Localization of Point Sources with Random Spatial Position and Random Discipline of Pulse Generation

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Abstract. The methods and speed-optimal algorithms oriented to spatial localization of pulsed-point sources manifesting themselves at random time by generation of instantaneous delta pulses are discussed. Optimal search procedures have been proposed that are focused on the localization of random pulsed-point objects in standard and advanced search modes (for example, in the absence of a priori information about the intensity of the source or when its density is unknown within the search interval).

Keywords: optimal search, random pulsed-point source, localization accuracy, receiver.

1 Introduction

Research on the optimal search for random pulse-point objects is subject of current interest for many scientific and technical disciplines. The need to conduct them arises in the design of various electron-optical converters and detectors [1]; in the tasks of the suppression of impulse noise on noisy and low-contrast images [2]; in the development of methods for tech troubleshooting, appearing in a form of the alternating equipment failures [3]; in problems of detecting radioactive sources using systems consisting of one or several sensors [4], in radio physics and radio astronomy, when searching for sources of gravitational waves [5] and in many other applications. This paper presents the methods and algorithms for the speed-optimal search for point Poisson sources that manifest themselves at random time by generation of instantaneous delta pulses. The optimal search algorithm should, as a rule, satisfy one of two requirements: minimize the total search effort required to detect an object; or maximize the total probability of detection in the presence of limited search effort.

The point-pulsed source will be understood below as an object of negligibly small angular dimensions (mathematical point), having a random distribution on the abscissa axis with a priori probability density $f(x)$ and radiating infinitely short pulses (δ-functions) with Poisson intensity λ. Thus, the time intervals between pulses are a random variable t with an exponential probability density $h(t) = \lambda\exp(-\lambda t)$. The search for an object is carried out with the help of a recording device having a tunable «window» with an arbitrarily time function. The pulse is fixed if the active object that initiated the pulse is in the «view window» of the recording device. Otherwise, the pulse is considered to be missed. When registering the pulse, the window narrows, and (as a result) the position of the source is determined more accurately. It is required for the minimum (in statistical terms) time to find a source with an accuracy of ε

2 Single-step search algorithms

Introducing the binary function x

$$u(x,t)=\begin{cases}1, & \text{if the point } x \text{ at the time moment } t \text{ is in} \\ & \text{the view window of the receiving device,} \\ 0, & \text{otherwise,}\end{cases}$$

describing the view window at time t, we obtain the average time from the start of the search to the registration of the first pulse:

$$\langle\tau\rangle = \lambda\int_0^\infty dt\int_0^\infty dx\left[tf(x)u(x,t)\exp(-\lambda\int_0^t u(x,\xi)d\xi)\right].$$

For the random priori distribution of a pulsed source on the *x*-axis, the construction of even a one-step (which ends immediately when the first pulse is registered) time-optimal search procedure causes considerable difficulties. In one-step *periodic* search algorithms, the relative load $\varphi(x)$ on the point *x* (that is, the relative time it's located in the view window) remains constant throughout the entire search time. With this approach, the problem is to find the function $\varphi(x)$, minimizing the average search time

$$\langle\tau\rangle = \frac{1}{\lambda}\int\frac{f(x)}{\varphi(x)}dx \tag{1}$$

when the following conditions are met

$$\int\varphi(x)dx = \varepsilon, \tag{2}$$

$$0 \le \varphi(x) \le 1. \tag{3}$$

Optimization of expression (1) with constraints (2) - (3) belongs to non-linear programming problems [6-7]. To solve it, we will use the method of Lagrange multipliers [8] and we will look for the function $\varphi(x)$ minimizing the expression

$$\int\left[\frac{f(x)}{\varphi(x)} + \mu\varphi(x)\right]dx.$$

Differentiating by φ and taking into account the constraint (2), we obtain

$$\varphi(x) = {\varepsilon\sqrt{f(x)}}\Big/{\int\sqrt{f(x)}dx} \quad . \tag{4}$$

If for any *x* condition (3) is not violated, then function (4) is a solution to the formulated extremal problem. If there exist such domains *x*, where the solution $\varphi(x) > 1$, then inside these areas it is necessary to put $\varphi(x) = 1$, and to recalculate (for the remaining points) the indefinite factor μ taking into account the already changed conditions (2) and (3). After that, any binary function $u(x, t)$ can be selected as the optimal search strategy if it satisfies the relations

$$\int u(x,t)dx = \varepsilon; \qquad \int u(x,\xi)d\xi = \varphi(x)t.$$

In the general case, building an optimal (not necessarily periodic) one-step search algorithm is associated with finding such a function $\varphi(x, t)$ – the relative load on point *x* at time *t*, – which minimizes the average localization time

$$\langle\tau\rangle = \int dt\int dx f(x)\exp(-\lambda\int_0^t\varphi(x,\xi)d\xi)$$

provided that

$$0 \le \varphi(x,t) \le 1 \tag{5}$$

and for any *t*

$$\int\varphi(x,t)dx = \varepsilon. \tag{6}$$

To simplify further calculations, we introduce a function $\alpha(x,t) = \int_0^t\varphi(x,\xi)d\xi$ corresponding to the total time for point *x* stays in the view window (for the entire period from the beginning of the search to the time *t*). To take into account constraints (5) and (6), we again introduce the Lagrange multiplier $\mu(t)$. Then the task of building an optimal search strategy will be reduced to finding the function $\alpha(x,t)$ minimizing the functional $\int dt\int dx[\exp(-\lambda\alpha(x,t)f(x) + \mu(t)\alpha(x,t)]$, provided that

$$\int_{-\infty}^{\infty} \alpha(x,t)dx = \varepsilon t, \tag{7}$$

$$0 \le \alpha(x,t) \le t.$$

The solution to this variational problem is the function

$$\alpha(x,t) = \begin{cases} 0, \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)} < 0; \\ \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)}, 0 \le \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)} \le t; \\ t, t < \frac{1}{\lambda}\ln\frac{\lambda f(x)}{\mu(t)}, \end{cases} \tag{8}$$

where the multiplier $\mu(t)$ is determined from relation (7), and any binary function can be chosen as the optimal search strategy $u(x, t)$ that satisfies the conditions

$$\int_0^t u(x,\xi)d\xi = \alpha(x,t); \qquad \int u(x,t)dx = \varepsilon.$$

The use of optimal search algorithms in practice leads to certain difficulties. The fact is that in those cases when source's priori distribution density function differs from the uniform one, both of the proposed optimal one-step search algorithms cannot be physically realized by moving the singly connected (non-separable) scanning window. Therefore, in actual search procedures, it is advisable to perform a one-step procedure according to the following scheme. The interval (0, L) is pre-divided into a series of discrete elements having a length ε, and the a priori given density $f(x)$ is "stepwise" approximated on each of them. The value of ε is considered to be sufficiently small (which meets the high requirements for localization accuracy) so that the variation of the function $f(x)$ within one discrete can be neglected. The search must begin with "observation" of the highest "peak", within which the amplitude function $f(x)$ is maximum, then after the time t_1, the window is alternately set under the two highest "peaks", then after the time t_2, three elements are alternately observed, etc. All switching moments t_i are determined in exact accordance with the above relation (8), which is the basis for constructing an optimal search strategy.

It should be noted that the search algorithm under discussion assumes that the intensity of the source λ is known in advance. If such a priori information is not available, it is possible to recommend a periodic procedure that does not depend on this intensity. In accordance with this strategy, the integrals of the density $f(x)$ are initially calculated in each ε-discrete. If the total number of ε-increments (into which the initial search interval is divided) is m, and the "step" integrals over each of them are equal to $P_1, P_2, ..., P_m$, then the view window should cycle through all discretes with relative load

$$\beta_i = \sqrt{P_i} \Big/ \sum_{j=1}^{m} \sqrt{P_j} \quad , \quad (j=1,...,m).$$

These β_i values are easy to obtain if the method of Lagrange multipliers is used again to minimize the average search time:

$$\langle \tau \rangle = (1/\lambda)(P_1/\beta_1 + P_2/\beta_2 + ... + P_m/\beta_m) \Rightarrow \min, \qquad (\beta_1 + ... + \beta_m = 1).$$

3 Multi-step search algorithms

With high requirements for localization accuracy, one-step algorithms are far from optimal. So, when constructing an optimal search procedure, we cannot limit ourselves to one-step algorithms, and should consider a search procedure consisting of several steps. For the average time of the m-step localization procedure, the ratio is

$$\langle \tau \rangle = \sum_{k=1}^{m} \lambda^k \int_0^\infty dx f(x) \int ... \int t_k \times \\ \times \left\{ \prod_{l=1}^{k} \left[dt_l u_l(x, \sum_{s=1}^{l} t_s, t_1, ..., t_{l-1}) \exp(-\lambda \int_{\sum_{s=1}^{l-1} t_s}^{\sum_{s=1}^{l} t_s} u_l(x, \xi, t_1, ... t_{l-1}) d\xi) \right] \right\}, \tag{9}$$

$$\int u_m(x,t,t_1,...,t_{m-1})dx = \varepsilon,$$

where $u_i(x,t,t_1,...,t_{i-1})$ is the function that sets the view window at the i-th stage, provided that the intervals recorded between the previous (i-1) pulses were respectively $t_1,t_2,...,t_{i-1}$. It is not always possible to find extremals that deliver a minimum to the localization time (9) in the general case (when the probability density $f(x)$ is arbitrary). Therefore, we have developed a universal procedure to search for a source in the case when there is no priori information about the intensity of the source (so, we can assume that the source has a uniform distribution over the search interval). Due to the limited scope of this message, only the resulting table summarizing the parameters of the optimal multi-step search for a random uniformly distributed point source for systems with one receiver is given, and all necessary analytical and numerical calculations are omitted. More detailed information on this can be found in [9-10].

Table 1. Parameters of the optimal procedure for finding a random uniformly distributed pulse source on the interval $(0,L)$ depending on the required localization accuracy ε.

(ε/L) (required localization accuracy)	m_{opt} (optimal number of steps)	$W_m, m=\overline{1,m_{opt}}$ (system search window at each stage m_{opt} of the optimal search)	$<\tau>$ (average time of localization)
$\frac{1}{4} \le (\varepsilon/L) < 1$	1	$W_1 = \varepsilon$	$\frac{1}{\lambda}(\varepsilon/L)^{-1}$
$\left(\frac{2}{3}\right)^6 \le (\varepsilon/L) \le \frac{1}{4}$	2	$W_1 = (\varepsilon/L)^{\frac{1}{2}} \times L$ $W_2 = (\varepsilon/L) \times L = \varepsilon$	$\frac{2}{\lambda}(\varepsilon/L)^{-\frac{1}{2}}$
$\left(\frac{3}{4}\right)^{12} \le (\varepsilon/L) \le \left(\frac{2}{3}\right)^6$	3	$W_1 = (\varepsilon/L)^{\frac{1}{3}} \times L$ $W_2 = (\varepsilon/L)^{\frac{2}{3}} \times L$ $W_3 = (\varepsilon/L) \times L = \varepsilon$	$\frac{3}{\lambda}(\varepsilon/L)^{-\frac{1}{3}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\left(\frac{m}{m+1}\right)^{m(m+1)} \le (\varepsilon/L) \le \left(\frac{m-1}{m}\right)^{m(m-1)}$	m	$W_1 = (\varepsilon/L)^{\frac{1}{m}} \times L$ $W_2 = (\varepsilon/L)^{\frac{2}{m}} \times L$ $\vdots$ $W_i = (\varepsilon/L)^{\frac{i}{m}} \times L$ $\vdots$ $W_m = (\varepsilon/L)^{\frac{m}{m}} \times L = \varepsilon$	$\frac{m}{\lambda}(\varepsilon/L)^{-\frac{1}{m}}$

Taking into account that

$$\lim_{m\to\infty}\left(\frac{m}{m+1}\right)^m = e^{-1}$$

when $(\varepsilon/L) \to 0$, we get the following asymptotic relations for systems with one receiver:

$$m_{opt} \approx \ln(L/\varepsilon); \quad W_i \approx e^{-i} \times L, i = \overline{1,m_{opt}}; \quad <\tau>_{opt} \approx \frac{e\ln(L/\varepsilon)}{\lambda}.$$

The results presented above are the set of speed-optimal algorithms to localize random pulsed-point sources using system with one receiver. The studies were carried out both for single-step search procedures (in the case of an arbitrary probability density of the random source distribution on the search interval) and for multi-stage localization algorithms (for those cases when a random point-impulse object has a uniform distribution on the search interval).

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