The Concept of Nearest Fuzzy Sets (Kronecker Proximity) and its Application for Data Representation and Decision Making Under Uncertainty

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Abstract

In the article is considered the concept of nearest fuzzy sets (Kronecker proximity) and its application for data representation and decision making under conditions of uncertainty.

Representation of uncertainty in the form of a subset of ordered pairs, one of the components of which is the weight function determined by experts, is the most common. However, this approach has certain limitations. For certain problems, the assignment of membership function determined by an expert is almost impossible due to limited human expertise. The requirement to obtain a solution is not removed, the assignment of a random heuristic membership function does not guarantee the rationality of the decision.For solving such problems proposed the usage of a structure - a subset of ordered pairs, which, on the one hand, assumes almost unlimited variability, on the other hand - a tensor model (Kronecker-product of a subset of ordered pairs) allows to reveal hidden knowledge and thus expand the range of solutions tasks under conditions of uncertainty. A fuzzy set (a subset of ordered pairs) can be represented as a 2-D tensor (the tensor product of the components of fuzzy sets). The 2-D tensor in turn can be decomposed by a singular decomposition into a new subset of ordered pairs that is close (in the sense of f-norm and defuzzyfied value) to the initial fuzzy set.

Keywords [1]

Fuzzy set, uncertainty, hidden knowledge, tensor, tensor decomposition, Kronecker product, Kronecker algebra

1. Introduction

The problem of control under conditions of uncertainty has not only become significantly more complicated recently but has received a number of new problems, attempts to solve which by standard methods for fuzzy set theory are experiencing certain difficulties. It is objectively related to 2 factors:

- a necessity to automate the process of forming a standard fuzzy set;
- a necessity to work with a fundamentally new object - BIG DATA. It significantly complicates and prevents the usage of human intelligence to the extent and form required by the theory of fuzzy sets (FST).

It is no coincidence that in [1] special attention is paid to the issue of identifying close (to the existing) structures for which a mathematical model can be developed, the properties of which can be transferred to the real structure. In this regard, the FST is an exemplary example, the heuristic fuzzy extension principle allowed to transfer almost all standard mathematics in an environment of uncertainty. In the FST, the concept of proximity is realized by introducing the concept of the nearest crisp set. In [2] it was shown that the closest crisp set to the fuzzy set (FS)

$$\tilde{A} = \left\{ a / \mu_{\tilde{A}} \right\},\ a \in A, \mu_{\tilde{A}} \to [0,1]$$

in terms of the smallest Euclidean distance between sets or in terms of the smallest norm (according to FS) will be, in particular, a subset that has the following properties:

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$$\chi_A(x_i) = \left(0, if\, \mu_{\tilde{A}}(x_i) < 0.5;\;\; 1,\;\; if\, \mu_{\tilde{A}}(x_i)\; > 0.5; 0\, or\;\; 1,\;\;\; if\, \mu_{\tilde{A}}(x_i) = 0.5.\right)$$

Note that $\chi_A(x_i)$ - is a characteristic function by definition. The concept of proximity can be extended to FS. For example, all fuzzy sets formed to model the statement < **approximately a**> on a universal set (US) X=[x^{min}, x^{max}], $x^{min} \leq a \leq x^{max}$ will not only be close to the norm, but have almost the same defuzzyfied values. This is confirmed by the examples given in the following context.

In [3], the expediency of using the concept of proximity for conditions of uncertainty is presented as the impossibility of accurately modeling complex natural processes or engineering systems using the traditional nonlinear mathematical approach with limited prior knowledge. Ideally, the analyst uses the information and knowledge from previous experiments or tests with the system to develop a model and predict the results. However, the new systems, in particular, which is clearly seen in the BIG DATA example, could contain distorted or missing data, or the performing costs could be limited. This means that prior knowledge and information are fundamentally unavailable.

A lack of data about the system or knowledge about it largely makes the model development by standard methods extremely difficult and often impossible, even with the usage of FST, because the formation of the linguistic basis of the system can not be appropriate without additional observations. In this context, the authors understand the lack of data about the system or knowledge about it as follows: the system is presented as an initial set of data that has data gaps or distortions, which makes it difficult (generally impossible) to determine the universal set and system modeling by FST methods. Note that it is these features of uncertainty that are least taken into account in FST.

The object of the research is the modeling of structured uncertainty, which assumes that only the primary information is known about the studied object in the form of a subset of initial data, which may be additionally distorted or missing, where it is impossible to accurately apply statistics and probabilities because of uncertainty in the values distribution or super-large size of the statistical sample

The subject of the research – algorithms, and methods of representation of structural uncertainty and decision-making on the basis of tensor structures (Kronecker product) and methods of tensor decompositions. The purpose of the work is to develop methods and algorithms for forming subsets of ordered pairs (sequences) to represent structured uncertainty and decision making.

The main tasks are to substantiate the nearest fuzzy sets concept (on the basis of Kronecker-product) and to develop methods (algorithms) for the formation of subsets of ordered pairs (SOP) for situations:

- taking into account the objective uncertainty (blurring of the universal set by using special matrices (Toeplitz, Hankel);
- using the Monte Carlo method to represent uncertainty.

2. Problem statement

2.1 List of main symbols and abbreviations

In table 1 are presented the main abbreviations, that are used in the article.m In table 2 are presented the main nomenclatures that is used in the article.

2.2 Main statement

In [4-6], the tensor models (TM) of FS are obtained as a tensor product of FS components:

$$\tilde{a} = \left\{ a / \mu^{(a)} \right\},\; a \in A,\; \mu^{(a)} \to [0,1] :\; \mathbf{T}^{(a)} = a \otimes \mu^{(a)} \in \mathrm{R}^{n \times n},$$

where n is the number of α-levels. One of the important characteristics $\mathbf{T}^{(a)}$ is that the singular decomposition $[usv] = svd(T^{(a)})$ allows obtaining 2 SOP:

$$^{(new)}\tilde{a}_1 = \left\{ a / {}^{(new)}\mu^{(a)} \right\}, a \in A, {}^{(new)}\mu^{(a)} \to [0,1],\; {}^{(new)}\tilde{a}_2 = \left\{ a / \nu^{(a)} \right\}, a \in A,\;\; \nu^{(a)} \to [0,1],$$

which have the property of proximity in the understanding of F-norms and defuzzyfied values, ie

$$\left\| {}^{(new)}\tilde{a}_1 \right\|_F \cong \|\tilde{a}\|_F ,\left\| {}^{(new)}\tilde{a}_2 \right\|_F \cong \|\tilde{a}\|_F ; \tag{1}$$

$$\operatorname{def}\left({}^{(new)}\tilde{a}_1\right) \cong \operatorname{def}\left(\tilde{a}\right), \operatorname{def}\left({}^{(new)}\tilde{a}_2\right) \cong \operatorname{def}\left(\tilde{a}\right), \quad (2)$$

Table 1
Abbreviations

Abbreviation	Explanation of the meaning of abbreviations
US	Universal set
SOP	A subset of ordered pairs
NKP	The nearest Kronecker product
MF	Membership function
KP	Kronecker product
TP	Tensor product
ISD	An initial set of data
FN	Fuzzy number
FST	Fuzzy set theory
FS	Fuzzy set

Table 2
Nomenclature

Symbol	Definition
A, A, a, a	Tensor, matrix, (column) vector, scalar
R	The set of real numbers
$\circ$	Outer product
Vec( )	Vectorization operator
$\otimes$	Kronecker product
$\times_n$	n-mode product
$A_{(n)}$	n-mode matricization of tensor **A**
$\mathbf{A}^{-1}$	Inverse of A
$\lVert A \rVert_F$	Frobenius norm - $trace\ (A^T A)^{1/2}$
A(:,i)	Spans the entire *i* th column of **A** (same for tensors)
A(i,:)	Spans the entire *i* th row of **A** (same for tensors)
reshape ( )	Rearrange the entries of a given matrix or tensor to a given set of dimensions
$\tilde{a}$	Type-1 fuzzy set: $\tilde{a} = \left\{a / \mu^{(a)}\right\}\ or\ \left(a_1\ \mu^{(a_1)}; \cdots; a_n\ \mu^{(a_n)}\right) \in \mathrm{R}^{n \times 2}, a \in A, \mu^{(a)} \to [0,1]$

Membership functions (MF) ${}^{(new)}\mu^{(a)}$ and $\mu^{(a)}$ are almost identical in shape, MF $\nu^{(a)}$ has a sigmoid-type shape. Note, that the SOPs obtained as a result of singular decompositions of 2D and 3D tensor models of uncertainty, as a rule, do not have corresponding analytical models.

Based on the obtained results, the search for hidden information under the accepted conditions of uncertainty is formed as follows:

- based on the known (or set) the initial data set (IDS) (taking into account the missing or distortion of the unstructured IDS) is implemented the procedure of structuring the IDS (2D or 3D tensor). The subsequent procedure of tensor decomposition of structured IDS allows to determine missing (distorted) elements [7] and a new SOP by solving the corresponding optimization problem;

- the obtained 2D or 3D tensors allow to determine the max / min-elements, to agree on the degree of its sampling

$$A = \left[a^{(\min)} : \Delta_A : a^{(\max)} \right] \in R^{1 \times n}$$

and to implement the procedure of objective blurring [8], to obtain tensors

$$T^{(toeplitz)} = toeplitz(A),\ T^{(hankel)} = hankel(A), \quad \left(T^{(toeplitz)}, T^{(hankel)} \right) \in R^{n \times n};$$

singular decomposition

$$[usv] = svd\left(T^{(toeplitz)}\right), [usv] = svd\left(T^{(hankel)}\right)$$

allows to calculate new SOPs;

- SOP formation on the Monte Carlo principles,

$$^{(m)}\tilde{a} = \begin{pmatrix} ^{(m)}a^{(\min)} & \mu^{\left(^{(m)}a^{(\min)}\right)} \\ \vdots & \\ ^{(m)}a^{(\max)} & \mu^{\left(^{(m)}a^{(\max)}\right)} \end{pmatrix}, \tag{3}$$

where $\mu^{(m)_a} = sort(rand(1,n))$

The formed set of SOP along with FS, formed in accordance with the recommendations of FST, is a multi-fuzzy set $^{(m)}\tilde{\mathbf{A}}$ у $\mathbf{A}$, which is defined as a set of ordered sequences:

$$^{(m)}\tilde{\mathbf{A}} = \{< a, \mu^{\left(^{(m)}a\right)}, \mu(a), \nu^{(new)}\mu(a), \ldots >: a \in A\},$$

where $\mu_i \in i$, for $i \in N$. Note that the multi-fuzzy set (MFS) gives a guaranteed solution, while the standard FS gives a separate solution, experiments show that in many cases these decisions, if they are made on the basis of defuzzified values, are close.

3. Review of the literature

The problem of finding the nearest fuzzy sets is not new for FST, it is mostly formulated in terms of the problem of approximation. In particular, this in many cases concerns fuzzy numbers, because the influence of FST on classical mathematics and as a consequence of the creation of fuzzy mathematics was extremely large. In particular, in [10] the nearest real interval is calculated, which allows to calculate the triangular (symmetric) fuzzy number (FN) closest (relative to the Euclidean average distance) to the trapezoidal (symmetric) FN, preserving the ambiguity. A list of approximations of fuzzy numbers under conditions is also given. In [11] is considered the approximation of the interval representation of the FN, which can have many applications. For example, it can be applied to the comparison of fuzzy numbers by the order relation defined on the set of interval numbers, the crisp approximation of fuzzy sets is called the nearest interval approximation of fuzzy set.

Discrete fuzzy sets are considered in detail and an important conclusion is made that a crisp approximation of this fuzzy set is not the only one. This paper proposes a method for finding the parametric interval of the FN approximation under two conditions: firstly, the interval is a continuous interval-approximation operator, and secondly, the parametric distance between this interval and the approximate number is minimal and continuous. The main purpose of the cited work is to illustrate how the parametric interval of the FN can be used as a crisp approximation of the FN.

In [12] it is convincingly shown that the need to make crisp decisions in uncertain (fuzzy) situations leads to the need for "approximation". The authors note that natural approximation ideas, such as the usage of a α-slice for a given α value, often do not enough for solving problems, and there is a need to find new approximations that are not only workable but also optimal in in a certain (reasonable) sense.

In solving problems under conditions of uncertainty in the form of vagueness, the traditional fuzzy technique leads to a vague recommendation, for example, "it is rather reasonable to make a decision **A**". This fuzzy recommendation can be described by a fuzzy set (membership function), which assigns to each possible situation x degree $\mu(x)$, as far as solution **A** is reasonable in this particular situation. Such a fuzzy representation is rational when making decisions, but if there is a need to create an automatic decision-making system, the user must turn a fuzzy recommendation $\mu(x)$ into a crisp recommendation $\chi(x)$, which for each *x* either recommends decision **A(** $\chi(x)=1$), or the opposite decision ($\chi(x)=0$). Therefore, this crisp recommendation can be characterized as a crisp set *S*, which $\chi(x)$ is a characteristic function.

There is a need to have such a crisp recommendation that allows you to obtain the best reproduction the original initial fuzzy recommendation $\mu(x)$. In other words, a crisp set S should be in some reasonable sense the optimal approximation to the original fuzzy set $\mu(x)$. If we assume the possibility of not making any decisions, then there is a need to approximate the output fuzzy set by a "shadowed" set, ie a set in which the characteristic function can take three possible values: "yes" ($\chi(x)=1$), "no" ($\chi(x)=0$) and “undefined” ($\chi(x)=[0,1]$).

In [2] it is shown that FST belongs to the group of sciences, where the use of the concept of a structure determines that the object under conditions of uncertainty has the structure of FS, which practically determines the solution of the problem. In turn, FS can be represented in the form of a tensor (Kronecker-product) structure. In [13] it was shown that the basic models of tensor (ie multidimensional arrays) decompositions and factorization, although proposed and known long ago, have found wide application in solving problems, in particular, classification, forecasting, multidimensional clustering, etc., but in FST is practically not used. Due to their multidimensional nature, tensors provide powerful tools for analyzing and merging massive data along with a mathematical basis for identifying hidden complex data structures.

Modern methods of solving problems (including management) under conditions of uncertainty are mostly based on FST, although this should apply only to the group of non-factors that are inaccurate and unclear. In FST often uses the concept of structure, in particular, fuzzy structure, these concepts are defined in the notation inherent in FST. In a several sources, in particular [18], FST is divided into 2 parts: the fuzzy set theory and the technique of using FS (crisp matrices-fuzzy matrices, crisp equations - fuzzy equations). Note that the representation of a crisp mathematical object into the rank of "fuzzy", if it is not dictated by the conditions of the problem, but due only to the purpose of obtaining a "technical" solution, can not always be realized.

In [1], different types of hidden and approximate structures in the matrix and their key role in different applications are considered. Matrices with a special structure often arise in scientific and engineering problems and have long been the subject of numerical linear algebra, as well as matrix and operator theory. For example, this category includes Toeplitz matrices, which are used to implement procedures for blurring audio and video signals. Relatively recently, research has begun on classes of matrices whose structural properties may be unobvious.

Structural analysis of specific complex types of uncertainty leads to the fact that the initial data sets structured in the form of a matrix (tensor), which at first seem completely unstructured, on closer inspection reveal a significant number of features of the hidden structure, for example, in the distribution of nonzero elements and eigenvalues, the existence of subsets of ordered pairs, etc. Knowledge of these properties (signs) can be crucial in the development of effective numerical methods for solving such problems. FST arose, not least, due to vagueness of the initial data to be processed: low (or insufficient) accuracy of data definition, missing data, its distortion, and so on. This means that in many cases the data set can be structured (represented as a matrix) only "approximately"; for example, an array of data represented as a matrix, which may in some way be close to a matrix having the desired structure. In such cases, it may be possible to develop algorithms that use the "almost structure" present in the matrix; for example, efficient preprocessors can be designed for a neighboring structural problem and applied to the original problem.

In other cases, solving the nearest problem for which there are effective algorithms may be a sufficient approximation to the solution of the original problem, and the main difficulty may be to identify the "nearest" structured problem. The presence of a structure (or a structure close to it) of the object of a study may not be immediately obvious in this problem. In general, referring to [14] in this context are discussed mainly structured matrix calculations that emerge as a result of applying tensor analysis, in particular tensor decompositions.

In addition to the above, in accordance with [1], the emphasis is on the calculation of structured matrices that arise in the context of various "svd-like" tensor decompositions. Kronecker products and low-level manipulations are central. Note that the "exploitation of structure in matrix calculations" refers to the demonstration of the idea of "hidden" structure, which in turn in this context should be understood as "non-obvious structure". In each case, the usage of a hidden structure has important implications in terms of calculations.

Summarizing the results of the literature review, according to [15], the influence of FST on standard mathematics was in a number of cases rather contradictory. In this paper, it is shown that the reverse side of the usage of fuzzy models (standard mathematical models containing fuzzy numbers (or variables)) revealed a contradiction between the solutions obtained using new methods and the results of classical theories, loss of the solutions stability, a violation of natural relations in models in which fuzzy, as mentioned above, are only the parameters, unjustified expansion of the degree of fuzziness of the result, increasing the computational complexity of problems, and so on.

The fuzzy mathematics usually deals with fuzzy numbers, which limits the possibilities of FST, but gives formal rigor to the models, unfortunately, without expanding their capabilities, because there are questions about the models of FN representation, which allow to store the original expert information and provide necessary qualitative properties of solutions (stability, preservation of crisp mathematical relations, etc.). Example: systems of fuzzy equations, which are formed on the basis of fuzzification of crisp parameters, do not have a practical ability to store the original expert information, because there are several types of solutions (weak, strong, possible, etc.) [16], very often the problem is only in search acceptable solution.

This also applies to the method of efficient numerical implementation of solving problems with fuzzy parameters, based on the corresponding algebraic structures. Recall that the FN is a kind of fuzzy quantity, the membership function of which $\mu_{A\sim}(x)$: $R \rightarrow [0; 1]]$ has the following properties: piecewise continuity, convexity $\forall(x_1, x_2) \in \mathbb{R}$; $\forall\gamma \in [0; 1]$ $\mu_{A\sim}(\gamma x_1 + (1 - \gamma) x_2) > \min\{\mu_{A\sim}(x_1), \mu_{A\sim}(x_2)\}$; normality $\sup x \in \mathbb{R} (\mu_{A\sim}(x)) = 1$. Fuzzy numbers are considered as an object of interest from the point of view of practical application precisely because of the continuity of membership functions, but obvious disadvantages do not allow to expand the range of new real problems solved under conditions uncertainty in the FN environment.In many cases it is necessary to take into account FN with discrete MF, when performing operations on which problems arise, in particular, the result of arithmetic operations performed on random fuzzy values according to the Zade generalization principle, it is not always possible to obtain FN. The result of the multiplication of FN with a discrete MF can be a subnormal fuzzy set.

4. Materials and methods.

Modern methods of solving problems (including management) under conditions of uncertainty are mostly based on the FST, although this should apply only to the group of non-factors that are inaccurate and unclear [17]. The concept of structure, in particular, fuzzy structure, groupoid, is often used in FST. In general, the expression "object has the FS structure (or close to FS)" is widely used. Earlier it was noted that in a number of sources, in particular [18], the FST is divided into 2 parts: the fuzzy set theory and the technique (of applying) of FS. Mathematical structure on the set in mathematics is a general name for additional mathematical objects, given on the set. For defining the mathematical structures should be determined the relations for the elements of the set. Examples of mathematical structures are algebraic structures (groups, rings, fields, vector spaces, etc.), measures, metric structures, topologies, orders, differential structures, categories, etc. The set may have more than one structure at a time. For example, an order generates a topology; a set with a topology can be a group, then it is called a topological group. Mappings between sets that store structures (so that the structures defined for the first set are mapped to equivalent structures in the second set) are called morphisms. For example, homomorphism - preserves algebraic structures, homeomorphism - preserves topological structures; diffeomorphism - preserves differential structures.

To determine the structure itself, specify the relations in which the elements of these sets are located. Then it is postulated that these relations satisfy certain conditions, which are axioms of this structure.

In [19,20] the possibility of representing FS in the form of a hierarchical structure (binary tree) was considered for the first time. The dendrogram encoded by the binary alphabet is a 2-adic number that can be used as its characteristic. Comparison of hierarchical clustering of fuzzy data and their defuzzification, performed at the level of 2-adic trees, allows us to conclude about the presence (absence) of structural proximity of objects. In [1] is reviewed different types of hidden and approximate structures, first of all, in matrices and their key role in different programs. In consonance with [1, 14] in this context are considered the structured matrix calculations arising from tensor analysis, especially, tensor decompositions, and the main role is played by low-rank and Kronecker structures.

In addition to the above, according to [1], the emphasis is on the calculation of structured matrices that arise in the context of various svd-like matrix (tensor) decompositions. Kronecker products and low-level manipulations are central. Recall that the main object of FST, the FS

$$\tilde{a} = \left\{ a / \mu^{(a)} \right\}, a \in A, \mu^{(a)} \to [0,1]$$

provides representation in matrix form:

$$\tilde{a} = \begin{pmatrix} a_i & \mu^{(a_i)} \end{pmatrix}_{i=1}^{n} \in \mathrm{R}^{n \times 2},$$

which allows obtaining a representation in the form of Kronecker product

$$\mathbf{T}^{(a)} = a \otimes \mu^{(a)} \in \mathrm{R}^{n \times n},$$

where n is the number of α - levels. In turn $\mathbf{T}^{(a)}$, it can be represented as a 3D tensor

$$\mathcal{T}^{(a)} = reshape\left(\mathbf{T}^{(a)}, [m, p, f] \right), \ \mathcal{T}^{(a)} \in \mathrm{R}^{m \times p \times f} \ ; \ m \cdot p \cdot f = n \cdot n.$$

In [9], multi-fuzzy sets are considered as a method of constructing more general fuzzy sets by using ordinary fuzzy sets as building blocks. The concept of multi-fuzzy sets is introduced in terms of the use of ordered sequences of membership functions. The family of operations T, S, M of multi-fuzzy sets is introduced by the usage of coordinated t-norms, s-norms and aggregation operations.

Definition [9]. Let X be a nonempty set, N - a set of all natural numbers, {Li: i ∈ N} - a family of complete lattices. Multi-fuzzy set ${}^{(m)}\tilde{\mathbf{A}}$ in X is a set of ordered sequences: ${}^{(m)}\tilde{\mathbf{A}}$ = {⟨x, μ_1 (x), μ_2 (x), ..., μ_i (x), ...⟩: x ∈ X}, where μ_i ∈ i, for i ∈ N.

Note 1: [9]. The function μA = ⟨μ1, μ2, ...⟩ is called a function with several members of the multi FS ${}^{(m)}\tilde{\mathbf{A}}$. If the sequences of membership functions have only *k*-terms (finite number of terms), k is called the dimension ${}^{(m)}\tilde{\mathbf{A}}$ and is denoted as $\mathbf{M}^{\mathbf{k}}\mathbf{FS}(X)$.

Note that the notation X, N, I and I^{X} have the following meaning: nonempty set, which is called universal, the set of all natural numbers, the unit interval [0, 1] and the set of all functions from X to I, respectively. I^{k} means I × I × ... × I (k-times), for any positive integer k.

The proximity (minimum distance) of FS

$$\tilde{x} \overset{\Delta}{=} \left\{ x / \mu^{(x)} \right\} \text{ and } \tilde{y} \overset{\Delta}{=} \left\{ y / \mu^{(y)} \right\}$$

is estimated: a) by the proximity of their tensor models, b) by the proximity of objects

$$\tilde{x}_{(matr)} \in \mathrm{R}^{n \times 2} \text{ and } \tilde{y}_{(matr)} \in \mathrm{R}^{n \times 2}.$$

Recall that the problem of minimizing $\phi(B,C) = \|A - B \otimes C\|_F$ is considered in vectorized form, ie

$$\phi(B,C) = \|A - B \otimes C\|_F = \left\| \hat{A}_{CS} - \underbrace{\underbrace{vec(B)}_{US} \otimes \underbrace{vec(C)}_{MF}}_{\text{Standart } MF} \right\|_F \qquad (4)$$

where, for example, for the matrix $A \in \mathrm{R}^{3 \times 2}$ we have:.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \to \hat{A} = = [A_{11} \ A_{21} \ A_{31} \ A_{12} \ A_{22} \ A_{32}]^T.$$

The solution is to calculate $SVD\ U^T\hat{A}V = \sum$ (where U, V- left and right singular matrices, respectively, $\sum$ is a matrix of singular values,) and is set

$$vec\left(B^{(opt)}\right) = \left(\sigma_1\right)^{1/2} U(:,1), \quad vec\left(C^{(opt)}\right) = \left(\sigma_1\right)^{1/2} V(:,1),$$

where *U (:, 1), V (:, 1)* are left and right singular vectors. Note that the optimum does not change if

$$B^{(opt)} = B^{(opt)} \cdot \lambda,\ C^{(opt)} = C^{(opt)} \cdot \frac{1}{\lambda} = \quad \text{for } \lambda>0$$

SOP is formed in the form

$$\left(\underbrace{\sigma_1 \cdot U(:,1) \cdot \max\left(abs\left(V(:,1)\right)\right)}_{Universal\quad set} \quad \underbrace{V(:,1) \cdot \frac{1}{\max\left(abs\left(V(:,1)\right)\right)}}_{Membership\quad function} \right) \in R_+^{n\times 2} \qquad (5)$$

Definition. FS $\tilde{x}$ and $\tilde{y}$ are close (according to the understanding of the F-norm), if for $\mathbf{T}^{(x)} = x \otimes \mu^{(x)} \in R^{n\times n}$ and $\mathbf{T}^{(y)} = y \otimes \mu^{(y)} \in R^{n\times n}$ there is statement

$$\left\|\mathbf{T}^{(x)}\right\|_F - \left\|\mathbf{T}^{(y)}\right\|_F = = \left\|\mathbf{T}^{(x)} - \mathbf{T}^{(y)}\right\|_F \to \min,$$

here $\otimes$ is the symbol of the Kronecker product, n is the number of α -levels in the FS, the F-norm of the matrix $\mathbf{A} \in R^{n\times m}$ (calculated as

$$\|\mathbf{A}\|_F = tr\left(\mathbf{A} \quad \cdot \mathbf{A}^T\right)^{1/2},$$

where *tr()* is the trace (sum of diagonal elements) of the matrix).

Note, that the FS representation $\tilde{x}$ and $\tilde{y}$ in the form:

$$\tilde{x}_{(matr)} = \begin{pmatrix} x_1 & \mu^{(x_1)} \\ \vdots & \vdots \\ x_n & \mu^{(x_n)} \end{pmatrix} \in R^{n\times 2}, \quad \tilde{y}_{(matr)} = \begin{pmatrix} y_1 & \mu^{(y_1)} \\ \vdots & \vdots \\ y_n & \mu^{(y_n)} \end{pmatrix} \in R^{n\times 2}, \qquad (6)$$

allows us to consider that for close $\tilde{x}$ and $\tilde{y}$, takes place $\|\tilde{x}\|_F \cong \|\tilde{y}\|_F$.

Assertion. Singular decomposition of the tensor model of FS $\mathbf{T}^{(x)} \in R^{n\times n}$ - the procedure

$$[u\ s\ v] = svd\left(\mathbf{T}^{(x)}\right)$$

allows to form subsets of ordered pairs:

$$\text{Pr\_PVP}_1\text{=([abs(u(:,1))*s(1,1)*max(abs(v(:,1))) (abs(v(:,1)))/max(abs(v(:,1)))]),}$$

$$\text{Pr\_PVP}_2\text{=sort([abs(u(:,1))*s(1,1)*max(abs(v(:,1))) (abs(v(:,1)))/max(abs(v(:,1)))]),}$$

which have the following properties:

$$\left\|\text{Pr\_PVP}_1\right\|_F = \left\|\tilde{x}_{matr}\right\|_F, \left\|\text{Pr\_PVP}_1(:,1) \otimes \text{Pr\_PVP}_1(:,2)\right\|_F = \left\|\tilde{x}_{matr}(:,1) \otimes \tilde{x}_{matr}(:,2)\right\|_F,$$

$$def\left(\text{Pr\_PVP}_1\right) = def\left(\tilde{x}_{matr}\right);$$

$$def\left(\text{Pr\_PVP}_1\right)\text{=sum(Pr\_PVP}_1\text{(:,1).*Pr\_PVP}_1\text{(:,2))/sum(Pr\_PVP}_1\text{(:,2)),}$$

$$def\left(\tilde{x}_{matr}\right)\text{=sum(}\tilde{x}_{matr}\text{(:,1).*}\tilde{x}_{matr}\text{(:,2))/sum(}\tilde{x}_{matr}\text{(:,2))}.$$

SOP $\mathrm{Pr\_PVP}_2$ has similar properties except that $\mathrm{Pr\_PVP}_1$ practically coincides with the initial FS, $\mathrm{Pr\_PVP}_2$ has a sigmoid-type shape, in addition

$$def\left(\mathrm{Pr\_PVP}_2\right) \cong def\left(\tilde{\mathrm{x}}_{matr}\right).$$

Let's pay attention to such conditions:

- the nearest (to the fuzzy) crisp set is unique, at the same time there can be several nearest fuzzy sets, which, in our opinion, is quite consistent with the nature of FS, because the same fuzzy statement depends on the opinion of experts can be modeled by several types of FS; in addition, this circumstance is influenced by the property of the SVD solution as an optimization problem, which follows from [14];
- in [22, 23] it is shown that uncertainty has objective and subjective components, although FST proposes to use only the subjective component to represent uncertainty, it often happens that these components coincide; the concept of the proximity of FS makes it possible to use objectively calculated SOP, which is calculated for the following situations:

1. blurring of the universal set by using special matrices (Toeplitz, Hankel);
2. the singular decomposition of the set of initial data;
3. the SOP formation due to the calculated US and randomly assigned sequence of ordered values obtained using a random value generator, namely

$$\left\{\underbrace{\left[x^{(\min)}:\Delta x:x^{(\max)}\right]}_{n} / \mathrm{sort}\left(rand(1,n)\right)\right\}.$$

5. Results

5.1 Experiments

The experimental part of the research, which is presented in the following context, consists of 2 parts:

- the general part of the analysis of the proximity of SOP;
- specific examples of the application of the identification of hidden knowledge for decision-making under conditions of uncertainty.

The file pricladi_nearest_set_1.m, fragments of which are presented in the paper, pursues the goal - to show that the FS, which is formed by heuristic rules (selected from the MatLab libraries) and has the standard properties is not a universal apparatus for modeling uncertainty. The main factor for modeling the uncertainty is the fact that there is a subset of ordered pairs, which is calculated according to formal rules; in contrast to FS, the formally calculated SOP does not explicitly reproduce the significance (weight) of a particular component of FS in the truth of the statement, which is modeled under conditions of uncertainty in a number of cases and always has a sigmoid-like shape.

Note that in this context FS and SOP are semantically different, although in [2] these two concepts are equivalent. Thus, an object

$$\tilde{x} \stackrel{\Delta}{=} \left\{x/\mu^{(x)}\right\} \quad x \in X, \mu^{(x)} \in [0,1] \text{ is an FS, if } \mu^{(x)}$$

is a heuristically defined set of values that clearly reproduces the significance (weight) of an element $x \in X$, for instance, in the truth of a fuzzy statement that simulates a given FS; if $\mu^{(x)}$ is calculated formally, on the basis of the alternative least squares method and does not explicitly reproduce the weight properties, its values are obtained according to the alternative least squares method [21], then $\tilde{x}$ - SOP.

5.2 Experiments Results

Comparison of the quality of uncertainty modeling by standard methods of the FST and with the help of SOP, calculated on the basis of tensor models.

1. The initial set of data - measurements in the range [4-6] with an accuracy of 20% (Fig. 1). Uncertainty is modeled by a standard Gaussian type FS.
2. Modeling results (Fig. 2). The tensor model of FS with a Gaussian MF show in Fig.3.

x =										
5.63	5.94	5.7	4.09	4.37	5	5.09	4.5	4.76	4.67	4.9
5.81	5.91	5.87	4.19	4.98	5.92	4.28	5.23	5.14	4.32	4.17
4.25	4.97	5.36	5.65	4.89	4.68	4.3	4.95	4.15	5.59	4.46
5.83	5.6	5.52	5.39	5.29	5.17	4.52	4.7	4.11	4.62	5.83
5.26	4.28	5.49	4.63	5.42	4.45	5.68	5.66	5.06	5.06	4.3
4.2	4.84	4.78	5.9	5.51	5.5	4.51	5.17	5.56	4.33	5.65
4.56	5.83	5.31	4.07	4.55	4.51	5.63	5.1	5.87	5.2	5.08
5.09	5.58	4.34	4.88	5.36	5.01	4.49	5.83	4.26	4.53	5.99
5.92	5.92	5.41	5.53	4.33	5.78	4.7	5.51	4.94	5.38	4.89
4.32	4.07	4.55	5.59	4.24	5.92	4.39	5.51	4.02	5.5	4.21

Figure 1: The initial set of data

Model FS			fs2gaus =							
4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6
0.61	0.73	0.84	0.92	0.98	1	0.98	0.92	0.84	0.73	0.61
F-norm init_set_ x		55.42	fs2gaus and		16.95	defuzzyfied value		5		

Figure 2: Modeling results

tx2 =										
2.43	2.9	3.34	3.69	3.92	4	3.92	3.69	3.34	2.9	2.43
2.55	3.05	3.51	3.88	4.12	4.2	4.12	3.88	3.51	3.05	2.55
2.67	3.2	3.68	4.06	4.31	4.4	4.31	4.06	3.68	3.2	2.67
2.79	3.34	3.84	4.25	4.51	4.6	4.51	4.25	3.84	3.34	2.79
2.91	3.49	4.01	4.43	4.7	4.8	4.7	4.43	4.01	3.49	2.91
3.03	3.63	4.18	4.62	4.9	5	4.9	4.62	4.18	3.63	3.03
3.15	3.78	4.34	4.8	5.1	5.2	5.1	4.8	4.34	3.78	3.15
3.28	3.92	4.51	4.98	5.29	5.4	5.29	4.98	4.51	3.92	3.28
3.4	4.07	4.68	5.17	5.49	5.6	5.49	5.17	4.68	4.07	3.4
3.52	4.21	4.84	5.35	5.69	5.8	5.69	5.35	4.84	4.21	3.52
3.64	4.36	5.01	5.54	5.88	6	5.88	5.54	5.01	4.36	3.64

Figure 3: The tensor model of FS with a Gaussian MF

SOP №1, calculated as a result of singular decomposition of the tensor model of FS show in Fig. 4. SOP №2, calculated as a result of singular decomposition of the tensor model of FS with Gaussian MF show in Fig. 5. Conclusion: the result of singular decomposition is the closest (to the initial FS) of sigmoid-like SOP, where F-norms and defuzzyfied values practically coincide. Objective blurring the US using a Toeplitz matrix (Fig. 6). Analyzing the obtained result, it is necessary to recognize the practical coincidence of F-norm and defuzzyfied value with similar values calculated on the basis of standard (heuristic) MF, but the fact of significant reduction of the range of possible result values can in some cases play a key role in decision making. The nearest SOP has the form show in Fig. 7.

The formation of SOP with a randomly assigned MF in the range [0,1] (Fig. 8). F-normes and defuzzyfied values show in Fig. 8. These simulation results indicate the possibility of applying the approach based on the Monte Carlo method for decision-making under conditions of uncertainty, accuracy (if this concept can be applied in the case, where it is an approximate solution) is 10%. A singular decomposition of the structured initial data set. SOP №1, calculated as a result of singular decomposition of the structured IDS show in Fig. 10.

Pr_PVP1 =										
4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6
0.61	0.61	0.73	0.73	0.84	0.84	0.92	0.92	0.98	0.98	1
Characteristics of SOP №1										
The norm TD component SOP №1			The norm of the initial tensor model			The norm of SOP№1			Defuzzyfied values of SOP 1	
46.72			46.72		16.95			5.1		

Figure 4: Calculated as a result of singular decomposition of the tensor model of FS (SOP №1)

Pr_PVP2 =										
4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6
0.61	0.73	0.84	0.92	0.98	1	0.98	0.92	0.84	0.73	0.61
Characteristics of SOP №1										
The norm TD component SOP №2			The norm of the initial tensor model			The norm of SOP№2			Defuzzyfied values of SOP 2	
46.72			46.72			16.95			5	

Figure 5: Calculated as a result of singular decomposition of the tensor model of FS (SOP №1)

tx2t =										
4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6
4.2	4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8
4.4	4.2	4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6
4.6	4.4	4.2	4	4.2	4.4	4.6	4.8	5	5.2	5.4
4.8	4.6	4.4	4.2	4	4.2	4.4	4.6	4.8	5	5.2
5	4.8	4.6	4.4	4.2	4	4.2	4.4	4.6	4.8	5
5.2	5	4.8	4.6	4.4	4.2	4	4.2	4.4	4.6	4.8
5.4	5.2	5	4.8	4.6	4.4	4.2	4	4.2	4.4	4.6
5.6	5.4	5.2	5	4.8	4.6	4.4	4.2	4	4.2	4.4
5.8	5.6	5.4	5.2	5	4.8	4.6	4.4	4.2	4	4.2
6	5.8	5.6	5.4	5.2	5	4.8	4.6	4.4	4.2	4

Figure 6: Objective blurring the US using a Toeplitz matrix

Pr_PVP2 =										
4.81	4.83	4.83	4.88	4.88	4.97	4.97	5.11	5.11	5.28	5.28
0.91	0.91	0.91	0.93	0.93	0.94	0.94	0.97	0.97	1	1
Characteristics of SOP №1										
F-norm: Kron-prod component SOP			tx2t			component SOP			Defuzzyfied value	
52.06			52.31			16.87			5	

Figure 7: The nearest SOP has the form

6. Conclusions

The problem of control under uncertainty has received an extremely powerful stimulus in the form of fuzzy set theory. Representation of uncertainty in the form of a subset of ordered pairs, one of the components of which is a weight function determined by an expert, is the most common, although there is no reason to consider it the most rational. FS with MF assigned by an expert from a set of "standard" MF is practically dominant in modern decision-making practice. However, currently, there is a class of problems that requires new approaches for solving management problems under conditions of uncertainty, using the fuzzy set theory. Features of the new tasks are the following:

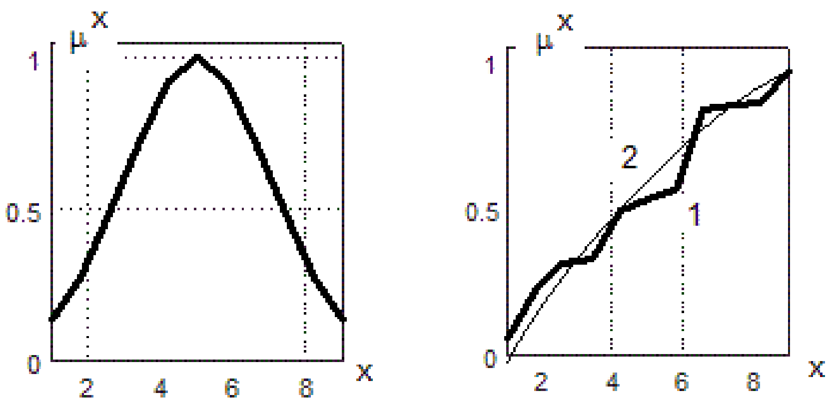
- ignoring the existence of objective uncertainty, the attempt to present all the uncertainty only as its subjective component is potentially contradictory. The fact that nowadays there are practically absent researches which would show the examples when the wrong choice of subjective MF leads to the need to resolve the problem with the new MF;

- should be paid special attention to the results of the work of Nobel laureates D. Kahneman and A. Tversky [24], who showed that only 15-25% of expert (heuristic) decisions are rational, others - anomalous; in other words, the solution of the problem under conditions of uncertainty by using only one MF is not always justified;
- in a number of problems determining the expert rational MF is almost impossible due to the limited capabilities of the human expert, but the requirement to obtain a solution is not removed, the assignment of almost random heuristic MF does not guarantee the rationality of the solution;
- the variety of forms of uncertainty objectively implies the fundamental impossibility of data processing only with the help of the FS apparatus with expertly selected (heuristic MF). This situation is partially compensated by the use of combinations of different types of FS; for example, in [19] is offered several combinations of different types of fuzzy sets, in particular, interval-valued FS (IVFS), soft sets, Atanassov's intuitionistic fuzzy sets (AIFSs) , rough sets, HFS, etc ;
- FS, which has the structure of SOP (in particular, multi-fuzzy FS, the generalized form of which

$$^{(m)}\tilde{\mathbf{A}} = \left\{ \left\langle x, \underbrace{\{\mu_1(x),\ \mu_2(x),\ \ldots,\ \mu_i(x),\ \ldots\}}_{\hat{\mu}(x)} : x \in X \right\rangle \right\}, \qquad (\forall i)\,\mu_i(x) \to [0,1];$$

z2(:,:,1) =		z2(:,:,2) =		z2(:,:,3) =		z2(:,:,1000) =	
4	0.22	4	0.17	4	0.09	4	0.05
4.2	0.38	4.2	0.33	4.2	0.11	4.2	0.12
4.4	0.39	4.4	0.44	4.4	0.14	4.4	0.18
4.6	0.4	4.6	0.51	4.6	0.15	4.6	0.29
4.8	0.43	4.8	0.59	4.8	0.2	4.8	0.33
5	0.76	5	0.67	5	0.32	5	0.54
5.2	0.77	5.2	0.77	5.2	0.53	5.2	0.66
5.4	0.79	5.4	0.83	5.4	0.41	5.4	0.71
5.6	0.81	5.6	0.86	5.6	0.75	5.6	0.9
5.8	0.89	5.8	0.88	5.8	0.79	5.8	0.99
6	0.95	6	0.99	6	0.83	6	1
min/max		F-normes		min/max		defuzz.values	
16.78		16.89		5.23		5.5	
mean defuzzyfied values =				5.34			

Figure 8: The nearest SOP has the form



a b

Figure 9: a – standart FS x2=[1:8/10:9]'; y2= gaussmf(x2, [2.0 5]); fs2gaus=[x2 y2]; b – nearest (to FS «a»), formed as SOP from ordered set of random numbers from [0, 1] on US x2.

x =										
5.63	5.94	5.7	4.09	4.37	5	5.09	4.5	4.76	4.67	4.9
5.81	5.91	5.87	4.19	4.98	5.92	4.28	5.23	5.14	4.32	4.17
4.25	4.97	5.36	5.65	4.89	4.68	4.3	4.95	4.15	5.59	4.46
5.83	5.6	5.52	5.39	5.29	5.17	4.52	4.7	4.11	4.62	5.83
5.26	4.28	5.49	4.63	5.42	4.45	5.68	5.66	5.06	5.06	4.3
4.2	4.84	4.78	5.9	5.51	5.5	4.51	5.17	5.56	4.33	5.65
4.56	5.83	5.31	4.07	4.55	4.51	5.63	5.1	5.87	5.2	5.08
5.09	5.58	4.34	4.88	5.36	5.01	4.49	5.83	4.26	4.53	5.99
5.92	5.92	5.41	4.76	5.31	5.4	5.86	4.57	5.14	5.31	4.16
5.93	5.31	4.06	5.53	4.33	5.78	4.7	5.51	4.94	5.38	4.89
4.32	4.07	4.55	5.59	4.24	5.92	4.39	5.51	4.02	5.5	4.21
Pr_PVP1 =										
5.01	5.1	5.24	5.29	5.31	5.33	5.35	5.36	5.4	5.42	5.53
0.91	0.92	0.92	0.93	0.93	0.94	0.97	0.97	0.97	0.98	1
Characteristics of SOP №1										
TD norm of SOP components №1					The norm of initial tensor model				The SOP №1 norm	
			55.42		46.72				17.88	
Defaulted values			5.31							

Figure 10: Calculated as a result of singular decomposition of the structured IDS

tensor model of MFS

$$\mathbf{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)} = \left(x \otimes \left\langle \hat{\mu}(x) \right\rangle\right) \in \mathrm{R}^{n \times k \cdot n}$$

allows to obtain by singular decomposition the SOP, similar to type-1 FS and thus obtain an approximate solution (defuzzyfied value), which will allow having an approximate (alternative) solution. Also an alternative solution can be a representation ${}^{(m)}\tilde{\mathbf{A}}$ as a 3D tensor

$$\mathcal{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)},$$

which consists of frontal slices

$$\mathbf{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)}(:,:,1) = \left(x \otimes \mu_1(x)\right) \in \mathrm{R}^{n \times n}, \quad \mathbf{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)}(:,:,1) = \left(x \otimes \mu_1(x)\right) \in \mathrm{R}^{n \times n},$$

$$\mathbf{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)}(:,:,2) = \left(x \otimes \mu_2(x)\right) \in \mathrm{R}^{n \times n}, \quad \mathbf{T}^{\left({}^{(m)}\tilde{\mathbf{A}}\right)}(:,:,i) = \left(x \otimes \mu_i(x)\right) \in \mathrm{R}^{n \times n},$$

the usage of high-order singular decomposition allows you to get (calculate) SOP.

The FST has a powerful structure - a subset of ordered pairs, which, on the one hand, assumes almost unlimited variability, on the other hand - the tensor model (Kronecker-product of SOP components) allows to reveal hidden knowledge and thus expands the range of problems solved under conditions of uncertainty. In the general case, the solution of any problem of decision-making under conditions of uncertainty should be solved for three cases:

- FS with heuristically assigned MF;
- SOP, computed on the basis of objective blurring of US, pre-calculated on the basis of a structured initial data set, taking into account missing or distorted data;
- SOP, computed on the basis of randomly assigned MF, preordered according to US (sigmoid-type) MF.

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