Team Semantics for Spatial Reasoning: Locality and Separability

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Abstract. Team Semantics is a generalization of Tarski's semantics for First Order Logic in which formulas are satisfied or not satisfied by sets of assignments. Despite being reducible to Tarskian semantics over First Order Logic, Team Semantics permits to extend it in novel ways, like for instance by means of new types of atoms that express dependencies between different assignments.

In this work I will discuss the applicability of Team Semantics to spatial reasoning. I will argue that Team Semantics is a highly appropriate framework for reasoning about notions such as locality, in which the value of some variable at some point is affected only by the values of other variables in a certain neighbourhood of that point, and separability of spaces into regions with different properties.

Keywords. Team Semantics, Dependence Logic, Spatial Logic, Locality

1. Introduction

Team Semantics [30] generalizes Tarski's usual semantics for First Order Logic by letting formulas be satisfied or not satisfied by sets of assignments (called *Teams* for historical reasons), rather than by single assignments. This semantics was originally introduced in order to find a compositional semantics that is equivalent to the game-theoretical semantics for Independence-Friendly Logic [28,29,41], an extension of First Order Logic – roughly equivalent to Branching Quantifier Logic [27] – which allows for more general patterns of dependence and independence between quantifiers.

With Väänänen's development of *Dependence Logic* [44], however, it became clear that Team Semantics is a powerful and valuable generalization of Tarskian semantics in its own right, independently from its original application to Independence-Friendly Logic. Within the framework of Team Semantics it is possible to extend the language of First Order Logic by adding operators or atoms that express *dependencies* between multiple assignments, which is of course impossible in Tarskian Semantics. In particular, in terms of Team Semantics Independence-Friendly Logic – the original motivation for its development – is in very close correspondence with the logic obtained by adding to First Order Logic *functional dependence atoms*, whose semantics correspond precisely to database-theoretic *functional dependencies* [1]. It was soon recognized that other database-theoretic notions can be similarly added to the First Order Logic via Team

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Semantics [23,11,19], and the study and classification of the logics obtained in this way has blossomed into one of the central topics of research in the area.

The same type of "lifting" operation that leads from Tarskian First Order semantics to Team Semantics can be also applied to other forms of compositional semantics, such as those of Propositional Logic [47,48], Modal Logic [45,9,7,33], Computation Tree Logic (CTL) [36] and more recently Linear Temporal Logic (LTL) [37]. This later work, in particular, showed that by generalizing the semantics of LTL to sets of traces it is possible to obtain an effective framework for the specification of *hyperproperties* (i.e. system properties – such as "the system terminates within a bounded amount of time" – that cannot be verified by considering each possible trace in isolation but only by considering the set of all possible traces as a whole). This framework is incomparable with HyperLTL [4], the most common extension of LTL for the specification of hyperproperties; it can express properties of practical importance, such as uniform termination, which cannot be expressed in it; and it has better computational properties (in particular, the satisfiability problem for HyperLTL is undecidable [13] whereas the one for Team LTL is in PSPACE).

In contrast to this interest in the use of variants of Team Semantics for the specification and verification of temporal dependencies, to my knowledge there has been surprisingly little research on the use of Team Semantics for the specification of *spatial* dependencies. Yet, those dependencies certainly do exist and are of considerable practical interest: to mention one possible example, the results of geological surveys within a certain radius from some site of potential interest in Central Asia may be useful to predict the existence or non-existence of oil in it, but the results of surveys in New Zealand or in Argentina likely are of no immediate relevance for that.

This work constitutes an exploration of the possibilities of Team Semantics as a framework for the specification and verification of spatial dependencies. Rather than starting from a spatial logic and "teamifying" it in the usual way (i.e. by lifting the semantics to *sets* of the relevant meaning-carrying entities), here we will begin from the usual – and well-studied – Team Semantics of First Order Logic and add spatial information to it. In this way, we will obtain a formalism that is capable of expressing sophisticated spatial dependencies but is still very close to the usual first order Team Semantics.[2] This has clear advantages, as this semantics has been the object of intense research in the last few years, in particular insofar as the study and classification of computationally treatable fragments (see e.g. [32,8,16,17,26,18,40]) and proof systems (see e.g. [35,25,39]) are concerned; and it is the hope of the author that the present work will showcase the possibilities of Team Semantics in this context.

2. Preliminaries: Team Semantics

In this section, we will briefly recall the definition of (first order) Team Semantics, as well as some basic results regarding its properties. As we will see, for First Order Logic proper Team Semantics is reducible, in a very strong sense, to the usual Tarskian semantics; however, the greater richness of the meaning-carrying entities (which will be *sets* of assignments, called *Teams* for historical reasons, rather than single assignments), as well as the higher order quantification implicit in the rules **TS-**$\vee$ and **TS-**$\exists$ for disjunction

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[2]In fact, it will not be difficult to see that it will be reducible to *two-sorted* first order Team Semantics.

and existential quantification, make it possible to make use of Team Semantics to extend First Order Logic in novel and interesting ways.

2.1. Definitions

Definition 1 (Team). Let $\mathfrak{M}$ be a first order model with domain M, and let $V \subseteq \mathtt{Var}$ be a finite set of variables. A *Team* over $\mathfrak{M}$ with domain V is a set of assignments $s : V \to M$.

Definition 2 (Splitting). Let X, Y and Z be teams over a model $\mathfrak{M}$ with the same domain $\mathtt{Dom}(X) = \mathtt{Dom}(Y) = \mathtt{Dom}(Z) = V$. Then we say that X *splits* into Y and Z if $X = Y \cup Z$.

In general, in the above definition we do not require Y and Z to be disjoint.[3] We will also need to be able to talk about *updates* of a team along one variable:

Definition 3 (Restriction). Let X be a team over a model $\mathfrak{M}$ with domain $\mathtt{Dom}(X)$, and let $V \subseteq \mathtt{Dom}(X)$ be a subset of its variables. Then its *restriction* $X_{|V}$ is the team $\{s_{|V} : s \in X\}$ of the restrictions of its assignments to V, where for each $s \in X$ we have that $s_{|V}$ is the assignment with domain V such that $s_{|V}(v) = s(v)$ for all $s \in V$.

Definition 4 (Team Update). Let X be a team over a model $\mathfrak{M}$ with domain $\mathtt{Dom}(X)$, let $v \in \mathtt{Var}$ be any variable, and let Y be a team over the same $\mathfrak{M}$ with domain $\mathtt{Dom}(X) \cup \{v\}$. Then we say that Y is an *update* (or *supplementation*) of X along v if X and Y agree on all variables aside from v, that is, if and only if $X_{|\mathtt{Dom}(X) \setminus \{v\}} = Y_{|\mathtt{Dom}(X) \setminus \{v\}}$.

The following result is trivial:

Proposition 1. *Let X be a team over a model $\mathfrak{M}$ with domain $Dom(X)$, let $v \in Var$ be any variable and let Y be a team over the same $\mathfrak{M}$ with domain $Dom(X) \cup \{v\}$. Then Y is an update of X along v if and only if there exists a function F, sending each assignment $s \in X$ into a nonempty set of elements $F(s) \subseteq M$, $F(s) \neq \emptyset$, such that*

$$Y = X[F/v] = \{s[m/v] : s \in X, m \in F(s)\}$$

where $s[m/v]$ agrees with s on $Dom(s) \setminus \{v\}$ and sends v to m.

A particular type of team update that will be useful to consider is the *most general update* of a team along a variable, also called the *duplication* of a team along a variable:

Definition 5 (Most General Update). Let X be a team over a model $\mathfrak{M}$ with domain $\mathtt{Dom}(X)$, let v be a variable (not necessarily in $\mathtt{Dom}(X)$) and let Y be a team over the same $\mathfrak{M}$ with domain $\mathtt{Dom}(X) \cup \{v\}$. Then Y is the *most general update* of X along v if

1. Y is an update of X along v;
2. For all updates Y' of X along v we have that $Y' \subseteq Y$.

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[3]This is related to the distinction between *lax* and *strict* semantics [14], or - in game theoretical terms - to the distinction between nondeterministic and deterministic properties. Much of the recent work in the area has focused on the lax (not-necessarily-disjoint) case, essentially because in the other the value of a formula may be affected by the values of variables that do not appear in it.

$$\begin{array}{c|cc} & x & y \\ \hline s_0 & a & b \\ s_1 & a & c \\ s_2 & b & c \end{array} \quad = \quad \begin{array}{c|cc} & x & y \\ \hline s_0 & a & b \\ s_1 & a & c \end{array} \quad \cup \quad \begin{array}{c|cc} & x & y \\ \hline s_1 & a & c \\ s_2 & b & c \end{array}$$

$$\begin{array}{c|c} & x \\ \hline s_0' & a \\ s_1' & b \end{array} \left[\begin{array}{c} s_0' \mapsto \{b,c\} \\ s_1' \mapsto \{c\} \end{array} \Big/ y \right] = \begin{array}{c|cc} & x & y \\ \hline s_0 & a & b \\ s_1 & a & c \\ s_2 & b & c \end{array} \qquad \begin{array}{c|c} & x \\ \hline s_0' & a \\ s_1' & b \end{array} \left[M \Big/ y \right] = \begin{array}{c|cc} & x & y \\ \hline s_0'' & a & a \\ s_1'' & a & b \\ s_2'' & a & c \\ s_3'' & b & a \\ s_4'' & b & b \\ s_5'' & b & c \end{array}$$

Figure 1. Splitting, Update and Most General Update over a model $\mathfrak{M}$ with domain $M = \{a,b,c\}$.

Proposition 2. *Let X be a team over a model $\mathfrak{M}$ with domain $\mathrm{Dom}(X)$ and let v a variable. Then the most general update of X along v exists, is unique, and is given by*

$$X[M/v] = \{s[m/v] : s \in X, m \in M\}.$$

We now have all ingredients to define the Team Semantics over First Order Logic. We will assume, for simplicity, that all expressions are in Negation Normal Form:

Definition 6 (Team Semantics for First Order Logic). Let $\mathfrak{M}$ be a first order model with domain M, let X be a team over it, and let $\phi(\vec{x})$ be a first order formula in Negation Normal Form over the signature of $\mathfrak{M}$ and with free variables in $\mathrm{Dom}(X)$. Then we say that X *satisfies* ϕ in $\mathfrak{M}$, and we write $\mathfrak{M} \models_X \phi$, if and only if this follows from the rules

TS-lit: For all first order literals α, $\mathfrak{M} \models_X \alpha$ if and only if, for all assignments $s \in X$, $\mathfrak{M} \models_s \alpha$ in the sense of the usual Tarskian Semantics;

TS-$\vee$: For all NNF formulas ϕ and ψ, $\mathfrak{M} \models_X \phi \vee \psi$ if and only if $X = Y \cup Z$ for two subteams Y and Z such that $\mathfrak{M} \models_Y \phi$ and $\mathfrak{M} \models_Z \psi$;

TS-$\wedge$: For all NNF formulas ϕ and ψ, $\mathfrak{M} \models_X \phi \wedge \psi$ if and only if $\mathfrak{M} \models_X \phi$ and $\mathfrak{M} \models_X \psi$;

TS-$\exists$: For all NNF formulas ϕ and all variables $v \in \mathtt{Var}$, $\mathfrak{M} \models_X \exists v \phi$ if and only if there exists some update $X[F/v]$ of X along v such that $\mathfrak{M} \models_{X[F/v]} \phi$;

TS-$\forall$: For all NNF formulas ϕ and all variables $v \in \mathtt{Var}$, $\mathfrak{M} \models_X \forall v \phi$ if and only if $\mathfrak{M} \models_{X[M/v]} \phi$ for the most general update $X[M/v]$ of X along v.

If ϕ has no free variables, we say that ϕ is *true* in $\mathfrak{M}$ according to Team Semantics if and only if $\mathfrak{M} \models_{\{\varepsilon\}} \phi$ where $\varepsilon : \emptyset \to M$ is the empty assignment.

2.2. Some Results

As mentioned before, over First Order Logic proper Team Semantics is reducible to Tarskian Semantics. More precisely, the following result holds:

Proposition 3. *Let $\mathfrak{M}$ be a first order model, let X be a team over it, and let ϕ be a first order formula in Negation Normal Form over the signature of $\mathfrak{M}$ and with free variables*

in Dom(X). Then $\mathfrak{M} \models_X \phi$ if and only if, for all $s \in X$, $\mathfrak{M} \models_s \phi$ according to the usual rules of Tarski's semantics. In particular, if ϕ is a sentence, ϕ is true in $\mathfrak{M}$ according to Team Semantics if and only if it is true in $\mathfrak{M}$ according to Tarskian Semantics.

Does this result imply that Team Semantics is an unnecessarily complicated, but fundamentally equivalent, variant of Tarskian Semantics? Well, no: as already mentioned, the richer structure of teams over assignments allows one to extend First Order Logic with Team Semantics in novel ways that have no obvious analogue in Tarskian semantics. Perhaps the easiest – and certainly the most studied – way to do so is to add to the language of First Order Logic new *dependency atoms*, expressing dependencies between the values that variables take in different assignments, like the following ones:

Definition 7 (Functional Dependence, Inclusion, Exclusion, and Independence Atoms). For all models $\mathfrak{M}$, all teams X and all tuples of variables[4] $\vec{x}$ and $\vec{y}$,

TS-fdep: $\mathfrak{M} \models_X =(\vec{x}; \vec{y})$ if any two $s, s' \in X$ which agree on $\vec{x}$ also agree on $\vec{y}$;

TS-inc: $\mathfrak{M} \models_X \vec{x} \subseteq \vec{y}$ if the tuples $\vec{x}$ and $\vec{y}$ have the same length and, furthermore, every possible value of $\vec{x}$ in X is also a possible value for $\vec{y}$ in X;

TS-exc: $\mathfrak{M} \models_X \vec{x} | \vec{y}$ if the tuples $\vec{x}$ and $\vec{y}$ have the same length and, furthermore, no possible value of $\vec{x}$ in X is also a possible value for $\vec{y}$ in X;

TS-ind: $\mathfrak{M} \models_X \vec{x} \perp \vec{y}$ if for any $s, s' \in X$ there exists some $s'' \in X$ with $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$ (that is, all possible values for $\vec{x}$ and $\vec{y}$ in X may occur together in it).

The logics obtained by adding these atoms to the language of First Order Logic are called (functional) *Dependence Logic*, *Inclusion Logic*, *Exclusion Logic* and (non-conditional) *Independence Logic*[5] respectively, and they are formalisms deserving of investigation in their own right. Here we mention briefly that every sentence of functional dependence, exclusion, or independence logic is equivalent to some sentence of existential second order logic Σ_1^1, and that conversely every Σ_1^1 sentence is equivalent to some sentence of any of these logics [44,24,14]; but that, on the other hand, Inclusion Logic corresponds to the positive fragment of Greatest Fixed Point Logic [20], and hence captures PTIME over finite ordered models by [31,46]. On the level of formulas, however, independence logic differs from functional and exclusion logic (which are however equivalent): very briefly, it was proved that every Σ_1^1-definable property of teams[6] that is true of the empty team corresponds to the satisfaction conditions to some Independence Logic formula, and vice versa [14], whereas for functional and exclusion logic the above is true only if we further require that these relations are additionally *downwards closed* (i.e., whenever they hold of a team they also hold of all its subteams [34]).

Team Semantics allows also to extend First Order Logic in new ways via extra connectives, such as the contradictory negation $\sim$ (such that $\mathfrak{M} \models_X \sim \phi$ if and only if $\mathfrak{M} \not\models_X \phi$ – note, this is *not* equivalent to the usual "dual negation") or various types of *generalized quantifier* [10,12,38,2]; and furthermore, "weighted" or probabilistic variants of Team Semantics have been also considered [43,21,5,6].

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[4]Or, more in general, terms; but for simplicity we will only consider dependence atoms applied to variables.

[5]There are also *conditional independence atoms* $\vec{x} \perp_{\vec{z}} \vec{y}$, which state that the possible values of $\vec{x}$ and $\vec{y}$ in X are informationally independent *for any fixed value of* $\vec{z}$; but as pointed out in [22], these atoms can be defined in terms of non-conditional independence atoms.

[6]Or, to be more precise, every Σ_1^1-definable property of the relations corresponding to teams.

2.3. The Doxastic Interpretation of Team Semantics

As briefly shown above, Team Semantics is a natural generalization of Tarskian Semantics which is of significant theoretical interest, as it makes it possible to extend First Order Logic in novel ways (the classification of which is still largely incomplete). However, some uncertainty would be understandable at this point regarding the *meaning* of Team Semantics. What *are* teams, exactly? The rules of Definition 6 may well arise naturally from the analysis of non-deterministic strategies in the Game-Theoretic Semantics of First Order Logic, and they may well be appropriate for providing a compositional semantics for logics such as Branching Quantifier Logic or Independence-Friendly Logic; but do they have an actual and understandable *meaning*, or are they mere technical tricks of a semantics that – regardless of its nice formal features – does not admit much of an interpretation? This is a question that is of central importance for this work, since we intend to discuss the applicability of Team Semantics to spatial reasoning.

As it was discussed at length in [15], and as we will now briefly see, Team Semantics admits a natural interpretation in terms of *doxastic states*. If an assignment represents a potential *state of things*, a team can easily represent the *belief set* of an agent – that is, the set of all states of things that an agent believes possible. Then Proposition 3, for instance, can be interpreted as showing that, for all first order ϕ, $\mathfrak{M} \models_X \phi$ if and only if an agent that believes that the true state of things lies in X can be sure that ϕ will be true of this true state; a functional dependence atom $=(\vec{x}; \vec{y})$ states that the agent could infer the true value of $\vec{y}$ from the true value of $\vec{x}$; inclusion and exclusion atoms $\vec{x} \subseteq \vec{y}$ and $\vec{x}|\vec{y}$ assert respectively that the agent considers every possible value of $\vec{x}$ a possible/impossible value for $\vec{y}$; and an independence atom $\vec{x} \perp \vec{y}$ asserts that learning the true value of $\vec{x}$ would provide the agent with no new information whatsoever regarding the value of $\vec{y}$.

Splitting a team into two as per rule **TS-∨** can be seen as a form of *case-based reasoning*: if the agent believes that the true state is in $X = Y \cup Z$, they can conclude that the true state is in Y or in Z (or possibly in both). Note that this would not be the case for the "Boolean disjunction"

TS-⊔: $\mathfrak{M} \models_X \phi \sqcup \psi$ iff $\mathfrak{M} \models_X \phi$ or $\mathfrak{M} \models_X \psi$

which would instead assert that, knowing that the true state is in X, the agent can conclude that ϕ is true or that ψ is true. In other words, the difference between $\phi \vee \psi$ and $\phi \sqcup \psi$ is the same as that between $\mathbf{K}(\phi \vee \psi)$ ("the agent knows that ϕ or ψ is true") and $\mathbf{K}(\phi) \vee \mathbf{K}(\psi)$ ("the agent knows that ϕ is true or the agent knows that ψ is true"). Thus, for instance, $x = y \vee x \neq y$ is satisfied by any team whose domain contains the variables x and y, but there are teams (for instance, $X = \{(x : 0, y : 0), (x : 0, y : 1)\}$) which do not satisfy $x = y \sqcup x \neq y$. By combining splitting and dependency atoms we can obtain interesting effects: for example, $=(x; y) \vee =(x; y)$ is not equivalent to $=(x; y)$, and it asserts that any value of x corresponds to at most *two* values of y – or, to put the matter into more explicitly doxastic terms, that the agent believes that two scenarios are possible, and that in either scenario they could learn y given x.

What about quantifiers and variable updates? By definition, $\mathfrak{M} \models_X \exists v \phi$ if and only if there exists some possible belief state Y, which disagrees from X at most with respect to the variable v, in which ϕ holds. In other words, the agent could learn something about the possible values of the variable v – but about that variable alone – after which they would agree that ϕ holds. The most general update $X[M/v]$, on the other hand, represents

an "agnostic update" after which the agent believes that the variable v could take any value at all regardless of the values of the other variables; and thus, $\mathfrak{M} \models_X \forall v \phi$ if this agent – after disregarding anything about the value of v – believes that ϕ.

Conjunctions and first order literals pose no difficulties; and, thus, we obtained a doxastic interpretation for all expressions of our language. This interpretation can be extended much further, and we refer to [15] for more details. We now have enough background to begin exploring the application of Team Semantics to spatial reasoning.

3. Spatial Team Semantics

As we just discussed, the assignments in a team can be understood as possible states of things, or equivalently as possible (and not necessarily consistent) *observations*. It is thus entirely natural to think of these observations as *located* into space. This can be done easily by adding a special *location variable* $\ell \notin \mathtt{Var}$ to all assignments:

Definition 8 (Located Assignments). Let $\mathfrak{M}$ be a first order model with domain M, let $V \subseteq \mathtt{Var}$ be a finite set of variables and let $\mathbb{H}$ be an arbitrary real Hilbert Space[7]. A $\mathbb{H}$-*located assignment* over $\mathfrak{M}$ with domain V is a function $s : V \cup \{\ell\} \to M \cup \mathbb{H}$, where $s(\ell) \in \mathbb{H}$ and $s(v) \in M$ for all $v \in V$.

Definition 9 (Located Teams, Location Range). Let $\mathfrak{M}$ be a first order model with domain M, let $V \subseteq \mathtt{Var}$ and let $\mathbb{H}$ be a real Hilbert Space. Then a $\mathbb{H}$-located team over $\mathfrak{M}$ with domain V is a set X of $\mathbb{H}$-located assignments over $\mathfrak{M}$ with domain V. Its *location range* $X(\ell)$ is the set $\{s(\ell) : s \in X\} \subseteq \mathbb{H}$ of the positions of all its assignments.

The doxastic interpretation of Section 2.3 can be extended to located teams in the obvious way: in brief, a located team still represent a set of possible observations, but now each observation is also *situated* in a particular point of the space $\mathbb{H}$ (see e.g. Figure 2). The semantics of Definition 6, as well as the dependency atoms of Definition 7 and all the other operators and connectives studied in the context of Team Semantics, can be applied to located teams without any change whatsoever; however, the fact that every assignment is now made to correspond to a particular point of $\mathbb{H}$ allows us to consider new kinds of operators and dependencies over located teams. For example, one may note that we did not require that distinct assignments have different positions (this can be used to represent e.g. ambiguous data about a location). However, a simple dependence atom can be added to state that the values of certain variables are *spatially determined*:

TS-sdet: $\mathfrak{M} \models_X \, = (\ell; \vec{x})$ iff, for any two $s, s' \in X$, if $s(\ell) = s'(\ell)$ then $s(\vec{x}) = s'(\vec{x})$.

This spatial determination operator is obviously just a minor variant of the functional dependency atom of Definition 7; and we may likewise add an "independence atom" $\ell \perp x$ to state that the range of the possible values for x is the same for any possible location ℓ.

Can we do anything else with locations? Well, to begin with, let us consider the disjunction operator. As before, the interpretation is clear: $\mathfrak{M} \models_X \phi \lor \psi$ if we can split the set of located observations X into two (possibly overlapping) subsets Y and Z that satisfy

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[7]For most intended applications, we can assume that $\mathbb{H} = \mathbb{R}^2$ or $\mathbb{H} = \mathbb{R}^3$, but the definitions of this work apply equally well to the case of a general Hilbert space over $\mathbb{R}$.

$$X = \begin{array}{c|cccc} & \ell & v_1 & v_2 & v_3 \\ \hline s_0 & (0.2, 0.4) & 1 & 0 & 1 \\ s_1 & (0.2, 0.4) & 1 & 1 & 0 \\ s_2 & (0.6, 0.2) & 2 & 1 & 0 \\ s_3 & (0.5, 0.9) & 3 & 0 & 1 \\ s_4 & (0.5, 0.9) & 3 & 1 & 0 \\ s_5 & (0.7, 0.6) & 3 & 1 & 0 \end{array}$$
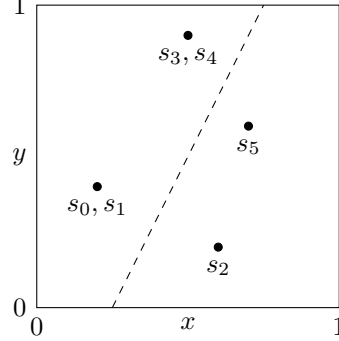

Figure 2. A $\mathbb{R}^2$-located team with domain $\{v_1, v_2, v_3\}$ over a model $\mathfrak{M}$ with four elements 0, 1, 2 and 3. Note that $\mathfrak{M} \models_X =(\ell; v_1)$, because any two assignments that are in the same position agree about v_1; $M \not\models_X =(\ell; v_2)$, because for example s_0 and s_1 have the same position but disagree about v_2; $\mathfrak{M} \models_X =(\ell; v_2) \otimes_{\text{un}} v_3 = 0$, because we can split X into $Y = \{s_0, s_2, s_3, s_5\}$ and $Z = \{s_1, s_2, s_4, s_5\}$ and we have that $X(\ell) = Y(\ell) = Z(\ell)$ (no location is lost in any subteam), $\mathfrak{M} \models_Y =(\ell; v_2)$ (in Y, the value of v_2 is a function of the position) and $\mathfrak{M} \models_Z v_3 = 0$ (v_3 is 0 for all assignments of Z); and $\mathfrak{M} \models_X \ell \perp v_2 \otimes_{\text{lin}} v_1 \neq v_2$, because the linear operator $\mathfrak{h} : \mathbb{R}^2 \to \mathbb{R}$ given by $\mathfrak{h}(x, y) = y + 0.5 - 2x$ separates X into two other regions $Y' = \{s \in X : \mathfrak{h}(s(\ell)) \geq 0\} = \{s_0, s_1, s_3, s_4\}$ and $Z' = \{s \in X : \mathfrak{h}(s(\ell)) < 0\} = \{s_2, s_5\}$ such that $\mathfrak{M} \models_{Y'} \ell \perp v_2$ (in Y', the set of all the possible values of v_2 is the same – that is, $\{0,1\}$ – in all positions) and $\mathfrak{M} \models_{Z'} v_1 \neq v_2$.

the conditions described by ϕ and ψ respectively. This remains a perfectly legitimate connective, with obvious uses – for example, an expression of the form $=(\ell; x) \vee =(\ell; x)$ will say that every location corresponds to at most two values for the variable x. However, the fact that each assignment has a location permits us to think about how this location affects which "sides" of a split an assignment would be put into. Many possible choices can be considered here, and we will just mention two simple ones that are of obvious interest: on one hand, we might want to require that the split is *independent* on the location, so that both Y and Z contain assignments in all locations in which X contains assignments, and on the other we might want instead to require that Y and Z are *linearly separable* on the basis of location. This justifies the two following new connectives (see Figure 2 and its caption for examples):

Definition 10 (Location-Uniform and Linear Splits). For any model $\mathfrak{M}$, real Hilbert space $\mathbb{H}$, $\mathbb{H}$-located team X over $\mathfrak{M}$, and formulas ϕ, ψ over the signature of $\mathfrak{M}$ and with free variables in $\text{Dom}(X)$,

TS-$\otimes_{\text{un}}$: $\mathfrak{M} \models_X \phi \otimes_{\text{un}} \psi$ if and only if there exist two $\mathbb{H}$-located teams $Y, Z \subseteq X$ such that $Y(\ell) = Z(\ell) = X(\ell)$, $Y \cup Z = X$, $\mathfrak{M} \models_Y \phi$ and $\mathfrak{M} \models_Z \psi$;[8]

TS-$\otimes_{\text{lin}}$: $\mathfrak{M} \models_X \phi \otimes_{\text{lin}} \psi$ if and only if there exists a linear operator $\mathfrak{h} : \mathbb{H} \to \mathbb{R}$ such that, for $Y = \{s \in X : \mathfrak{h}(s(\ell)) \geq 0\}$ and $Z = \{s \in X : \mathfrak{h}(s(\ell)) < 0\}$, it holds that $\mathfrak{M} \models_Y \phi$ and $\mathfrak{M} \models_Z \psi$.

What else? Quite a bit. For example, we might want to say that the value of a variable in a location is determined by the values of certain other variables inside some *range*:

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[8]As an aside, it is not difficult to see that this is essentially a spatial version of the *value-preserving disjunctions* of [42], and as such it can be defined in terms of independence atoms.
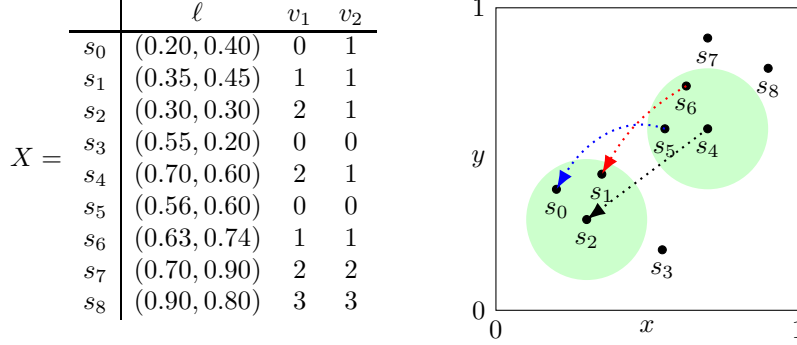
$$X = \begin{array}{c|ccc} & \ell & v_1 & v_2 \\ \hline s_0 & (0.20, 0.40) & 0 & 1 \\ s_1 & (0.35, 0.45) & 1 & 1 \\ s_2 & (0.30, 0.30) & 2 & 1 \\ s_3 & (0.55, 0.20) & 0 & 0 \\ s_4 & (0.70, 0.60) & 2 & 1 \\ s_5 & (0.56, 0.60) & 0 & 0 \\ s_6 & (0.63, 0.74) & 1 & 1 \\ s_7 & (0.70, 0.90) & 2 & 2 \\ s_8 & (0.90, 0.80) & 3 & 3 \end{array}$$

Figure 3. Local similarity. The locations of the table on the left are approximated to two decimals. $s_2 \sim_{(X, v_1, 0.2)} s_4$: indeed, there exists an orthogonal transformation (a rotation by $-\pi/4$ that corresponds to sending s_4 to s_2, s_5 to s_0 and s_6 to s_1, and each of these pairs agree on v_1. The values and locations of s_3, s_7 and s_8 are irrelevant, because their distances from s_2 and s_4 is more than 0.2. Note, furthermore, that it is *not* true that $s_2 \sim_{(X, v_1 v_2, 0.2)} s_4$, because s_0 and s_5 disagree about the value of v_2.

Definition 11 (Local Restriction). Let $\mathbb{H}$ be a real Hilbert space and let X be a $\mathbb{H}$-located team over some model $\mathfrak{M}$, and let $s \in X$ be a located assignment of X. Furthermore, let $\delta \in (0, \infty)$ be a positive real number, and let $V \subseteq \text{Dom}(X)$ be a set of variables in the domain of X. Then the (V, δ)-local restriction of X around s is the located team

$$X_{|V, \delta, s} = \{ s'_{|V}[s'(\ell) - s(\ell)/\ell] : s' \in X, d(s'(\ell), s(\ell)) \leq \delta \}$$

where, as usual, $s'_{|V}$ is the restriction of s to the variables of V (as well as ℓ); $d(p, q) = \|p - q\| = \sqrt{\langle p - q, p - q \rangle}$ is the distance associated to $\mathbb{H}$ via its inner product $\langle \cdot, \cdot \rangle$; and we subtracted the location of s from the coordinates of all points so that s lies at the origin of $X_{|V, \delta, s}$.[9]

Definition 12 (Local Similarity). Let $\mathbb{H}$ be a real Hilbert space, let X be a $\mathbb{H}$-located team over some $\mathfrak{M}$ and let $s, s' \in X$. Furthermore, let $\delta \in (0, \infty)$ and let $V \subseteq \text{Dom}(X)$. Then we say that s and s' are (V, δ)-locally similar in X, and we write $s \sim_{(X, V, \delta)} s'$, if and only if there exist an orthogonal operator[10] $\mathfrak{o} : X_{|V, \delta, s}(\ell) \to X_{|V, \delta, s'}(\ell)$ and a function h from $X_{|V, \delta, s}$ onto $X_{|V, \delta, s'}$ such that $h(s) = s'$ and such that, for all $s'' \in X_{|V, \delta, s}$.

1. $s''(v) = h(s'')(v)$ for all $v \in V$ (h preserves the values of the variables in V);
2. $\mathfrak{o}(s''(\ell)) = h(s'')(\ell)$ (h transforms the locations according to $\mathfrak{o}$).

If $\vec{x}$ is a tuple of (possibly repeating) variables, we will write $s \sim_{(X, \vec{x}, \delta)} s'$ as a shorthand for $s \sim_{(X, \text{Var}(\vec{x}), \delta)} s'$, where $\text{Var}(\vec{x}) = \{ v \in \text{Var} : v \text{ occurs in } \vec{x} \}$.

See Figure 3 for a simple example of local similarity in a $\mathbb{R}^2$-located team. Now we can define the following *local dependency atom*, having *range* $\delta \in (0, \infty)$:

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[9]This is so that in Definition 12 we will not need to worry about translations.

[10]Very briefly, this means that $\mathfrak{o}$ preserves inner products (and, consequently, also norms and distances). An example of such an operator in $\mathbb{R}^n$ would be a rotation or a reflection; some *non-examples* would be a scaling operation, a translation, or any non-continuous transformation. If we wanted to exclude reflections, it would suffice to require that $\mathfrak{o}$ belongs in the group $SO(\mathbb{H})$ of the *orientation-preserving* orthogonal operators for $\mathbb{H}$.

$$X = \begin{array}{c|cccc} & \ell & m & t & r \\ \hline s_0 & 0 & 0.5 & 2.5 & L \\ s_1 & 1 & 0.8 & 4.5 & M \\ s_2 & 2 & 1.2 & 5.5 & M \\ s_3 & 3 & 2 & 6.1 & H \\ s_4 & 4 & 1 & 6.5 & H \\ s_5 & 5 & 1.1 & 6.1 & H \\ s_6 & 6 & 1.2 & 4.6 & M \\ s_7 & 7 & 0.8 & 3.6 & L \\ s_8 & 8 & 0.5 & 2.5 & L \end{array}$$
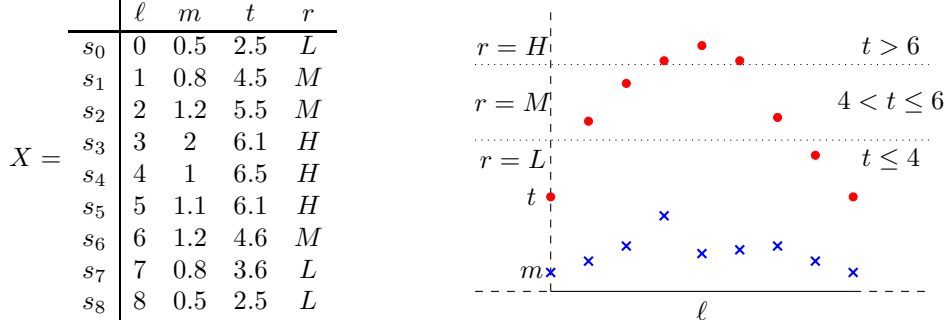

Figure 4. Local dependencies. The variable m (blue crosses) represents the result of various measurements (e.g. seismographic ones), in various points ℓ of the space (for simplicity we work in one dimension: $\mathbb{H} = \mathbb{R}$). The variable t (red dots) represents the sum of the measurements m at distance ≤ 2 from the current position: $s(t) = \sum \{s'(m) : |s(\ell) - s'(\ell)| \leq 2\}$. The variable r represents whether the risk a possible event (e.g. building damage) is estimated to be high (H, if $t > 6$), medium (M, if $4 < t \leq 6$), or low (L, if $t \leq 4$). The local dependency $=(o : 2.0; r)$ holds by construction, since $s(r)$ does not depend on the values of $s'(m)$ for $|s(\ell) - s'(\ell)| > 2$. However, $=(o : 1.0; r)$ fails: for instance, $s_1 \sim_{(X,m,1)} s_7$ (pick $h(s_0) = s_8$, $h(s_1) = s_7$ and $h(s_2) = s_6$ – note, this corresponds to a reflection $\mathfrak{o} : X_{|o,1.0,s_1}(\ell) \to X_{|o,1.0,s_7}(\ell)$), but s_1 and s_7 do not agree on r.

TS-locdep: $\mathfrak{M} \models_X =(\vec{x} : \delta; \vec{y})$ if, for any two $s, s' \in X$, if $s \sim_{(X, \vec{x}, \delta)} s'$ then $s(\vec{y}) = s'(\vec{y})$.

According to the above definition, $\mathfrak{M} \models_X =(\vec{x} : \delta; \vec{y})$ if and only if any two local assignments whose δ-neighbourhoods are the same (up to orthogonal transformations, e.g. rotations or reflections) insofar as the values of the variables in $\vec{x}$ are concerned must also agree about the value of $\vec{y}$. Figure 4 shows a toy example of such a dependency in the case of risk assessment via seismographic data analysis.

All of this could be generalized in several ways: for instance, it would not be difficult to let tuples of variables $\vec{x}_1 \ldots \vec{x}_n$ influence the value of $\vec{y}$ within different radii $\delta_1 \ldots \delta_n$, or we could weaken the similarity condition by allowing a certain degree of "error" in the mappings of the locations. However, the above should suffice to exemplify the possibilities of team semantics insofar as spatial reasoning is concerned. We leave a more detailed examination of the possibilities – and, most importantly, of the limitations and the computational costs of various choices of connectives, atoms and operators – to future work.

4. Conclusions

In this work, we explored some of the possible ways in which the formalism of Team Semantics could be used to model spatial reasoning. Of course, this is a very preliminary sort of work, intended essentially to showcase the possibilities of such an approach and – hopefully – to convince the reader that this is a research direction that is worthy of being investigated further. The obvious next steps would be to compare the expressive properties of this type of approach to those of other formalisms for spatial reasoning and investigate the expressive properties of particular selections of connectives of operators.

It could be also interesting to add a temporal dimension to this framework, either by replacing assignments with *traces* as it was done in [37] or by adding an a special temporal parameter $\tau \notin \texttt{Var} \cup \{\ell\}$ to each assignment, much as we added the spatial parameter

ℓ. This second approach could perhaps be argued to be more in keeping with the original intuitions of Team Semantics and with its doxastic interpretation, as assignments are intended to represent single observations or possible states of things and the relationships between them should be expressed in terms of dependencies; and, once again, keeping as close as standard First Order Team Semantics as possible has the considerable advantage of letting us make use of the not insubstantial amount of work already done in the area.

Another idea worth investigating would be the combination of such an approach with weighted variants of Team Semantics such as the ones discussed in [5,6]. The resulting framework would be an extremely powerful one, integrating spatial, probabilistic, and possibly also temporal reasoning into a single package. Aside from applications in spatial reasoning, such a framework could also be used for the representation and modelling of statistical learning, for instance by using the linear split connective (and/or more sophisticated variants thereof) for representing concepts such as the learnability of certain properties under certain conditions. This could also have intriguing connections with the recent work on causal reasoning via Team Semantics by Barbero and Sandu [3].

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