Z}$ for some positive integers $N_1,\ldots,N_m$. We fix an isomorphism and identify the two groups. Recall that a character of a finite abelian group G is a group homomorphism $\phi : G \to \mathbb{C}^\times$. The set of characters of G is denoted by $\hat{G}$, which forms a group by point-wise multiplication. It is well-known that $\hat{G}$ is isomorphic to G as a group.

Specifically, for each $w = (w_1,\ldots,w_m) \in G$, the function

$$\mathsf{ch}_w : (x_1,\ldots,x_m) \mapsto \exp\left(2\pi i\frac{x_1w_1}{N_1}\right)\cdots\exp\left(2\pi i\frac{x_mw_m}{N_m}\right)$$

is a character of G. In fact the map $w \mapsto \mathsf{ch}_w$ defines a group isomorphism from G to $\hat{G}$. We identify G with $\hat{G}$ by this isomorphism.

Let G be a finite abelian group and $F : G \to \mathbb{C}$ be a function. Then, the Fourier transform of F over G is a function $\mathcal{F}_GF : G \to \mathbb{C}$ defined by

$$\mathcal{F}_GF(w) := \sum_{x\in G}\frac{1}{\sqrt{|G|}}\cdot\overline{\mathsf{ch}_w(x)}\cdot F(x).$$

The inverse transform of $\mathcal{F}_G$, denoted by $\mathcal{F}_G^*$, is given by $\mathcal{F}_G^*F(x) = \sum_{w\in G}\frac{1}{\sqrt{|G|}}\cdot \mathsf{ch}_x(w)\cdot F(w)$.

We naturally identify a function from G to $\mathbb{C}$ (resp., the set of the functions from G to $\mathbb{C}$) with a vector in the $|G|$-dimensional vector space $\mathbb{C}^{|G|}$ (resp., the vector space $\mathbb{C}^{|G|}$). Moreover, we assume that $\mathbb{C}^{|G|}$ is endowed with the standard Hermitian inner product. Then $\mathcal{F}_G$ can be regarded as a unitary operator.

8.2 Linear Correlations

Let G, H be finite abelian groups and $f : G \to H$ be a function. For $\alpha \in G$ and $\beta \in H$, the (generalized) linear correlation $\mathrm{Cor}(f;\alpha,\beta)$ is defined as

$$\mathrm{Cor}(f;\alpha,\beta) := \sum_{x\in G}\frac{1}{|G|}\overline{\mathsf{ch}_\beta(f(x))}\cdot\mathsf{ch}_\alpha(x).$$

We call α (resp., β) an input mask (resp., output mask).

Let $f_{\mathrm{emb}} : G\oplus H \to \mathbb{C}$ be the function defined by $f_{\mathrm{emb}}(x,y) = 1$ if $y = f(x)$ and $f_{\mathrm{emb}}(x,y) = 0$ if $y \neq f(x)$. Then, some straightforward calculation shows that

$$\left(\left(\mathcal{F}_G^*\otimes\mathcal{F}_H\right)f_{\mathrm{emb}}\right)(\alpha,\beta) = \sqrt{|G|/|H|}\cdot\mathrm{Cor}(f;\alpha,\beta) \tag{7}$$

holds. (This corresponds to Eq. (3) for the linear cryptanalysis over $(\mathbb{Z}/2\mathbb{Z})^{\oplus n}$.)

8.3 Extension of CEA

For an arbitrary finite abelian group G, we assume that elements of G are appropriately encoded into n-bit strings for some n s.t. $|G| \leq 2^n$. Let $\psi : G \to \mathbb{C}$ be a function satisfying $\sum_{x \in G} |\psi(x)|^2 = 1$, and $|\psi\rangle := \sum_x \psi(x) |x\rangle$. Recall that the Quantum Fourier Transform (QFT) over an abelian group G, denoted by QFT_G, is defined by

$$\mathrm{QFT}_G |\psi\rangle = \sum_x (\mathcal{F}_G^* \psi)(x) |x\rangle . \tag{8}$$

With these notations, the extension of CEA on a function $f : G \to H$ (G and H are finite abelian groups) is obtained by replacing the Hadamard transform in CEA with the QFT (or its inverse) over G and H. Specifically, the extended algorithm runs as follows.

Extended Version of CEA.

(a) Prepare the initial state $|0_G\rangle |0_H\rangle$.
(b) Apply QFT_G on the first (left) register.
(c) Apply U_f on the state (i.e., make a quantum query to f).
(d) Apply $\mathrm{QFT}_G \otimes \mathrm{QFT}_H^*$ on the state.
(e) Measure the entire state by the computational basis and return the observed result $(\alpha, \beta) \in G \oplus H$.

We also use the symbol CEA^f to denote the extended algorithm.

The following proposition is an extension of Proposition 3.

Proposition 8. *The quantum state of* CEA^f *before the final measurement is*

$$\sum_{\alpha \in G, \beta \in H} \frac{\mathrm{Cor}(f; \alpha, \beta)}{\sqrt{|H|}} |\alpha\rangle |\beta\rangle . \tag{9}$$

In particular, for any subset $S \subset G \oplus H$,

$$\Pr\left[(\alpha, \beta) \leftarrow \mathsf{CEA}^f : (\alpha, \beta) \in S\right] = \sum_{(\alpha, \beta) \in S} \frac{\mathrm{Cor}(f; \alpha, \beta)^2}{|H|} \tag{10}$$

holds.

Proof. The quantum state of CEA^f before the final measurement is

$$\begin{aligned}
&(\mathrm{QFT}_G \otimes \mathrm{QFT}_H^*) U_f \left(\mathrm{QFT}_G \otimes I_n\right) |0_G\rangle |0_H\rangle \\
&\quad = (\mathrm{QFT}_G \otimes \mathrm{QFT}_H^*) U_f \sum_{x \in G} \frac{1}{\sqrt{|G|}} |x\rangle |0_H\rangle \\
&\quad = (\mathrm{QFT}_G \otimes \mathrm{QFT}_H^*) \sum_{x \in G} \frac{1}{\sqrt{|G|}} |x\rangle |f(x)\rangle \\
&\quad \overset{\text{Def. of } f_{\mathrm{emb}}}{=} (\mathrm{QFT}_G \otimes \mathrm{QFT}_H^*) \sum_{x \in G, y \in H} \frac{f_{\mathrm{emb}}(x,y)}{\sqrt{|G|}} |x\rangle |y\rangle \\
&\quad \overset{\text{Def. of QFT}}{=} \sum_{\alpha \in G, \beta \in H} \frac{((\mathcal{F}_G^* \otimes \mathcal{F}_H) f_{\mathrm{emb}})(\alpha, \beta)}{\sqrt{|G|}} |\alpha\rangle |\beta\rangle \\
&\quad \overset{\text{Eq. (7)}}{=} \sum_{\alpha \in G, \beta \in H} \frac{\mathrm{Cor}(f; \alpha, \beta)}{\sqrt{|H|}} |\alpha\rangle |\beta\rangle .
\end{aligned}$$

Hence we have Eq. (9). Eq. (10) immediately follows from Eq. (9). □

8.4 Quantum Speed-up for Generalized Linear Distinguishers

Let $f : G \to H$ be a function, where G and H are finite abelian groups. Here we define linearly independent masks and linearly completely dependent masks.

1. Suppose G and H are decomposed as $G = G_1 \oplus G_2$ and $H = H_1 \oplus H_2$. Then, we say the set $G_1 \oplus H_1 (\subset G \oplus H)$ is a set of linearly independent input-output masks.
2. Suppose again the decomposition $G = G_1 \oplus G_2$ and $H = H_1 \oplus H_2$, and assume that there is a group isomorphism $\phi : G_1 \to H_1$. Then we say that the set $\{(g, \phi(g)) | g \in G_1\}$ is a set of linearly completely dependent input-output masks.

We show distinguishers when input-output masks are linearly independent or completely dependent, which correspond to $\mathcal{A}_2$ and $\mathcal{A}_3$ in Section 5. We provide only rough ideas and heuristic estimations, and omit detailed analysis.

Distinguisher for Linearly Independent Input-Output Masks. Suppose $f_K : G \to H$ is a keyed function, G and H are decomposed as $G = G_1 \oplus G_2$, $H = H_1 \oplus H_2$, and $\sum_{\alpha \in G_1, \beta \in H_1} \mathrm{Cor}(f_K; \alpha, \beta)^2 / |H_1| \gg \frac{1}{|G_2|}$ holds. Let $f_K^{(1)} : G \to H_1$ be the projection of f_K onto H_1, and $F : G_1 \oplus H_1 \to \{0,1\}$ be the Boolean function such that $F(\alpha, \beta) = 1$ iff $(\alpha, \beta) \in G_1 \oplus H_1$. Then,

$$p_{\mathrm{real}} := \Pr\left[(\alpha, \beta) \leftarrow \mathsf{CEA}^{f_K^{(1)}} : F(\alpha, \beta) = 1\right] = \sum_{(\alpha, \beta) \in G_1 \oplus H_1} \frac{\mathrm{Cor}(f_K; \alpha, \beta)^2}{|H_1|}$$

follows from Proposition 8. Meanwhile, for a random function $\mathsf{RF} : G \to H$,

$$p_{\mathrm{ideal}} := \Pr\left[(\alpha,\beta) \leftarrow \mathsf{CEA}^{\mathsf{RF}^{(1)}} : F(\alpha,\beta) = 1\right] = \sum_{(\alpha,\beta)\in G_1\oplus H_1} \frac{\mathrm{Cor}(\mathsf{RF};\alpha,\beta)^2}{|H_1|}$$
$$\approx \sum_{(\alpha,\beta)\in G_1\oplus H_1} \frac{1}{|G|}\cdot\frac{1}{|H_1|} = \frac{1}{|G_2|}.$$

(We heuristically assume the third equality approximately holds due to [6, Theorem 3.2].) Since $p_{\mathrm{real}} \gg p_{\mathrm{ideal}}$ by assumption, we can distinguish f_K from RF by applying the QAA on $\mathsf{CEA}^{f_K^{(1)}}$ (or $\mathsf{CEA}^{\mathsf{RF}^{(1)}}$) and F with $O(\sqrt{1/p_{\mathrm{real}}})$ iterations.

Distinguisher for Linearly Completely Dependent Input-Output Masks. Again, let $f_K : G \to H$ be a keyed function, G and H are decomposed as $G = G_1 \oplus G_2$, $H = H_1 \oplus H_2$. Moreover, assume there is a group isomorphism $\phi : G_1 \to H_1$ and $\sum_{\alpha\in G_1} \mathrm{Cor}(f_K;\alpha,\phi(\alpha))^2/|H_1| \gg \frac{1}{|G|}$ holds. Let $F : G_1 \oplus H_1 \to \{0,1\}$ be the binary function such that $F(\alpha,\beta) = 1$ iff $\alpha \in G_1$ and $\beta = \phi(\alpha)$. Then, from Proposition 8,

$$p_{\mathrm{real}} := \Pr\left[(\alpha,\beta) \leftarrow \mathsf{CEA}^{f_K^{(1)}} : F(\alpha,\beta) = 1\right] = \sum_{\alpha\in G_1} \frac{\mathrm{Cor}(f_K;\alpha,\phi(\alpha))^2}{|H_1|}$$

follows. On the other hand, for a random function $\mathsf{RF} : G \to H$ we have

$$p_{\mathrm{ideal}} := \Pr\left[(\alpha,\beta) \leftarrow \mathsf{CEA}^{\mathsf{RF}^{(1)}} : F(\alpha,\beta) = 1\right] = \sum_{\alpha\in G_1} \frac{\mathrm{Cor}(\mathsf{RF};\alpha,\phi(\alpha))^2}{|H_1|}$$
$$\approx \sum_{\alpha\in G_1} \frac{1}{|G|}\cdot\frac{1}{|H_1|} = \frac{1}{|G|}.$$

Since $p_{\mathrm{real}} \gg p_{\mathrm{ideal}}$ holds by assumption, we can distinguish f_K from RF by applying the QAA on $\mathsf{CEA}^{f_K^{(1)}}$ (or $\mathsf{CEA}^{\mathsf{RF}^{(1)}}$) and F with $O(\sqrt{1/p_{\mathrm{real}}})$ iterations.

Application to the FF3-1 Structure. Beyne [6] showed generalized linear distinguishers on the FF3-1 structure in addition to linear distinguishers on FEA. The FF3-1 structure is almost the same as the FEA-1 structure (see Fig. 1), but the XOR operations in FEA-1 are replaced with modular additions in FF3-1. Thus, generalized linear distinguisher is more suitable for the FF3-1 structure.

The (generalized) linear approximation for FF3-1 in [6] is similar to the multidimensional linear approximation for FEA-1, but underlying groups are different from $\mathbb{Z}_2^n$. In fact, firstly a keyed function $F_K : \mathbb{Z}/2^{n/2}\mathbb{Z} \oplus \mathbb{Z}_2^t \to \mathbb{Z}/2^{n/2}\mathbb{Z}$ is built from the FF3-1 structure by fixing some inputs (here, input means plaintext and tweak) and truncating some outputs, and the distinguisher is applied F_K. The set (sub-group) of input-output masks is given by $\left\{((\alpha,0),\alpha) \in (\mathbb{Z}/2^{n/2}\mathbb{Z} \oplus \mathbb{Z}_2^t) \oplus \mathbb{Z}/2^{n/2}\mathbb{Z} \middle| \alpha \in \mathbb{Z}/2^{n/2}\mathbb{Z}\right\}$. In particular, the input-output masks are linearly completely dependent. The corresponding sum of the

squared correlation is estimated as $\sum_{\alpha \in \mathbb{Z}/2^{n/2}\mathbb{Z}} \mathrm{Cor}(F_K; (\alpha, 0), \alpha)^2 \approx 2^{-n(r/4-1)}$, and the classical distinguishing complexity is $O(2^{(r/4-3/4)n})$.

On the other hand, if we apply the quantum distinguisher explained above, we achieve the complexity $O(2^{(r/8-1/4)n})$.

9 Concluding Remarks

This paper showed a quantum speed-up for the multidimensional (zero correlation) linear distinguishers for the first time in such a way to to exploit multidimensional approximation in a non-trivial way. Firstly, we observed that there is a close relationship between the subroutine of Simon's algorithm and linear correlations of functions via Fourier transform. Specifically, a slightly modified version of the subroutine, which we call CEA, returns input and output linear masks of a target function with probability proportional to the squared linear correlation. Combining CEA with QAA, we achieved a quantum speed-up for multidimensional linear distinguishers. It is interesting that, only with a slight modification made, the subroutine of Simon's algorithm can be used to speed-up such a statistical attack. Our technique is naturally extended to generalized linear distinguishers on arbitrary finite abelian groups by replacing the Hadamard transform in CEA with general QFT. We also showed that CEA similarly speeds-up multidimensional zero correlation linear distinguishers, as well as some integral distinguishers via the correspondence shown by Bogdanov et al. and Sun et al [8, 60]. Especially, we observe that a more than quadratic speed-up is possible if an integral property holds on multiple mutually orthogonal vector spaces of the same dimension, and even a single-query distinguisher for a toy example of a 4-bit cell SPN cipher with the same integral property as the 2.5-round AES.

Future Directions. An important future work is to investigate how to extend our technique to key-recovery attacks, or combine it with Schrottenloher's [57].

All the distinguishers in this paper can be extended to key-recovery attacks just by guessing sub-keys of additional rounds using Grover's algorithm. Suppose we would like to recover the key of an $(r + r')$-round cipher and there is a (quantum) r-round distinguisher on the cipher running in time T. In addition, assume that we can apply the distinguisher on the intermediate r rounds if we know a k-bit subkey K' in the remaining r'-rounds. Then, roughly speaking, by just guessing the subkey K' with the Grover search while checking if a key-guess is correct with the distinguisher, we achieve an $(r + r')$-round quantum key-recovery attack of time complexity $O(T \cdot 2^{k/2})$.

Still this idea is too naive, compared to classical key-recovery attacks using sophisticated techniques such as the FFT [23, 61, 30]. As mentioned in Section 1.2, the recent work by Schrottenloher [57] has shown how to combine such key-recovery techniques using the FFT with the QFT, taking multiple linear approximations into account. However, in Schrottenloher's attack, multiple approximations contribute to only the precision of the attack by a constant factor,

and does not contribute much to reducing time complexity. It is definitely an important and interesting future work to investigate theoretical relationships between our technique with Schrottenloher's and study how to reduce the time complexity of key-recovery exploiting multidimensional approximations.

Another important future work is to study quantum speed-up for integral distinguishers based on zero-sum properties. As mentioned before, our quantum integral distinguishers are applicable only if the distinguishers are based on a balanced functions and not a zero-sum property. However, distinguishers based on zero-sum properties often break more rounds than those on balanced functions, especially when extended to key-recovery attacks. Since multiple integral properties sometimes could lead to a more than quadratic speed-up, a quantum attack breaking more rounds of a cipher than classical attacks may be found by investigating this direction.

It would be also of interest to investigate how the super-quadratic speed-up in Section 7.1 can be reproduced more broadly. We observe that the following two things are essential in achieving that speed-up: (i) There exist multiple properties that are similar to each other, but only one of them can be exploited at a time in the classical setting. (For the 2.5-round integral property of AES-like ciphers, there are 16 choices on which input cell to activate, but the existence of multiple choices is not exploited in the classical distinguisher.) (ii) The properties are translated/embedded into quantum amplitude in some sense (by using CEA, through the correspondence between integral and zero-correlation linear properties). So, if we find some classical properties satisfying (i) and a quantum technique enabling (ii), we might be able to reproduce similar quantum speed-ups, not only for linear/integral distinguishers but also for some other techniques.

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Structure	Rounds	Round Functions	Attack Type	Complexity	Reference
Balanced Feistel	4		CPA	$O(2^{n/2})$	[55]
Balanced Feistel	4		QCCA	$O(\text{poly}(n))$	[40]
Balanced Feistel	5		CPA	$O(2^{n})$	[56]
Balanced Feistel	5	bij.	KPA	$O(2^{2n/3})$	[44, 62]
Balanced Feistel	5		QCPA	$O(2^{2n/3})$	[21]
Balanced Feistel	5	bij.	QCPA	$O(2^{n/2})$	**Ours**
k-branch Type-I Generalized Feistel	k^2-k+1	bij.	QCCA	$O(\text{poly}(n))$	[53]
k-branch Type-I Generalized Feistel	k^2+k-1		CPA	$O(2^{(1-\frac{1}{k})n})$	[52]
4-branch Type-I Generalized Feistel	18	bij.	KPA	$O(2^{3n/4})$	[9] (combined with [8])
k-branch Type-I Generalized Feistel	k^2+k-2	bij.	QCPA	$O(2^{n/2})$	**Ours**
k-branch Type-II Generalized Feistel	$k+1$	bij.	QCPA	$O(\text{poly}(n))$	[28]
k-branch Type-II Generalized Feistel	$2k+1$		CPA	$O(2^{(1-\frac{1}{k})n})$	[52]
4-branch Type-II Generalized Feistel	9		KPA	$O(2^{3n/4})$	[9] (combined with [8])
k-branch Type-II Generalized Feistel	$2k+1$	bij.	QCPA	$O(2^{n/2})$	**Ours**

Table 2: Comparison of classical and quantum attacks on balanced and Type-I/II generalized Feistel structures. "bij." means that the attack assumes round functions are bijective. The parameters k for Type-II generalized Feistel are even numbers. QCPA (resp., QCCA) denotes quantum superposition (Q2) chosen plaintext attack (resp., quantum superposition chosen ciphertext attack). All of our attacks appearing in this table are multidimensional zero correlation linear distinguishers.

A Pearson's Chi-Squared Goodness-of-Fit Test and Distinguishers

Given a list L of elements of a finite set S, the test statistic of Pearson's chi-squared goodness-of-fit test (on the uniform distribution over S) is defined as $\chi^2 := |S| \cdot |\mathsf{L}| \cdot \sum_{z \in S} \left(\frac{\mathsf{num}(z)}{|\mathsf{L}|} - \frac{1}{|S|} \right)^2$, where $\mathsf{num}(z)$ denotes the number of times that z appears in L.

Suppose that entries of L are sampled independently from the uniform distribution or another distribution D, and let χ^2_U (resp., χ^2_D) be the value of the test statistic when the distribution is uniform (resp., D). Then χ^2_U approximately

follows the chi-squared distribution with $|\mathsf{L}|-1$ degrees of freedom, of which the mean is $\mu_U := |\mathsf{L}|-1$ and the standard deviation is $\sigma_U := \sqrt{2(|\mathsf{L}|-1)}$. If $|\mathsf{L}|$ is sufficiently large, we can assume that χ^2_U approximately follows the normal distribution $\mathcal{N}\left(|\mathsf{L}|-1, \sqrt{2(|\mathsf{L}|-1)}\right)$, which implies $\Pr\left[\chi^2_U \geq \mu_U + \sigma_U\right] \gtrapprox 1/6$. Thus, if we know that the value of χ^2_D is larger than $\mu_U + \sigma_U$ quite often, then we can distinguish which distribution the entries of L follow with a sufficiently large advantage by checking whether the test statistic is larger than $\mu_U + \sigma_U$ or not.

B Proof of Proposition 2

First, we restate the algorithm and the statement of the proposition. Recall that s is an arbitrary positive integer constant and $p_U := \Pr\left[x \xleftarrow{\text{measure}} U\,|0^n\rangle : F(x)=1\right]$.

QAA for Distinguisher (Algorithm $\mathcal{QD}$).

1. For $j = 1, \ldots, s$, do:
   (a) Choose i from the set of integers from 0 to $\left\lfloor \frac{1}{\sin\left(2\cdot\arcsin\left(\sqrt{t}\right)\right)} \right\rfloor$ uniformly at random.
   (b) Apply $Q(U,F)^i U$ to $|0^n\rangle$ and measure the entire state by the computational basis, and let x be the outcome.
   (c) Compute $F(x)$. If $F(x)=1$, return 1 and abort.
2. Return 0.

Proposition 9 (Restatement of Proposition 2). *Suppose $1/4 > t > 0$. Then, for any constant s, $\mathcal{QD}$ applies U, U^*, and $\mathcal{S}_F$ at most $s(\frac{1}{\sqrt{t}}+1)$ times and (1) returns 1 with probability at least $(1-(\frac{3}{4})^s)\cdot \Pr_{U\sim D_1}[1/4 > p_U \geq t]$ if U is chosen according to D_1 and (2) returns 1 with probability at most $s\cdot(16t'/t + 20t'/\sqrt{t}) + \Pr_{U\sim D_2}[t' < p_U]$ for any $t' > 0$ satisfying $4\sqrt{t'/t} + 2\sqrt{t'} < \pi/2$ if U is chosen according to D_2.*

We use the following lemma from [17].

Lemma 1 (Lemma 2 in [17]). *Assume $0 < p_U < 1/2$, and let N be an integer satisfying $N \geq \frac{1}{\sin\left(2\cdot\arcsin\left(\sqrt{p_U}\right)\right)}$. Then $\Pr\left[i \xleftarrow{\$} \{0,\ldots,N-1\}, x \xleftarrow{measure} Q(U,F)^i U\,|0^n\rangle : F(x)=1\right] \geq 1/4$ holds.*

In fact Lemma 2 in [17] proves the claim only when $U = H^{\otimes n}$ but it is straightforward to check that the proof is valid for arbitrary U (due to Lemma 1 in [18]). We assume $0 < p_U < 1/4$ so that $0 < 2\cdot\arcsin\left(\sqrt{p_U}\right) < \pi/2$ will hold.

Proof (of Proposition 2). The claim for the number of applications of U, U^*, and $\mathcal{S}_F$ immediately follows from the definition of the algorithm and $Q(U,F)$ because

$$\frac{1}{\sin\left(2\cdot\arcsin\left(\sqrt{t}\right)\right)} \leq \frac{1}{\sin(2\sqrt{t})} \leq \frac{1}{2\sqrt{t} - \frac{4}{3}t\sqrt{t}} \leq \frac{1}{\sqrt{t}}$$

holds, where we used $\arcsin(x) \geq x$, $\sin(x) \geq x - x^3/6$, and $t < 1/4$.

Next, we lower bound the success probability when U is chosen according to D_1. Recall that the algorithm iteratively picks a random i and measure the state $Q(U,F)^i U\,|0^n\rangle$. Now, assume $1/4 > p_U \geq t$. Then, by Lemma 1, the probability that the algorithm fails to find x satisfying $F(x) = 1$ at the j-th iteration of the algorithm is at most $3/4$ for each $j = 1, \ldots, s$. Thus, assuming $1/4 > p_U \geq t$, the probability that the algorithm succeeds to find x satisfying $F(x) = 1$ after s iteration is at least $(1 - (3/4)^s)$.

Hence, when U is chosen according to D_1, the probability that the algorithm finds x satisfying $F(x) = 1$ is lower bounded by $(1 - (3/4)^s) \cdot \Pr_{U \sim D_1}[1/4 > p_U \geq t]$.

Next, we upper bound the success probability when U is chosen according to D_2. Recall that t' is a positive value satisfying $4\sqrt{t'/t} + 2\sqrt{t'} \leq \pi/2$. Now, assume $p_U \leq t'$. Then, for any i between 0 and $\frac{1}{\sin(2 \cdot \arcsin(\sqrt{t}))}$ we have

$$i \leq \frac{1}{\sin(2 \cdot \arcsin(\sqrt{t}))} \overset{(x/2 \leq \sin(x))}{\leq} \frac{1}{\arcsin(\sqrt{t})} \overset{(x \leq \arcsin(x))}{\leq} \frac{1}{\sqrt{t}}$$

and thus

$$(2i+1) \cdot \arcsin(\sqrt{p_U}) \leq (\frac{2}{\sqrt{t}} + 1) \cdot \arcsin(\sqrt{t'}) \overset{(2x \geq \arcsin(x))}{\leq} 4\sqrt{\frac{t'}{t}} + 2\sqrt{t'} \quad (\leq \pi/2)$$

holds. Hence, by Proposition 1, the probability that the algorithm finds x satisfying $F(x) = 1$ at the j-th iteration of the algorithm is at most

$$\begin{aligned}\sin^2((2i+1)\arcsin\sqrt{p_U}) &\leq \sin^2\left(4\sqrt{t'/t} + 2\sqrt{t'}\right) \\ &\overset{(\sin(x) \leq x)}{\leq} 16t'/t + 16t'/\sqrt{t} + 4t' \\ &\leq 16t'/t + 20t'/\sqrt{t}.\end{aligned}$$

Thus, assuming $t' \geq p_U$, the probability that the algorithm succeeds to find x satisfying $F(x) = 1$ after s iteration is at most $s \cdot (16t'/t + 20t'/\sqrt{t})$.

Therefore, when U is chosen according to D_2, the probability that the algorithm finds x satisfying $F(x) = 1$ is upper bounded bounded by

$$\begin{aligned}s \cdot (16t'/t + 20t'/\sqrt{t}) &\Pr_{U \sim D_2}[t' \geq p_U] + \Pr_{U \sim D_2}[t' < p_U] \\ &\leq s \cdot (16t'/t + 20t'/\sqrt{t}) + \Pr_{U \sim D_2}[t' < p_U],\end{aligned}$$

which completes the proof. □

C Proof of Proposition 4

Proof. Here we provide a proof only for $\mathcal{A}_2$. The proofs for $\mathcal{A}_1$ and $\mathcal{A}_3$ are obtained similarly by changing parameters appropriately. First, note that

$$p_f := \Pr\left[(\alpha,\beta) \xleftarrow{\text{measure}} \mathsf{CEA}^f\,|0^m\rangle\,|0^n\rangle : F(x) = 1\right] = \sum_{(\alpha,\beta) \in V - \{\mathbf{0}\}} \frac{\mathrm{Cor}(f;\alpha,\beta)^2}{2^w}$$

holds for arbitrary function f by Lemma 3 (in this proof, f is now a truncated version of a block cipher $\mathsf{msb}_w[E_K]$ or a randomly chosen permutation $\mathsf{msb}_w[P]$). p_U in Proposition 2 corresponds to p_f here.

If the oracle given to $\mathcal{A}_2$ is the real cipher E_K, then Proposition 2 guarantees that $\mathcal{A}_2$ returns 1 (i.e., it judges the oracle is the real cipher E_K) with probability at least $(1-(3/4)^s)p \geq p/2$.

If the oracle given to $\mathcal{A}_2$ is a random permutation P, then Claim 1 guarantees that $\mathbb{E}_P[p_P]$ and $\mathbf{Var}_P[p_P]$ are approximately upper bounded as $\mu_P := \mathbb{E}_P[p_P] \lessapprox 2^{\dim(V)-n-w}$ and $\sigma_P := \sqrt{\mathbf{Var}_P[p_P]} \lessapprox \sqrt{2^{\dim(V)+1-2n-2w}}$. Hence we have

$$\Pr_P\left[p_P > (n+1)\cdot\frac{2^{\dim(V)}}{2^{n+w}}\right] \lessapprox \Pr_P\left[p_P > \mu_P + n\cdot 2^{\frac{\dim(V)-1}{2}}\sigma_P\right] \overset{\text{Chebyshev's inequality}}{\leq} 2^{-\dim(V)+1}\cdot n^{-2}.$$

By the claim on D_2 in Proposition 2 with $t' = (n+1)\cdot\frac{2^{\dim(V)}}{2^{n+w}}$, the probability that $\mathcal{A}_2$ returns 1 is at most[21]

$$3\cdot(16t'/c + 20t'/\sqrt{c}) + 2^{-\dim(V)+1}\cdot n^{-2} \lessapprox \frac{2^{\dim(V)+7}(n+1)}{2^{n+w}\cdot c} + 2^{-\dim(V)+1}\cdot n^{-2}.$$

In addition, by definition of $\mathcal{QD}$ and $\mathcal{A}_2$, $\mathcal{A}_2$ makes at most $6\sqrt{2^w/c}$ quantum queries to E_K or P and the costs for other operations are negligibly small. Hence the running time of $\mathcal{A}_2$ is at most $6\sqrt{2^w/c}$. □

D About Why Claim 1 is Plausible

We argue that Claim 1 is plausible due to the following four facts.

(i) A previous work [2, Theorem 4] proves a weaker statement where "$2^v - 2^u - 2^w + 1$ degrees of freedom" in the above claim is weaken to "*at most* $2^v - 2^u - 2^w + 1$ degrees of freedom".
(ii) The same work conjectures that the claim holds [2, Conjecture 1], showing some experimental results supporting the conjecture.
(iii) If a random variable X follows the χ^2 distribution with $2^v - 2^u - 2^w + 1$ degrees of freedom, then $\mathbb{E}[X] = 2^v - 2^u - 2^w + 1$.
(iv) We can formally prove that $\mathbb{E}_P[2^n\cdot\mathrm{Cap}(p_S^P)]$ is equal to $\frac{2^n}{2^n-1}(2^v-2^u-2^w+1)$, which is quite close to $2^v - 2^u - 2^w + 1$. Namely, the following proposition holds.

Proposition 10. *For a randomly chosen permutation P, $\mathbb{E}[2^n\cdot\mathrm{Cap}(p_S^P)] = \frac{2^n}{2^n-1}(2^v - 2^u - 2^w + 1)$ holds.*

[21] We need the condition $c \gg 2^{-n-w+\dim(V)}$ so that $4\sqrt{t'/c} + 2\sqrt{t'} \ll \pi/2$ will hold and we can apply the claim on D_2 here.

The goal of the remaining part of the section is to show this proposition. This section follows the notations used in Claim 1. That is, $V \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$ is a vector space and S be an arbitrary basis of V, and parameters v, u, and w are defined as $v := \dim(V)$, $u := \dim(V \cap \mathbb{F}_2^n \times \{0^n\})$, and $w := \dim(V \cap \{0^n\} \times \mathbb{F}_2^n)$.

Since $2^n \cdot \mathrm{Cap}(p_S^P) = 2^n \cdot \sum_{(\alpha,\beta)\in V-\{\mathbf{0}\}} \mathrm{Cor}(P;\alpha,\beta)^2$ holds, it suffices to show the following proposition.

Proposition 11. *Let $f : \{0,1\}^m \to \{0,1\}^n$ be a random balanced function. (Namely, f is chosen uniformly at random from the set of all balanced functions from $\{0,1\}^m$ to $\{0,1\}^n$. If $m = n$, f is just a random permutation.) Then, for arbitrary $\alpha \in \{0,1\}^m$ and $\beta \in \{0,1\}^n$,*

$$\mathbb{E}_f\left[\mathrm{Cor}(f;\alpha,\beta)^2\right] = \begin{cases} 1 & \text{if } \alpha = 0^m \text{ and } \beta = 0^n \\ 0 & \text{if } \alpha \neq 0^m \text{ and } \beta = 0^n, \text{ or } \alpha = 0^m \text{ and } \beta \neq 0^m \\ \frac{1}{2^m-1} & \text{if } \alpha \neq 0^m \text{ and } \beta \neq 0^n \end{cases}$$

holds. Here, the expectation value is taken over the random choice of f.

Proof. When $\alpha = 0^m$ and $\beta = 0^n$ (resp., $\alpha \neq 0^m$ and $\beta = 0^n$), the correlation $\mathrm{Cor}(f;\alpha,\beta)$ is zero (resp., 1) for arbitrary function f. When $\alpha = 0^m$ and $\beta \neq 0^n$,

$$\begin{aligned}\mathrm{Cor}(f;\alpha,\beta) &= \sum_{x\in\{0,1\}^m} \frac{(-1)^{\alpha\cdot x \oplus \beta\cdot f(x)}}{2^m} = \sum_{x\in\{0,1\}^m} \frac{(-1)^{\beta\cdot f(x)}}{2^m} \\ &= \sum_{y\in\{0,1\}^n} \sum_{x\in f^{-1}(y)} \frac{(-1)^{\beta\cdot y}}{2^m} = \sum_{y\in\{0,1\}^n} 2^{m-n}\frac{(-1)^{\beta\cdot y}}{2^m} = 0\end{aligned}$$

always holds for arbitrary balanced function f (we used the balancedness of f for the second last equality).

Next, we show the claim when $\alpha \neq 0^m$ and $\beta \neq 0^m$. Let $\mathsf{Perm}(m)$ be the set of all permutations over $\{0,1\}^m$ and $\mathsf{Reg}(m,n)$ denote the set of all balanced functions from $\{0,1\}^m$ to $\{0,1\}^n$. $\chi_{\alpha,\beta}(f,x)$ denote the indicator function such

that $\chi_{\alpha,\beta}(f,x)=1$ iff $\alpha\cdot x=\beta\cdot f(x)$. Then

$$\begin{aligned}
\mathrm{Cor}(f;\alpha,\beta)^2 &= \left(\Pr_x[\alpha\cdot x=\beta\cdot f(x)]-\Pr_x[\alpha\cdot x\neq\beta\cdot f(x)]\right)^2\\
&= \left(2\Pr_x[\alpha\cdot x=\beta\cdot f(x)]-1\right)^2\\
&= \left(\frac{2\cdot\#\left\{x\in\{0,1\}^m|\alpha\cdot x=\beta\cdot f(x)\right\}-2^m}{2^m}\right)^2\\
&= \frac{1}{2^{2m}}\left(2\sum_{x\in\{0,1\}^m}\chi_{\alpha,\beta}(f,x)-2^m\right)^2\\
&= \frac{1}{2^{2m}}\left(4\sum_{x,x'\in\{0,1\}^m}\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')-2^{m+2}\sum_{x\in\{0,1\}^m}\chi_{\alpha,\beta}(f,x)+2^{2m}\right)\\
&= \frac{1}{2^{2m}}\left(4\sum_{\substack{x,x'\in\{0,1\}^m\\x\neq x'}}\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')-(2^{m+2}-4)\sum_{x\in\{0,1\}^m}\chi_{\alpha,\beta}(f,x)+2^{2m}\right)
\end{aligned}\tag{11}$$

holds. Now, for each $x\in\{0,1\}^m$ we have

$$\mathbb{E}_f\left[\chi_{\alpha,\beta}(f,x)\right]=\Pr_f\left[\chi_{\alpha,\beta}(f,x)=1\right]=\frac{1}{2}\tag{12}$$

because, for each fixed tuple (α,β,x) with $\beta\neq 0^n$, the number of $f\in\mathsf{Reg}(m,n)$ satisfying $\alpha\cdot x=\beta\cdot f(x)$ is equal to the number of f satisfying $\alpha\cdot x\neq\beta\cdot f(x)$. For any permutation $P\in\mathsf{Perm}(m)$, let P_{tr} be the truncated function of P obtained by discarding the rightmost $(m-n)$ bits. Then, for arbitrary distinct $x,x'\in\{0,1\}^m$ we have

$$\begin{aligned}
&\mathbb{E}_f\left[\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')\right]\\
&\quad= \Pr_{f\xleftarrow{\$}\mathsf{Reg}(m,n)}\left[\chi_{\alpha,\beta}(f,x)=1\wedge\chi_{\alpha,\beta}(f,x')=1\right]\\
&\quad= \Pr_{P\leftarrow\mathsf{Perm}(m)}\left[\chi_{\alpha,\beta||0^{m-n}}(P_{tr},x)=1\wedge\chi_{\alpha,\beta||0^{m-n}}(P_{tr},x')=1\right]\\
&\quad= \Pr_{P\xleftarrow{\$}\mathsf{Perm}(m)}\left[\chi_{\alpha,\beta||0^{m-n}}(P_{tr},x')=1|\chi_{\alpha,\beta||0^{m-n}}(P_{tr},x)=1\right]\\
&\qquad\qquad\cdot\Pr_{P\xleftarrow{\$}\mathsf{Perm}(m)}\left[\chi_{\alpha,\beta||0^{m-n}}(P_{tr},x)=1\right]\\
&\quad= \begin{cases}\frac{2^{m-1}-1}{2^m-1}\cdot\frac{1}{2} & \text{if } \alpha\cdot x=\alpha\cdot x'\\ \frac{2^{m-1}}{2^m-1}\cdot\frac{1}{2} & \text{if } \alpha\cdot x\neq\alpha\cdot x'.\end{cases}
\end{aligned}$$

In addition, since $\alpha \neq 0^m$ we have

$$\{(x, x') \in \{0,1\}^m \times \{0,1\}^m | x \neq x' \wedge \alpha \cdot x = \alpha \cdot x'\} = 2^m \cdot (2^{m-1} - 1),$$
$$\{(x, x') \in \{0,1\}^m \times \{0,1\}^m | x \neq x' \wedge \alpha \cdot x \neq \alpha \cdot x'\} = 2^m \cdot 2^{m-1}.$$

Therefore

$$\begin{aligned}
&\sum_{\substack{x,x' \in \{0,1\}^m \\ x \neq x'}} \mathbb{E}_f \left[\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')\right] \\
&= \sum_{\substack{x,x' \in \{0,1\}^m \\ x \neq x' \wedge \alpha \cdot x = \alpha \cdot x'}} \mathbb{E}_f \left[\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')\right] + \sum_{\substack{x,x' \in \{0,1\}^m \\ x \neq x' \wedge \alpha \cdot x \neq \alpha \cdot x'}} \mathbb{E}_f \left[\chi_{\alpha,\beta}(f,x)\chi_{\alpha,\beta}(f,x')\right] \\
&= \sum_{\substack{x,x' \in \{0,1\}^m \\ x \neq x' \wedge \alpha \cdot x = \alpha \cdot x'}} \frac{2^{m-1}-1}{2^m - 1} \cdot \frac{1}{2} + \sum_{\substack{x,x' \in \{0,1\}^m \\ x \neq x' \wedge \alpha \cdot x \neq \alpha \cdot x'}} \frac{2^{m-1}}{2^m - 1} \cdot \frac{1}{2} \\
&= \frac{2^m (2^{m-1}-1)^2}{2 \cdot (2^m - 1)} + \frac{2^{3m-2}}{2 \cdot (2^m - 1)} \\
&= \frac{2^{3m-2} - 2^{2m-1} + 2^{m-1}}{2^m - 1} \qquad (13)
\end{aligned}$$

holds. From Eq. (11)-(13), it follows that

$$\begin{aligned}
\mathbb{E}_f \left[\mathrm{Cor}(f; \alpha, \beta)\right] &= \frac{1}{2^{2m}} \left(4 \cdot \frac{2^{3m-2} - 2^{2m-1} + 2^{m-1}}{2^m - 1} - (2^{m+2} - 4) \cdot \frac{2^m}{2} + 2^{2m} \right) \\
&= \frac{1}{2^m - 1},
\end{aligned}$$

which completes the proof. □

E Non-Asymptotic Complexity of Quantum One-Dimensional Linear Distinguishers

As mentioned in Section 5, the asymptotic complexity of our technique is the same as Kaplan et al.'s, but the non-asymptotic complexity become smaller in a specific situation. This section explains the details of the comparison.

Assume that we have a one-dimensional linear approximation of a cipher E_K, and the absolute value of the linear correlation is concentrated in a very narrow range around a known value $c \gg 2^{-n/2}$. Let α and β be the input-ouput masks of the approximation, and $f(x, y)$ be the Boolean function such that $f(x, y) = 1$ iff $(x, y) = (\alpha, 1)$.

In the above situation, by using CEA, we can distinguish E_K by making $(\sqrt{2}\pi) \cdot (1/c)$ queries: If we apply CEA on the Boolean function $x \mapsto \beta \cdot E_K(x)$ and measure the final state, the probability that we get $(\alpha, 1)$ is $q = c^2/2$. Thus, by applying QAA on f with $U = \mathsf{CEA}^{\beta \cdot E_K}$ and $\lceil (\pi/4) \arcsin(\sqrt{q}) \rceil$ iterations, we

get $(\alpha, 1)$ with a high probability. Since the function $\beta \cdot E_K(x)$ can be computed on quantum circuit by making at most two queries to E_K, the total number of queries to E_K by QAA is $2 \cdot 2 \cdot \lceil (\pi/4) \arcsin(\sqrt{q}) \rceil + 1 \approx (\sqrt{2}\pi) \cdot (1/c)$. (When the above procedure is run against a random permutation instead of E_K, the probability that QAA outputs $(\alpha, 1)$ is negligible.)

On the other hand, Kaplan et al.'s distinguisher in [43] makes $2(\sqrt{2}\pi) \cdot (1/c)$ queries: For a permutation P, let $F^P : \{0,1\}^n \to \{0,1\}$ be the Boolean function such that $F^P(x) = 1$ iff $\alpha \cdot x = \beta \cdot P(x)$. In addition, define $p := |(F^P)^{-1}(1)|/2^n$. If P is a random permutation, then p is very close to $1/2$. If $P = E_K$, then p is close to $1/2 \pm c/2$. Kaplan et al.'s attack distinguishes E_K by estimating p to check whether it is close to $1/2$ or not. To achieve a constant order of advantage, the precision of the estimation must be less than c. Theorem 3 of [43] implies that, for an estimation with precision c by quantum counting, at least $(2\pi\sqrt{p}) \cdot (1/c) \approx (2\sqrt{2}\pi) \cdot (1/c)$ queries to F are required. Hence we can deduce that Kaplan et al.'s distinguisher makes at least $(2\sqrt{2}\pi) \cdot (1/c)$ queries to the encryption oracle.

Note that the above arguments do not work if (the absolute value of) the variance of the linear correlation is not small. If the variance is large, we do not observe evident difference between non-asymptotic complexity between Kaplan et al.'s and ours because we have to run QAA multiple times as $\mathcal{A}_3$ does, unlike the above our attack running QAA only once.

F How to Efficiently Compute F in Distinguishers

This section explains how to compute Boolean functions F used in our quantum multidimensional (zero-correlation) linear distinguishers on a quantum circuit F.

The functions F in the distinguishers satisfy that $F(\alpha, \beta) = 1$ iff $(\alpha, \beta) \in V - \{\mathbf{0}\}$ for some vector space $V \subset \mathbb{F}_2^m \times \mathbb{F}_2^n$. We explain how to judge whether a given (α, β) belongs to V in the following two cases.

Case 1: V is the set of n-bit strings of which specific bit positions are zero (e.g., the set of strings of the form $0^{n/2}||x$ for some $x \in \mathbb{F}_2^{n/2}$).
Case 2: V is a general sub-vector-space of $\mathbb{F}_2^n$.

For Case 1, whether $(\alpha, \beta) \in V$ can be judged just by checking whether some specific bits are zero.

For Case 2, we choose and fix a $\mathbb{F}_2$-linear isomorphism (invertible matrix) M such that $M(V)$ becomes a space of Case 1. To check whether $(\alpha, \beta) \in V$, we perform the following procedure.

(i) compute $v := M(\alpha, \beta)$
(ii) check whether $v \in M(V)$
(iii) uncompute (i)

Note that (ii) is exactly the judgement for Case 1 (because $M(V)$ is a vector space of Case 1). (i) and (iii) may require additional computational costs, but they will be less than the cost for a few computations of a linear layer of a typical block cipher. (In any case, we can implement a quantum circuit by using some CNOT/CCNOT gates.)

G Proof of Proposition 5

Proof (of Proposition 5). We give a proof only for $\mathcal{B}_2$. The proof for $\mathcal{B}_3$ is obtained by replacing w with $\dim(V)$.

It immediately follows from the definition of $\mathcal{B}_2$, Proposition 2, and Proposition 3 that $\mathcal{B}_2$ always returns 1 when the given oracle is E_K and the running time of $\mathcal{B}_2$ is at most $2\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor + 1$ encryptions by E_K.

Below we lower bound the probability that $\mathcal{B}_2$ returns 0 when the oracle is a random permutation P. First, note that

$$p_f := \Pr\left[(\alpha,\beta) \xleftarrow{\text{measure}} \mathsf{CEA}^f\,|0^m\rangle\,|0^n\rangle : F(x) = 1\right] = \sum_{(\alpha,\beta)\in V-\{\mathbf{0}\}} \frac{\mathrm{Cor}(f;\alpha,\beta)^2}{2^w}$$

holds for arbitrary function f by Lemma 3 (f is now a truncated version of a randomly chosen permutation $\mathsf{msb}_w[P]$). If the oracle given to $\mathcal{B}_2$ is a random permutation P, then Claim 1 guarantees that $\mathbb{E}_P[p_P]$ and $\mathbf{Var}_P[p_P]$ are approximately upper bounded as $\mu_P := \mathbb{E}_P[p_P] \lessapprox 2^{\dim(V)-n-w} = 2^{u-n}$ and $\sigma_P := \sqrt{\mathbf{Var}_P[p_P]} \lessapprox \sqrt{2^{\dim(V)+1-2n-2w}} = 2^{\frac{-\dim(V)+1}{2}} \cdot \mu_P$. Hence we have

$$\Pr_P\left[|p_P - \mu_P| > \frac{1}{2}\mu_P\right] \lessapprox \Pr_P\left[|p_P - \mu_P| > 2^{\frac{\dim(V)-1}{2}}\sigma_P\right] \overset{\text{Chebyshev's inequality}}{\leq} 2^{-\dim(V)+1}.$$

Especially, $\Pr_P\left[\frac{1}{2}2^{u-n} \le p_P \le 2\cdot 2^{u-n}\right] \ge 1 - 2^{-\dim(V)+1}$ holds. In addition, for each P such that $\frac{1}{2}2^{u-n} \le p_P \le 2\cdot 2^{u-n}$, we have

$$\begin{aligned}&\Pr\left[(\alpha,\beta) \xleftarrow{\text{measure}} Q(\mathsf{CEA}^{\mathsf{msb}_w[P]}, F)^{\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor}\mathsf{CEA}^{\mathsf{msb}_w[P]}\,|0^n\rangle\,|0^w\rangle\right]\\ &\quad= \sin^2\left(\left(2\left\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\right\rfloor + 1\right)\arcsin(\sqrt{p_P})\right)\sin^2\left(\frac{\pi}{2}\sqrt{2^{n-u}}\cdot\sqrt{p_P}\right) \ge \frac{1}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}&\Pr\left[(\alpha,\beta) \xleftarrow{\text{measure}} Q(\mathsf{CEA}^{\mathsf{msb}_w[P]}, F)^{\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor}\mathsf{CEA}^{\mathsf{msb}_w[P]}\,|0^n\rangle\,|0^w\rangle\right]\\ &\quad\ge \Pr\left[(\alpha,\beta) \leftarrow Q(\mathsf{CEA}^{\mathsf{msb}_w[P]}, F)^{\lfloor\frac{\pi}{4}\sqrt{2^{n-u}}\rfloor}\mathsf{CEA}^{\mathsf{msb}_w[P]}\,|0^n\rangle\,|0^w\rangle\,\middle|\right.\\ &\qquad\left.\frac{1}{2}2^{u-n} \le p_P \le 2\cdot 2^{u-n}\right]\cdot\Pr_P\left[\frac{1}{2}2^{u-n} \le p_P \le 2\cdot 2^{u-n}\right]\\ &\quad\gtrapprox \frac{1}{2}\cdot\left(1 - 2^{-\dim(V)+1}\right)\end{aligned}$$

holds, which completes the proof. □