MVA2000 IAPR Workshop on Machine Vision Applications, Nov. 28-30, 2000, The University of Tokyo, Japan

13—2 A Decomposition Approach to Geometric Fitting

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Abstract

Many geometric fitting problems encountered in computer vision are of non-linear nature and have a large number of parameters. Typically, iterative techniques such as the Levenberg-Marquardt algorithm are used to compute the optimum of highly complex optimization functions. Here we are faced with problems of good initial estimates and high risk of getting in local optimums. In this paper we propose a decomposition approach to alleviating these problems by reducing the number of optimization parameters. This way not only the optimization process itself is simplified, it tends also easier to investigate low-dimensional optimization spaces. We apply the decomposition approach to solving various fitting problems from both 2D and 3D domain. Particularly for cylinder fitting, we are able to reduce the number of parameters from five or six of earlier methods to only two. Experiments have demonstrated the advantages of our approach.

1 Introduction

Geometric fitting is of fundamental importance to many tasks in computer vision. Usually, a segmentation step is carried out to group image points into sets, each belonging to a different image feature. Then, geometric fitting follows to compute the optimal mathematical representation of an appropriate type for each set of image points. This way we are able to generate a compact representation of images which is the basis for various tasks such object recognition and reverse engineering.

Many fitting problems encountered in computer vision are of non-linear nature and have a large number of parameters. Typically, iterative techniques such as the Levenberg-Marquardt algorithm are used to compute the optimum of highly complex optimization functions. Here we are faced with two difficulties. First, any such iterative algorithm requires good initial estimate of the optimum which is then refined. It is not always easy to find reasonable initial estimates. Secondly, a large number of parameters results in a high-dimensional optimization space which may have a very complex topography and therefore brings a high risk of getting in local optimums. In this case we often have troubles to analyze the optimization space, for instance by means of visualization. Consequently, it may even be impossible to specify maximal estimation errors of the initial parameter values such that the iterative process is guaranteed to reach the global optimum. This uncertainty is characteristic to high-dimensional non-linear optimization tasks.

A key issue to alleviate the problems discussed above lies in the reduction of the optimization space dimensions. A smaller number of parameters tends to ease the initial estimation of parameters. In this case the optimization space may become simple enough to be analyzed to get an insight into its topography. This will certainly help the initial estimation as well. In this paper we propose a novel decomposition approach to reducing the number of optimization parameters. Based on a simple idea, this technique turns out to be applicable to various geometric fitting problems from both 2D and 3D domain.

2 Decomposition principle

We assume that n data points $p_i, i = 1, 2, \ldots, n$, are to be fitted by a representation function $f(\vec{a}) = 0$ specified by a k-dimensional parameter vector $\vec{a} \in \Re^k$. Let $d(p, f(\vec{a}))$ be some distance measure of a point p to the function $f(\vec{a}) = 0$. Then, geometric fitting is formulated as:

$$\vec{a}_{\text{opt}} = \arg \min_{\vec{a} \in \Re^k} \sum_{i=1}^{n} d(p_i, f(\vec{a})) \qquad (1)$$

That is, we look for an optimal parameter vector $\vec{a}_{\text{opt}}$ among all possibilities out of $\Re^k$ such that the sum of distances is minimized. The formulation (1) implies a simultaneous optimization of all k parameters.

Now we take a slightly different view of the geometric fitting problem. We decompose the k parameters into a subset $\vec{a}_1$ of k_1 parameters and another subset $\vec{a}_2$ of k_2 ($\equiv k - k_1$) parameters and obtain thus $\vec{a} = \vec{a}_1\vec{a}_2$ after some rearrangement of the parameter order in $\vec{a}$. Then, the optimization task (1) can be stated as:

$$\begin{aligned} \vec{a}_{\text{opt}} &= \arg \min_{\vec{a}_1 \in \Re^{k_1}} \Big[\min_{\vec{a}_2 \in \Re^{k_2}} \sum_{i=1}^{n} d(p_i, f(\vec{a}_1\vec{a}_2)) \Big] \qquad (2) \\ &= \arg \min_{\vec{a}_1 \in \Re^{k_1}} A(\vec{a}_1) \end{aligned}$$

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where the term in the brackets, $A(\vec{a}_1)$, simply means the minimum sum of distances for a fixed $\vec{a}_1$ and all possibilities of $\vec{a}_2$. There are $2^n - 2$ such decompositions in total. Sometimes we can find a decomposition $\vec{a} = \vec{a}_1\vec{a}_2$ such that the optimization subtask $A(\vec{a}_1)$ with k_2 parameters becomes a trivial one. This is the case when it can be analytically solved to obtain a function of $\vec{a}_1$ or a non-iterative procedure is available to compute the minimum for a particular $\vec{a}_1$. In each situation we have basically reduced the initial optimization problem of k parameters to a simpler one of k_1 ($< k$) parameters.

3 An illustrative example

We illustrate the decomposition principle introduced above by considering the task of finding the best fit line of n data points p_i in the 2D plane. The distance function $d()$ is chosen to be the squared orthogonal distance of p_i to the fit line. Based on the representation $x \cos\alpha + y \sin\alpha = \rho$, which is often used for the Hough transform, we obtain:

$$\vec{a}_{\text{opt}} = \arg \min_{(\alpha,\rho)\in\Re^2} \sum_{i=1}^{n} (x_i \cos\alpha + y_i \sin\alpha - \rho)^2.$$

There exist only two decomposition possibilities. Taking the decomposition $\vec{a}_1 = (\alpha)$, $\vec{a}_2 = (\rho)$, the alternative formulation (2) of the optimization task turns into:

$$\begin{aligned}\vec{a}_{\text{opt}} &= \arg\min_{\alpha\in\Re}[\min_{\rho\in\Re} \sum_{i=1}^{n} (x_i \cos\alpha + y_i \sin\alpha - \rho)^2] \\ &= \arg\min_{\alpha\in\Re} A(\alpha)\end{aligned}$$

The optimization subtask $A(\alpha)$ is trivially solvable, yielding

$$\rho_{\text{opt}}(\alpha) = \frac{1}{n} \sum_{i=1}^{n} (x_i \cos\alpha + y_i \sin\alpha).$$

Now we are faced with:

$$\vec{a}_{\text{opt}} = \arg\min_{\alpha\in\Re} \sum_{i=1}^{n} (x_i \cos\alpha + y_i \sin\alpha - \rho_{\text{opt}}(\alpha))^2$$

which contains a single parameter α. The optimization term here can be shown to be a sine function and, therefore, the final minimization problem does not cause any difficulty. Note that the example is only used to illustrate the decomposition approach. The 2D line fitting problem represents a special case of general super-plane fitting in arbitrary dimensions based on squared orthogonal distance function. For this purpose solutions are known using eigen computation (see [6] for the case of three dimensions).

4 2D line fitting

The decomposition approach can be shown to be applicable to a variety of geometric fitting problems, including fitting of 2D/3D lines and cylinders. In this section we consider the 2D line fitting problem once more. In contrast to the example in the last section, the sum of orthogonal distances is minimized:

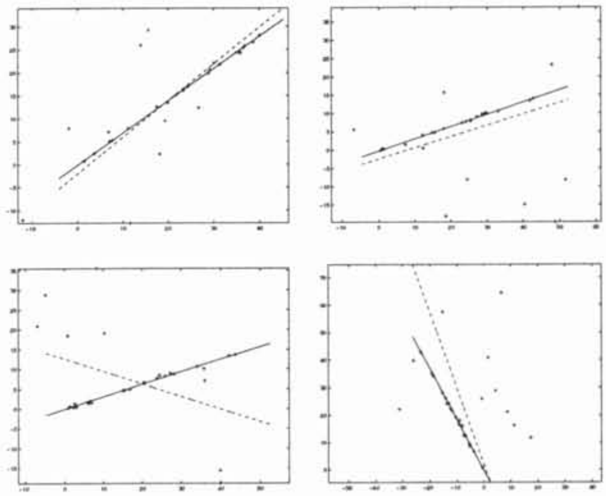

Figure 1: Line fitting by L_1 approximation (decomposition approach, solid line) and L_2 approximation (eigen computation, dashed line).

$$\vec{a}_{\text{opt}} = \arg \min_{(\alpha,\rho)\in\Re^2} \sum_{i=1}^{n} |x_i \cos\alpha + y_i \sin\alpha - \rho|.$$

That is, the L_1-norm is applied here which is well-known to be more robust than the L_2-norm used in the illustrative example when dealing with outliers. The decomposition $\vec{a}_1 = (\alpha)$, $\vec{a}_2 = (\rho)$, results in:

$$\vec{a}_{\text{opt}} = \arg\min_{\alpha\in\Re}[\min_{\rho\in\Re} \sum_{i=1}^{n} |x_i \cos\alpha + y_i \sin\alpha - \rho|].$$

The term $A(\alpha)$ in the brackets is minimal at:

$$\rho_{\text{opt}}(\alpha) = \text{median}_{1\le i\le n}(x_i \cos\alpha + y_i \sin\alpha).$$

Finally, we obtain a one-dimensional optimization problem:

$$\vec{a}_{\text{opt}} = \arg\min_{\alpha\in\Re} \sum_{i=1}^{n} |x_i \cos\alpha + y_i \sin\alpha - \rho_{\text{opt}}(\alpha)|.$$

This final task can be solved using any non-linear least-squares optimization method.

We have conducted a series of experiments to verify that the decomposition approach works well to find the best fit line based on the L_1-norm. All experiments described in this paper were done in MATLAB and non-linear optimization was solved using the Levenberg-Marquardt algorithm. Straight lines of random α values are generated. Thirty points on a straight line are computed and distorted in each dimension by a Gaussian noise of zero mean and standard deviation 0.3. Then, ten randomly

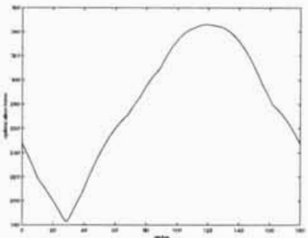

Figure 2: Optimization space of 2D line fitting.

chosen points are displaced in each dimension up to a distance 30. Both the L_1 approximation based on the decomposition approach and the eigen computation for the L_2-norm are implemented. The results for four straight lines are shown in Figure 1. In all cases the L_1 approximation provides precise solutions, while the L_2-norm is very sensitive to outliers. We have recorded the results of thirty random runs of the program. For the parameter α we get an average estimation error 0.45^o at a standard deviation 0.30^o using L_1 approximation. On the other hand, the L_2-norm results in an average estimation error 9.89^o at a standard deviation 6.33^o.

An advantage of the decomposition approach is that the optimization space can be more easily investigated with a smaller number of parameters. In this application we have reduced the original problem of two parameters to the single one α. The optimization space corresponding to the right test instance in the bottom row of Figure 1 is shown in Figure 2. It has a simple shape and a clear global optimum around the ground truth $\alpha = 28.58^o$. An immediate consequence of such an optimization space is that the initialization is not critical at all. In all experiments we have simply initialized α by 0^o.

5 Cylinder fitting

Common quadric surfaces such as cylinders are found in most manufactured parts. A reliable estimation of the parameters of cylinders is therefore an essential requirement in object recognition and reverse engineering. Several researchers have tackled this problem by finding the best quadric surface by means of various distance functions, not necessarily resulting in true cylinders. There exist only very few works on faithful cylinder fitting [2, 5].

A cylinder is represented by a fix point (x_0, y_0, z_0), the directional unit vector $\vec{v}$ on its symmetry axis, and the radius r. The directional unit vector $\vec{v}$ is specified by two angles α and β, where α is the angle between the projection of $\vec{v}$ onto the xy-plane and the x-axis, β is the angle between $\vec{v}$ and the z-axis. Based on the decomposition $\vec{a}_1 = (\alpha, \beta)$, $\vec{a}_2 = (x_0, y_0, z_0, r)$, the fitting task becomes:

$$\vec{a}_{\text{opt}} = \arg\min_{\alpha,\beta}[\min_{x_0,y_0,z_o,r} \sum_{i=1}^{n} d(p_i, f(\alpha, \beta, x_0, y_0, z_0, r))]$$

where $d()$ gives an appropriate distance of a point to the cylinder. An optimal solution $(\alpha_{\text{opt}}, \beta_{\text{opt}})$ means that we obtain a circle when projecting the given data points onto a plane that is orthogonal to $\vec{v}$ and passes through the origin. As a consequence, the optimization subtask $A(\alpha, \beta)$ in the brackets can be interpreted as finding the best circle for the data points after the projection based on (α, β).

Many solutions exist for the simpler circle approximation problem. Of particular interest here are the methods proposed in [7, 8]. Thomas and Chan [8] finds the best circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ of n points $(x_i, y_i), i = 1, 2, \ldots, n$, by minimizing:

$$\sum_{i=1}^{n}[(x_i - x_0)^2 + (y_i - y_0)^2 - r^2]^2$$

for which an analytical solution exists. In [7] the term:

$$\sum_{i=1}^{n}[\frac{(x_i - x_0)^2 + (y_i - y_0)^2 - r^2}{r}]^2$$

is minimized based on eigen computation. Note that the latter distance function is a better approximation of the true Euclidean distance than the first one. We may use either of the circle fitting methods to compute $A(\alpha, \beta)$. Now we have only two parameters α and β instead of six to solve the initial cylinder fitting problem.

For computing $A(\alpha, \beta)$ the original data points p_i are projected onto the auxiliary plane defined above which is spanned by two orthogonal unit vectors:

$$\begin{aligned} \vec{n}_\alpha &= (-\sin\alpha, \cos\alpha, 0) \\ \vec{n}_\beta &= (\cos\alpha\cos\beta, \sin\alpha\cos\beta, -\sin\beta) \end{aligned}$$

obtained by partial derivatives of $\vec{v} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$. This results in the projected data points p_i^*:

$$p_i^* = (p_i \cdot \vec{n}_\alpha, p_i \cdot \vec{n}_\beta).$$

The circle approximation is effectively applied to p_i^*.

We have used the popular range image set acquired by a Technical Arts scanner at the Michigan State University. Images containing cylindrical surfaces are segmented into regions by edge detection [3] and a subsequent contour closure [4]. The cylindrical regions are manually determined and fed to the cylinder fitting program. Figure 3 shows an example of a segmented image. For this image the brightest region is used for cylinder fitting. For the brightest region the radius estimation error is 0.35mm and 0.26mm using the circle fitting method from [7] and [8], respectively. The estimated symmetry axis is $\alpha_{\text{opt}} = 127^o, \beta_{\text{opt}} = 79^o$. The corresponding optimization space is drawn in Figure 4. In both cases there seems to have two areas of high optimality on the left and right side. The reason for this phenomenon is that (α, β) and $(\alpha + \pi, \pi - \beta)$ represent the same symmetry axis. Therefore, besides the area of high optimality around $(\alpha_{\text{opt}}, \beta_{\text{opt}})$ there must be another such area around $\alpha = 307^o$, $\beta = 101^o$, which can be clearly observed in Figure 4. Because

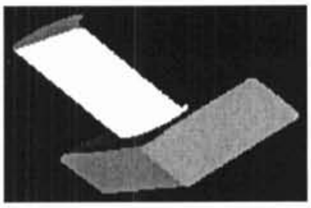

Figure 3: Region segmentation of a range image. The brightest region is used for cylinder fitting.

of the cyclic nature of angles the whole optimization space is a repetition of a local space of simple shape. Accordingly, we generally have no difficulty to initialize the parameters α and β. In our experiments we have simply used 0^o for this purpose.

At this place a comparison of our decomposition approach with other faithful cylinder fitting methods is in order. Flynn and Jain [2] bases their fitting on a cylinder representation of five parameters. It is easy to see that using the circle fitting method from [8] we have minimized exactly the same optimization term as in [2]. Using the circle fitting method from [7], on the other hand, the optimization term which we minimize is exactly identical to that one in [5], where the authors choose a cylinder representation of six parameters. Therefore, we are able to reduce the number of parameters from five respectively six to only two in solving the same optimization problem. This difference is remarkable.

6 Conclusions

Geometric fitting is of fundamental importance to many tasks in computer vision. Many fitting problems encountered here are of non-linear nature and have a large number of parameters. Typically, iterative techniques such as the Levenberg-Marquardt algorithm are used to compute the optimum of highly complex optimization functions. Generally, we are faced with problems of good initial estimates and high risk of getting in local optimums. In this paper we have proposed a decomposition approach to alleviating these problems by reducing the number of optimization parameters. This way not only the optimization process itself is simplified, it tends also easier to investigate low-dimensional optimization spaces. We have applied the decomposition approach to solving different fitting problems from both 2D and 3D domain. Particularly for cylinder fitting, we are able to reduce the number of parameters from five or six of earlier methods to only two. Experiments have demonstrated the advantages of our approach.

The decomposition approach is very general. Currently, we are investigating its application to other geometric fitting problems. In Section 5 we have considered fitting of circular cylinders. Basically, it is easy to extend it to elliptic cylinders. For this purpose we only need to replace the circle fitting by an ellipse fitting, for instance by the eigen computation method from [1]. Another possibility of extension is the fitting of cones or even solids of revolution. Finally, the decomposition is by no means restricted to geometric fitting. It is general enough to be potentially applicable to optimization problems from other domains.

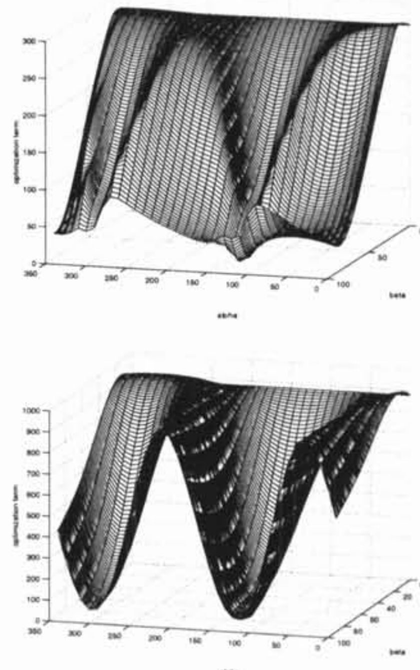

Figure 4: Optimization space of cylinder fitting using circle fitting method from [7] (top) and [8] (bottom).

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