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Lower bounds of the Laplacian graph eigenvalues

by Aleksandar Torgašev [a] and Miroslav Petrović [b]

[a] *Mathematical Faculty, Studentski trg 16a, 11000 Belgrade, Serbia and Montenegro*
[b] *Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Serbia and Montenegro*

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ABSTRACT

In this paper we prove that all positive eigenvalues of the Laplacian of an arbitrary simple graph have some positive lower bounds. For a fixed integer $k \geqslant 1$ we call a graph without isolated vertices k-minimal if its kth greatest Laplacian eigenvalue reaches this lower bound. We describe all 1-minimal and 2-minimal graphs and we prove that for every $k \geqslant 3$ the path P_{k+1} on $k+1$ vertices is the unique k-minimal graph.

1. INTRODUCTION

Let G be a simple graph on n vertices and the vertex set $V(G) = \{v_1, \dots, v_n\}$. Next, let $A(G) = [a_{ij}]$ be its (0, 1) adjacency matrix, and $D(G) = \operatorname{diag}(d_1, \dots, d_n)$ be the diagonal matrix with degrees $d_1, \dots, d_n$ of its vertices $v_1, \dots, v_n$. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph G. It is symmetric, singular and positive semidefinite. Its eigenvalues are all real and nonnegative and form the Laplacian spectrum $\{\lambda_1, \dots, \lambda_n\}$ of the graph G. We shall always assume that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. It is well known that $\lambda_n = 0$ and the multiplicity of 0 is equal to the number of (connected) components of G. The Laplacian eigenvalue $\lambda_{n-1}(G)$ of any graph G on n vertices is called by M. Fiedler *the algebraic connectivity* of G and denoted by $a(G)$. It is known that $a(G) > 0$ if and only if G is a connected graph.

For any integer $n \geqslant 1$, P_n is the path on n vertices. As is well known, the Laplacian spectrum of the path P_n on n vertices reads $\{2(1 + \cos\frac{\pi i}{n}) \mid i = 1, \dots, n\}$,

E-mails: torgasev@matf.bg.ac.yu (A. Torgašev), petrovic@knez.uis.kg.ac.yu (M. Petrović).

wherefrom $a(P_n) = \lambda_{n-1}(P_n) = 2(1 - \cos\frac{\pi}{n})$. For any $k \in N = \{1, 2, \ldots\}$ denote:

$$A_k = a(P_{k+1}) = 2\left(1 - \cos\frac{\pi}{k+1}\right).$$

The Laplacian spectrum of graphs is widely investigated in the literature, and some related papers are quoted in the list of references. In the sequel we quote some known results which we shall use later. The most important result for us was proved by M. Fiedler in [5].

Theorem 1 [5]. *For every connected graph G on $n \geqslant 2$ vertices*

(1) $$a(G) = \lambda_{n-1}(G) \geqslant A_{n-1}.$$

Equality holds in (1) *if and only if G is the path P_n on n vertices.*

Theorem 2 [7]. *If a graph H on m vertices is a subgraph* (*not necessary induced*) *of a graph G, then for each $i = 1, \ldots, m$ we have*

$$\lambda_i(H) \leqslant \lambda_i(G).$$

Further we recall the notion of the direct sum of two graphs G_1 and G_2. If G_1 is a graph whose vertex set is $V(G_1)$ and the edge set is $E(G_1)$ and G_2 is a graph whose vertex set is $V(G_2)$ and the edge set is $E(G_2)$, and if $V(G_1) \cap V(G_2) = \emptyset$, then their direct sum $G_1 \dotplus G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$. It is well known that the Laplacian spectrum of the graph $G_1 \dotplus G_2$ is the union of Laplacian spectra of its summands G_1 and G_2. The last property is also true for any finite direct sum of graphs $G_1 \dotplus \cdots \dotplus G_m$ $(m \geqslant 2)$.

If E_m $(m \geqslant 0)$ is the void graph on m vertices, then for any graph G and any $m \geqslant 0$, the positive Laplacian eigenvalues of G and $G \dotplus E_m$ are the same. Hence, we shall always assume that all considered graphs have no isolated vertices.

2. MAIN RESULTS

Proposition 1. *Let G be an arbitrary graph and $\lambda_k(G) > 0$ for some $k \in N$.*

(a) *If G is a connected graph, then*

(2) $$\lambda_k(G) \geqslant A_k,$$

with equality in (2) *at least for the path P_{k+1}.*

(b) *If G is a disconnected graph and $k \geqslant 2$ then $\lambda_k(G) > A_k$.*

Proof. (a) First assume that G is a connected graph with n vertices and $\lambda_k(G) > 0$. Then obviously $n > k$. Now it is easy to see that there is at least one connected graph H_{k+1} with $k + 1$ vertices which is an induced subgraph of G. By Theorem 2 we then have that

(3) $$\lambda_k(G) \geqslant \lambda_k(H_{k+1}).$$

Since H_{k+1} is a connected graph, Theorem 1 gives

$$\lambda_k(H_{k+1}) \geqslant A_k. \tag{4}$$

Inequality (2) follows immediately from (3) and (4).

The path P_{k+1} obviously satisfies equality in (2) for every $k \in N$.

(b) Next assume that G is a disconnected graph on $n > k$ vertices, and $\lambda_k(G) > 0$. Denote its components by $G_1, \ldots, G_m$ $(m \geqslant 2)$. Since the Laplacian spectrum of G is the union of the Laplacian spectra of the graphs $G_1, \ldots, G_m$, we conclude that there is some $i \leqslant m$ and $j \leqslant k$ such that $\lambda_k(G) = \lambda_j(G_i)$. We shall distinguish two cases.

1^0. $\lambda_k(G) \geqslant 2$. Then $\lambda_k(G) \geqslant A_1 = 2 > A_k$ because $k \geqslant 2$.

2^0. $\lambda_k(G) < 2$. In this case $j < k$ because $\lambda_1(G_1), \ldots, \lambda_1(G_m) \geqslant A_1 = 2$, so by the statement (a) $\lambda_k(G) = \lambda_j(G_i) \geqslant A_j > A_k$. □

By inequality (2) we can say that for every integer k the interval $(0, A_k)$ is "forbidden" for the Laplace eigenvalue $\lambda_k(G)$ of any graph G of order $n = |G| > k$.

Applying Proposition 1 to the complementary graph $\overline{G}$ of a graph G, we can easily prove the following upper bound for the eigenvalue $\lambda_k(G)$.

Corollary 1. *If a graph G of order $n > k$ satisfies $\lambda_k(G) < n$, then*

$$\lambda_k(G) \leqslant B_{n,k} = n - A_{n-k}. \tag{5}$$

Equality holds in (5) if, for instance, G is isomorphic to the graph obtained by removing the path P_{n-k+1} from the complete graph K_n $(n \geqslant 2)$.

Note that condition $\lambda_k(G) < n$ in this corollary means that complementary graph $\overline{G}$ has at most k connected components.

By this corollary we see that interval $(B_{n,k}, n)$ is also forbidden for the Laplace eigenvalue $\lambda_k(G)$ $(k < n = |G| \geqslant 2)$.

In particular, taking $k = 1$ and $k = n - 1$ in (5) we get

$$\lambda_1(G) \leqslant n - 2 + 2\cos\frac{\pi}{n} \quad (n \geqslant 2),$$

if $\lambda_1(G) < n$, and

$$\lambda_{n-1}(G) \leqslant n - 2 \quad (n \geqslant 2),$$

if $\lambda_{n-1}(G) < n$, that is if G is not the complete graph K_n on n vertices $(n \geqslant 2)$.

Next we are interested in finding all graphs G which have the property $\lambda_k(G) = A_k$ for a fixed $k \in N$. We shall call a (connected or disconnected) graph G without isolated vertices *k-minimal* if $\lambda_k(G) = A_k$. So, P_{k+1} is at least one k-minimal graph. Since $A_k > 0$ it follows that each k-minimal graph G has the order $|G| \geqslant k + 1$.

By Proposition 1 (b) every k-minimal graph is connected for every $k \geqslant 2$, but it is not necessary true for $k = 1$. We shall also see that k-minimal graphs are not unique for $k = 1$ and $k = 2$.

Proposition 2. *All* 1*-minimal graphs are of the form* $K_2 \dotplus \cdots \dotplus K_2 = mK_2$ *for some* $m \geqslant 1$.

Proof. It is easy to see that any graph of the form $G = mK_2$ $(m \geqslant 1)$ satisfies $\lambda_1(G) = 2$, so it is 1-minimal.

Next we prove that mK_2 $(m \geqslant 1)$ are the only 1-minimal graphs. If G is a connected 1-minimal graph, then $|G| = 2$, because otherwise $\lambda_1(G) > 2$, a contradiction. Therefore $G = K_2$. If G is a disconnected 1-minimal graph and $G_1, \ldots, G_m$ are its components, then $G = G_1 \dotplus \cdots \dotplus G_m$ and

$$\lambda_1(G) = \max\{\lambda_1(G_1), \ldots, \lambda_1(G_m)\} = 2.$$

We conclude that $\lambda_1(G_i) = 2$ $(i = 1, \ldots, m)$ and therefore $G_1 = \cdots = G_m = K_2$. Thus, $G = mK_2$ for some $m \geqslant 2$. □

Proposition 3. *A graph G is* 2*-minimal if and only if G is a star* $K_{1,m}$ $(m \geqslant 2)$.

Proof. As is well known, the star $K_{1,m}$ on $m + 1$ vertices has the Laplacian spectrum $\{m + 1 > \underbrace{1 = \cdots = 1}_{m-1} > 0\}$, so $\lambda_2(K_{1,m}) = 1$ for $m \geqslant 2$.

Conversely, let G be a 2-minimal graph. Then it is connected by Proposition 1(b), and $|G| \geqslant 3$. Let $d = d(G)$ be the diameter of G.

If $d = 1$, then $G = K_n$ $(n \geqslant 3)$ and $\lambda_2(G) = \lambda_2(K_n) \geqslant \lambda_2(K_3) > A_2$, which is a contradiction.

If $d \geqslant 3$, then G contains the path P_4 as an induced subgraph and $\lambda_2(G) \geqslant \lambda_2(P_4) = 2 > A_2$, which is again a contradiction.

Hence, we conclude that $d = 2$ and the proof is complete. □

Next proposition gives an important property of k-minimal graphs $(k \geqslant 3)$.

Proposition 4. *If G is a k-minimal graph* $(k \geqslant 3)$, *then* $n = |G| = k + 1$.

Proof. On the contrary, suppose that G is a k-minimal graph $(k \geqslant 3)$ and $n = |G| \geqslant k + 2$. Then G is connected by Proposition 1(b), and there is a connected graph H_{k+2} with $k + 2$ vertices which is an induced subgraph of G. Let T_{k+2} be a spanning tree of the graph H_{k+2}.

If T_{k+2} is the star $K_{1,k+1}$, then

$$\lambda_k(G) \geqslant \lambda_k(H_{k+2}) \geqslant \lambda_k(K_{1,k+1}) = 1 > 2 - \sqrt{2} = A_3 \geqslant A_k,$$

a contradiction.

If $T_{k+2} \neq K_{1,k+1}$ then there is a bridge e in T_{k+2} such that $T_{k+2} - e = T_p \dotplus T_q$ where T_p and T_q are subtrees on p and q vertices respectively, and $p, q \geqslant 2$, $p + q = k + 2$. Then

$$\begin{aligned}\lambda_k(G) &\geqslant \lambda_k(T_{k+2}) \geqslant \lambda_k(T_p \dotplus T_q) = \min\{\lambda_{p-1}(T_p), \lambda_{q-1}(T_q)\}\\ &\geqslant \min\{A_{p-1}, A_{q-1}\} \geqslant A_{k-1} > A_k,\end{aligned}$$

because $p-1, q-1 \leqslant k-1$. But this again gives the contradiction $\lambda_k(G) > A_k$. Therefore $n = |G| \leqslant k+1$. But since $n \geqslant k+1$, we finally get $n = |G| = k+1$. $\square$

By Proposition 4 we see that k-minimal graphs for $k \geqslant 3$ always lie in the class of graphs with $k+1$ vertices, so $\lambda_k(G)$ of such graphs coincides with their algebraic connectivities.

Theorem 2 and Proposition 4 finally give:

Proposition 5. *For every integer $k \geqslant 3$, the path P_{k+1} is the unique k-minimal graph.*

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