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Electronic Notes in Theoretical Computer Science 273 (2011) 21–31

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Constructive Description Logics Hybrid-Style

Valeria de Paiva [1]

School of Computer Science
Birmingham University, United Kingdom

Edward Hermann Haeusler [2]

Dep. Informática
PUC-Rio, Brazil

Alexandre Rademaker [3]

EMAp
Fundação Getulio Vargas, Brazil

Abstract

Constructive modal logics come in several different flavours and constructive description logics, not surprisingly, do the same. We introduce an intuitionistic description logic, which we call $i\mathcal{ALC}$ (for intuitionistic $\mathcal{ALC}$, since $\mathcal{ALC}$ is the name of the canonical description logic system) and provide axioms, a Natural Deduction formulation and a sequent calculus for it. The system $i\mathcal{ALC}$ is related to Simpson's constructive modal logic IK the same way Mendler and Scheele's $c\mathcal{ALC}$ is related to constructive CK and in the same way classical multimodal K is related to ALC. In the system $i\mathcal{ALC}$, as well as in $c\mathcal{ALC}$, the classical principles of the excluded middle $C \sqcup \neg C = \mathsf{T}$, double negation $\neg\neg C = C$ and the definitions of the modalities $\exists R.C = \neg\forall R.\neg C$ and $\forall R.C = \neg\exists R.\neg C$ are no longer validities, but simply non-trivial TBox statements used to axiomatize specific application scenarios. Meanwhile in $i\mathcal{ALC}$, like in classical ALC, we have that the distribution of existential roles over disjunction i.e. $\exists R.(C \sqcup D) = \exists R.C \sqcup \exists R.D$ and (the nullary case) $\exists R.\bot = \bot$ hold, which is *not* true for $c\mathcal{ALC}$. We intend to use $i\mathcal{ALC}$ for modelling juridical Artificial Intelligence (AI) systems and we describe briefly how.

1 Introduction

Description Logics are an important knowledge representation formalism, unifying and giving a logical basis to the well known AI frame-based systems of the eighties. Description logics are very popular right now. Given the existent and proposed applications of the Semantic Web, there has been a fair amount of work into finding

[1] Email: valeria.depaiva@gmail.com

[2] Email: hermann@inf.puc-rio.br

[3] Email: alexandre.rademaker@fgv.br

doi:10.1016/j.entcs.2011.06.010

the most well-behaved system of description logic that has the broadest application, for any specific domain. As discussed in [7], considering versions of *constructive* description logics makes sense, both from a theoretical and from a practical viewpoint. There are several possible and sensible ways of defining *constructive* description logics, whether your motivation is natural language semantics (as in [7]) or Legal AI (as in [9]).

Description logics tend to come in families of logical systems, depending on which concept constructors you allow in the logic. Since description logics came into existence as fragments of first-order logic chosen to find the best trade-off possible between expressiveness and tractability of the fragment, several systems were discussed and in the taxonomy of systems that emerged the one called $\mathcal{ALC}$ (for Attributive Language with Complements) has come to be known as the canonical one. The basic building blocks of description logics are *concepts*, *roles* and *individuals*. Think of concepts as unary predicates in usual first-order logic and of roles as binary predicates, used to modify the concepts.

As far as *constructive* description logics are concerned, Mendler and Scheele have worked out a very compelling system $c\mathcal{ALC}$ [12], based on the constructive modal logic CK [2]), a favorite [4] system of ours. However in this note we follow a different path and describe a constructive version of $\mathcal{ALC}$, based on the framework for constructive modal logics developed by Simpson (the system IK) in his phd thesis [17]. We call our system $i\mathcal{ALC}$ for Intuitionistic $\mathcal{ALC}$. (For a proof-theoretic comparison between the constructive modal logics CK and IK one can see [14]).

Our motivation, besides Simpson's work, is the framework developed by Braüner and de Paiva in [5] for constructive Hybrid Logics. We reason that having already frameworks for constructive modal and constructive hybrid logics in the labelled style of Simpson, we might end up with the best style of constructive description logics, in terms of both solid foundations and ease of implementation. Since submitting this paper we have been told about the master thesis of Clément [6] which follows broadly similar lines. Clément proves soundness and completeness of the system called IALC and then provides a focused version of this system, a very interesting development, as focused systems are, apparently, very useful for proof search.

We first recall Simpson's framework for constructive modal logics and Braüner and de Paiva's system for constructive hybrid logics. Then we introduce our version of intuitionistic description logic, denoted $i\mathcal{ALC}$. We briefly describe the immediate properties of this system and most importantly we discuss a case study of the use of $i\mathcal{ALC}$ in legal AI and conclude with some interesting directions of further work.

2 Constructive modal and hybrid logics

Traditionally modal logics are classical propositional logics augmented with modalities for necessity, possibility, obligations, provability etc. While by no means the most popular ones, there are several reasonable systems of *constructive* modal logics

[4] This system has categorical semantics, which are not very easy to obtain for modal logics.

in the literature too. Surprisingly, very little is known about the inter-relationships between several of these systems. Many of these systems take the semantics of propositional modal logics in terms of Kripke frames as their fundamental intuition and modify it to account for an intuitionistic basis, instead of the classical one. In this paper we are mostly concerned with the framework proposed by Simpson [17]. This consists of a series of Natural Deduction systems, which arise from interpreting the usual possible worlds definitions in an intuitionistic meta-theory. The main benefit of these Natural Deduction systems over axiomatizations is their susceptibility to proof-theoretic techniques. Strong normalization and confuence results are proved for all of the systems described. On the downside the basic structure of Natural Deduction needs to be extended to deal with assumptions of the form *world x is R-related to world y*, which is written as a second kind of formula xRy.

Building up from Simpson's framework for constructive modal logics, in [5], Braüner and de Paiva introduced *intuitionistic hybrid logics*, denoted by IHL. Hybrid logics add to usual modal logics a new kind of propositional symbols, the *nominals*, and also the so-called *satisfaction operators*. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. If x is a nominal and X is an arbitrary formula, then a new formula $x\colon X$ called a satisfaction statement can be formed. The part $x\colon$ of $x\colon X$ is called a satisfaction operator. The satisfaction statement $x\colon X$ expresses that the formula X is true at one particular world, namely the world at which the nominal x is true. Constructing a system of intuitionistic hybrid logic, based on Simpson's Natural Deduction is relatively straightforward. The hard work is to prove that the whole machinery of nominals and satisfaction operators is orthogonal enough to the intuitionistic characteristics of the basis and that we can still have the expected proof-theoretical properties of the hybrid system, as desired. In hindsight one can see that Simpson's formulation of modal logic (called here IML, for intuitionistic modal logic) shares with hybrid formalisms the idea of making the possible-world semantics part of the deductive system. While IML makes the relationship between worlds (e.g., xRy) part of the deductive system, IHL goes one step further and sees the worlds themselves x, y as part of the deductive system, (as they are now nominals) and the satisfaction relation itself as part of the deductive system, as it is now a syntactic operator, with modality-like properties.

Out of these tightly connected systems of intuitionistic modal logic IML and intuitionistic hybrid logics IHL, we want to carve out our system of intuitionistic description logic $i\mathcal{ALC}$. However, for some of our applications, we prefer to work with a sequent calculus, as opposed to a Natural Deduction system. For this reason, we make use of Negri's well-developed proof theory for modal systems [13].

3 Towards the system $i\mathcal{ALC}$

Like classical $\mathcal{ALC}$ [1] the intuitionistic version $i\mathcal{ALC}$ is a basic description language whose concept constructors are described by the following grammar:

$$C, D ::= A \mid \bot \mid \top \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C$$

where A stands for an atomic concept and R for an atomic role, given an initial set of role names and of atomic concepts names. This syntax is more general than standard $\mathcal{ALC}$ in that it includes subsumption $\sqsubseteq$ as a concept-forming operator. In a constructive setting subsumption behaves somewhat like strict implication. (We will have no use for nested subsumptions, but they do make the system easier to define, so we keep the general rules.) Negation can be represented via subsumption, as $\neg C$ can be defined as $C \sqsubseteq \bot$, but we find it convenient to keep it in the language. The constant $\top$ can also be omitted since it can be represented by $\neg\bot$.

Following Mendler and Scheele we say a constructive interpretation of $i\mathcal{ALC}$ is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of entities in which each entity represents a partially defined individual; a refinement pre-ordering $\preceq^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, i.e., a reflexive and transitive relation; and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name R to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each atomic concept A to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under refinement, i.e., $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is lifted from atomic $\bot, A$ to arbitrary concepts via:

$$
\begin{aligned}
\top^{\mathcal{I}} &=_{df} \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta^{\mathcal{I}}. x \preceq y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta^{\mathcal{I}}. (x \preceq y \text{ and } y \in C^{\mathcal{I}}) \Rightarrow y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta^{\mathcal{I}}. x \preceq y \Rightarrow \exists z \in \Delta^{\mathcal{I}}. (y,z) \in R^{\mathcal{I}} \text{ and } z \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta^{\mathcal{I}}. x \preceq y \Rightarrow \forall z \in \Delta^{\mathcal{I}}. (y,z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}
\end{aligned}
$$

Our setting is a simplification of Mendler and Scheele's where we dispense with infallible entities, since our system $i\mathcal{ALC}$ satisfies $\exists R.\bot = \bot$, just like classical $\mathcal{ALC}$.

Semantic validity can be introduced as follows: say "x satisfies C in the interpretation $\mathcal{I}$", written as $\mathcal{I}, x \models C$, if x is in the interpretation of C, $x \in C^{\mathcal{I}}$. Say $\mathcal{I} \models C$ if this happens for all x in $\Delta^{\mathcal{I}}$. Finally say $\models C$ if for all interpretations $\mathcal{I}$ we have $\mathcal{I} \models C$. These definitions are usually extended to sets of concepts.

Typical reasoning in description logics is done via TBoxes and ABoxes. If we use Γ for a TBox, i.e. a collection of concepts and subsumptions and Θ for an ABox, a collection of instantiations of concepts then we can say $\Theta, \Gamma \models C$ if for all interpretations $\mathcal{I}$, which are models of all the concepts in Γ it is the case that every x in $\mathcal{I}$ which satisfy the axioms in Θ must also satisfy C, or

$$\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \text{ and } \mathcal{I}, x \models \Gamma) \text{ implies } \mathcal{I}, x \models C$$

A Hilbert-style axiomatization of $i\mathcal{ALC}$ is easy to provide. It consists of all axioms of intuitionistic propositional logic plus the axioms and rules displayed in Figure 1.

Proving soundness and completeness of the Hilbert version of $i\mathcal{ALC}$ above, as it is done by Mendler and Scheele [12, p. 7] poses no problems. Repeating their work we can say: Let the symbol $\vdash_H$ denote a Hilbert deduction, that is $\Gamma \vdash_H C$ if there

$$\text{all axioms of propositional intuitionistic logic} \qquad (\mathsf{IPL})$$
$$\forall R.(C \sqsubseteq D) \sqsubseteq (\forall R.C \sqsubseteq \forall R.D) \qquad (\forall\mathsf{K})$$
$$\exists R.(C \sqsubseteq D) \sqsubseteq (\exists R.C \sqsubseteq \exists R.D) \qquad (\exists\mathsf{K})$$
$$\exists R.(C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D) \qquad (\mathsf{DIST})$$
$$\exists R.\bot \sqsubseteq \bot \qquad (\mathsf{DIST0})$$
$$(\exists R.C \sqsubseteq \forall R.C) \sqsubseteq \forall R.(C \sqsubseteq D) \qquad (\mathsf{DISTmix})$$
$$\text{If } C \text{ is a theorem then } \forall R.C \text{ is a theorem too.} \qquad (\mathsf{DISTmix})$$
$$\text{If } C \text{ and } C \sqsubseteq D \text{ are theorems, } D \text{ is a theorem too.} \qquad (\mathsf{MP})$$

Fig. 1. The System $i\mathcal{ALC}$: Hilbert-style

exists a derivation $C_0, C_1, C_2, \ldots, C_n$ where the last concept $C_n = C$ and each C_i is either a hypothesis (C_i is in Γ) or is a substitution instance from one of the axioms above or obtained via the rules MP and Nec from earlier concepts $C_j, j \leq i$. The Hilbert calculus implements TBox reasoning and we have, just doing cut-and-paste from [12]:

Theorem 3.1 *The Hilbert calculus described in Figure 1 is sound and complete for TBox reasoning, that is* $\Gamma, \emptyset \models C$ *if and only if* $\Gamma \vdash_H C$.

4 A sequent calculus for $i\mathcal{ALC}$

Working to give a Gentzen sequent-style presentation for $i\mathcal{ALC}$ we move first to a labelled system in the style of Simpson's framework. Simpson's original system is a Natural Deduction system, where the rules for modalities are meant to capture exactly the intuitions of possible worlds. Restricting Simpson's IK to the description logic fragment gives the rules in Figure 2, where we elided the rules for $\sqcup$ and $\sqcap$, which are well-known.

A sequent calculus version of Simpson's rules is discussed by Negri [13] (in the classical case) and we prefer to use the sequent calculus. We have to adapt Negri's sequent calculus to the description logic fragment and to make it intuionistic, which seems easy enough. The rules are in the Figure 3 below. Note that our version, which is constructive, has restrictions to a single conclusion formula in the rules for subsumption and universal-quantification-role on the right, which are essential to keep the system intuitionistic, in the propositional setting.

There are two main modifications from usual, non-labelled sequent calculus for modal logic. First, of course we need to add the labels, which intuitively describe the world where the concept is being asserted. Thus $x\colon C$ means that the concept C is asserted to exist in the world x. Secondly we have the relational kind of premises in the deductive system, assertions of the form xRy, which mean that the role R relates worlds x and y. Both of these additions would seem sensible in the description logic setting: it is reassuring to see the same rules for roles in Straccia's 4-valued Description Logic [18].

$$\frac{}{\Gamma \Rightarrow\ \top} \qquad \frac{\Gamma \Rightarrow\ x\colon \bot}{\Gamma \Rightarrow\ y\colon C}$$

$$\frac{\Gamma \Rightarrow\ x\colon C \sqsubseteq D \qquad \Gamma \Rightarrow\ x\colon C}{\Gamma \Rightarrow\ x\colon D}\ \sqsubseteq\text{-e} \qquad \frac{\Gamma, x\colon C \Rightarrow\ x\colon D}{\Gamma \Rightarrow\ x\colon C \sqsubseteq D}\ \sqsubseteq\text{-i}$$

$$\frac{\Gamma \Rightarrow\ \forall R.C \qquad \Gamma \Rightarrow\ xRy}{\Gamma \Rightarrow\ y\colon C}\ \forall\text{-e} \qquad \frac{\Gamma, xRy \Rightarrow\ y\colon C}{\Gamma \Rightarrow\ x\colon \forall R.C}\ \forall\text{-i}$$

$$\frac{\Gamma \Rightarrow\ x : \exists R.C \qquad \Gamma, y\colon C, xRy \Rightarrow\ z\colon D}{\Gamma \Rightarrow\ z\colon D}\ \exists\text{-e}$$

$$\frac{\Gamma \Rightarrow\ y\colon C \qquad \Gamma \Rightarrow\ xRy}{\Gamma \Rightarrow\ x\colon \exists R.C}\ \exists\text{-i}$$

Fig. 2. Natural Deduction $i\mathcal{ALC}$ Rules

The rules for the propositional connectives $(\sqcap, \sqcup)$ are basically the same as for classical $\mathcal{ALC}$, we just have to add worlds everyehere, but these do not change with the application of rules. (Similarly the rules for subsumption $\sqsubseteq$ are just the rules for intuitionistic implication with worlds added). The main modification comes for the modal (or role quantification) rules, which now follow exactly the intuitions of Kripke relational semantics. Since the intuitive semantics of box (necessity) says

$$x \models \Box C \text{ iff } \text{ for all } y, xRy \text{ implies } y \models C$$

and we are reading $\forall R.C$ as $\Box C$ (following Schild [16]) we derive a rule that says if $y\colon C$ can be derived for an arbitrary y that is R-related to x then $x\colon \forall R.C$ holds, or

$$\frac{\Gamma, xRy \Rightarrow\ y\colon C}{\Gamma \Rightarrow\ x\colon \forall R.C}\ \forall\text{-r}$$

The fact that y must be arbitrary is reflected in the usual condition that y is not (free) in Γ. Reading the semantics again, the converse gives us the left rule for the universal role,

$$\frac{\Gamma, x\colon \forall R.C, y\colon C, xRy \Rightarrow\ \Delta}{\Gamma, x\colon \forall R.C, xRy \Rightarrow\ \Delta}\ \forall\text{-l}$$

as if $x\colon \forall R.C$ and y is accessible from x then $y\colon C$, where we need to repeat the formula $x\colon \forall R.C$ to make the rule invertible. Similar reasoning, from the intended semantics, get us the rules for existential quantification. We say

$$x \models \Diamond C \text{ iff } \text{ there exists } y, xRy \text{ and } y \models C$$

The left to right direction gives us

$$\frac{\Gamma, xRy, y\colon C \Rightarrow\ \Delta}{\Gamma, x\colon \exists R.C \Rightarrow\ \Delta}\ \exists\text{-l}$$

while the right to left direction gives the binary rule

$$\frac{\Gamma \Rightarrow \ \Delta, xRy \qquad \Gamma \Rightarrow \ \Delta, y\colon C}{\Gamma \Rightarrow \ \Delta, x\colon \exists R.C}$$

which is turned into the equivalent unary rule

$$\frac{\Gamma, xRy \Rightarrow \ \Delta, y\colon C, x\colon \exists R.C}{\Gamma, xRy \Rightarrow \ \Delta, x\colon \exists R.C}\ \exists\text{-r}$$

where the concept $x\colon \exists R.C$ is repeated in the antecedent, just for invertibility reasons. As traditional in first-order logic, the rules $\forall$-r and $\exists$-l have the side condition that y is not in the conclusion.

$$\frac{}{xRy, \Gamma \Rightarrow \ \Delta, xRy} \qquad \frac{}{\Gamma, x\colon C \Rightarrow \ x\colon C, \Delta}$$

$$\frac{}{\Gamma \Rightarrow \ \top} \qquad \frac{}{\Gamma, x\colon \bot \Rightarrow \ \Delta}$$

$$\frac{\Gamma \Rightarrow \ x\colon C \qquad \Gamma, x\colon D \Rightarrow \ \Delta}{\Gamma, x\colon C \sqsubseteq D \Rightarrow \ \Delta}\ \sqsubseteq\text{-l} \qquad \frac{\Gamma, x\colon C \Rightarrow \ x\colon D}{\Gamma \Rightarrow \ x\colon C \sqsubseteq D}\ \sqsubseteq\text{-r}$$

$$\frac{\Gamma, x\colon C, x\colon D \Rightarrow \ \Delta}{\Gamma, x\colon (C \sqcap D) \Rightarrow \ \Delta}\ \sqcap\text{-l} \qquad \frac{\Gamma \Rightarrow \ x\colon C, \Delta \qquad \Gamma \Rightarrow \ x\colon D, \Delta}{\Gamma \Rightarrow \ x\colon (C \sqcap D), \Delta}\ \sqcap\text{-r}$$

$$\frac{\Gamma, x\colon C \Rightarrow \ \Delta \qquad \Gamma, x\colon D \Rightarrow \ \Delta}{\Gamma, x\colon (C \sqcup D) \Rightarrow \ \Delta}\ \sqcup\text{-l} \qquad \frac{\Gamma \Rightarrow \ x\colon C, x\colon D, \Delta}{\Gamma \Rightarrow \ x\colon (C \sqcup D), \Delta}\ \sqcup\text{-r}$$

$$\frac{\Gamma, x\colon \forall R.C, y\colon C, xRy \Rightarrow \ \Delta}{\Gamma, x\colon \forall R.C, xRy \Rightarrow \ \Delta}\ \forall\text{-l} \qquad \frac{\Gamma, xRy \Rightarrow \ y\colon C}{\Gamma \Rightarrow \ x\colon \forall R.C}\ \forall\text{-r}$$

$$\frac{\Gamma, xRy, y\colon C \Rightarrow \ \Delta}{\Gamma, x\colon \exists R.C \Rightarrow \ \Delta}\ \exists\text{-l} \qquad \frac{\Gamma \Rightarrow \ \Delta, y\colon C, x\colon \exists R.C}{\Gamma \Rightarrow \ \Delta, x\colon \exists R.C}\ \exists\text{-r}$$

Fig. 3. $i\mathcal{ALC}$ Sequent Rules

The system $i\mathcal{ALC}$ described here is related to Simpson's IK the same way Mendler and Scheele's $c\mathcal{ALC}$ [12] is related to constructive CK [2] and [11] and in the same way classical multimodal K is related to ALC[16]. In the system $i\mathcal{ALC}$ we defined, as well as in $c\mathcal{ALC}$, the classical principles of the excluded middle $C \sqcup \neg C = \mathsf{T}$, double negation $\neg\neg C = C$ and the definitions of the modalities $\exists R.C = \neg\forall R.\neg C$ and $\forall R.C = \neg\exists R.\neg C$ are no longer validities, but non-trivial TBox statements used to axiomatize specific application scenarios. Meanwhile in $i\mathcal{ALC}$, like in classical ALC, we have that the distribution of existential roles over disjunction i.e. $\exists R.(C \sqcup D) = \exists R.C \sqcup \exists R.D$ and in the nullary case $\exists R.\bot = \bot$ hold, which is not true for $c\mathcal{ALC}$.

5 Properties of $i\mathcal{ALC}$

Soundness and completeness of a sequent calculus version of $i\mathcal{ALC}$ are easy to prove. This is similar to the proof indicated in [12, page 10], although the sequent calculus we proposed in Figure 3 is different from theirs. Our sequents are simpler, as we do not insist in carrying negative information along derivations. Our modal rules are different enough. Nonetheless we have:

Theorem 5.1 *The sequent calculus for $i\mathcal{ALC}$ in Figure 3 and the Hilbert calculus described in Figure 1 are equivalent. For any TBox Θ and concept C, we have that $\Theta, \emptyset \vdash_H C$ if and only if the sequent $\Theta \Rightarrow C$ has a derivation using the rules in Figure 3.*

The proof of soundness and completeness of the sequent calculus for $i\mathcal{ALC}$ does not come straight from Straccia's work, as our rules for roles are the same, but our semantics are different. (He insists on 4-valuedness, we want constructiveness.)

Theorem 5.2 *The sequent calculus described in Figure 3 is sound and complete for TBox reasoning, that is $\Theta, \emptyset \models C$ if and only if $\Theta \vdash_S C$.*

We still have to contend with the criticism levelled by Bozzato et al in [4] that a constructive description logic ought to satisfy the *finite model property*, which is not clear from our (original) formulation. Bozzato et al have a formulation of constructive hybrid logic based on closing the logic down under *Kuroda's axiom*, by construction. Other researchers (including Mendler and Scheele) have proved the finite model property and decidability for (variants) of the description logics we consider. In particular Simpson has proved the finite model property for IK [17, page 157] but for his birelational models. We leave as future work to do the same for $i\mathcal{ALC}$.

6 Applying $i\mathcal{ALC}$

Mendler and Scheele cite auditing of business as their motivational application. We envisage applying our system to legal AI, as one of us (Hausler) is tasked with developing prototypes for legal AI systems. We have presented a simplified case study of this application in [9] and [8]. We repeat its rationale here, in a simplified form.

One of the main problems from legal theory is to make precise the use of the term "law". In fact, the problem of individuation, namely, the problem of deciding what counts as the unit of law, seems to be one of the fundamental open questions in jurisprudence. That is, any discussion of law classification must be preceded by an answer to the question "What is to count as one complete law?" ([15]). There are two main approaches to this question. One is to take as the law all (existing) legally valid statments as a single, whole entity. This totality is called "the law". This approach is predominant in legal philosophy and jurisprudence owing its significance to the Legal Positivism tradition initiated by Hans Kelsen (for a contemporary reference see [10]). The coherence of "the law" plays a central role

in this approach, whilst a debate on whether coherence is built-in by the restrictions induced by Nature in an evolutionary way, or whether it should be object of knowledge management, seems to be a long and classical debate. The other approach to law definition is to take into account all legally valid statements as being *individual laws*. This view, in essence, is harder to be shared with jurisprudence principles, since those principles are firstly concerned with justifying the law. This latter approach seems to be more suitable to Legal AI. It is also considered by legal theoreticians, at least partially, whenever they start considering ontological commitments, such as, taking some legal relations as primitive ones (Hohfeld, 1919), *primary and secondary rule* (Hart, 1961) or even a two-level logic to deal with different aspects of law (see *logic-of-imperation/logic-of-obligation* from [3]). In fact, many Knowledge Engineering (KE) groups pursue the definition of legal ontologies on this basis. We also follow this route. It is important to note that the pure use of a deontic logic has been shown to be inadequate to accomplish this task. In [19] it is shown that deontic logic does not properly distinguish the normative status of a situation from the normative status of a norm (rule).

From the semantic point of view, $i\mathcal{ALC}$ seems to be adequate to model the legal theoretic approach pursued by KE as described above. Let us consider an $i\mathcal{ALC}$ model having as individuals each of the possible *legal statements*. The $\preceq$ relation is the natural hierarchy existing between individual *legal statements*. For example, sometimes conflicts between *legal statements* are solved by inspecting the age of the laws, the difference between enforcement scope of each law, etc. Any of these relations can be considered transitive and reflexive. If C is a concept symbol in the description logic language, its semantics is a subset of *legal statements* representing a *kind of* legal situation. Roles in the description logic language are associated to relations between these *legal situations*, imposed by the relationship between each pair of individual *legal stls -l atements*.

The main reason to use an intuitionistic logic in legal reasoning is to have the ability to express partiality and incomplete information, beyond the standard open world assumption. Because the semantic meaning of our concepts should be context dependent, we need a constructive version of undefinedness that allows for intrinsic refinement of concepts. Classical description logic assumes that each concept is static and that at the outset either it includes a given entity or not. This corresponds to a binary truth interpretation. If we trade this static setting for a constructive notion of truth we believe this will provide us with a well-understood and more sophisticated way of dealing with refinement of concepts. Of course it remains to be seen if we can keep the other features that made description logics as useful as they have proved themselves, so far.

7 Conclusions

We presented rules leading up to a new system of constructive description logic called $i\mathcal{ALC}$. This system is the natural restriction of Simpson's framework for constructive modal logic to the description logic setting and fits in naturally with

Brauner and de Paiva's intuitionistic hybrid logic IHL. The results we prove about the system are not very surprising. What is unsettling is the number of design choices left to us and the difficulty in obtaining hard criteria for choosing between the multiple systems available. We have not much to report on this, yet.

Besides further investigating the relationship between systems based on IK and systems based on CK, especially their semantics counterparts, we would like to implement a framework that would allow us to construct proofs in the three systems $i\mathcal{ALC}$, IK and IHL. The main application we envisage for our system at the moment is in knowledge engineering of juridical systems, as one of us (Hauesler) leads a project on this topic. This project is just beginning and time will tell if the initial intuitions concerning simplicity of modeling coupled with ease of implementation will bear fruit or not.

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